EIGENVALUE DISTRIBUTION OF SECOND-ORDER DYNAMICS EQUATIONS ON TIME SCALES CONSIDERED AS FRACTALS

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Abstract. Let $\mathbb{T} \subset [a, b]$ be a time scale with $a, b \in \mathbb{T}$. In this paper we study the asymptotic distribution of eigenvalues of the following linear problem $-u^{\Delta\Delta} = \lambda u^{\sigma}$, with mixed boundary conditions $\alpha u(a) + \beta u^{\Delta}(a) = 0 = \gamma u(\rho(b)) + \delta u^{\Delta}(\rho(b))$. It is known that there exists a sequence of simple eigenvalues $\{\lambda_k\}_k$; we consider the spectral counting function $N(\lambda, \mathbb{T}) = \#\{k : \lambda_k \leq \lambda\}$, and we seek for its asymptotic expansion as a power of λ. Let d be the Minkowski (or box) dimension of $\mathbb{T}$, which gives the order of growth of the number $K(\mathbb{T}, \varepsilon)$ of intervals of length ε needed to cover $\mathbb{T}$, namely $K(\mathbb{T}, \varepsilon) \approx \varepsilon^d$. We prove an upper bound of $N(\lambda)$ which involves the Minkowski dimension, $N(\lambda, \mathbb{T}) \leq C\lambda^{d/2}$, where C is a positive constant depending only on the Minkowski content of $\mathbb{T}$ (roughly speaking, its d-volume, although the Minkowski content is not a measure). We also consider certain limiting cases ($d = 0$, infinite Minkowski content), and we show a family of self similar fractal sets where $N(\lambda, \mathbb{T})$ admits two-side estimates.

1. Introduction

In this paper we study the following eigenvalue problem:

$$-u^{\Delta\Delta} = \lambda u^{\sigma}, \tag{1.1}$$

in a time scale $\mathbb{T} \subset [a, b]$, with boundary conditions:

$$\alpha u(a) + \beta u^{\Delta}(a) = 0 = \gamma u(\rho(b)) + \delta u^{\Delta}(\rho(b)) \tag{1.2}$$

where $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) \neq 0$. Here, Δ stands for the usual derivative on the time scale $\mathbb{T}$, and σ and ρ are the jump operators

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

assuming that $\inf \emptyset = \sup \mathbb{T}$, and $\sup \emptyset = \inf \mathbb{T}$. We refer the interested reader to [1] for the properties of calculus and differential equations on time scales.

There exists a large and growing literature for eigenvalue problems in time scales, see for example [4], [10], [11], [21]. The existence of a discrete set of eigenvalues for problem (1.1) was proved in [2], and a variational characterization of them in terms of a Rayleigh type quotient was given. Moreover, it is possible to work on Sobolev spaces defined in [3], [28] recovering the usual variational setting on Hilbert spaces (although the problem is not self-adjoint), obtaining several properties of the eigenvalue problem that are well known when $\mathbb{T} = [a, b]$, such as the simplicity of eigenfunctions, the increasing number of zeros of eigenfunctions (that is, u_k has

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k nodal domains, or $k+1$ generalized zeros $x_0 = a < x_1 < \ldots < x_k = b$ where not necessarily $x_i \in \mathbb{T}$), monotonicity with respect to the domain, and comparison principles among several others, see the recent work [13].

Calculus on time scales was introduced by Hilger [20], and a large body of theory unifying and generalizing the theories of difference and differential equations was developed by Agarwal, Anderson, Bohner, Guseinov, Henderson, Peterson among others, see [7], [8] and the references therein.

However, we can expect that the strong differences between the discrete and the continuous calculus, together with other pathological behaviors, must appear somewhere in the theory, depending perhaps on finer details associated to topological properties of $\mathbb{T}$.

In this work we focus on the dimension of $\mathbb{T}$ considered as a fractal set. Let us observe that there are several notions of dimension of sets, introduced by Hausdorff, Minkowski and Bouligand among others, with associated measures or contents. In the last decades, they were incorporated into the theory of fractal sets (see [16], [24]), and now they are widely used to classify the complexity of closed sets $\mathbb{T} \subset [a, b]$. Hence, it is natural to ask if they are related to differential equations defined on time scales $\mathbb{T}$.

Let us note that the eigenvalue problem (1.1) for difference equations (that is, when $\mathbb{T}$ is a finite set) has only finite eigenvalues, and infinitely many when $\mathbb{T} = [a, b]$ (see also examples 3.15 and 3.17 in [13]). Then, by defining the spectral counting function $N(\lambda, \mathbb{T}) = \#\{k : \lambda_k \leq \lambda\}$ and after computing its asymptotic expansion as λ goes to infinity, it is well known that we have for those particular cases the same asymptotic expansion for $N(\lambda, \mathbb{T})$:

$$N(\lambda, \mathbb{T}) = c\lambda^{d(\mathbb{T})/2} + O(\lambda^{d(\partial\mathbb{T})/2}),$$

where the parameters involved in the formula reveal some geometric properties of $\mathbb{T}$. That is, $d(\mathbb{T})$ is the (topological) dimension of the time scale (which is 0 for finite points, and 1 for $\mathbb{T} = [a, b]$; $d(\partial\mathbb{T})$ is the dimension of the boundary of $\mathbb{T}$, which is 0 in both cases; and c is a positive constant depending only on d and $|\mathbb{T}|_d$, the $d-$dimensional measure of the time scale (the cardinal measure of $\mathbb{T}$ for $d = 0$, the Lebesgue measure of $\mathbb{T}$ for $d = 1$), see for example [12]. Moreover, 2 is the order of the operator (for problems involving the one dimensional p-laplacian, $-(|u'|^{p-2}u')'$, it is replaced by p, see [17], [18]). This formula was obtained first by H. Weyl, and it was generalized for the laplacian operator Δ on domains $\Omega \subset \mathbb{R}^N$. For the case of an open set Ω with irregular boundary $\partial\Omega$ it is known that the Minkowski dimension $d(\partial\Omega)$ appears on the bounds of the remainder term $R(\lambda, \Omega) = N(\lambda, \Omega) - c\lambda^{N/2}$. This fact has been firstly observed in [9] and proved in [23] (see also the references therein).

We conjecture that a similar formula holds in the context of time scales, which in certain sense correspond to a transition from the discrete to the continuous case. Following the classical and beautiful work of Kac [22], we are asking if we can hear the dimension of the time scale $\mathbb{T}$. This work is the first step towards its proof, and we will show that there exists an upper bound for the eigenvalue counting function depending on topological properties of $\mathbb{T}$, namely its fractal Minkowski or box dimension d and its Minkowski content $M_d(\mathbb{T})$ (see Section §2 for the definitions and auxiliary tools).

Our main result is the following:

Theorem 1.1. *Let $d \in (0,1]$ be the Minkowski dimension of $\mathbb{T}$, and let $M_d^*(\mathbb{T}) < \infty$ be the upper Minkowski content of $\mathbb{T}$. Let $\{\lambda_n\}_n$ be the sequence of eigenvalues of problem* (1.1) *with Dirichlet or Neumann boundary conditions. Then, there exists a positive constant C depending only on $M_d^*(\mathbb{T})$ such that*

$$N(\lambda, \mathbb{T}) \leq C\lambda^{d/2}.$$

Indeed, Theorem 1.1 is valid for different boundary conditions like (1.2). We will write $N_D(\lambda, \mathbb{T})$ or $N_N(\lambda, \mathbb{T})$ whenever we need to stress that they are the eigenvalue counting functions of problem (1.1) with Dirichlet or Neumann boundary condition respectively.

Usually, it is difficult to obtain an upper bound for $N(\lambda)$, since it is equivalent to obtain lower bounds of eigenvalues. However, we have the following Lyapunov inequality obtained in [19]

Theorem 1.2 (Theorem 1.1 of [19])**.** *Suppose that $q > 0$ and*

$$\left[\int_a^{\sigma^2(b)} \Delta t\right] \int_a^{\sigma(b)} q(t)\Delta t \leq 4.$$

Then, $u^{\Delta\Delta} + q(t)u^{\sigma} = 0$ is disconjugate on $[a, \sigma^2(b)]$.

The Lyapunov inequality is an useful tool in eigenvalue problems, see for example [14] [25], [26]; for time scales was proved in [6], although generalized zeros were not considered there, see also [5]. Clearly, replacing q by any eigenvalue λ_k, we obtain the bound

$$\frac{4}{(\int_a^{\sigma^2(b)} \Delta t)(\int_a^{\sigma(b)} \Delta t)} \leq \lambda_k, \tag{1.3}$$

since the equation is not disconjugate (the associated eigenfunction has at least two generalized zeros). We will use a slightly different version as a lower bound of the fifth eigenvalue of a Neumann problem, namely

$$\frac{4}{(b-a)^2} \leq \lambda_5$$

Clearly, this bound is far from being optimal, since it is the Lyapunov inequality when $\mathbb{T} = [a, b]$ and gives a lower bound for the first Dirichlet eigenvalue. Surprisingly, this 'bad' approximation (1.3) will be enough for our purposes, and it is our main tool in order to prove Theorem 1.1 together with a generalization of the Dirichlet-Neumann bracketing of Courant [12]:

Theorem 1.3. *Let $\mathbb{T}$ be a time scale in $[a, b]$, and let us consider $\mathbb{T}_1 = \mathbb{T} \cap [a, c]$ and $\mathbb{T}_2 = \mathbb{T} \cap [\sigma(c), b]$ for any $c \in (a, b)$. Then,*

$$N_D(\lambda, \mathbb{T}_1 \cup \mathbb{T}_2) \leq N_D(\lambda, \mathbb{T}) \leq N_N(\lambda, \mathbb{T}) \leq N_N(\lambda, \mathbb{T}_1 \cup \mathbb{T}_2).$$

Moreover,

$$N(\lambda, \mathbb{T}_1 \cup \mathbb{T}_2) \approx N(\lambda, \mathbb{T}_1) + N(\lambda, \mathbb{T}_2).$$

We will sketch its proof in Section §2.

Let us note that in Theorem 1.1 we have excluded the case $d = 0$. When $\mathbb{T}$ is a finite set of points, we have finitely many eigenvalues, and the limiting cases of the previous theorems suggest that whenever we increase the number of points, new eigenvalues enter from infinity, which is compatible with the fact that the eigenvalues of a finite difference approximation of a differential equation approach the lowest eigenvalues of the continuous problem. However, when $\mathbb{T}$ is not finite but still $d(\mathbb{T}) = 0$, we may have a sequence of eigenvalues $\{\lambda_n\}$ going to infinity faster than any power of n. We can expect a nonclassical asymptotic behavior in this situation, and further work will be needed in order to settle completely this case.

Let us remember the $o-$ notation, $f(\lambda) = o(g(\lambda))$ means that $f/g \to 0$ when $\lambda \to \infty$. We have the following weaker results:

Theorem 1.4. *Let $d = 0$ be the Minkowski dimension of $\mathbb{T}$. Then, for all $\delta > 0$ we have*

$$N(\lambda) = o(\lambda^{\delta/2})$$

when $\lambda \to \infty$.

Moreover, another special case occurs when the upper Minkowski content of $\mathbb{T}$ is not finite. In this case we have:

Theorem 1.5. *Let $d \in (0, 1)$ be the Minkowski dimension of $\mathbb{T}$, and $M_d^*(\mathbb{T}) = \infty$. Then, for all $\delta > 0$ we have*

$$N(\lambda) = o(\lambda^{d/2+\delta})$$

when $\lambda \to \infty$.

Finally, for certain self similar fractals, like the ternary Cantor set $\mathcal{C}$, it is possible to find two-side estimates for $N(\lambda, \mathcal{C})$ and the eigenvalues. We will show that in this case we have, for any $\varepsilon > 0$ arbitrarily small,

$$c_1 \lambda^{2\ln(3)/\ln(2)-\varepsilon} \le N(\lambda, \mathcal{C}) \le c_2 \lambda^{2\ln(3)/\ln(2)},$$

where the constants c_1 and c_2 depend only on $d = \ln(2)/\ln(3)$ and the first Dirichlet eigenvalue of $\mathcal{C}$. This example can be easily generalized to other self similar sets.

The paper is organized as follows. In Section §2 we will introduce the necessary definitions and some auxiliary results, and we will prove Theorem 1.3. In Section §3 we will prove Theorems 1.1 and 1.5. Section §4 is devoted to an example of two-side estimates for self similar fractal sets.

2. Preliminary results

2.1. **Minkowski dimension and content.** Given $A \subset \mathbb{R}$, we denote the tubular neighborhood of radius ε as A_ε, i. e.,

$$A_\varepsilon = \{x \in \mathbb{R} : \text{dist}(x, A) \le \varepsilon\},$$

and $|A|_1$ its Lebesgue measure.

We define the Minkowski dimension of $\mathbb{T}$ as

$$d = \dim(T) = \inf\{\delta \ge 0 \ : \ \lim_{\varepsilon \to 0^+} \varepsilon^{-(1-\delta)} |\mathbb{T}_\varepsilon|_1 = 0\}.$$

We define the Minkowski content of $\mathbb{T}$ as the limit (whenever it exists):

$$M(\mathbb{T}, d) = \lim_{\varepsilon \to 0^+} \varepsilon^{-(1-d)} |\mathbb{T}_\varepsilon|_1, \tag{2.1}$$

and in that case, we will say that $\mathbb{T}$ is d-Minkowski measurable (despite the fact that the Minkowski content is not a measure, since it is not σ-additive).

When $\mathbb{T}$ is not Minkowsi measurable we can still define $M^*(\mathbb{T},d)$ (resp., $M_*(\mathbb{T},d)$), the $d-$dimensional upper (resp., lower) Minkowski content, replacing the limit in (2.1) by an upper (resp., lower) limit.

Sometimes it is convenient to use an equivalent characterization of the Minkowski dimension in terms of coverings:

Proposition 2.1. *Let $K(\mathbb{T},\varepsilon)$ be the minimal number of disjoint intervals of length ε which are needed to cover $\mathbb{T}$. Then,*

$$d = \dim(\mathbb{T}) = \inf\left\{\delta \geq 0 \ : \limsup_{\varepsilon\to 0^+} \varepsilon^\delta K(\mathbb{T},\varepsilon) = 0\right\},$$

and the upper Minkowski content is

$$M_d^*(\mathbb{T}) = \limsup_{\varepsilon\to 0^+} \varepsilon^d K(\mathbb{T},\varepsilon).$$

Namely, in order to compute the Minkowski content of a set, it is enough to cover a set with boxes of diameter ε and to count how many of them intersect the set. The Minkowski dimension is obtained in this context as

$$d = \dim(\mathbb{T}) = \lim_{\varepsilon\to 0^+} \frac{\ln(K(\mathbb{T},\varepsilon))}{\ln(1/\varepsilon)},$$

whenever the limit exists. This characterization is very useful from a computational point of view, and the name of box dimension follows from it. We refer the reader to [16] for the proof and other properties of the Minkowski dimension and content, see also [15] for some criteria about Minkowski measurability.

2.2. **Auxiliary Results.** The results of this subsection are gathered from [2] and [3]; we include it without proofs for the sake of completeness.

Given a time scale $\mathbb{T}$ and any interval $[a,b]$ with $a,b\in\mathbb{T}$, we will call $J=[a,b]\cap\mathbb{T}$, and $J^0=[a,b)\cap\mathbb{T}$. Also, J^{k^j} is defined as

$$J^{k^j} = [a,\rho^j(b)]\cap\mathbb{T}.$$

We will say that u belongs to the Sobolev space $W_\Delta^{1,2}(J)$ if and only if $u\in L_\Delta^2(J^0)$ and there exists $g: J^1\to\mathbb{R}$ such that $g\in L_\Delta^2(J^0)$ and

$$\int_{J^0} (u\cdot\varphi^\Delta)(s)\Delta s = -\int_{J^0}(g\cdot\varphi^\sigma)(s)\Delta s$$

for all $\varphi\in C_{0,rd}^1(J^1)$ where

$$C_{0,rd}^1(J^1) = \{f : J\to\mathbb{R} : f\in C_{rd}^1(J^1), f(a)=0=f(b)\}.$$

We will need also the Sobolev space $W_{0,\Delta}^{1,2}(J)$, defined as the completion of $C_{0,rd}^1(J^1)$ in $W_\Delta^{1,2}(J)$ with the norm $\|.\|_{W_\Delta^{1,2}(J)}$ given by

$$\|u\|_{W_\Delta^{1,2}(J)} = \|u\|_{L_\Delta^2} + \|u^\Delta\|_{L_\Delta^2}.$$

Concerning the existence of eigenvalues of problem (1.1), we have the following theorem (see also [13]):

Theorem 2.2 (Theorem 1 in [2])**.** *The eigenvalues of problem* (1.1) *may be arranged as $-\infty<\lambda_1<\lambda_2<\lambda_3<\dots$, and the eigenfunction corresponding to λ_{k+1} has exactly k generalized zeros in the open interval (a,b).*

We define the Rayleigh quotient

$$R(u) = -\frac{\int_a^{\rho(b)} u^{\Delta\Delta} u^\sigma \ \Delta t}{\int_a^{\rho(b)} |u^\sigma|^2 \ \Delta t},$$

and let us remember that the Dirichlet eigenvalues are obtained as follows:

$$\lambda_{k+1} = \min_{u \in W_0^{1,2}([a,b]), u \perp \{\varphi_1, \dots \varphi_k\}} R(u)$$

where $\varphi_1, \dots \varphi_k$ are the first k eigenfunctions (see Theorem 2 in [2], Theorem 3.10 in [13]).

The Neumann problem can be studied in a similar way, considering now the space $W^{1,2}([a,b])$. As usual, we may introduce intermediate spaces to handle different boundary conditions (1.2), although we will need only these two spaces.

2.3. **Dirichlet Neumann Bracketing.** This section is devoted to the proof of the so called Dirichlet-Neumann bracketing method in Theorem 1.3.

Given a time scale $\mathbb{T}$ in $[a,b]$, we consider $\mathbb{T}_1 = \mathbb{T} \cap [a,c]$ and $\mathbb{T}_2 = \mathbb{T} \cap [\sigma(c), b]$ for any $c \in (a,b) \cap \mathbb{T}$. When c is right scattered, we have $\sigma(c) = c$.

We may consider the Sobolev spaces $W_\Delta^{1,2}([a,c])$, $W_\Delta^{1,2}([\sigma(c), b])$, and let us note that we have a continuous restriction operator

$$P : W_\Delta^{1,2}([a,b]) \to W_\Delta^{1,2}([a,c]) \oplus W_\Delta^{1,2}([\sigma(c), b]),$$

namely,

$$P(u) = (u|_{[a,c]}, u|_{[\sigma(c),b]}).$$

On the other hand, given $(u_1, u_2) \in W_{0,\Delta}^{1,2}([a,c]) \oplus W_{0,\Delta}^{1,2}([\sigma(c), b])$, we have the extension operator

$$E : W_{0,\Delta}^{1,2}([a,c]) \oplus W_{0,\Delta}^{1,2}([\sigma(c), b]) \to W_{0,\Delta}^{1,2}([a,b]),$$

which is well defined since $u_1(c) = 0 = u_2(\sigma(c))$.

Both operators, P and E define strict inclusions, and enable us to write

$$W_{0,\Delta}^{1,2}([a,c]) \oplus W_{0,\Delta}^{1,2}([\sigma(c), b]) \subset W_{0,\Delta}^{1,2}([a,b]) \subset$$
$$\subset W_\Delta^{1,2}([a,b]) \subset W_\Delta^{1,2}([a,c]) \oplus W_\Delta^{1,2}([\sigma(c), b]).$$

Then, it is clear that by minimizing the Rayleigh quotient on each space from the left to the right, the first eigenvalue not increases. Indeed, this is also true for all the eigenvalues, and the k^{th} eigenvalue of the problem in $W_{0,\Delta}^{1,2}([a,c]) \oplus W_{0,\Delta}^{1,2}([\sigma(c), b])$ is greater than k^{th} eigenvalue in $W_{0,\Delta}^{1,2}([a,b])$, and so on. This fact follows by an equivalent characterization of eigenvalues, namely

$$\lambda_k = \inf_{L_k \subset W} \sup_{u \in L_k} R(u)$$

where L_k runs over all the k-dimensional subspaces of a given space W. We omit the proof of this fact, which is the same as in the continuous case.

Hence, we can prove Theorem 1.3 for the Dirichlet (resp., Neumann) eigenvalue problem in $W_{0,\Delta}^{1,2}([a,c]) \oplus W_{0,\Delta}^{1,2}([\sigma(c), b])$ (resp., $W_\Delta^{1,2}([a,c]) \oplus W_\Delta^{1,2}([\sigma(c), b])$) as in Proposition 2.4 of [18], since a simple argument with test functions shows that the eigenvalues correspond to the ones of the same equation in each interval separately.

So, we have

$$N(\lambda, \mathbb{T}) \approx N(\lambda, \mathbb{T}_1) + N(\lambda, \mathbb{T}_2),$$

and

$$N_D(\lambda, \mathbb{T}_1 \cup \mathbb{T}_2) \le N_D(\lambda, \mathbb{T}) \le N_N(\lambda, \mathbb{T}) \le N_N(\lambda, \mathbb{T}_1 \cup \mathbb{T}_2),$$

and the Theorem is proved.

Remark 2.3. There exist an interpretation of this result in terms of the Sturm-Liouville oscillation theory (and an alternative proof). For a fixed λ, let λ_k be the greatest eigenvalue lower or equal than λ. Since the k^{th} eigenfunction has $k-1$ generalized zeros, and given a partition of $\mathbb{T}$ as before, we will have j and $k-j$ zeros in each subinterval (perhaps one of the generalized zeros belongs to the gap between c and $\sigma(c)$, but we can disregard it when k -and λ- goes to infinity). Hence, counting j zeros in $\mathbb{T}_1$ and $k-j$ in $\mathbb{T}_2$ is closely related to the existence of j (resp., $k-j$) eigenvalues on $\mathbb{T}_1$ (resp., $\mathbb{T}_2$) lower than λ. For a detailed analysis of this argument for a singular ordinary differential equation on $[0,\infty)$ see [27].

Remark 2.4. Let us note that this reduces our problem to estimate the eigenvalues on disjoint subintervals of $\mathbb{T}$. Moreover, we can divide $\mathbb{T}$ in any finite number of subintervals and the same result follows by induction.

3. Proof of the Main Theorems

In this section we will prove Theorems 1.1, 1.4 and 1.5.

3.1. **Proof of Theorem 1.1.** For simplicity, we will divide the proof in several parts. Our first task is to find a lower bound for the fifth Neumann eigenvalue.

Proposition 3.1. *Let λ_5 be the the fifth Neumann eigenvalue of problem* (1.1) *on $[a,b]\cap\mathbb{T}$, i.e., satisfying the boundary conditions* (1.2) *with $\alpha=\gamma=0$ and $\beta=\delta=1$. Then,*

$$\lambda_5 > \frac{4}{(b-a)^2}.$$

Proof. Let φ_5 be a Neumann eigenfunction corresponding to λ_5. We know that φ_5 has four generalized zeros on the open interval (a,b) from Theorem 2.2.

Hence, we will consider the following cases:

- b is left dense: we choose any point $\hat{b}\in\mathbb{T}$ between the fourth generalized zero and b.
- b is left scattered: we consider now the point $b_1=\rho(b)$, and we have again two cases:
  - b_1 is left dense: we choose any point $\hat{b}\in\mathbb{T}$ between the third generalized zero and b_1 (let us observe that the fourth generalized zero can be b_1).
  - b_1 is left scattered: we consider now the point $b_2=\rho(b_1)$, and we have again two cases:
    * b_2 is left dense: we choose any point $\hat{b}\in\mathbb{T}$ between the second generalized zero and b_2.
    * b_2 is left scattered: we choose $\hat{b}=\rho(b_2)$.

Hence, we always have that φ_5 satisfies $\varphi_5^{\Delta\Delta}+\lambda_5\varphi_5^{\sigma}=0$ on $[a,\hat{b}]$, and $\sigma^2(\hat{b})\le b$. From Theorem 1.2, we know that

$$\left[\int_a^{\sigma^2(\hat{b})}\Delta t\right]\int_a^{\sigma(\hat{b})}\lambda_5\Delta t\le 4.$$

implies that $\varphi_5^{\Delta\Delta}+\lambda_5\varphi_5^{\sigma}=0$ is disconjugate on $[a,\sigma^2(\hat{b})]$. However, by the previous construction we have at least two generalized zeros on $(a,\hat{b})$, and φ_5 cannot be disconjugate on this interval. Hence, we have

$$\left[\int_a^{\sigma^2(\hat{b})}\Delta t\right]\int_a^{\sigma(\hat{b})}\lambda_5\Delta t>4.$$

Since $\sigma^2(\hat{b})\le b$, we have the desired inequality, and the proof is complete. $\square$

Remark 3.2. Let us note that we need at least five points on $[a,b]\cap\mathbb{T}$. However, with four or less points we cannot have more than four eigenvalues, since in this case the problem is reduced to a discrete one and they correspond to the ones of a matrix at most in $\mathbb{R}^{4\times 4}$.

Our next result gives an upper bound for the number of intervals covering $\mathbb{T}$ given a sufficiently small length ε.

Proposition 3.3. *Let $d\in(0,1]$ be the Minkowski dimension of $\mathbb{T}$, and $M_d^*(\mathbb{T})<\infty$ be its upper Minkowski content of $\mathbb{T}$. Let $K(\mathbb{T},\varepsilon)$ be the number of disjoint intervals of length ε which are needed to cover $\mathbb{T}$. Then, given $\delta>0$, there exists a positive ε_0 such that, for any $\varepsilon<\varepsilon_0$,*

$$K(\mathbb{T},\varepsilon)\le\varepsilon^{-d}(M_d^*(\mathbb{T})+\delta).$$

The proof follows immediately from the characterization of the Minkowski dimension and content given in Proposition 2.1.

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. From the previous proposition, fix ε_0 such that

$$K(\mathbb{T},\varepsilon)\le\varepsilon^{-d}(M_d^*(\mathbb{T})+1)$$

for any covering with intervals of length $\varepsilon<\varepsilon_0$.

Next, choose any value of λ satisfying $2/\lambda^{1/2}<\varepsilon_0$, and calling $\varepsilon=2/\lambda^{1/2}$, let us cover $[a,b]\cap\mathbb{T}$ with intervals of length ε:

$$I_1=[a,a+\varepsilon];$$
$$I_2=[\sigma(a+\varepsilon),\sigma(a+\varepsilon)+\varepsilon];$$
$$I_3=[\sigma(\sigma(a+\varepsilon)+\varepsilon),...];...$$

Hence, we have a family $\{I_j\}_{1\le j\le K(\mathbb{T},\varepsilon)}$ and by Proposition 3.3, $K(\mathbb{T},\varepsilon)$ is bounded by above by $\varepsilon^{-d}(M_d^*(\mathbb{T})+1)$.

Now, by using the covering $\{I_j\}_{1\le j\le K(\mathbb{T},\varepsilon)}$, we can use the Dirichlet Neuman bracketing given in Theorem 1.3. We have

$$N(\lambda,\mathbb{T})\le\sum_{1\le j\le K(\mathbb{T},\varepsilon)}N_N(\lambda,I_j\cap\mathbb{T}).$$

We bound the number of Neumann eigenvalues in each time scale $I_j\cap\mathbb{T}$ by using Proposition 3.1. Let us call $a=\inf\{t\in I_j\cap\mathbb{T}\}$, and $b=\sup\{t\in I_j\cap\mathbb{T}\}$. If $I_j\cap\mathbb{T}$ has three points or less it is clear that $N_N(\lambda,I_j\cap\mathbb{T})\le 4$, since the fifth eigenfunction has four generalized zeros. If $I_j\cap\mathbb{T}$ has more than three points, the fifth Neumann eigenvalue is greater than λ (since the equation is disconjugate in $[a,\sigma^2(\hat{b})]$, and $\sigma^2(\hat{b})\le b$). In both cases, we obtain

$$N_N(\lambda,I_j\cap\mathbb{T})\le 4$$

for $1 \le j \le K(\mathbb{T}, \varepsilon)$. That is,

$$N(\lambda, \mathbb{T}) \le 4K(\mathbb{T}, \varepsilon).$$

Since $K(\mathbb{T}, \varepsilon) \le \varepsilon^{-d}(M_d^*(\mathbb{T}) + 1) = 2^{-d}\lambda^{d/2}(M_d^*(\mathbb{T}) + 1)$, we obtain the upper bound

$$N(\lambda, \mathbb{T}) \le (M_d^*(\mathbb{T}) + 1)2^{2-d}\lambda^{d/2},$$

and the Theorem is proved. □

Corollary 3.4. *Let $\mathbb{T}$ be a time scale as before. The eigenvalue counting function $N(\lambda, \mathbb{T})$ of problem* (1.1) *with boundary conditions* (1.2) *satisfies*

$$N(\lambda, \mathbb{T}) \le C\lambda^{d/2}.$$

Proof. Let us observe that any intermediate space $V_\Delta^{1,2}$ of functions satisfying the boundary conditions (1.1) is a subspace of $W_\Delta^{1,2}$, and we can apply the Dirichlet Neumann bracketing exactly as before. □

Corollary 3.5. *Let $\mathbb{T}$ be a time scale as before. Then,*

$$cn^{2/d} \le \lambda_n,$$

where c is a positive constant.

Proof. It follows immediately from the fact that $n = N(\lambda_n, \mathbb{T}) \le C\lambda_n^{d/2}$. □

Remark 3.6. Let us note that the constant $M_d^*(\mathbb{T}) + 1$ can be refined to $M_d^*(\mathbb{T})$, taking δ arbitrarily small. Also, when $\mathbb{T}$ is Minkowski measurable, we can replace the upper content by the content $M_d(\mathbb{T})$, following the ideas below in the proofs of Theorems 1.4 and 1.5.

Remark 3.7. In other words, we can hear the Minkowski dimension of $\mathbb{T}$ from this upper bound for $N(\lambda, \mathbb{T})$. The order of growth of $N(\lambda, \mathbb{T})$ is known to be optimal for $d = 1$, and we will show in Section §4 below that the same is true for self similar Cantor sets. However, we cannot hear the Minkowski content due to the presence of the factor 2^{d-2}, which it is known that is not the correct one for $d = 1$.

3.2. **Proof of Theorem 1.4.** We wish to show that, given any $\delta > 0$, we have

$$\lim_{\lambda\to\infty} \frac{N(\lambda, \mathbb{T})}{\lambda^{\delta/2}} = 0,$$

that is, we wish to show that for any given positive small constant c, there exists λ_c such that

$$\frac{N(\lambda, \mathbb{T})}{\lambda^{\delta/2}} \le c$$

for $\lambda \ge \lambda_c$.

Hence, let us fix an arbitrarily small value c. Since the Minkowski dimension of $\mathbb{T}$ is zero, and

$$d = \dim(\mathbb{T}) = \inf\left\{\delta \ge 0 \ : \limsup_{\varepsilon\to 0^+} \varepsilon^\delta K(\mathbb{T}, \varepsilon) = 0\right\},$$

where ε is the length of the intervals which cover $\mathbb{T}$, given $\delta > 0$ there exists a critical length ε_0 such that $\varepsilon^\delta K(\mathbb{T}, \varepsilon) < c/4$ for any $\varepsilon < \varepsilon_0$.

Now, we determine λ_0 in much the same way as in Theorem 1.1. We choose as λ_0 the value $4\varepsilon_0^{-2}$. So, for any $\lambda > \lambda_0$, by using a covering with intervals of length $\varepsilon = 2/\lambda^{1/2} < \varepsilon_0$, we have at most four eigenvalues in each interval.

Therefore,

$$N(\lambda,\mathbb{T}) \leq \sum_{1\leq j\leq K(\mathbb{T},\varepsilon)} N_N(\lambda, I_j\cap \mathbb{T}) \leq 4K(\mathbb{T},\varepsilon) \leq c\varepsilon^{-\delta} \leq c2^{-d}\lambda^{\delta/2},$$

which gives

$$\frac{N(\lambda,\mathbb{T})}{\lambda^{\delta/2}} \leq \frac{c2^{-d}\lambda^{\delta/2}}{\lambda^{\delta/2}} = c2^{-d} < c$$

and the proof is finished.

3.3. **Proof of Theorem 1.5.** In order to prove this theorem, we only need to note that if d is the Minkowski dimension of $\mathbb{T}$, for all $\delta > 0$ we have

$$\limsup_{\varepsilon\to 0^+} \varepsilon^{d+\delta} K(\mathbb{T},\varepsilon) = 0,$$

and the proof runs exactly as the previous one.

4. An Example of Two-Side Estimates

Let $\mathcal{C}$ be the ternary Cantor set, i.e., the invariant set on $[0,1]$ of the transformations $f_1(x) = x/3$, $f_2(x) = x/3 + 2/3$. Its Minkowski dimension is $d = \ln(2)/\ln(3)$.

Let us call $\mu = \lambda_1$ the first Dirichlet eigenvalue of problem (1.1) when $\mathbb{T} = \mathcal{C}$, and φ the corresponding eigenfunction. Clearly, the second eigenvalue can be bounded above by $3^2\mu$, since the function

$$\psi(x) = \begin{cases} \varphi(3x) & x \in f_1(\mathcal{C}) \\ -\varphi(3x) & x\in f_2(\mathcal{C}) \end{cases}$$

belongs to $W_0^{1,2}(\mathcal{C})$ and is an admissible function for the variational characterization of λ_2.

By Theorem 1.1, we have $N(\lambda,\mathcal{C}) \leq C\lambda^{d/2}$. However, we will derive this upper bound again in a simpler way. Our main objective is to find a similar bound from below for $N(\lambda,\mathcal{C})$.

Theorem 4.1. *Given $\varepsilon > 0$, there exist λ_0 and positive constants c_1, c_2 such that*

$$c_2\lambda^{d/2-\varepsilon} \leq N(\lambda,\mathcal{C}) \leq c_1\lambda^{d/2}$$

for any $\lambda \geq \lambda_0$.

Proof. Given $\varepsilon > 0$, choose a number $K_0 \in \mathbb{N}$ big enough such that

$$\frac{\ln(2)}{\ln(3)} - 2\varepsilon \leq \frac{\ln(2^K)}{\ln(3+3^K)} \leq \frac{\ln(2^K)}{\ln(3^K)} = \frac{\ln(2)}{\ln(3)}$$

for any $K \geq K_0$.

Now, fix a value of λ and K such that $\mu 3^{2K} \leq \lambda \leq \mu 3^{2(K+1)}$ with $K \geq K_0$; and let us cover $\mathcal{C}$ with 2^K disjoint intervals of length 3^{-K}.

We observe that the intersection of any of these intervals with $\mathcal{C}$ can be obtained by applying K times the functions f_1, f_2 to $\mathcal{C}$. In particular, each of them is a translation of the scaled set

$$3^{-K}\mathcal{C} = f_1^{(K)}(\mathcal{C}) = f_1\circ ...\circ f_1(\mathcal{C}) \quad (K \text{ times}).$$

From the Dirichlet Neumann bracketing we obtain the following estimates:

$$2^K N(\lambda, f_1^{(K)}(\mathcal{C})) \leq N(\lambda,\mathcal{C}) \leq 2^{K+1}N(\lambda, f_1^{(K+1)}(\mathcal{C}))$$

Let us observe that $\mu 3^{2K}$ is the first eigenvalue on $f_1^{(K)}(\mathcal{C})$. Since $\mu 3^{2K} \leq \lambda \leq \mu 3^{2(K+1)}$, we have

$$N(\lambda, f_1^{(K+1)}(\mathcal{C})) \leq 1 \leq N(\lambda, f_1^{(K)}(\mathcal{C})),$$

that is,

$$2^K \leq N(\lambda, \mathcal{C}) \leq 2^{K+1}$$

By the characterization of Minkowski dimension in terms of coverings,

$$\frac{\ln(2)}{\ln(3)} = \dim(\mathcal{C}) = \lim_{\varepsilon \to 0^+} \frac{\ln(K(\mathcal{C}, \varepsilon))}{\ln(1/\varepsilon)},$$

and given ε, K_0 we have

$$\frac{\ln(2)}{\ln(3)} - 2\varepsilon \leq \frac{\ln(2^{K-1})}{\ln(3^K)} \leq \frac{\ln(2^K)}{\ln(3^K)} = \frac{\ln(2)}{\ln(3)}$$

for $K \geq K_0$.

By using that

$$3^K \leq (\lambda/\mu)^{1/2} \leq 3^{K+1}$$

we obtain

$$\ln(3^K) \leq \ln(\lambda/\mu)^{1/2} \leq \ln(3) + \ln(3^K)$$

$$\frac{\ln(2^K)}{\ln(3^K)} \geq \frac{\ln(2^K)}{\ln(\lambda/\mu)^{1/2}} \geq \frac{\ln(2^K)}{\ln(3) + \ln(3^K)}$$

and then

$$\frac{\ln(2)}{\ln(3)} \geq \frac{\ln(2^K)}{\ln(\lambda/\mu)^{1/2}} \geq \frac{\ln(2)}{\ln(3)} - 2\varepsilon.$$

Since $d = \ln(2)/\ln(3)$, we have the following inequality:

$$\ln(\lambda/\mu)^{d/2} \geq \ln(2^K) \geq \ln(\lambda/\mu)^{d/2-\varepsilon},$$

and finally,

$$(\lambda/\mu)^{d/2} \geq 2^K \geq (\lambda/\mu)^{d/2-\varepsilon},$$

which implies, for $N(\lambda)$,

$$\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^{d/2-\varepsilon} \leq N(\lambda, \mathcal{C}) \leq \left(\frac{\lambda}{\mu}\right)^{d/2}$$

and so completes the proof. □

Remark 4.2. Let us note that the following inequality

$$0 < c \leq \liminf_{n\to\infty} \frac{\lambda_n}{n^{2\ln(3)/\ln(2)}} \leq \limsup_{n\to\infty} \frac{\lambda_n}{n^{2\ln(3)/\ln(2)-\delta}} \leq C < \infty$$

holds, where δ depends only on ε, since

$$\frac{1}{2}\left(\frac{\lambda_n}{\mu}\right)^{d/2-\varepsilon} \leq N(\lambda_n, \mathcal{C}) = n \leq \left(\frac{\lambda_n}{\mu}\right)^{d/2}.$$

Remark 4.3. Let us observe that this proof does not give the exact order of growth of $N(\lambda)$, since we can have a nonclassical asymptotic behavior like $\lambda^{d/2}/\ln(\lambda)$. However, it is possible to read off the fractal dimension of $\mathcal{C}$ from the asymptotic expansion of $N(\lambda)$.

Remark 4.4. This example can be generalized to other Cantor sets $\mathcal{C}_{m,n}$ defined as the complement on $[0, m.n(n-m)^{-1}]$ of $\cup_k \Omega_k$, where Ω_k consist of m^k intervals of lengths n^{1-k} where $m < n$.

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