EIGENVALUE HOMOGENIZATION FOR QUASILINEAR ELLIPTIC OPERATORS

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Abstract. In this work we study the homogenization problem for (nonlinear) eigenvalues of quasilinear elliptic operators. We prove convergence of the first and second eigenvalues and, in the case where the operator is independent of ε, convergence of the full (variational) spectrum together with an explicit order of convergence in k and in ε.

1. Introduction

In this paper we study the asymptotic behavior (as $\varepsilon \to 0$) of the eigenvalues of the following problems

$$\begin{cases} -div(a_\varepsilon(x, \nabla u^\varepsilon)) = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ε is a positive real number, and λ^ε is the eigenvalue parameter.

The weight functions $\rho_\varepsilon(x)$ are assumed to be positive and uniformly bounded away from zero and infinity and the family of operators $a_\varepsilon(x, \xi)$ have precise hypotheses that are stated below, but the prototypical example is

$$-div(a_\varepsilon(x, \nabla u^\varepsilon)) = -div(A^\varepsilon(x)|\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon), \tag{1.2}$$

with $1 < p < +\infty$, and $A^\varepsilon(x)$ is a family of uniformly elliptic matrices (both in $x \in \Omega$ and in $\varepsilon > 0$).

The study of this type of problems have a long history due to its relevance in different fields of applications. The problem of finding the asymptotic behavior of the eigenvalues of (1.1) is an important part of what is called *Homogenization Theory*. Homogenization Theory is applied in composite materials in which the physical parameters such as conductivity and elasticity are oscillating. Homogenization Theory try to get a good approximation of the macroscopic behavior of the heterogeneous material by letting the parameter $\varepsilon \to 0$. The main references for the homogenization theory of periodic structures are the books by Bensoussan-Lions-Papanicolaou [6], Sanchez–Palencia [26], Oleĭnik-Shamaev-Yosifian [24] among others.

In the linear setting (i.e., $a_\varepsilon(x, \xi)$ as in (1.2) with $p = 2$) this problem is well understood. It is known that, up to a subsequence, there exists a limit operator

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$a_h(x,\xi) = A^h(x)\xi$ and a limit function $\bar\rho$ such that the spectrum of (1.1) converges to that of the limit problem.

$$\begin{cases} -div(a_h(x,\nabla u)) = \lambda\bar\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

In the important case of periodic homogenization, i.e. when $\rho_\varepsilon(x) = \rho(x/\varepsilon)$ and $A_\varepsilon(x) = A(x/\varepsilon)$ where $\rho(x)$ and $A(x)$ are $Q-$periodic functions, Q being the unit cube in $\mathbb{R}^N$, the limit problem can be fully characterized and so the entire sequence $\varepsilon \to 0$ is convergent. See [20, 21].

In the general nonlinear setting, recently Baffico, Conca and Donato [5], relying on the $G-$convergence results of Chiadó Piat, Dal Maso and Defranceschi [11] for monotone operators, study the convergence problem of the principal eigenvalue of (1.1). The concept of $G-$convergence of linear elliptic second order operators was introduced by Spagnolo in [27]. See Section 2 for the precise definitions.

Up to our knowledge, no further investigation was made in the quasilinear non-uniformly elliptic case. One of the reasons why in [5] only the principal eigenvalue was studied is that, as long as we know, no results are available for higher order eigenvalues of (1.1).

The principal eigenvalue of (1.1) was studied by Kawohl, Lucia and Prashanth in [18] where, among other things, they prove its existence together with the simplicity and the positivity of the associated eigenfunction.

In order to continue with this investigation, in Section 3, we extend some results for higher order eigenvalues that are well known in the $p-$Laplacian case, to (1.1). Namely, the isolation of the principal eigenvalue, the existence of a sequence of (variational) eigenvalues growing to $+\infty$ and a variational characterization of the second eigenvalue.

Using the results of Section 3, in Section 4 we give a new simpler proof of the convergence of the principal eigenvalues of (1.1) to the principal eigenvalue of the limit problem (1.3). Moreover we can prove the convergence of the second eigenvalues of (1.1) to the second eigenvalue of (1.3). These two results rely on a more general one that says that the limit of any sequence of eigenvalues of (1.1) is an eigenvalue of (1.3). Although this result was already proved in [5], we provide here a simplified proof of this fact.

Convergence of eigenvalues in the multidimensional linear case was studied in 1976 by Boccardo and Marcellini [7] for general bounded matrices. Kesavan [20] studied the problem in an periodic frame.

Now, we turn our attention to the order of convergence of the eigenvalues. Clearly, the question of order of convergence cannot be treated with the previous generality. To this end, we restrict ourselves to the problems

$$\begin{cases} -div(a(x,\nabla u^\varepsilon)\nabla u^\varepsilon) = \lambda^\varepsilon\rho_\varepsilon|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where the family of weight functions ρ_ε are given in terms of a single bounded $Q-$periodic function ρ in the form $\rho_\varepsilon(x) := \rho(x/\varepsilon)$, Q being the unit cube of $\mathbb{R}^N$.

The limit problem is then given by

$$\begin{cases} -div(a(x,\nabla u)\nabla u) = \lambda\bar{\rho}|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\bar{\rho}$ is the average of ρ in Q.

The first result in this problem, for the linear case, can be found in Chapter III, section 2 of [24]. By estimating the eigenvalues of the inverse operator, which is compact, and using tools from functional analysis in Hilbert spaces, they deduce that

$$|\lambda_k^\varepsilon - \lambda_k| \leq \frac{C\lambda_k^\varepsilon(\lambda_k)^2}{1-\lambda_k\beta_k^\varepsilon}\,\varepsilon^{\frac{1}{2}}.$$

Here, C is a positive constant, and β_ε^k satisfies

$$0 \leq \beta_\varepsilon^k < \lambda_k^{-1},$$

and

$$\lim_{\varepsilon\to 0}\beta_\varepsilon^k = 0$$

for each $k \geq 1$.

The problem, again in the linear setting and in dimension $N = 1$, with $a = 1$, was recently studied by Castro and Zuazua in [9, 10]. In those articles the authors, using the so-called WKB method which relays on asymptotic expansions of the solutions of the problem, and the explicit knowledge of the eigenfunctions and eigenvalues of the constant coefficient limit problem, proved

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^4\varepsilon$$

and they also presented a variety of results on correctors for the eigenfunction approximation. Let us mention that their method needs higher regularity on the weight ρ, which must belong at least to $C^2(\Omega)$ and that the bound holds for $k \sim \varepsilon^{-1}$.

More recently, Kenig, Lin and Shen [19] studied the linear problem in any dimension (allowing an ε dependance in the diffusion matrix of the elliptic operator) and proved that for Lipschitz domains Ω one has

$$|\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon|\log(\varepsilon)|^{\frac{1}{2}+\sigma}$$

for any $\sigma > 0$, C depending on k and σ.

Moreover, the authors show that if the domain Ω is more regular ($C^{1,1}$ is enough) they can get rid of the logarithmic term in the above estimate. However, no explicit dependance of C on k is obtained in that work.

In this paper, in Section 5, we analyze the order of convergence of eigenvalues of (1.4) to the ones of (1.5) and prove that,

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{\frac{2p}{N}}\varepsilon$$

with C independent of k and ε. In this result, by λ_k^ε and λ_k we refer to the variational eigenvalues of problems (1.4) and (1.5) respectively.

Some remarks are in order:

(1) Classical estimates on the eigenvalues of second order, N-dimensional problems, show that λ_k and λ_k^ε behaves like $ck^{\frac{2}{N}}$, with c depending only on the coefficients of the operator and N. Hence, the order of growth of the right-hand side in the estimate of [24] is

$$\frac{\lambda_k^\varepsilon(\lambda_k)^2\varepsilon^{\frac{1}{2}}}{1-\lambda_k\beta_k^\varepsilon} \sim \frac{k^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}}{1-\lambda_k\beta_k^\varepsilon} \geq k^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}.$$

Moreover, the constant involved in their bound are unknown.

(2) If we specialize our result to the one dimensional linear case, we recover the estimate obtained in [10]. Moreover, we are considering more general weights ρ since very low regularity is needed and the estimate is valid for any k. On the other hand, no corrector results are presented here.

(3) In our result very low regularity on the domain Ω is assumed in this work. We only required the validity of the Hardy inequality (see [23])

$$\int_\Omega \frac{|u|^p}{d^p} \leq C\int_\Omega |\nabla u|^p,$$

where $d(x) = dist(x, \partial\Omega)$ and $u \in W_0^{1,p}(\Omega)$. For instance, Lipschitz regularity will do. So we get an improvement of the result in [19]. However, we recall that the result in [19] allows for a dependence in ε on the operator. Nevertheless, our result includes nonlinear eigenvalue problems, such as the $p-$Laplacian eigenvalues.

Organization of the paper. The rest of the paper is organized as follows: In Section 2, we collect some preliminary results on monotone operators that are needed in order to deal with (1.1) and also we recall the definition and some properties of $G-$convergence. In Section 3 we study the eigenvalue problem (1.1) for a fixed ε and prove the isolation of the first eigenvalue together with a variational characterization of the second eigenvalue (Theorems 3.4 and 3.5 respectively). In Section 4 we study the convergence of the eigenvalues of (1.1) and show that the first and second eigenvalues converges to the ones of the limit problem (1.3) (Theorems 4.4 and 4.6 respectively). Finally, in Section 5, we address the problem of the rate of convergence of the eigenvalues, the main result being Theorem 5.6.

2. Preliminary results

In this section we review some results gathered from the literature, enabling us to clearly state our results and making the paper self-contained.

2.1. **Monotone operators.**

We consider $\mathcal{A}\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by

$$\mathcal{A}u := -div(a(x, \nabla u)),$$

where $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies, for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, the following conditions:

(H0) *measurability:* $a(\cdot,\cdot)$ is a Carathéodory function, i.e. $a(x,\cdot)$ is continuous a.e. $x \in \Omega$, and $a(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^N$.

(H1) *monotonicity:* $0 \leq (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)$.

(H2) *coercivity:* $\alpha|\xi|^p \leq a(x,\xi)\xi$.

(H3) *continuity:* $a(x,\xi) \leq \beta|\xi|^{p-1}$.
(H4) $p-$*homogeneity:* $a(x,t\xi) = t^{p-1}a(x,\xi)$ for every $t > 0$.
(H5) *oddness:* $a(x,-\xi) = -a(x,\xi)$.

Let us introduce $\Psi(x,\xi_1,\xi_2) = a(x,\xi_1)\xi_1 + a(x,\xi_2)\xi_2$ for all $\xi_1, \xi_2 \in \mathbb{R}^N$, and all $x \in \Omega$; and let $\delta = min\{p/2, (p-1)\}$.

(H6) *equi-continuity:*

$$|a(x,\xi_1) - a(x,\xi_2)| \leq c\Psi(x,\xi_1,\xi_2)^{(p-1-\delta)/p}(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)^{\delta/p}$$

(H7) *cyclical monotonicity:* $\sum_{i=1}^k a(x,\xi_i)(\xi_{i+1} - \xi_i) \leq 0$, for all $k \geq 1$, and $\xi_1, \dots, \xi_{k+1}$, with $\xi_1 = \xi_{k+1}$.
(H8) *strict monotonicity:* let $\gamma = \max(2,p)$, then

$$\alpha|\xi_1 - \xi_2|^\gamma \Psi(x,\xi_1,\xi_2)^{1-(\gamma/p)} \leq (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2).$$

See [5], Section 3.4 where a detailed discussion on the relation and implications of every condition (H0)–(H8) is given.

In particular, under these conditions, we have the following Proposition:

Proposition 2.1 ([5], Lemma 3.3)**.** *Given* $a(x,\xi)$ *satisfying* (H0)–(H8) *there exists a unique Carathéodory function* Φ *which is even,* $p-$*homogeneous strictly convex and differentiable in the variable* ξ *satisfying*

$$\alpha|\xi|^p \leq \Phi(x,\xi) \leq \beta|\xi|^p \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$ *a.e.* $x \in \Omega$ *such that*

$$\nabla_\xi \Phi(x,\xi) = pa(x,\xi)$$

and normalized such that $\Phi(x,0) = 0$.

2.2. **Definition of G-convergence.**

Definition 2.2. We say that the family of operators $\mathcal{A}_\varepsilon u := -div(a_\varepsilon(x,\nabla u))$ G-converges to $\mathcal{A}u := -div(a(x,\nabla u))$ if for every $f \in W^{-1,p'}(\Omega)$ and for every f_ε strongly convergent to f in $W^{-1,p'}(\Omega)$, the solutions u^ε of the problem

$$\begin{cases} -div(a_\varepsilon(x,\nabla u^\varepsilon)) = f_\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

satisfy the following conditions

$$\begin{aligned} u^\varepsilon &\rightharpoonup u && \text{weakly in } W_0^{1,p}(\Omega), \\ a_\varepsilon(x,\nabla u^\varepsilon) &\rightharpoonup a(x,\nabla u) && \text{weakly in } (L^p(\Omega))^N, \end{aligned}$$

where u is the solution to the equation

$$\begin{cases} -div(a(x,\nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For instance, in the linear periodic case, the family $-div(A(x/\varepsilon)\nabla u)$ G-converges to a limit operator $-div(A^*\nabla u)$ where A^* is a constant matrix which can be characterized in terms of A and certain auxiliary functions. See for example [12].

It is shown in [5] that properties (H0)–(H8) are stable under $G-$convergence, i.e.

Theorem 2.3 ([5], Theorem 2.3)**.** *If* $\mathcal{A}_\varepsilon u := -div(a_\varepsilon(x, \nabla u))$ $G-$*converges to* $\mathcal{A}u := -div(a(x, \nabla u))$ *and* $a_\varepsilon(x, \xi)$ *satisfies* (H0)–(H8)*, then* $a(x, \xi)$ *also satisfies* (H0)–(H8)*.*

In the periodic case, i.e. when $a_\varepsilon(x, \xi) = a(x/\varepsilon, \xi)$, and $a(\cdot, \xi)$ is $Q-$periodic for every $\xi \in \mathbb{R}^N$, one has that $\mathcal{A}_\varepsilon$ $G-$converges to the homogenized operator $\mathcal{A}_h$ given by $\mathcal{A}_h u = -div(a_h(\nabla u))$, where $a_h : \mathbb{R}^N \to \mathbb{R}^N$ can be characterized by

$$a_h(\xi) = \lim_{s\to\infty} \frac{1}{s^N} \int_{Q_s(z_s)} a(x, \nabla \chi_s^\xi + \xi)dx \tag{2.2}$$

where $\xi \in \mathbb{R}^N$, $Q_s(z_s)$ is the cube of side length s centered at z_s for any family $\{z_s\}_{s>0}$ in $\mathbb{R}^N$, and χ_s^ξ is the solution of the following auxiliary problem

$$\begin{cases} -div(a(x, \nabla \chi_s^\xi + \xi)) = 0 & \text{in } Q_s(z_s) \\ \chi_s^\xi \in W_0^{1,p}(Q_s(z)), \end{cases} \tag{2.3}$$

see [8] for the proof.

In the general case, one has the following compactness result due to [11]

Proposition 2.4 ([11], Theorem 4.1)**.** *Assume that* $a_\varepsilon(x, \xi)$ *satisfies* (H1)–(H3) *then, up to a subsequence,* $\mathcal{A}_\varepsilon$ $G-$*converges to a maximal monotone operator* $\mathcal{A}$ *whose coefficient* $a(x, \xi)$ *also satisfies* (H1)–(H3)

3. Properties of the eigenvalues and eigenfunctions

This section is devoted to the study of the following (nonlinear) eigenvalue problem

$$\begin{cases} -div(a(x, \nabla u)) = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $a(x, \xi)$ verifies (H0)–(H8) and

$$0 < \rho^- \le \rho(x) \le \rho^+ < \infty \qquad \text{a.e. in } \Omega. \tag{3.2}$$

The purpose of the section is to extend to (3.1) the results that are well-known for the $p-$Laplacian case, i.e. the existence of a sequence of variational eigenvalues, the simplicity and isolation of the first eigenvalue, etc.

The methods in the proofs here very much resembles the ones used for the $p-$Laplacian and we refer the reader to the articles [2, 3, 4, 17, 22].

We denote by

$$\Sigma := \{\lambda \in \mathbb{R} \colon \text{there exists } u \in W_0^{1,p}, \text{ nontrivial solution to (3.1)}\},$$

the spectrum of (3.1). It is immediate to check that $\Sigma \subset (0, +\infty)$ and that it is closed.

By means of the critical point theory of Ljusternik–Schnirelmann it is straight forward to see that we can obtain a discrete sequence of *variational* eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ tending to $+\infty$ (see [11]). We denote by Σ_{var} the sequence of variational eigenvalues.

The question of whether $\Sigma_{\rm var} = \Sigma$ or not is only known to hold in the liner setting and also for the $p-$Laplacian in one space dimension. It is an open problem in any other situation. See [13]. See also [14] where this fact is proved for (3.1) in one space dimension.

The kth–variational eigenvalue is given by

$$\lambda_k = \inf_{C\in\Gamma_k} \sup_{v\in C} \frac{\int_\Omega \Phi(x,\nabla v)}{\int_\Omega \rho|v|^p}$$

where $\Phi(x,\xi)$ is the potential function given in Proposition 2.1,

$$\Gamma_k = \{C \subset W_0^{1,p}(\Omega) : C \text{ compact}, \ C = -C, \ \gamma(C) \geq k\}$$

and $\gamma(C)$ is the Kranoselskii genus, see [25] for the definition and properties of γ.

The following maximum principle for quasilinear operators was proved in [18] and it will be most useful in the sequel.

Theorem 3.1 ([18], Proposition 3.2). *Assume that $u \in W^{1,p}_{loc}(\Omega)$ satisfies*

$$\int_\Omega a(x,\nabla u)\nabla\phi + \rho|u|^{p-2}u\phi \geq 0, \quad \forall\phi \in C_0^\infty(\Omega), \ \phi \geq 0.$$

Consider its zero set

$$\mathfrak{Z} := \{x \in \Omega \colon \tilde{u}(x) = 0\},$$

where $\tilde u$ is the $p-$quasi continuous representative of u.

Then, either $Cap_p(\mathfrak{Z}) = 0$ or $u = 0$.

The positivity of the first eigenfunction together with the simplicity of the first eigenvalue was proved in [18].

Theorem 3.2 ([18], Section 6.2). *Let u_1 be an eigenfunction corresponding to λ_1, then u_1 does not changes sign on Ω. Also, the first eigenvalue is simple, that is, any other eigenfunction u associated to λ_1 is a multiple of u_1.*

Next, we show that the first eigenvalue λ_1 is isolated in Σ. The key step in the proof of the isolation is the next result:

Proposition 3.3. *Let $\lambda \in \Sigma$ and let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$. Then, w changes sign on Ω, that is $u^+ \neq 0$ and $u^- \neq 0$. Moreover, there exists a positive constant C independent of w and λ such that*

$$|\Omega^+| \geq C\lambda^{-\gamma}, \quad |\Omega^-| \geq C\lambda^{-\gamma},$$

where $\Omega^\pm$ denotes de positivity and the negativity set of w respectively, γ is a positive parameter, and C depends on N, p, ρ^+ and the coercivity constant α in (H2). *Here, $\gamma = (N-p)/p$ if $p < N$, $\gamma = 1$ if $p = N$, and $\gamma = (p-N)/N$ if $p > N$.*

Proof. Let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$ and let u be an eigenfunction corresponding to λ_1.

Assume that w does not changes sign on Ω. We can assume that $w \geq 0$ and $u \geq 0$ in Ω. For each $k \in \mathbb{N}$, let us truncate u as follows:

$$u_k(x) := \min\{u(x), k\}$$

and for each $\varepsilon > 0$ we consider the function $u_k^p/(w+\varepsilon)^{p-1} \in W_0^{1,p}(\Omega)$. We get

$$\int_\Omega a(x,\nabla u)\nabla u - a(x,\nabla w)\nabla\Big(\frac{u_k^p}{(w+\varepsilon)^{p-1}}\Big) = \int_\Omega \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \tag{3.3}$$

We claim that the integral in the left hand side in (3.3) is non-negative. Indeed, let Φ be the potential function given by Proposition 2.1. Then, as Φ is $p-$homogeneous in the second variable we have (see [18], p.19, 5.15)

$$\begin{aligned} &a(x,\nabla u)\nabla u - a(x,\nabla w)\nabla\Big(\frac{u_k^p}{(w+\varepsilon)^{p-1}}\Big) = \\ &p\Big\{\Phi(x,\nabla u) + (p-1)\Phi(x,\frac{u_k}{w+\varepsilon}\nabla w) - a(x,\frac{u_k}{w+\varepsilon}\nabla w)\nabla u_k\Big\}. \end{aligned} \tag{3.4}$$

By using the property that $\xi \mapsto \Phi(x,\xi)$ is convex, we easily deduce that (3.4) is nonnegative. Therefore, coming back to (3.3) we get

$$\int_\Omega \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \geq 0. \tag{3.5}$$

Since by the strong maximum principle for quasilinear operators (Theorem 3.1) the set $\{\tilde{w}=0\}$, where $\tilde{w}$ is the $p-$quasi continuous representative of w, is of measure zero then (3.5) is equivalent to

$$\int_{\{w>0\}} \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \geq 0. \tag{3.6}$$

Now, letting $\varepsilon \to 0$ and $k \to \infty$ in (3.6), we get

$$(\lambda_1 - \lambda)\int_\Omega \rho|u|^p \geq 0$$

which is a contradiction. Therefore w changes sign on Ω.

The second part of the proof follows almost exactly as in the $p-$Laplacian case. Let us suppose first that $p < N$. In fact, as w changes sign, we can use w^+ as a test function in the equation satisfied by w to obtain

$$\begin{aligned} \int_\Omega a(x,\nabla w)\nabla w^+ &= \lambda\int_\Omega \rho|w|^{p-2}ww^+ \\ &= \lambda\int_{\Omega^+}\rho|w|^p \\ &\leq \lambda\rho^+\int_{\Omega^+}|w|^p \\ &\leq \lambda\rho^+\|w^+\|^p_{L^{p^*}(\Omega)}|\Omega^+|^{p/(N-p)} \\ &\leq \lambda\rho^+K_p|\Omega^+|^{p/(N-p)}\int_\Omega|\nabla w^+|^p, \end{aligned}$$

where K_p is the optimal constant in the Sobolev-Poincaré inequality.

Now, by (H2), it follows that

$$\int_\Omega a(x,\nabla w)\nabla w^+ \geq \alpha\int_\Omega |\nabla w^+|^p.$$

Combining these two inequalities, we obtain

$$|\Omega^+| \geq \Big(\frac{\alpha}{K_p \lambda \rho^+}\Big)^{(N-p)/p}.$$

The estimate for $|\Omega^-|$ follows in the same way.

The remaining cases are similar: $p = N$ follows by using the Sobolev's inclusion $W_0^{1,N}(\Omega) \subset L^N(\Omega)$, and the case $p > N$ follows from Morrey's inequality. □

Now we are ready to prove the isolation of λ_1.

Theorem 3.4. *The first eigenvalue λ_1 is isolated. That is, there exists $\delta > 0$ such that $(\lambda_1, \lambda_1 + \delta) \cap \Sigma = \emptyset$.*

Proof. Assume by contradiction that there exists a sequence $\lambda_j \in \Sigma$ such that $\lambda_j \to \lambda_1$ as $j \to \infty$. Let u_j be the associated eigenfunctions normalized such that

$$\int_\Omega \rho |u_j|^p = 1.$$

By (H2) it follows that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ so, passing to a subsequence if necessary, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} u_j &\rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega)\\ u_j &\to u \qquad \text{strongly in } L^p(\Omega) \end{aligned}$$

Now, as the functional

$$v \mapsto \int_\Omega \Phi(x, \nabla v)$$

is weakly sequentially lower semicontinuous (see [5]), it follows that u is an eigenfunction associated to λ_1.

Now, by Theorem 3.2, we can assume that $u \geq 0$ and by Proposition 3.3 we have $|\{u = 0\}| > 0$. But this is a contradiction to the strong maximum principle in [18], Theorem 3.1. □

As a consequence of Theorem 3.4 it makes sense to define the second eigenvalue Λ_2 as the infimum of the eigenvalues greater than λ_1. Next, we show that this second eigenvalue Λ_2 coincides with the second variational eigenvalue λ_2. This result is known to hold for the $p-$Laplacian (see [3]) and we extended here for the general case (3.1).

Theorem 3.5. *Let λ_2 be the second variational eigenvalue, and let Λ_2 be defined as*

$$\Lambda_2 = \inf\{\lambda > \lambda_1 \colon \lambda \in \Sigma\}.$$

Then

$$\lambda_2 = \Lambda_2.$$

Proof. The proof of this Theorem follows closely the one in [15] where the analogous result for the Steklov problem for the $p-$Laplacian is analyzed.

Let us call

$$\mu = \inf\left\{ \int_\Omega \Phi(x, \nabla u) \colon \|\rho u\|^p_{L^p(\Omega)} = 1 \text{ and } |\Omega^\pm| > c_{\lambda_2} \right\},$$

where $c_{\lambda_2} := C\lambda_2^{-\gamma}$ and C, γ are given by Proposition 3.3.

If we take u_2 an eigenfunction of (3.1) associated with Λ_2 such that $\|\rho u\|^p_{L^p(\Omega)} = 1$, by Theorem 3.3, we have that u_2 is admissible in the variational characterization of μ. It follows that $\mu \leq \Lambda_2$. The proof will follows if we show that $\mu \geq \lambda_2$. The inverse of μ can be written as

$$\frac{1}{\mu} = \sup\left\{\int_\Omega \rho|u|^p \colon \int_\Omega \Phi(x, \nabla u) = 1 \text{ and } |\Omega^\pm| > c_{\lambda_2}\right\}.$$

The supremum is attained by a function $w \in W_0^{1,p}(\Omega)$ such that $\int_\Omega \Phi(x, \nabla w) = 1$ and $|\Omega^\pm| > c_{\lambda_2}$. As w^+ and w^- are not identically zero, if we consider the set

$$C = span\{w^+, w^-\} \cap \{u \in W_0^{1,p}(\Omega) \colon \|u\|_{W_0^{1,p}(\Omega)} = 1\},$$

then $\gamma(C) = 2$. Hence, we obtain

$$\frac{1}{\lambda_2} \geq \inf_{u\in C} \int_\Omega \rho|u|^p \tag{3.7}$$

but, as w^+ and w^- have disjoint support, it follows that the infimum (3.7) can be computed by minimizing the two variable function

$$G(a, b) := |a|^p \int_\Omega \rho|w^+|^p + |b|^p \int_\Omega \rho|w^-|^p$$

with the restriction

$$H(a, b) := |a|^p \int_\Omega \Phi(x, \nabla w^+) + |b|^p \int_\Omega \Phi(x, \nabla w^-) = 1.$$

Now, an easy computation shows that

$$\frac{1}{\lambda_2} \geq \min\left\{\frac{\int_\Omega \rho|w^+|^p}{\int_\Omega \Phi(x, \nabla w^+)}, \frac{\int_\Omega \rho|w^-|^p}{\int_\Omega \Phi(x, \nabla w^-)}\right\}.$$

We can assume that the minimum in the above inequality is realized with w^+. Then, for $t > -1$ the fuction $w+tw^+$ is admissible in the variational characterization of μ, hence if we denote

$$Q(t) := \frac{\int_\Omega \rho|w + tw^+|^p}{\int_\Omega \Phi(x, \nabla w + t\nabla w^+)},$$

we get

$$0 = Q'(0) = p\int_\Omega \rho|w|^{p-2}ww^+ - \frac{p}{\mu}\int_\Omega a(x, \nabla w)\nabla w^+,$$

therefore

$$\frac{\int_\Omega \rho|w^+|^p}{\int_\Omega \Phi(x, \nabla w^+)} = \frac{1}{\mu}$$

and the result follows. □

4. Convergence of eigenvalues

In this section we analyze the convergence of the spectrum Σ_ε of (1.1) to the spectrum Σ_h of the homogenized limit problem (1.3)

In the linear case, it is known (see [1]) that the G-convergence of the operators implies the convergence of their spectra in the sense that the kth–variational eigenvalue λ_k^ε converges to the kth–variational eigenvalue of the limit problem.

We want to study the convergence of the spectrum in the non-linear case. We begin with a general result for bounded sequences of eigenvalues. This result was already proved in [5] but we include here a simpler proof for the reader's convenience.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded. Let $\lambda^\varepsilon \in \Sigma_\varepsilon$ be a sequence of eigenvalues of the problems* (1.1) *with $\{u^\varepsilon\}_{\varepsilon>0}$ associated normalized eigenfunctions.*

Assume that the sequence of eigenvalues is convergent

$$\lim_{\varepsilon\to 0^+} \lambda^\varepsilon = \lambda.$$

Then, $\lambda \in \Sigma_h$ and there exists a sequence $\varepsilon_j \to 0^+$ such that

$$u^{\varepsilon_j} \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega)$$

with u a normalized eigenfunction associated to λ.

Remark 4.2. In most applications, we take the sequence λ^ε to be the sequence of the kth–variational eigenvalue of (1.1). In this case, it is not difficult to check that the sequence $\{\lambda_k^\varepsilon\}_{\varepsilon>0}$ is bounded and so, up to a subsequence, convergent.

In fact, by using the variational characterization of λ_k^ε, (2.1) and our assumptions on ρ we have that

$$\frac{\alpha}{\rho^+}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} \le \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p} \le \frac{\beta}{\rho^-}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p},$$

therefore

$$\frac{\alpha}{\rho^+}\mu_k \le \lambda_k^\varepsilon \le \frac{\beta}{\rho^-}\mu_k$$

where μ_k is the kth variational eigenvalue of the $p-$Laplacian.

Proof. As λ_ε is bounded and u^ε is normalized, by (H2) it follows that the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.

Therefore, up to some sequence $\varepsilon_j \to 0$, we have that

$$\begin{aligned} u^{\varepsilon_j} &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega) \\ u^{\varepsilon_j} &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned} \tag{4.1}$$

with u also normalized.

We define the sequence of functions $f_\varepsilon := \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon$. By using the fact that $\rho_\varepsilon \rightharpoonup \bar\rho$ *-weakly in $L^\infty(\Omega)$ together with (4.1) it follows that

$$f_{\varepsilon_j} \rightharpoonup f := \lambda\bar\rho|u|^{p-2}u \quad \text{weakly in } L^p(\Omega)$$

and therefore

$$f_{\varepsilon_j} \to f \quad \text{strongly in } W^{-1,p'}(\Omega).$$

By Proposition 2.4 we deduce that u^{ε_j} converges weakly in $W_0^{1,p}(\Omega)$ to the unique solution v of the homogenized problem

$$\begin{cases} -div(a_h(x, \nabla v)) = \lambda\bar{\rho}|u|^{p-2}u & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of the limit, $v = u$ is a normalized eigenfunction of the homogenized problem. □

Remark 4.3. In the case where the sequence λ^ε is the sequence of the kth–variational eigenvalues of (1.1) it would be desirable to prove that it converges to the kth–variational eigenvalue of the homogenized problem (1.3) (see Remark 4.2).

Unfortunately, we are able to prove this fact only for the first and second variational eigenvalues in the general setting.

In the one dimensional case, one can be more precise and this fact holds true. See [14].

In section 5, we address this problem in the more specific situation of $a_\varepsilon(x, \xi) = a(x, \xi)$ and $\rho_\varepsilon(x) = \rho(x/\varepsilon)$ and prove that this fact also holds true and, moreover, we provide with an estimate for the error term $|\lambda_k^\varepsilon - \lambda_k|$.

4.1. **Convergence of the first and second eigenvalue.** The first eigenvalue of (1.1) is the infimum of the Rayleigh quotient

$$\lambda_1^\varepsilon = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega \Phi_\varepsilon(x, \nabla v)}{\int_\Omega \rho_\varepsilon |v|^p}.$$

In the following result we prove the convergence of λ_1^ε when ε tends to zero.

Theorem 4.4. *Let be λ_1^ε the first eigenvalue of* (1.1) *and λ_1 the first eigenvalue of the limit problem* (1.3)*, then*

$$\lim_{\varepsilon \to 0} \lambda_1^\varepsilon = \lambda_1.$$

Moreover, if u_1^ε and u_1 are the (normalized) nonnegative eigenfunctions of (1.1) *and* (1.3) *associated to λ_1^ε and λ_1 respectively, then*

$$u_1^\varepsilon \rightharpoonup u_1 \quad \textit{weakly in } W_0^{1,p}(\Omega).$$

Remark 4.5. In [5] using the theory of convergence of monotone operators the authors obtain the conclusions of Theorem 4.4. We propose here a simple proof of this result which exploits the fact that the first eigenfunction has constant sign.

Proof. Let u_1^ε be the nonnegative normalized eigenfunction associated to λ_1^ε, the uniqueness of u_1^ε follows from Theorem 3.2.

By Theorem 4.1, up to some sequence, u_1^ε converges weakly in $W_0^{1,p}(\Omega)$ to u, an eigenfunction of the homogenized eigenvalue problem associated to $\lambda = \lim_{\varepsilon\to 0} \lambda_1^\varepsilon$.

But then, u is a nonnegative normalized eigenfunction of the homogenized problem (1.3) and so $u = u_1$. Therefore $\lambda = \lambda_1$ and the uniqueness imply that the whole sequences λ_1^ε and u_1^ε are convergent. □

Now we turn our attention to the second eigenvalue. For this purpose we use the fact that eigenfunctions associated to the second variational eigenvalue of problems (1.1) and (1.3) have, at least, two nodal domains (cf. Proposition 3.3).

Theorem 4.6. *Let λ_2^ε be the second eigenvalue of* (1.1) *and λ_2 be the second eigenvalue of the homogenized problem* (1.3)*. Then*

$$\lim_{\varepsilon\to 0} \lambda_2^\varepsilon = \lambda_2$$

Proof. Let u_2 be a normalized eigenfunction associated to λ_2 and let $\Omega^\pm$ be the positivity and the negativity sets of u_2 respectively.

We denote by $u_\pm^\varepsilon$ the first eigenfunction of (1.1) in $\Omega^\pm$ respectively. Extending $u_\pm^\varepsilon$ to Ω by 0, these function have disjoint supports and therefore they are linearly independent in $W_0^{1,p}(\Omega)$.

Let S be the unit sphere in $W_0^{1,p}(\Omega)$ and we define the set C_2^ε as

$$C_2^\varepsilon := \operatorname{span}\{u_+^\varepsilon, u_-^\varepsilon\} \cap S.$$

Clearly C_2^ε is compact, symmetric and $\gamma(C_2^\varepsilon) = 2$. Hence,

$$\lambda_2^\varepsilon = \inf_{C\in\Gamma_2} \sup_{v\in C} \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p} \le \sup_{v\in C_2^\varepsilon} \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p}$$

As C_2^ε is compact, the supremum is achieved for some $v^\varepsilon \in C_2^\varepsilon$ which can be written as

$$v^\varepsilon = a_\varepsilon u_+^\varepsilon + b_\varepsilon u_-^\varepsilon$$

with $a_\varepsilon, b_\varepsilon \in \mathbb{R}$ such that $|a_\varepsilon|^p + |b_\varepsilon|^p = 1$. Since the functions u_+^ε and u_-^ε have disjoint supports, we obtain, using the $p-$homogeneity of Φ_ε (see Proposition 2.1),

$$\lambda_2^\varepsilon \le \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v^\varepsilon)}{\int_\Omega \rho_\varepsilon |v^\varepsilon|^p} = \frac{|a_\varepsilon|^p \int_{\Omega^+} \Phi_\varepsilon(x,\nabla u_+^\varepsilon) + |b_\varepsilon|^p \int_{\Omega^-} \Phi_\varepsilon(x,\nabla u_-^\varepsilon)}{\int_\Omega \rho_\varepsilon |v^\varepsilon|^p}$$

Using the definition of $u_\pm^\varepsilon$, the above inequality can be rewritten as

$$\lambda_2^\varepsilon \le \frac{|a_\varepsilon|^p \lambda_{1,+}^\varepsilon \int_{\Omega^+} \rho_\varepsilon |u_+^\varepsilon|^p + |b_\varepsilon|^p \lambda_{1,-}^\varepsilon \int_{\Omega^-} \rho_\varepsilon |u_-^\varepsilon|^p}{\int_\Omega \rho_\varepsilon |v^\varepsilon|^p} \le \max\{\lambda_{1,+}^\varepsilon, \lambda_{1,-}^\varepsilon\} \tag{4.2}$$

where $\lambda_{1,\pm}^\varepsilon$ is the first eigenvalue of (1.1) in the nodal domain $\Omega^\pm$ respectively.

Now, using Theorem 4.4, we have that $\lambda_{1,\pm}^\varepsilon \to \lambda_{1,\pm}$ respectively, where $\lambda_{1,\pm}$ are the first eigenvalues of (1.3) in the domains $\Omega^\pm$ respectively. Moreover, we observe that these eigenvalues $\lambda_{1,\pm}$ are both equal to the second eigenvalue λ_2 in Ω, therefore from (4.2), we get

$$\lambda_2^\varepsilon \le \lambda_2 + \delta$$

for δ arbitrarily small and ε tending to zero. So,

$$\limsup_{\varepsilon\to 0} \lambda_2^\varepsilon \le \lambda_2 \tag{4.3}$$

On the other hand, suppose that $\lim_{\varepsilon\to 0} \lambda_2^\varepsilon = \lambda$ where $\lambda \in \Sigma_h$. We claim that $\lambda > \lambda_1$.

In fact, we have that $u_2^\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ where u is a normalized eigenfunction associated to λ. As the measure of the positivity and negativity sets of u_2^ε are bounded below uniformly in $\varepsilon > 0$ (see Proposition 3.3), we have that either u changes sign or $|\{u = 0\}| > 0$. In any case, this implies our claim.

Then, as $\lambda > \lambda_1$ it must be $\lambda \geq \lambda_2$. Then

$$\lambda_2 \leq \lambda = \lim_{\varepsilon \to 0} \lambda_2^\varepsilon \tag{4.4}$$

Combining (4.3) and (4.4) we obtain the desired result. □

5. Rates of convergence

In this section we consider the eigenvalue problem in which the operator is independent on ε and the dependance on ε only appears in an oscillating weight ρ_ε.

We will prove that in this case the kth–variational eigenvalue of problem (1.4) converges to the kth–variational eigenvalue of the limit problem (1.5).

Moreover, our goal is to estimate the rate of convergence between the eigenvalues. That is, we want to find explicit bounds for the error $|\lambda_k^\varepsilon - \lambda_k|$.

We begin this section by proving some auxiliary results that are essential in the remaining of the paper.

We first prove a couple of lemmas in order to prove Theorem 5.5 which is a generalization for $p \neq 2$ of Oleinik's Lemma [24].

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and, for $\delta > 0$, let G_δ be a tubular neighborhood of $\partial\Omega$, i.e. $G_\delta = \{x \in \Omega \colon dist(x, \partial\Omega) < \delta\}$.*

Then

$$\|v\|^p_{L^p(G_\delta)} \leq C_{H,p}(\Omega)\delta^p \|\nabla v\|^p_{L^p(\Omega)},$$

for every $v \in W_0^{1,p}(\Omega)$, where $C_{H,p}(\Omega)$ is the best constant in the Hardy inequality (see [23])

$$\int_\Omega \frac{|v|^p}{d^p} \leq C_{H,p}(\Omega) \int_\Omega |\nabla u|^p \tag{5.1}$$

and $d(x) = dist(x, \partial\Omega)$.

Proof. The proof follows by noticing that if $x \in G_\delta$, then $d(x) \leq \delta$, so, by (5.1) we get

$$\int_{G_\delta} |v|^p = \int_{G_\delta} \frac{|v|^p}{d^p} d^p \leq \delta^p \int_\Omega \frac{|v|^p}{d^p} \leq C_{H,p}(\Omega)\delta^p \int_\Omega |\nabla u|^p.$$

The proof is now complete. □

Remark 5.2. Observe that the only requirement on the regularity of $\partial\Omega$ is the validity of Hardy's inequality (5.1). Therefore, much less than Lipschitz will do. We refer the reader to the book of Maz'ja [23].

Now we need an easy Lemma that computes the Poincaré constant on the cube of side ε in terms of the Poincaré constant of the unit cube. Although this result is well known and its proof follows directly by a change of variables, we choose to include it for the sake of completeness.

Lemma 5.3. *Let Q be the unit cube in $\mathbb{R}^N$ and let c_q be the Poincaré constant in the unit cube in L^q, $q \geq 1$, i.e.*

$$\|u - (u)_Q\|_{L^q(Q)} \leq c_q \|\nabla u\|_{L^q(Q)}, \qquad \text{for every } u \in W^{1,q}(Q),$$

where $(u)_Q$ is the average of u in Q. Then, for every $u \in W^{1,q}(Q_\varepsilon)$ we have

$$\|u - (u)_{Q_\varepsilon}\|_{L^q(Q_\varepsilon)} \leq c_q \varepsilon \|\nabla u\|_{L^q(Q_\varepsilon)},$$

where $Q_\varepsilon = \varepsilon Q$.

Proof. Let $u \in W^{1,q}(Q_\varepsilon)$. We can assume that $(u)_{Q_\varepsilon} = 0$. Now, if we denote $u_\varepsilon(y) = u(\varepsilon y)$, we have that $u_\varepsilon \in W^{1,q}(Q)$ and by the change of variables formula, we get

$$\int_{Q_\varepsilon} |u|^q = \int_Q |u_\varepsilon|^q \varepsilon^N \leq c_q^q \varepsilon^N \int_Q |\nabla u_\varepsilon|^q = c_q^q \varepsilon^q \int_{Q_\varepsilon} |\nabla u|^q.$$

The proof is now complete. □

The next Lemma is the final ingredient in the estimate of Theorem 5.5.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and denote by Q the unit cube in $\mathbb{R}^N$. Let $g \in L^\infty(\mathbb{R}^N)$ be a Q-periodic function such that $\bar{g} = 0$. Then the inequality*

$$\left| \int_{\Omega_1} g(\tfrac{x}{\varepsilon}) v \right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}$$

holds for every $v \in W_0^{1,1}(\Omega)$, where c_1 is the Poincaré constant given in Lemma 5.3 and $\Omega_1 \subset \Omega$ is given by

$$\Omega_1 = \bigcup Q_{z,\varepsilon}, \qquad Q_{z,\varepsilon} := \varepsilon(z + Q) \subset \Omega, \qquad z \in \mathbb{Z}^N.$$

Proof. Denote by I^ε the set of all $z \in \mathbb{Z}^N$ such that $Q_{z,\varepsilon} := \varepsilon(z + Q) \subset \Omega$. Let us consider the function $\bar{v}_\varepsilon$ given by the formula

$$\bar{v}_\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{Q_{z,\varepsilon}} v(y) dy$$

for $x \in Q_{z,\varepsilon}$. Then we have

$$\int_{\Omega_1} g_\varepsilon v = \int_{\Omega_1} g_\varepsilon (v - \bar{v}_\varepsilon) + \int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon. \tag{5.2}$$

Now, by Lema 5.3 we get

$$\begin{aligned} \|v - \bar{v}_\varepsilon\|_{L^1(\Omega_1)} &= \sum_{z \in I^\varepsilon} \int_{Q_{z,\varepsilon}} |v - \bar{v}_\varepsilon| dx \\ &\leq c_1 \varepsilon \sum_{z \in I^{z,\varepsilon}} \int_{Q_{z,\varepsilon}} |\nabla v(x)| dx \\ &\leq c_1 \varepsilon \|\nabla u\|_{L^1(\Omega)}. \end{aligned} \tag{5.3}$$

Finally, since $\bar{g} = 0$ and since g is $Q-$periodic, we get

$$\int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon = \sum_{z \in I^\varepsilon} \bar{v}_\varepsilon \mid_{Q_{z,\varepsilon}} \int_{Q_{z,\varepsilon}} g_\varepsilon = 0. \tag{5.4}$$

Now, combining (5.3) and (5.4) we can bound (5.2) by

$$\Big| \int_{\Omega_1} g_\varepsilon v \Big| \le \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}.$$

This finishes the proof. □

The next Theorem is essential to estimate the rate of convergence of the eigenvalues since it allows us to replace an integral involving a rapidly oscillating function with one that involves its average in the unit cube.

Theorem 5.5. *Let $g \in L^\infty(\mathbb{R}^N)$ be a $Q-$periodic function such that $0 < g^- \le g \le g^+ < \infty$. Then,*

$$\begin{aligned} \Big| \int_\Omega (g_\varepsilon(x) - \bar{g})|u|^p \Big| &\le \|g - \bar{g}\|_{L^\infty(\mathbb{R}^N)} \varepsilon \|\nabla u\|^p_{L^p(\Omega)} \Big[\frac{p}{\mu_1^{p-1}} c_1 + C_{H,p}(\Omega) N^{p/2} \varepsilon^{p-1} \Big] \\ &\le C\varepsilon \|\nabla u\|^p_{L^p(\Omega)}, \end{aligned}$$

for every $u \in W^{1,p}_0(\Omega)$. The constant $C_{H,p}(\Omega)$ is the best constant in Hardy's inequaity (5.1)*, c_1 is the optimal constant in Poincaré's inequality in $L^1(Q)$ and μ_1 is the first eigenvalue of the $p-$Laplacian in Ω.*

Proof. Let $\varepsilon > 0$ be fixed, and let Ω_1 be the set defined in Lemma 5.4.

Denote by $G := \Omega \setminus \Omega_1$ and observe that $G \subset G_{\sqrt{N}\varepsilon}$. In fact, with the notations of Lemma 5.4, if $x \in G$ then there exists a cube $Q = Q_{z,\varepsilon}$ such that $x \in Q$ and $Q \cap \partial\Omega \neq \emptyset$. Therefore, $dist(x, \partial\Omega) \le diam(Q) = \sqrt{N}\varepsilon$.

Now, denote by $h = g - \bar{g}$ and so, by Lemma 5.1,

$$\Big| \int_G h_\varepsilon |u|^p \Big| \le C_{H,p}(\Omega) \|h\|_{L^\infty(\mathbb{R}^N)} (\sqrt{N}\varepsilon)^p \|\nabla u\|^p_{L^p(\Omega)}. \tag{5.5}$$

Now, to bound the integral in Ω_1 we use Lemma 5.4 to obtain

$$\Big| \int_{\Omega_1} h_\varepsilon |u|^p \Big| \le \|h\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla(|u|^p)\|_{L^1(\Omega)}. \tag{5.6}$$

An easy computation shows that

$$\|\nabla(|u|^p)\|_{L^1(\Omega)} \le p\|u\|^{p-1}_{L^p(\Omega)} \|\nabla u\|_{L^p(\Omega)} \le \frac{p}{\mu_1^{p-1}} \|\nabla u\|^p_{L^p(\Omega)}. \tag{5.7}$$

Finally, combining (5.5), (5.6) and (5.7) we obtain the desired result. □

Now we are ready to prove the main result of this section.

Theorem 5.6. *Let λ_k^ε be the kth–variational eigenvalue associated to equation* (1.4) *and let be λ_k be the kth–variational eigenvalue associated to the limit problem* (1.5)*. Then there exists a constant $C > 0$ independent of the parameters ε and k such that*

$$|\lambda_k - \lambda_k^\varepsilon| \le C k^{\frac{2p}{N}} \varepsilon.$$

Proof. Let $\delta > 0$ and let $G_\delta^k \subset W_0^{1,p}(\Omega)$ be a compact, symmetric set of genus k such that

$$\lambda_k = \inf_{G\in\Gamma_k} \sup_{u\in G} \frac{\int_\Omega \Phi(x,\nabla u)}{\bar\rho \int_\Omega |u|^p} = \sup_{u\in G_\delta^k} \frac{\int_\Omega \Phi(x,\nabla u)}{\bar\rho \int_\Omega |u|^p} + O(\delta).$$

We use now the set G_δ^k, which is admissible in the variational characterization of the kth–eigenvalue of (1.4), in order to found a bound for it as follows,

$$\lambda_k^\varepsilon \le \sup_{u\in G_\delta^k} \frac{\int_\Omega \Phi(x,\nabla u)}{\int_\Omega \rho_\varepsilon |u|^p} = \sup_{u\in G_\delta^k} \frac{\int_\Omega \Phi(x,\nabla u)}{\bar\rho \int_\Omega |u|^p}\, \frac{\bar\rho \int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p}. \tag{5.8}$$

To bound λ_k^ε we look for bounds of the two quotients in (5.8). For every function $u \in G_\delta^k$ we have that

$$\frac{\int_\Omega \Phi(x,\nabla u)}{\bar\rho \int_\Omega |u|^p} \le \sup_{v\in G_\delta^k} \frac{\int_\Omega \Phi(x,\nabla v)}{\bar\rho \int_\Omega |v|^p} = \lambda_k + O(\delta). \tag{5.9}$$

Since $u \in G_\delta^k \subset W_0^{1,p}(\Omega)$, by Theorem 5.5 we obtain that

$$\frac{\bar\rho \int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p} \le 1 + C\varepsilon \frac{\|\nabla u\|_{L^p(\Omega)}^p}{\int_\Omega \rho_\varepsilon |u|^p}. \tag{5.10}$$

Now, by (3.2), (2.1) together with (5.9), we have

$$\begin{aligned} \frac{\|\nabla u\|_{L^p(\Omega)}^p}{\int_\Omega \rho_\varepsilon |u|^p} &\le \frac{\bar\rho}{\rho^-} \frac{\|\nabla u\|_{L^p(\Omega)}^p}{\int_\Omega \bar\rho |u|^p} \\ &\le \frac{\bar\rho}{\rho^-}\frac{1}{\alpha} \frac{\int_\Omega \Phi(x,\nabla u)}{\int_\Omega \bar\rho |u|^p} \\ &\le \frac{\bar\rho}{\rho^-}\frac{1}{\alpha}(\lambda_k + O(\delta)). \end{aligned} \tag{5.11}$$

Then combining (5.8), (5.9), (5.10) and (5.11) we find that

$$\lambda_k^\varepsilon \le (\lambda_k + O(\delta))\left(1 + C\varepsilon(\lambda_k + O(\delta))\right).$$

Letting $\delta \to 0$ we get

$$\lambda_k^\varepsilon - \lambda_k \le C\varepsilon\lambda_k^2. \tag{5.12}$$

In a similar way, interchanging the roles of λ_k and λ_k^ε, we obtain

$$\lambda_k - \lambda_k^\varepsilon \le C\varepsilon(\lambda_k^\varepsilon)^2. \tag{5.13}$$

So, from (5.12) and (5.13), we arrive at

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon \max\{\lambda_k, \lambda_k^\varepsilon\}^2.$$

In order to complete the proof of the Theorem, we need an estimate on λ_k and λ_k^ε. But this follows by comparison with the kth–variational eigenvalue of the $p-$Laplacian, μ_k and the bound on μ_k proved in [16].

In fact, from (2.1) we have

$$\frac{\alpha}{\bar\rho}\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \le \frac{\int_\Omega \Phi(x,\nabla u)}{\int_\Omega \bar\rho |u|^p} \le \frac{\beta}{\bar\rho}\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p},$$
$$\frac{\alpha}{\rho^+}\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \le \frac{\int_\Omega \Phi(x,\nabla u)}{\int_\Omega \rho_\varepsilon |u|^p} \le \frac{\beta}{\rho^-}\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p},$$

from where it follows that

$$\frac{\alpha}{\bar\rho}\mu_k \le \lambda_k \le \frac{\beta}{\bar\rho}\mu_k, \qquad \frac{\alpha}{\rho^+}\mu_k \le \lambda_k^\varepsilon \le \frac{\beta}{\rho^-}\mu_k.$$

Now, in [16], it is shown that

$$\mu_k \le Ck^{p/N}$$

and so the proof is complete. □

Remark 5.7. As we mentioned in the introduction, in the linear case and in one space dimension Castro and Zuazua [10] prove that, for $k < C\varepsilon^{-1}$,

$$|\lambda_k^\varepsilon - \lambda_k| \le Ck^4\varepsilon.$$

If we specialize our result to this case, we get the same bound. The advantage of our method is that very low regularity on ρ is needed (only L^∞). However, the method in [10], making use of the linearity of the problem, gives precise information about the behavior of the eigenfunctions u_k^ε.

Remark 5.8. In [19], Kenig, Lin and Shen studied the linear case in any space dimension (allowing a periodic oscillation diffusion matrix) and prove the bound

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon|\log\varepsilon|^{1+\sigma}.$$

for some $\sigma > 0$ and C depending on σ and k (The authors can get rid off the logarithmic term assuming more regularity on Ω). If we specialize our result to this case, we cannot treat an ε dependance on the operator, but we get an explicit dependance on k on the estimate and assuming very low regularity on Ω (Lipschitz is more than enough) we get a better dependance on ε.

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