Estimates for eigenvalues of quasilinear elliptic systems.

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The problem

In this work we analyze the problem

\[
\begin{cases}
-\Delta_p u = \lambda r(x) \alpha |u|^{\alpha - 2} |v|^\beta \\
-\Delta_q v = \lambda r(x) \beta |u|^\alpha |v|^\beta - 2 v
\end{cases}
\]

In \( \Omega \subset \mathbb{R}^N \) smooth and bounded, with homogeneous Dirichlet boundary conditions.
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\end{align*}
\]

In \( \Omega \subset \mathbb{R}^N \) smooth and bounded, with homogeneous Dirichlet boundary conditions.

\( r \in L^\infty \) and bounded away from 0.

\( \lambda \in \mathbb{R} \) is the eigenvalue parameter.

\[
\frac{\alpha}{p} + \frac{\beta}{q} = 1.
\]
History of the problem

[Boccardo - de Figueiredo, NoDEA 2002],
[Fleckinger et al., Adv. Diff. Eq. 1997],
[Allegretto - Huang, Nonlinear Anal. 1996],
.... (many others)
History of the problem

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Known results:

▶ Existence of a principal eigenvalue.
▶ Simplicity of the principal eigenvalue.
▶ Positivity of associated eigenfunction.
▶ Isolation of principal eigenvalue.
History of the problem (cont.)

- Existence of a sequence of eigenvalues \( \{ \lambda_k \} \) \( \rightarrow \) [De Napoli - Mariani, AAA 2002]
- Existence of generalized eigenvalues \( \rightarrow \) [De Napoli - Pinasco, JDE 2006].

\[
\lambda_k \leq \Lambda_{p,k} \left[ 1 + \left( \frac{p}{q} \right)^{q+1} \left( \inf_r \Lambda_{p,k} \right) \right] \left( \frac{q-p}{p} \right)
\]

where \( \Lambda_{p,k} \) is the \( k \)th (variational) eigenvalue of the \( p \)-laplacian with Dirichlet BC.
History of the problem
(cont.)

- Existence of a sequence of eigenvalues \( \{ \lambda_k \} \rightarrow \) [De Napoli - Mariani, AAA 2002]

- Existence of generalized eigenvalues \( \rightarrow \) [De Napoli - Pinasco, JDE 2006].

- Upper bounds for eigenvalues \( \rightarrow \) [De Napoli - Pinasco, JDE 2006].

\[
\lambda_k \leq \frac{\Lambda_{p,k}}{p} \left[ 1 + \left( \frac{p}{q} \right)^{q+1} \right] \left( \text{inf}_r \Lambda_{p,k} \right) \frac{q(p-q)}{p}
\]

where \( \Lambda_{p,k} \) is the \( k^{th} \) (variational) eigenvalue of the \( p \)–laplacian with Dirichlet BC.
Moreover, in 1D, using the asymptotic bound

\[ \Lambda_{p,k} \sim \left( \frac{\pi p}{\int_\Omega r^{1/p}} \right)^p k^p \]

One can obtain

\[ \lambda_k \leq \left( \frac{\pi p}{\int_\Omega r^{1/p}} \right)^p \frac{k^p}{p} . \]

for large enough \( k \).
Our objective is

Find explicit and asymptotic lower bounds for the $k^{th}$ (variational) eigenvalue of the system in $\mathbb{R}^N$. 

Applications:

- Bifurcation problems
- anti-maximum principles
Objective of the work

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Applications:
- Bifurcation problems
- Anti-maximum principles
- Existence / non-existence results

([Azizieh - Clément, JDE 2002], [Drabek et al., Diff. Int. Eq. 2003], [Stavrakakis - Zographopoulos, EJDE 1999], etc.)
The Spectral Counting Function

We introduce the *Spectral Counting Function* as

\[ N(\lambda) = \#\{k : \lambda_k \leq \lambda\} \]
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$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\}$$

Lower and upper bounds on eigenvalues can be translated in lower and upper bounds on $N(\lambda)$. For instance:

$$ck^b \leq \lambda_k \leq Ck^a \iff (C^{-1}\lambda)^{1/a} \leq N(\lambda) \leq (c^{-1}\lambda)^{1/b}$$
The 1D case

Theorem

Let $\Omega = (0, 1)$ and $N(\lambda)$ be the Spectral Counting Function. Then, as $\lambda \to \infty$,

1. If $q < p$,

$$c_1 \lambda^{1/p} \leq N(\lambda) \leq C_1 \lambda^{1/q} + C_2 \lambda^{1/p}.$$ 

2. If $q = p$,

$$c_1 \lambda^{1/p} \leq N(\lambda) \leq (C_1 + C_2) \lambda^{1/p}.$$ 

3. If $q = p$ and $\alpha = \beta$,

$$N(\lambda) \sim c_2 \lambda^{1/p}.$$
The 1D case
(cont.)

Remarks:
► In the 1D case, the variational eigenvalues exhaust the hole spectrum (see, for instance, [JFB - Pinasco, Ark. Math. 2003])
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The 1D case (cont.)

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- In the 1D case, the variational eigenvalues exhaust the hole spectrum (see, for instance, [JFB - Pinasco, Ark. Math. 2003]).

- The constants $c_1, c_2, C_1$ and $C_2$ are given explicitly in terms of $p, q, \alpha, \beta$ and the weight $r(x)$.

- A more precise lower bound can be given. In fact, let $S_p = \{\Lambda_{p,k}/\alpha\}$, $S_q = \{\Lambda_{q,k}/\beta\}$ and

$$S = S_p \cup S_q = \{\mu_k\}$$

Then $\mu_k \leq \lambda_k$. 
The N-dimensional case

Theorem

Let $\Omega \subset \mathbb{R}^N$ open and bdd. Then, as $\lambda \to \infty$,

1. If $q \leq p$,

$$\bar{c}_1 \lambda^{N/p} \leq N(\lambda) \leq \bar{C}_1 \lambda^{N/q} + \bar{C}_2 \lambda^{N/p}.$$ 

2. If $q = p$,

$$\bar{c}_1 \lambda^{N/p} \leq N(\lambda) \leq (\bar{C}_1 + \bar{C}_2) \lambda^{N/p}.$$
The N-dimensional case
(cont.)

Remarks:

▶ Again, the constants $\bar{c}_1, \bar{c}_2, \bar{C}_1$ and $C_2$ can be given explicitly in terms of $p, q, \alpha, \beta$ and $r(x)$.

▶ In the N-dimensional case, it is not known (even for a single equation) that the variational eigenvalues exhaust the hole spectrum.

▶ The analogous item 3. of the previous Theorem (i.e. $p = q$ and $\alpha = \beta$) we can only prove it for the linear system $p = q = 2$ and $\alpha = \beta = 1$, that correspond to the eigenvalues of the bi-laplacian with Navier BC.

\[
\begin{align*}
\Delta^2 u &= \lambda u \quad \text{in } \Omega \\
 u &= \Delta u = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Lemma

Let \( \Lambda_{p,k} \) be the \( k^{th} \) eigenvalue of the \( p \)-laplacian. Then there exists \( c_p, C_p \) such that

\[
c_p k^{p/N} \leq \Lambda_{p,k} \leq C_p k^{p/N}.
\]
Lemma

Let $\Lambda_{p,k}$ be the $k^{th}$ eigenvalue of the $p-$laplacian. Then there exists $c_p, C_p$ such that

$$c_p k^{p/N} \leq \Lambda_{p,k} \leq C_p k^{p/N}.$$ 

Proof: (case $r(x) \equiv 1$)

Let $Q_1 \subset \Omega \subset Q_2$ be two cubes. Then,

$$\Lambda_{p,k}(Q_1) \leq \Lambda_{p,k}(\Omega) \leq \Lambda_{p,k}(Q_2)$$

So we need to bound the eigenvalues of a cube.
We now define $\nu_{p,k}(Q)$ the eigenvalues of the pseudo $p-$laplacian in the cube $Q,$

\[
\begin{cases}
- \sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p-2} \partial_{x_i} u \right) = \nu |u|^{p-2} u & \text{on } Q \\
u = 0 \quad & \text{on } \partial Q
\end{cases}
\]
Auxiliary Results
Estimation of $\Lambda_{p,k}$ cont.

We now define $\nu_{p,k}(Q)$ the eigenvalues of the pseudo $p$–laplacian in the cube $Q$,

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u_{p,k} = \pi \frac{pN}{L^p} & \text{on } \partial Q
\end{cases}
\]

and observe that this eigenvalues $\nu_{p,k}$ can be computed by separation of variables.

In fact

\[ u_{p,1}(x) = \sin_p(\pi_p x_1 / L) \cdots \sin_p(\pi_p x_N / L), \quad \nu_{p,1} = \frac{\pi_p^p N}{L^p} \]

$L$ being the length of $Q$
Now, the result follows from comparison of the Rayleigh quotients, since

\[ \nu_{p,k} = \min \max \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \]

\[ \Lambda_{p,k} = \min \max \frac{\|\nabla u_2\|_p^p}{\|u\|_p^p} \]

and the norms in \( \mathbb{R}^N \), \( | \cdot |_p \) and \( | \cdot |_2 \) being equivalent. □
Lemma

Let \( S_p = \{ \Lambda_{p,k}/\alpha \} \), \( S_q = \{ \Lambda_{q,k}/\beta \} \) and \( S = S_p \cup S_q = \{ \mu_k \} \). Then, \( S \) consists exactly of the variational eigenvalues of the (uncoupled) system

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\begin{align*}
-\Delta_p u &= \lambda r(x) |u|^{p-2} u \quad \text{on } \Omega \\
-\Delta_q v &= \lambda r(x) |v|^{q-2} v \quad \text{on } \Omega
\end{align*}
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with Dirichlet BC.
Lemma

Let $S_p = \{\Lambda_p, k/\alpha\}$, $S_q = \{\Lambda_q, k/\beta\}$ and $S = S_p \cup S_q = \{\mu_k\}$. Then, $S$ consists exactly of the variational eigenvalues of the (uncoupled) system

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\end{aligned}
$$

with Dirichlet BC.

Proof.

Easy.

□
Proof of the main results

Upper bound

We will show the upper bound for $N(\lambda)$ which corresponds to the lower bound for $\lambda_k$. Also we consider the case $r(x) \equiv 1$. 
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First, by Young’s inequality,

$$\int_{\Omega} |u|^{\alpha} |v|^{\beta} \, dx \leq \frac{\alpha}{p} \int_{\Omega} |u|^p \, dx + \frac{\beta}{q} \int_{\Omega} |v|^q \, dx$$
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Therefore

\[
\frac{\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q \, dx}{\int_{\Omega} |u|^\alpha |v|^\beta \, dx} \geq \frac{\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q \, dx}{\frac{\alpha}{p} \int_{\Omega} |u|^p \, dx + \frac{\beta}{q} \int_{\Omega} |v|^q \, dx}
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Therefore

$$\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q \, dx \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q \, dx$$

So $\lambda_k \geq \mu_k$
Proof of the main results
Upper bound (cont.)

Then

\[ N(\lambda) = \# \{ k : \lambda_k \leq \lambda \} \leq \# \{ k : \mu_k \leq \lambda \} \]
Proof of the main results

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\[ = \# \{ k : \Lambda_{p,k}/\alpha \leq \lambda \} + \# \{ k : \Lambda_{q,k}/\beta \leq \lambda \} \]
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On the other hand,

\[ \Lambda_{p,k} \geq c_p k^p / N \]
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Therefore

\[ N_p(\alpha \lambda) \leq \left( \frac{\alpha \lambda}{c_p} \right)^{N/p} = \bar{c}_p \lambda^{N/p}. \]
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So

\[ N(\lambda) \leq \bar{c}_p \lambda^{N/p} + \bar{c}_q \lambda^{N/q}. \]
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Lower bound

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First, we cover $\Omega$ by a union of non-overlapping cubes $Q_i$ with sides of length $L$

$$\Omega \subset \bigcup_{i=1}^{J} Q_i \quad \ell(Q_i) = L.$$
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$$\Omega \subset \bigcup_{i=1}^{J} Q_i \quad \ell(Q_i) = L.$$ 

Now, it is easy to see that

$$N(\lambda) \geq \sum_{i=1}^{J} N(\lambda, Q_i)$$
Proof of the main results
Lower bound (cont.)

Then we need to estimate $N(\lambda, Q)$ for any cube $Q$ with $\ell(Q) = L$. 

We recall the following result:

$$\lambda_{1} \leq \Lambda_{p,1}[1 + (p/q)^{q+1} \Lambda_{p,1} (p-q)/p]$$

de Napoli - Pinasco, JDE 2006

But now, we observe that

$$\Lambda_{p,1}(Q) \leq \nu_{p,1} = \pi_{p} N L^{1/p}.$$ 

Combining these, we can choose $L = L(\lambda)$ such that $N(\lambda, Q) = 1$.

In fact $L = \pi_{p} (N/\lambda)^{1/p}$. 
Proof of the main results
Lower bound (cont.)

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$$\lambda_1 \leq \frac{\Lambda_{p,1}}{p} \left[ 1 + \left( \frac{p}{q} \right)^{q+1} \Lambda_{p,1}^{(p-q)/p} \right]$$

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Finally, it is easy to see that, for \( \lambda \to \infty \),

\[ J \sim c\lambda^{N/p}. \]
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Finally, it is easy to see that, for \( \lambda \to \infty \),

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That is,

\[ N(\lambda) \geq c\lambda^{N/p}. \]