UNIQUENESS OF LIMIT SOLUTIONS TO A FREE BOUNDARY PROBLEM FROM COMBUSTION

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Abstract. We investigate the uniqueness of limit solutions for a free boundary problem in heat propagation that appears as a limit of a parabolic system that arises in flame propagation.

1. Introduction

In this paper we consider the following problem arising in combustion theory

$$
\begin{cases} \Delta u^\varepsilon - u^\varepsilon_t = Y^\varepsilon f_\varepsilon(u^\varepsilon) & \text{in } \mathcal{D}, \\ \Delta Y^\varepsilon - Y^\varepsilon_t = Y^\varepsilon f_\varepsilon(u^\varepsilon) & \text{in } \mathcal{D}, \end{cases} \tag{1.1}
$$

where $\mathcal{D} \subset \mathbb{R}^{N+1}$.

This model appears in combustion theory in the analysis of the propagation of curved flames. It is derived in the framework of the theory of equidiffusional premixed flames analyzed in the relevant limit of high activation energy for Lewis number 1. In this application, Y^ε represents the fraction of some reactant (and hence it is assumed to be nonnegative), and u^ε is minus the temperature (more precisely, $u^\varepsilon = \lambda(T_f - T^\varepsilon)$ where T_f is the flame temperature and λ is a normalization factor). Observe that the term $Y^\varepsilon f_\varepsilon(u^\varepsilon)$ acts as an absorption term in the equation (1.1). Since $T^\varepsilon = T_f - (u^\varepsilon/\lambda)$, it is in fact a reaction term for the temperature. In the flame model, such a term represents the effect of the exothermic chemical reaction and f has accordingly a number of properties: it is a nonnegative Lipschitz continuous function which is positive in an interval $(-\infty, \varepsilon)$ and vanishes otherwise (i.e., reaction occurs only when $T > T_f - \frac{\varepsilon}{\lambda}$). The parameter ε is essentially the inverse of the activation energy of the chemical reaction. For the sake of simplicity we will assume that $f_\varepsilon(s) = \frac{1}{\varepsilon^2} f(\frac{s}{\varepsilon})$, where f is a nonnegative, Lipschitz continuous function with support in $(-\infty, 1]$.

For the derivation of the model, we cite [1].

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Here we are interested in high activation energy limits (i.e. $\varepsilon \to 0$). These limits, are currently the subject of active investigation, specially in the case $u^\varepsilon = Y^\varepsilon$. This is a natural assumption in the case of travelling waves.

In a previous paper [5] we have studied this problem in the case in which the initial values for u^ε and Y^ε – both converging to the same function u_0 – satisfy the condition

$$\frac{Y_0^\varepsilon(x) - u_0^\varepsilon(x)}{\varepsilon} \to w_0(x) \quad \text{uniformly in } \mathbb{R}^N \tag{1.2}$$

with $w_0 > -1$.

Problem (1.1) reduces to a single equation, namely

$$\Delta u^\varepsilon - u_t^\varepsilon = (u^\varepsilon + w^\varepsilon) f_\varepsilon(u^\varepsilon), \tag{P_ε}$$

where the function $w^\varepsilon(x,t)$ is the solution of the heat equation with initial datum $Y_0^\varepsilon(x) - u_0^\varepsilon(x)$. By (1.2) there exists the limit

$$\lim_{\varepsilon \to 0} \frac{w^\varepsilon(x,t)}{\varepsilon} = w_0(x,t) \tag{1.3}$$

and $w_0(x,t)$ is the solution of the heat equation with initial datum $w_0(x)$.

In this way, at least formally, the reaction term converges to a delta function and a free boundary problem appears. In fact, we have proved in [5] that every sequence of uniformly bounded solutions to (1.1), $\{u^{\varepsilon_n}\}$, with $\varepsilon_n \to 0$ has a subsequence $\{u^{\varepsilon_{n_k}}\}$ converging to a limit function u which is a solution of the following free boundary problem

$$\begin{cases} \Delta u - u_t = 0 & \text{in } \{u > 0\} \\ |\nabla u^+| = \sqrt{2M(x,t)} & \text{on } \partial\{u > 0\} \end{cases} \tag{P}$$

where $M(x,t) = \int_{-w_0(x,t)}^1 \big(s + w_0(x,t)\big) f(s)\,ds$.

We see that the free boundary condition strongly depends on the approximation u_0^ε, Y_0^ε of the initial datum u_0. In particular, the limit function u is different for different approximations of the initial datum u_0.

It is therefore natural to wonder whether the only condition that determines the limit function u is condition (1.2).

The purpose of this paper is to prove that this is indeed the case, at least under some monotonicity assumption on the initial value u_0. This monotonicity assumption is similar to that used to prove uniqueness of the limit for the case $u^\varepsilon = Y^\varepsilon$ in [9].

In fact, we follow here some of the ideas of [9] which are based on the fact that any limit function is a supersolution to (P). This is still true in our case. Unfortunately the simple construction in [9] of supersolutions of (P_ε) that approximate a strict classical supersolution of (P), when $w^\varepsilon = 0$, does not work in the general case unless one asks for a lot of complementary conditions on the reaction function f.

Therefore, we follow here the construction done in [7]. The proof that this construction works in based on blow up of the constructed functions. This technique was already seen to work very well for (P_ε) – under the condition (1.2) – in [5].

Our result can be summarized as saying that, *under suitable assumptions on the domain and on the initial datum* u_0*, there exists at most one limit solution to the free boundary problem* (P) *with nonvanishing gradient near its free boundary, as long as the approximate initial data – converging uniformly to* u_0 *with supports that converge to the support of* u_0 *– satisfy (1.2).*

Moreover, under the same geometric assumptions, *if there exists a classical solution to* (P)*, this is the only limit of solutions to* (P_ε) *with initial data satisfying the conditions above. In particular, it is the only classical solution to* (P).

As already stated, in the case $u^\varepsilon = Y^\varepsilon$, uniqueness results for limit solutions under geometric hypotheses similar to the ones made here can be found in [9]. Also in [7] the authors study the uniqueness and agreement between different concepts of solutions of problem (P) (again in the case $u^\varepsilon = Y^\varepsilon$) under the assumption of the existence of a classical solution and under different geometric assumptions. See also [8] for a similar result in the two-phase case.

Notation. Throughout the paper N will denote the spatial dimension. In addition, the following notation will be used:

For any $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ and $\tau > 0$, $B_\tau(x_0) := \{x \in \mathbb{R}^N / |x - x_0| < \tau\}$ and $B_\tau(x_0, t_0) := \{(x,t) \in \mathbb{R}^{N+1} / |x - x_0|^2 + |t - t_0|^2 < \tau^2\}$.

When necessary, we will denote points in $\mathbb{R}^N$ by $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$. Given a function v, we will denote $v^+ = \max(v, 0)$.

The symbols Δ and ∇ will denote the corresponding operators in the space variables; the symbol ∂_p applied to a domain will denote parabolic boundary.

Finally, we will say that u is supercaloric if $\Delta u - u_t \leq 0$, and u is subcaloric if $\Delta u - u_t \geq 0$.

Outline of the paper. An outline of the contents is as follows. In Section 2 we give precise definitions of classical sub- and supersolutions and prove a comparison result for problem (P) (Lemma 2.1). In Section 3 we state some auxiliary results. In Section 4 we prove that a strict classical supersolution to problem (P) is the uniform limit of a family of supersolutions to problem (P_ε) (Theorem 4.1) and as a consequence we obtain the boundedness of the support for limit solutions in the geometry under consideration (Proposition 4.1). Finally, in Section 5 we prove our main result (Theorem 5.1). We discuss in a final section (Section 6) the results proved in the paper as well as other possible geometries that can be considered.

2. Preliminaries

Following [9] we will define what we will understand by a classical supersolution of problem (P). Note that the meaning of *classical* here differes from the usual one since we are not assuming that the function be C^1 up to the free boundary or that the free boundary be C^1.

Definition 2.1. A continuous nonnegative function u in $\overline{Q}_T = \mathbb{R}^N \times [0,T]$, $T > 0$, is called a classical supersolution of (P) if $u \in C^1(\{u > 0\})$ and

(i) $\Delta u - u_t \le 0$ in $\Omega = \{u > 0\}$;
(ii) $\limsup_{\Omega \ni (y,s) \to (x,t)} |\nabla u(y,s)| \le \sqrt{2M(x,t)}$ for every $(x,t) \in \partial\Omega \cap Q_T$;
(iii) $u(\cdot, 0) \ge u_0$.

Respectively, u is a classical subsolution of (P) if conditions (i), (ii) and (iii) are satisfied with reversed inequalities and lim inf instead of lim sup in (ii).

A function u is a classical solution of (P) if it is both a classical subsolution and a classical supersolution of (P), $u \in C^1(\overline{\{u > 0\}})$ and the free boundary $\partial\{u > 0\} \cap Q_T$ is a C^1 surface.

Next, a classical supersolution u of (P) is a strict classical supersolution of (P) if there is a $\delta > 0$ such that the stronger inequalities

(ii') $\limsup_{\Omega \ni (y,s) \to (x,t)} |\nabla u(y,s)| \le \sqrt{2M(x,t) - \delta}$ for every $(x,t) \in \partial\Omega \cap Q_T$;
(iii') $u(\cdot, 0) \ge u_0 + \delta$ on $\Omega_0 = \{u_0 > 0\}$

hold. Analogously a strict classical subsolution is defined.

As a consequence of the results in [5], one can check that every limit solution $u = \lim_{j\to\infty} u^{\varepsilon_j}$ of (P) is a classical supersolution in the sense of Definition 2.1. In fact,

Proposition 2.1. *Let u^{ε_j} be solutions to (P_{ε_j}) – with w^{ε_j} satisfying (1.3) and $w_0 > -1$ – such that $u^{\varepsilon_j} \to u$ uniformly on compact sets and $\varepsilon_j \to 0$. Assume that the initial datum u_0 is Lipschitz continuous and that the approximations of the initial datum verify $|u_0^\varepsilon(x)|, |\nabla u_0^\varepsilon(x)| \le C$ and $u_0^\varepsilon \in C^1(\overline{\{u_0^\varepsilon > 0\}})$. Then u is a classical supersolution of (P).*

Proof. We have to verify conditions (i)-(iii) of Definition 2.1.

From our assumptions on the initial datum u_0, by Proposition 5.2.1 of [6], we have that $u^\varepsilon \to u$ uniformly on compact sets of $\overline{Q_T}$ so that u is continuous up to $t = 0$ and (iii) holds.

Now, (i) is proved in [5].

Finally, (ii) is a straightforward modification of Theorem 6.1 of [2] using Lemmas 2.1, 2.2 and 2.3 of [5] instead of Lemma 3.2 and Propositions 5.2 and 5.3 of [2] respectively. $\square$

Let us suppose that the initial datum u_0 of problem (P) is starshaped with respect to a point x_0, that we always assume to be 0, in the following sense: For every $\lambda \in (0,1)$ and $x \in \mathbb{R}^N$,

$$u_0(\lambda x) \ge u_0(x), \qquad \lambda\Omega_0 \subset\subset \Omega_0, \tag{2.1}$$

where $\Omega_0 = \{u_0 > 0\}$.

Also, assume that

$$w_0(\lambda x, 0) \le w_0(x, 0) \quad \text{if } x \in \mathbb{R}^N\ ,\ 0 < \lambda < 1. \tag{2.2}$$

Let u be a classical supersolution of (P). Let λ and λ' be two real numbers with $0<\lambda<\lambda'<1$. Define

$$u_\lambda(x,t)=\frac{1}{\lambda'}u(\lambda x,\lambda^2 t) \tag{2.3}$$

in Q_{T/λ^2}. The rescaling is taken so that u_λ satisfies the heat equation in

$$\Omega_\lambda=\{(x,t):\ (\lambda x,\lambda^2 t)\in\Omega\}. \tag{2.4}$$

Moreover, the fact that $0<\lambda<\lambda'<1$ makes u_λ a strict classical supersolution of (P).

In fact, let us first see that

$$M(\lambda x,\lambda^2 t)\le M(x,t)\qquad \text{if } 0<\lambda<1.$$

This is a consequence of the fact that the function

$$a\longrightarrow\int_{-a}^{1}(s+a)f(s)\,ds$$

is nondecreasing and

$$w_0(\lambda x,\lambda^2 t)\le w_0(x,t)\qquad \text{if } 0<\lambda<1. \tag{2.5}$$

In fact, the function $w_\lambda(x,t)=w_0(\lambda x,\lambda^2 t)$ is caloric and $w_\lambda(x,0)\le w_0(x,0)$ if $0<\lambda<1$ by hypothesis. Thus, by the comparison principle, $w_\lambda(x,t)\le w_0(x,t)$ in $\mathbb{R}^N\times(0,T)$.

Now, let $(x_0,t_0)\in\partial\{u_\lambda>0\}$. Then,

$$\begin{aligned}\limsup_{\Omega_\lambda\ni(x,t)\to(x_0,t_0)}|\nabla u_\lambda(x,t)| &= \limsup_{\Omega\ni(\lambda x,\lambda^2 t)\to(\lambda x_0,\lambda^2 t_0)}\Big|\frac{\lambda}{\lambda'}\nabla u(\lambda x,\lambda^2 t)\Big|\\ &\le\frac{\lambda}{\lambda'}\sqrt{2M(\lambda x_0,\lambda^2 t_0)}\le\sqrt{2M(x_0,t_0)}-\Big(1-\frac{\lambda}{\lambda'}\Big)\sqrt{2M_0},\end{aligned}$$

where $0<M_0<M(x,t)$ in $\mathbb{R}^N\times(0,T)$.

On the other hand, since $\lambda\Omega_0\subset\subset\Omega_0$, there holds that

$$u_0(\lambda x)\ge\gamma>0\qquad \text{if } x\in\Omega_0.$$

Thus, for $x\in\Omega_0$,

$$\begin{aligned}u_\lambda(x,0)=\frac{1}{\lambda'}u_0(\lambda x)&=u_0(\lambda x)+\Big(\frac{1}{\lambda'}-1\Big)u_0(\lambda x)\\ &\ge u_0(x)+\Big(\frac{1}{\lambda'}-1\Big)\gamma.\end{aligned}$$

The following comparison lemma for problem (P) can be proved as Lemma 2.4 in [9]. We omit the proof.

Lemma 2.1. *Let u_0 satisfy* (2.1) *and w_0 satisfy (2.2). Then every classical subsolution of (P) with bounded support, is smaller than every classical supersolution of (P). i.e. if u' is a classical subsolution such that Ω' is bounded and u is a classical supersolution then*

$$\Omega'\subset\Omega\quad\textit{and}\quad u'\le u,$$

where $\Omega' = \{u' > 0\}$ *and* $\Omega = \{u > 0\}$.

3. Auxiliary results

This section contains results on the following problem:

$$(P_0) \qquad \Delta u - u_t = (u + \omega_0) f(u),$$

where the function f is as in Section 1 and ω_0 is a constant, $\omega_0 > -1$. The results will be used in the next sections where (P_0) appears as a blow-up limit. The proofs are very similar to those of Lemmas 4.1, 4.3 and 4.4 in [7]. We leave the details to the reader.

Lemma 3.1. *Let* $a, b \geq 0$ *and let* ψ *be the classical solution to*

$$\begin{aligned} &\psi_{ss} = (\psi + \omega_0) f(\psi) \quad \text{for } s > 0,\\ &\psi(0) = a, \quad \psi_s(0) = -\sqrt{2b}. \end{aligned} \tag{3.1}$$

Let $B(\tau) = \int_{-\omega_0}^{\tau} (\rho + \omega_0) f(\rho)\, d\rho$.

$$\text{If } b = 0 \text{ and } a \in \{-\omega_0\} \cup [1, +\infty), \text{ then } \psi \equiv a. \tag{3.2}$$

$$\text{If } b = 0 \text{ and } a \in (-\omega_0, 1), \text{ then } \lim_{s\to+\infty} \psi(s) = +\infty. \tag{3.3}$$

$$\text{If } b \in \big(0,\, B(a)\big), \text{ then } \lim_{s\to+\infty} \psi(s) = +\infty. \tag{3.4}$$

$$\text{If } 0 < b = B(a), \text{ then } \psi_s < 0 \text{ and } \lim_{s\to+\infty} \psi(s) = -\omega_0. \tag{3.5}$$

$$\text{If } b \in \big(B(a),\, +\infty\big), \text{ then } \psi_s < 0 \text{ and } \lim_{s\to+\infty} \psi(s) = -\infty. \tag{3.6}$$

Lemma 3.2. *Let* $B(\tau)$ *be as in the previous Lemma and let* $\mathcal{R}_\gamma = \big\{(x,t) \in \mathbb{R}^{N+1} / x_1 > 0\,,\ -\infty < t \leq \gamma\big\}$, $0 \leq \theta < 1 + \omega_0$ *and let* $U \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\mathcal{R}_\gamma})$ *be such that*

$$\begin{aligned} \Delta U - U_t &= (U + \omega_0) f(U) && \text{in } \mathcal{R}_\gamma,\\ U &= 1 - \theta && \text{on } \{x_1 = 0\},\\ -\omega_0 \leq U &\leq 1 - \theta && \text{in } \overline{\mathcal{R}_\gamma}. \end{aligned}$$

1) If $\theta = 0$, *then* $|\nabla U| \leq \sqrt{2B(1)}$ *on* $\{x_1 = 0\}$.

2) If $0 < \theta < 1 + \omega_0$ *and* $0 < \sigma < B(1)$ *are such that* $\int_{-\omega_0}^{1-\theta} (\rho + \omega_0) f(\rho)\, d\rho = B(1) - \sigma$, *then* $|\nabla U| = \sqrt{2(B(1) - \sigma)}$ *on* $\{x_1 = 0\}$.

Finally, we state a compactness result.

Lemma 3.3. *Let* ε_j, γ_{ε_j} *and* τ_{ε_j} *be sequences such that* $\varepsilon_j > 0$, $\varepsilon_j \to 0$, $\gamma_{\varepsilon_j} > 0$, $\gamma_{\varepsilon_j} \to \gamma$, *with* $0 \leq \gamma \leq +\infty$, $\tau_{\varepsilon_j} > 0$, $\tau_{\varepsilon_j} \to \tau$ *with* $0 \leq \tau \leq +\infty$, *and such that* $\tau < +\infty$ *implies that* $\gamma = +\infty$. *Assume that* $w^{\varepsilon_j}/\varepsilon_j$ *converge to* w_0 *uniformly in compact sets of* $\mathbb{R}^N \times [0, T]$. *Let* $\rho > 0$ *and*

$$\mathcal{A}_{\varepsilon_j} = \left\{ (x,t) \ / \ |x| < \frac{\rho}{\varepsilon_j}\,,\ -\min(\tau_{\varepsilon_j}, \frac{\rho^2}{\varepsilon_j^2}) < t < \min(\gamma_{\varepsilon_j}, \frac{\rho^2}{\varepsilon_j^2}) \right\}.$$

Assume that $0 \le \theta < 1 + w_0(x_0, t_0)$ *and let* $\bar u^{\varepsilon_j}$ *be weak solutions to*

$$\Delta \bar u^{\varepsilon_j} - \bar u^{\varepsilon_j}_t = \left(\bar u^{\varepsilon_j} + \frac{w^{\varepsilon_j}(\varepsilon_j x + x_{\varepsilon_j}, \varepsilon_j^2 t + t_{\varepsilon_j})}{\varepsilon_j} \right) f(\bar u^{\varepsilon_j}) \ \text{in } \{x_1 > \bar h_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j},$$
$$\bar u^{\varepsilon_j} = 1 - \theta \quad \text{on } \{x_1 = \bar h_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j},$$
$$-\frac{w^{\varepsilon_j}(\varepsilon_j x + x_{\varepsilon_j}, \varepsilon_j^2 t + t_{\varepsilon_j})}{\varepsilon_j} \le \bar u^{\varepsilon_j} \le 1 - \theta \quad \text{in } \{x_1 \ge \bar h_{\varepsilon_j}(x',t)\} \cap \overline{\mathcal{A}_{\varepsilon_j}},$$

where $(x_{\varepsilon_j}, t_{\varepsilon_j}) \to (x_0, t_0)$*, with* $\bar u^{\varepsilon_j} \in C(\{x_1 \ge \bar h_{\varepsilon_j}(x',t)\} \cap \overline{\mathcal{A}_{\varepsilon_j}})$*, and* $\nabla \bar u^{\varepsilon_j} \in L^2$*. Here* $\bar h_{\varepsilon_j}$ *are continuous functions such that* $\bar h_{\varepsilon_j}(0,0) = 0$ *with* $\bar h_{\varepsilon_j} \to 0$ *uniformly on compact subsets of* $\mathbb{R}^{N-1} \times (-\tau, \gamma)$*. Moreover, we assume that* $\|\bar h_{\varepsilon_j}\|_{C^1(K)} + \|\nabla_{x'} \bar h_{\varepsilon_j}\|_{C^{\alpha, \frac{\alpha}{2}}(K)}$ *are uniformly bounded, for every compact set* $K \subset \mathbb{R}^{N-1} \times (-\tau, \gamma)$*.*

Then, there exists a function $\bar u$ *such that, for a subsequence,*

$$\bar u \in C^{2+\alpha, 1+\frac{\alpha}{2}}\big(\{x_1 \ge 0,\ \gamma > t > -\tau\}\big),$$
$$\bar u^{\varepsilon_j} \to \bar u \quad \text{uniformly on compact subsets of } \{x_1 > 0,\ \gamma > t > -\tau\},$$
$$\begin{aligned} &\Delta \bar u - \bar u_t = (\bar u + w_0(x_0,t_0)) f(\bar u) && \text{in } \{x_1 > 0,\ \gamma > t > -\tau\},\\ &\bar u = 1 - \theta && \text{on } \{x_1 = 0,\ \gamma > t > -\tau\},\\ &-w_0(x_0,t_0) \le \bar u \le 1 - \theta && \text{in } \{x_1 \ge 0,\ \gamma > t > -\tau\}. \end{aligned}$$

If $\gamma < +\infty$*, we require, in addition, that*

$$\|\bar h_{\varepsilon_j}(x', t + \gamma_{\varepsilon_j} - \gamma)\|_{C^1(K)} + \|\nabla_{x'} \bar h_{\varepsilon_j}(x', t + \gamma_{\varepsilon_j} - \gamma)\|_{C^{\alpha, \frac{\alpha}{2}}(K)}$$

be uniformly bounded for every compact set $K \subset \mathbb{R}^{N-1} \times (-\infty, \gamma]$*. And we deduce that*

$$\bar u \in C^{2+\alpha, 1+\frac{\alpha}{2}}\big(\{x_1 \ge 0,\ t \le \gamma\}\big).$$

If $\tau < +\infty$*, we let*

$$\mathcal{B}_{\varepsilon_j} = \left\{ x \ / \ |x| < \frac{\rho}{\varepsilon_j},\ x_1 > \bar h_{\varepsilon_j}(x', -\tau_{\varepsilon_j}) \right\},$$

and we require, in addition, that for every $R > 0$*,*

$$\|\bar u^{\varepsilon_j}(x, -\tau_{\varepsilon_j})\|_{C^\alpha\left(\overline{\mathcal{B}}_{\varepsilon_j} \cap \overline{B}_R(0)\right)} \le C_R,$$

and that there exists $r > 0$ *such that*

$$\|\bar u^{\varepsilon_j}(x, -\tau_{\varepsilon_j})\|_{C^{1+\alpha}\left(\overline{\mathcal{B}}_{\varepsilon_j} \cap \overline{B}_r(0)\right)} \le C_r.$$

Moreover, we assume that $\|\bar h_{\varepsilon_j}(x', t - \tau_{\varepsilon_j} + \tau)\|_{C^1(K)} + \|\nabla_{x'} \bar h_{\varepsilon_j}(x', t - \tau_{\varepsilon_j} + \tau)\|_{C^{\alpha, \frac{\alpha}{2}}(K)}$ *are uniformly bounded for every compact set* $K \subset \mathbb{R}^{N-1} \times [-\tau, +\infty)$*.*

Then, there holds that

$$\bar u \in C^{\alpha, \frac{\alpha}{2}}\big(\{x_1 \ge 0,\ t \ge -\tau\}\big),\ \nabla \bar u \in C\big(\{0 \le x_1 < r,\ t \ge -\tau\}\big),$$
$$\bar u^{\varepsilon_j}(x, -\tau_{\varepsilon_j}) \to \bar u(x, -\tau) \quad \text{uniformly on compact subsets of } \{x_1 > 0\}.$$

In any case (τ, γ be infinite or finite)

$$|\nabla \bar{u}^{\varepsilon_j}(0,0)| \to |\nabla \bar{u}(0,0)|.$$

4. Approximation result

In this section we prove that, under certain assumptions, a classical supersolution to problem (P) is the uniform limit of a family of supersolutions to problem (P_ε) (Theorem 4.1), and we prove an analogous result for subsolutions (Theorem 4.2). Also, we prove that for compactly supported initial data, limit solutions have bounded support (Proposition 4.1).

The following construction follows the lines of Theorem 5.2 in [7]. In our case we have to be more careful with the construction of the initial data.

Theorem 4.1. *Let $\widetilde{u}$ be a classical supersolution to (P) in Q_T with $\widetilde{u} \in C^1(\overline{\{\widetilde{u} > 0\}})$ and such that $\{\widetilde{u} > 0\}$ is bounded. Assume, in addition, that there exist $\delta_0, s_0 > 0$ such that*

$$\begin{aligned} &|\nabla \widetilde{u}^+| \le \sqrt{2M(x,t) - \delta_0} \qquad \text{on } Q \cap \partial\{\widetilde{u} > 0\}, \\ &|\nabla \widetilde{u}| > \delta_0 \qquad \text{in } Q \cap \{0 < \widetilde{u} < s_0\}. \end{aligned}$$

Let w^ε be a solution of the heat equation in $\mathbb{R}^N \times (0,T)$ such that $\frac{w^\varepsilon(x,t)}{\varepsilon} \to w_0(x,t)$ uniformly in $\mathbb{R}^N \times [0,T]$ with $w_0 \in C(\mathbb{R}^N \times [0,T])$ and $w_0 \ge -1 + \delta_1$ for a certain positive constant δ_1.

Then, there exists a family $u^\varepsilon \in C(\overline{Q_T})$, with $\nabla u^\varepsilon \in L^2_{\rm loc}(\overline{Q_T})$, of weak supersolutions to (P_ε) in Q_T, such that, as $\varepsilon \to 0$, $u^\varepsilon \to \widetilde{u}$ uniformly in $\overline{Q_T}$.

Proof. Step I. Construction of the family u^ε. Let $0 < \theta < \delta_1$ be such that

$$\int_{1-\theta}^{1} (s + W) f(s)\, ds = \frac{\delta_0}{8},$$

where W is a suitable uniform bound of $\|w^\varepsilon/\varepsilon\|_{L^\infty(\{\widetilde{u}>0\})}$. For every $\varepsilon > 0$ small, we define the domain $D^\varepsilon = \{\widetilde{u} < (1-\theta)\varepsilon\} \subset Q_T$.

Let z^ε be the bounded solution to

$$\Delta z^\varepsilon - z^\varepsilon_t = (z^\varepsilon + w^\varepsilon) f_\varepsilon(z^\varepsilon) \quad \text{in } D^\varepsilon,$$

with boundary data

$$z^\varepsilon(x,t) = \begin{cases} (1-\theta)\varepsilon & \text{on } \partial D^\varepsilon \cap t > 0, \\ z^\varepsilon_0(x) & \text{in } D^\varepsilon \cap \{t = 0\}. \end{cases}$$

In order to give the initial data z^ε_0, we let $\psi^\varepsilon(s,x)$ be the solution to (3.1) with

$$a = 1 - \theta, \quad b = \int_{-w^\varepsilon(x,0)/\varepsilon}^{1-\theta} (s + \frac{w^\varepsilon(x,0)}{\varepsilon}) f(s)\, ds, \quad \omega_0 = \frac{w^\varepsilon(x,0)}{\varepsilon}.$$

Assume first that $|\nabla\widetilde{u}|$ is smooth. Then we let

$$\varphi^\varepsilon(\xi,x)=\psi^\varepsilon\Big(\frac{1-\theta-\xi}{|\nabla\widetilde{u}(x,0)|},x\Big),$$

and we define

$$z_0^\varepsilon(x)=\varepsilon\varphi^\varepsilon\Big(\frac{1}{\varepsilon}\widetilde{u}(x,0),x\Big).$$

If $\widetilde{u}$ is not regular enough, we can replace $|\nabla\widetilde{u}(x,0)|$ by a smooth approximation $F_\varepsilon(x)$ so that the initial datum z_0^ε is $C^{1+\alpha}$. We leave the details to the reader.

Finally, we define the family u^ε as follows:

$$u^\varepsilon=\begin{cases}\widetilde{u} & \text{in } \{\widetilde{u}\geq(1-\theta)\varepsilon\},\\ z^\varepsilon & \text{in } \overline{D^\varepsilon}.\end{cases}$$

Step II. Passage to the limit. If $(x,0)\in\overline{D^\varepsilon}$, we have $0\leq\frac{1}{\varepsilon}\widetilde{u}(x,0)\leq1-\theta$. Since, from Lemma 3.1, we know that $-w^\varepsilon(x,0)/\varepsilon\leq\psi(s,x)\leq1-\theta$ for $s\geq0$, it follows that $-w^\varepsilon(x,0)\leq z^\varepsilon(x,0)\leq(1-\theta)\varepsilon$. Since $f_\varepsilon(s)\geq0$, constant functions larger than $-w^\varepsilon(x,t)$ are supersolutions to (P_ε). Therefore, $(1-\theta)\varepsilon$ is a supersolution if $\varepsilon<\varepsilon_1$ and we may apply the comparison principle for bounded super and subsolutions of (P_ε) to conclude that $-w^\varepsilon\leq z^\varepsilon\leq(1-\theta)\varepsilon$.

Hence,

$$\sup_{\overline{Q_T}}|u^\varepsilon-\widetilde{u}|=\sup_{D^\varepsilon}|z^\varepsilon-\widetilde{u}|\leq C\varepsilon$$

and therefore, the convergence of the family v^ε follows.

Step III. Let us show that there exists $\varepsilon_0>0$ such that the functions u^ε are supersolutions to (P_ε) for $\varepsilon<\varepsilon_0$.

If $u^\varepsilon>(1-\theta)\varepsilon$, then $u^\varepsilon=\widetilde{u}$, which by hypothesis is supercaloric. Since $f_\varepsilon(s)\geq0$ and $(1-\theta)\varepsilon\geq-w^\varepsilon$ if $\varepsilon<\varepsilon_1$, it follows that u^ε are supersolutions to (P_ε) here.

If $u^\varepsilon<(1-\theta)\varepsilon$, then we are in D^ε and therefore, by construction, u^ε are solutions to (P_ε).

That is, the u^ε's are continuous functions, and they are piecewise supersolutions to (P_ε). In order to see that u^ε are globally supersolutions to (P_ε), it suffices to see that the jumps of the gradients (which occur at smooth surfaces), have the right sign.

To this effect, we will show that there exists $\varepsilon_0>0$ such that

$$|\nabla u^\varepsilon|\geq\sqrt{2M(x,t)-\delta_0/2}\quad\text{on } \{\widetilde{u}=(1-\theta)\varepsilon\},\ \text{ for } \varepsilon<\varepsilon_0. \tag{4.1}$$

Assume that (4.1) does not hold. Then, for every $j\in\mathbb{N}$, there exist $\varepsilon_j>0$ and $(x_{\varepsilon_j},t_{\varepsilon_j})\in Q$, with

$$\varepsilon_j\to0\quad\text{and}\quad(x_{\varepsilon_j},t_{\varepsilon_j})\to(x_0,t_0)\in\partial\{\widetilde{u}>0\}\cap\{\widetilde{u}=0\},$$

such that

$$u^{\varepsilon_j}(x_{\varepsilon_j},t_{\varepsilon_j})=(1-\theta)\varepsilon_j\quad\text{and}\quad|\nabla u^{\varepsilon_j}(x_{\varepsilon_j},t_{\varepsilon_j})|<\sqrt{2M(x_{\varepsilon_j},t_{\varepsilon_j})-\delta_0/2}. \tag{4.2}$$

From now on we will drop the subscript j when referring to the sequences defined above and $\varepsilon \to 0$ will mean $j \to \infty$.

We can assume (performing a rotation in the space variables if necessary) that there exists a family g_ε of smooth functions such that, in a neighborhood of $(x_\varepsilon, t_\varepsilon)$,

$$
\begin{aligned}
&\{u^\varepsilon = (1-\theta)\varepsilon\} = \{(x,t) \,/\, x_1 - x_{\varepsilon 1} = g_\varepsilon(x' - x_\varepsilon{}', t - t_\varepsilon)\},\\
&\{u^\varepsilon < (1-\theta)\varepsilon\} = \{(x,t) \,/\, x_1 - x_{\varepsilon 1} > g_\varepsilon(x' - x_\varepsilon{}', t - t_\varepsilon)\},
\end{aligned} \tag{4.3}
$$

where there holds that

$$
g_\varepsilon(0,0) = 0, \quad |\nabla_{x'} g_\varepsilon(0,0)| \to 0, \quad \varepsilon \to 0.
$$

We can assume that (4.3) holds in $\big(B_\rho(x_\varepsilon) \times (t_\varepsilon - \rho^2, t_\varepsilon + \rho^2)\big) \cap \{0 \le t \le T\}$ for some $\rho > 0$.

Let us now define

$$
\bar u^\varepsilon(x,t) = \frac{1}{\varepsilon} u^\varepsilon(x_\varepsilon + \varepsilon x, t_\varepsilon + \varepsilon^2 t), \quad \bar g_\varepsilon(x',t) = \frac{1}{\varepsilon} g_\varepsilon(\varepsilon x', \varepsilon^2 t),
$$

and let

$$
\tau_\varepsilon = \frac{t_\varepsilon}{\varepsilon^2}\ ,\ \gamma_\varepsilon = \frac{T - t_\varepsilon}{\varepsilon^2}.
$$

We have, for a subsequence,

$$
\tau_\varepsilon \to \tau\ ,\ \gamma_\varepsilon \to \gamma
$$

where $0 \le \tau, \gamma \le +\infty$ and τ and γ cannot be both finite.

We now let

$$
\mathcal{A}_\varepsilon = \left\{ (x,t) \ /\ |x| < \frac{\rho}{\varepsilon},\ -\min(\tau_\varepsilon, \frac{\rho^2}{\varepsilon^2}) < t < \min(\gamma_\varepsilon, \frac{\rho^2}{\varepsilon^2}) \right\}.
$$

Then, the functions $\bar u^\varepsilon$ are weak solutions to

$$
\begin{aligned}
&\Delta \bar u^\varepsilon - \bar u^\varepsilon_t = \left(\bar u^\varepsilon + \frac{w^\varepsilon(x_\varepsilon + \varepsilon x, t_\varepsilon + \varepsilon^2 t)}{\varepsilon} \right) f(\bar u^\varepsilon) && \text{in } \{x_1 > \bar g_\varepsilon(x',t)\} \cap \mathcal{A}_\varepsilon,\\
&\bar u^\varepsilon = 1 - \theta && \text{on } \{x_1 = \bar g_\varepsilon(x',t)\} \cap \mathcal{A}_\varepsilon,\\
&-\frac{w^\varepsilon(x_\varepsilon + \varepsilon x, t_\varepsilon + \varepsilon^2 t)}{\varepsilon} \le \bar u^\varepsilon \le 1 - \theta && \text{in } \{x_1 \ge \bar g_\varepsilon(x',t)\} \cap \overline{\mathcal{A}_\varepsilon}.
\end{aligned}
$$

Note that we are under the hypotheses of Lemma 3.3. Then, there exists a function $\bar u$ such that, for a subsequence,

$$
\bar u \in C^{2+\alpha, 1+\frac{\alpha}{2}}\big(\{x_1 \ge 0,\ -\tau < t < \gamma\}\big),
$$

$$
\bar u^\varepsilon \to \bar u \quad \text{uniformly on compact subsets of } \{x_1 > 0,\ -\tau < t < \gamma\},
$$

$$
\begin{aligned}
&\Delta \bar u - \bar u_t = (\bar u + w_0(x_0,t_0)) f(\bar u) && \text{in } \{x_1 > 0,\ -\tau < t < \gamma\},\\
&\bar u = 1 - \theta && \text{on } \{x_1 = 0,\ -\tau < t < \gamma\},\\
&-w_0(x_0,t_0) \le \bar u \le 1 - \theta && \text{in } \{x_1 \ge 0,\ -\tau < t < \gamma\}.
\end{aligned}
$$

We will divide the remainder of the proof into two cases, depending on whether $\tau = +\infty$ or $\tau < +\infty$.

Case I. Assume $\tau = +\infty$.

In this case, Lemma 3.3 also gives

$$|\nabla \bar{u}^\varepsilon(0,0)| \to |\nabla \bar{u}(0,0)|.$$

On the other hand, $\bar{u}$ satisfies the hypotheses of Lemma 3.2 and therefore,

$$|\nabla \bar{u}| \geq \sqrt{2M(x_0,t_0) - \delta_0/4} \quad \text{on } \{x_1 = 0\},$$

which yields

$$|\nabla \bar{u}^\varepsilon(0,0)| \geq \sqrt{2M(x_0,t_0) - 3\delta_0/8},$$

for ε small. But this gives

$$|\nabla u^\varepsilon(x_\varepsilon,t_\varepsilon)| \geq \sqrt{2M(x_\varepsilon,t_\varepsilon) - \delta_0/2},$$

for ε small. This contradicts (4.2) and completes the proof in case $\tau = +\infty$.

Case II. Assume $\tau < +\infty$. (In this case $\gamma = +\infty$.)

There holds that $\bar{u}^\varepsilon(x, -\tau_\varepsilon) = \frac{1}{\varepsilon} u^\varepsilon(x_\varepsilon + \varepsilon x, 0)$, then

$$\bar{u}^\varepsilon(x,-\tau_\varepsilon) = \varphi^\varepsilon\Big(\frac{1}{\varepsilon}\widetilde{u}(x_\varepsilon + \varepsilon x, 0), x_\varepsilon + \varepsilon x\Big). \tag{4.4}$$

Here we want to apply the result of Lemma 3.3 corresponding to $\tau < +\infty$. In fact, we can see that there exist $C, r > 0$ such that $\|\bar{u}^\varepsilon(\cdot, -\tau_\varepsilon)\|_{C^{1+\alpha}\left(\overline{B}_r(0)\right)} \leq C$.

Now Lemma 3.3 gives, for a subsequence,

$$\bar{u} \in C^{\alpha, \frac{\alpha}{2}}\big(\{x_1 \geq 0,\, t \geq -\tau\}\big),$$
$$\bar{u}^\varepsilon(x,-\tau_\varepsilon) \to \bar{u}(x,-\tau) \quad \text{uniformly on compact subsets of } \{x_1 > 0\}.$$

Therefore, we get that (recall that in the case we are considering $t_0 = 0$),

$$\bar{u}(x,-\tau) = \bar{\varphi}\Big(1 - \theta - |\nabla \widetilde{u}^+(x_0,t_0)|\, x_1, x_0\Big).$$

where $\bar{\varphi}(s,x) = \psi\Big(\dfrac{1-\theta-s}{|\nabla \widetilde{u}(x,0)|}, x\Big)$ and $\psi(s,x)$ is the solution of (3.1) with

$$a = 1-\theta, \quad b = \int_{-w_0(x,0)}^{1-\theta} (s + w_0(x,0)) f(s)\, ds, \quad \omega_0 = w_0(x,0).$$

Thus,

$$\bar{u}(x,-\tau) = \psi(x_1,x_0).$$

Since the function $\psi(x_1,x_0)$ is a stationary solution to equation (P_0), bounded for $x_1 \geq 0$, and $\bar{u} = \psi$ on the parabolic boundary of the domain $\big\{x_1 > 0,\, t > -\tau\big\}$, we conclude that

$$\bar{u}(x,t) = \psi(x_1,x_0) \quad \text{in } \big\{x_1 \geq 0,\, t \geq -\tau\big\}.$$

It follows from Lemma 3.1 and the choice of θ that

$$\frac{1}{2}|\nabla \bar{u}(0,0)|^2 = \frac{1}{2}\big(\psi_s(0,x_0)\big)^2 = \int_{-w_0(x_0,t_0)}^{1-\theta} (s + w_0(x_0,t_0)) f(s)\, ds \geq M(x_0,t_0) - \frac{\delta_0}{8}.$$

This is,

$$|\nabla \bar u| \geq \sqrt{2M(x_0,t_0) - \delta_0/4} \quad \text{on } \{x_1 = 0, t \geq -\tau\}.$$

But Lemma 3.3 gives

$$|\nabla \bar u^\varepsilon(0,0)| \to |\nabla \bar u(0,0)|,$$

which yields

$$|\nabla \bar u^\varepsilon(0,0)| \geq \sqrt{2M(x_0,t_0) - 3\delta_0/8},$$

for ε small. Then,

$$|\nabla u^\varepsilon(x_\varepsilon,t_\varepsilon)| \geq \sqrt{2M(x_\varepsilon,t_\varepsilon) - \delta_0/2},$$

for ε small. This contradicts (4.2) and completes the proof in case $\tau < +\infty$. ☐

Remark 4.1. Observe that from the construction of u^ε done in the previous proof, it follows that

$$u^\varepsilon \equiv \widetilde u \qquad \text{in } \{\widetilde u > (1-\theta)\varepsilon\}.$$

Theorem 4.2. *Let $\widetilde u$ be a classical subsolution to (P) in Q_T with $\widetilde u \in C^1(\overline{\{\widetilde u > 0\}})$ such that $\{\widetilde u > 0\}$ is bounded. Assume, in addition, that there exist $\delta_0 > 0$ such that*

$$|\nabla \widetilde u^+| \geq \sqrt{2M(x,t) + \delta_0} \qquad \text{on } Q \cap \partial\{\widetilde u > 0\}.$$

Let w^ε be a solution of the heat equation in $\mathbb{R}^N \times (0,T)$ such that $\frac{w^\varepsilon(x,t)}{\varepsilon} \to w_0(x,t)$ uniformly in $\mathbb{R}^N \times [0,T]$. And assume, moreover that $w_0 \in C(\mathbb{R}^N \times [0,T])$ and $w_0(x,t) \geq -1 + \delta_1$ for a certain positive constant δ_1.

Then, there exists a family $u^\varepsilon \in C(\overline{Q_T})$, with $\nabla u^\varepsilon \in L^2_{\text{loc}}(\overline{Q_T})$, of weak subsolutions to (P_ε) in Q_T, such that, as $\varepsilon \to 0$, $u^\varepsilon \to \widetilde u$ uniformly in $\overline{Q_T}$.

Proof. The proof is analogous to Theorem 4.1. See [7] for a similar result in the case $w^\varepsilon = 0$. ☐

Finally, we end this Section by showing that, for compactly supported initial data, the support of a limit solution of problem (P) is bounded.

Proposition 4.1. *Let $u_0 \in C(\mathbb{R}^N)$ with compact support. Let u_0^ε converge uniformly to u_0 with supports converging to the support of u_0 and let w^ε be a solution of the heat equation in $\mathbb{R}^N \times (0,T)$ such that $\frac{w^\varepsilon(x,t)}{\varepsilon} \to w_0(x,t)$ uniformly in $\mathbb{R}^N \times [0,T]$. And assume, moreover that $w_0 \in C(\mathbb{R}^N \times [0,T])$ and $w_0(x,t) \geq -1 + \delta_1$ for a certain positive constant δ_1. Finally, let u^ε be the solution to (P_ε) with function w^ε and initial condition u_0^ε.*

Let $u = \lim u^{\varepsilon_j}$. Then $\{u > 0\}$ is bounded. Moreover, u vanishes in finite time.

Proof. Let $-1 < \omega_0 < w^\varepsilon(x,t)/\varepsilon$. Then it is easy to check that

$$M_{\omega_0} = \int_{-\omega_0}^1 (s + \omega_0) f(s)\, ds < M(x,t) = \int_{-w_0(x,t)}^1 (s + w_0(x,t)) f(s)\, ds. \tag{4.5}$$

Let us now consider the following self-similar function

$$V(x,t;T) = (T-t)^{1/2} h(|x|(T-t)^{-1/2}),$$

where $h = h(r)$ is a solution of

$$
\begin{aligned}
& h'' + \left(\frac{N-1}{r} + \frac{1}{2}r\right)h' + \frac{1}{2}h = 0, \quad 0 < r < R, \\
& h'(0) = 0, \quad h(r) > 0, \quad 0 \le r < R, \\
& h(R) = 0, \quad h'(R) = -\sqrt{2M_{\omega_0}}.
\end{aligned} \tag{4.6}
$$

It is proved in [4], Proposition 1.1, that there exists a unique $R > 0$ and a unique h solution of (4.6).

Moreover, it can be checked that if one picks T sufficiently large, then

$$V(x, 0; T) \ge u_0 + 1 \quad \text{in } \{u_0 > 0\},$$

and so $V(x, t; T)$ is a strict classical supersolution of (P) with bounded support and positive gradient near its free boundary.

Now, let u^{ε_j} be solutions to (P_{ε_j}) – with initial data $u_0^{\varepsilon_j}$ converging unifomly to u_0 such that support $u_0^{\varepsilon_j} \to$ support u_0 – such that $u = \lim u^{\varepsilon_j}$.

By Theorem 4.1, there exists a family v^{ε_j} of supersolutions of (P_{ε_j}) such that $v^{\varepsilon_j} \to V$ uniformly on compact sets, and $v^{\varepsilon_j}(x, 0) \ge u^{\varepsilon_j}(x, 0)$. Therefore, by the comparison principle, we obtain $u^{\varepsilon_j} \le v^{\varepsilon_j}$ and passing to the limit $u(x, t) \le V(x, t; T)$, and the result follows. □

5. Uniqueness of the limit solution

In this section we arrive at the main point of the article: we prove that, under certain assumptions, there exists a unique limit solution to the initial and boundary value problem associated to (P) as long as condition (1.2) is satisfied.

Let us begin with the following Proposition that is the key ingredient in the proof of our main result.

Proposition 5.1. *Let $\widetilde{u}$ be a strict classical supersolution to (P) with bounded support in $\mathbb{R}^N \times (0, T)$ such that there exists $s_0 > 0$ so that $|\nabla \widetilde{u}| > 0$ in $\{0 < \widetilde{u} < s_0\}$ and let $w^\varepsilon/\varepsilon$ be solutions to the heat equation in $\mathbb{R}^N \times (0, T)$ converging to w_0 uniformly with $w_0 \in C(\mathbb{R}^N \times [0, T])$ and $w_0 \ge -1 + \delta_1$ for a certain positive constant δ_1.*

Let u^ε be solutions to (P_ε) with function w^ε and initial condition u_0^ε, where u_0^ε are uniform approximations of u_0 with support $u_0^\varepsilon \to$ support u_0. *Then*

$$\limsup_{\varepsilon \to 0+} u^\varepsilon(x, t) \le \widetilde{u}(x, t)$$

for every $(x, t) \in Q_T$.

Proof. Let $\widetilde{u}$ be a strict classical supersolution of (P). Let us first, define the following regularization

$$u(x, t) = (\widetilde{u}(x, t + h) - \eta)^+,$$

for $h, \eta > 0$ small. So that u is a strict classical supersolution of (P) with C^1 free boundary, $C^1(\overline{\{u > 0\}})$ and $|\nabla u| > \delta_0 > 0$ in a neighborhood of its free boundary. So, by Theorem 4.1, there exists v^ε supersolution of (P_ε) such that $v^\varepsilon \to u$ uniformly in Q_{T-h}.

Now, using the comparison principle, we conclude that $u^\varepsilon \le v^\varepsilon$ in Q_{T-h}, and the Proposition now follows letting first $\varepsilon \to 0+$ and then $h, \eta \to 0+$. $\square$

Finally, we arrive at the main point of the paper: The uniqueness of limit solutions of (P).

Theorem 5.1. *Let the initial datum u_0 be Lipschitz, with compact support and satisfy the condition* (2.1). *Then there exists at most one limit solution such that its gradient does not vanish near its free boundary as long as the function w^ε in problem (P_ε) satisfies condition (1.3).*

More precisely, let $u_0^{\varepsilon_j}, \tilde u_0^{\varepsilon_k}$ be uniformly Lipschitz continuous in $\mathbb{R}^N$ with uniformly bounded Lipschitz norms and ε_j, $\varepsilon_k \to 0$. Assume that $u_0^{\varepsilon_j} \in C^1(\overline{\{u_0^{\varepsilon_j} > 0\}})$, $\tilde u_0^{\varepsilon_k} \in C^1(\overline{\{\tilde u_0^{\varepsilon_k} > 0\}})$, $u_0^{\varepsilon_j}, \tilde u_0^{\varepsilon_k} \to u_0$ uniformly and support $u_0^{\varepsilon_j}$, support $\tilde u_0^{\varepsilon_k} \to$ support u_0. *Let $w^{\varepsilon_j}/\varepsilon_j$ and $\tilde w^{\varepsilon_k}/\varepsilon_k$ be solutions of the heat equation converging to the same function $w_0 \in C(\overline{Q_T})$, uniformly bounded from below by $-1+\delta_1$ for a certain positive constant δ_1. Also, assume that w_0 satisfies the monotonicity condition (2.2).*

Let u^{ε_j} (resp. $\tilde u^{\varepsilon_k}$) be the solution to (P_{ε_j}) with function w^{ε_j} and initial datum $u_0^{\varepsilon_j}$ (resp. solution to (P_{ε_k}) with function $\tilde w^{\varepsilon_k}$ and initial datum $\tilde u_0^{\varepsilon_k}$). Let $u = \lim u^{\varepsilon_j}$ and $\tilde u = \lim \tilde u^{\varepsilon_k}$. If there exists $s_0 > 0$ such that $|\nabla \tilde u| > 0$ in $\{0 < \tilde u < s_0\}$.

Then, $u \le \tilde u$.

Proof. Since $\tilde u$ is a classical supersolution of (P), $\tilde u \in C^1(\{\tilde u > 0\})$ and, by Propositon 4.1, its support is bounded, the function $\tilde u_\lambda$ as defined in (2.3) satisfies the hypotheses of Proposition 5.1 in $Q_{T/\lambda^2} \supset Q_T$. So by letting $\lambda \to 1-$ we arrive at

$$u(x,t) \le \tilde u(x,t). \tag{5.1}$$

This finishes the proof.

$\square$

Theorem 5.2. *Let the initial datum u_0 be as in Theorem 5.1. Assume that there exists a classical solution v to (P) with initial datum u_0 and let $u_0^{\varepsilon_j}$ be uniformly Lipschitz continuous in $\mathbb{R}^N$ with $\varepsilon_j \to 0$, such that $u_0^{\varepsilon_j} \in C^1(\overline{\{u_0^{\varepsilon_j} > 0\}})$, $u_0^{\varepsilon_j} \to u_0$ uniformly and* support $u_0^{\varepsilon_j} \to$ support u_0. *Assume $w^{\varepsilon_j}/\varepsilon_j$ is a solution of the heat equation converging to w_0 uniformly with $w_0 \in C(\mathbb{R}^N \times [0,T])$ and $w_0 \ge -1 + \delta_1$ in $\mathbb{R}^N \times (0,T)$ for a certain $\delta_1 > 0$. Also, assume that w_0 satisfies the monotonicity condition (2.2).*

Let u^{ε_j} be the solution to (P_{ε_j}) with function w^{ε_j} and initial datum $u_0^{\varepsilon_j}$ and let $u = \lim u^{\varepsilon_j}$. Then, $u = v$.

In particular, there exists at most one classical solution to (P).

Proof. Since u is a classical supersolution to (P) and v is a classical subsolution, Lemma 2.1 applies and we get that $v \leq u$.

On the other hand, if we define v_λ as in (2.3), with $0 < \lambda < \lambda' < 1$, we have that v_λ satisfies the hypotheses of Proposition 5.1. Thus, there exists a family v^{ε_j} of supersolutions to (P_{ε_j}) with function w^{ε_j} such that, for a subsequence, $v^{\varepsilon_j} \to v$ with initial data converging uniformly to u_0. So by the comparison principle

$$u = \lim u^{\varepsilon_j} \leq \lim v^{\varepsilon_j} = v.$$

This finishes the proof. □

6. Conclusions

In this paper we have proved that the limits of sequences of solutions to (P_ε) with different constitutive functions w^ε and initial data u_0^ε coincide – as long as certain monotonicity assumptions are made – if the limit of $w^\varepsilon/\varepsilon$ and of u_0^ε are prescribed.

The monotonicity assumptions are necessary to provide strict classical supersolutions as close as we want to any classical supersolution. This kind of condition was also used with the same purpose – in the case in which $w^\varepsilon = 0$ – in [9] and [7]. In the latter, a different geometry was considered namely, the domain was a cylinder, Neumann boundary conditions were given on the boundary of the cylinder and monotonicity in the direction of the cylinder axis was assumed. In [7] it was proved that, if a classical solution exists and $w^\varepsilon = 0$, then it is equal to any limit of solutions to (P_ε).

In our case, this is with $w^\varepsilon \neq 0$ satisfying (1.3) and nondecreasing in the direction of the cylinder axis, the uniqueness result in the presence of a classical solution still holds.

The cylindrical geometry has the advantage of giving the condition of nonvanishing gradient in the positivity set of any limit solution. Since in dimension 2 one can prove that limit solutions are classical supersolutions up to the fixed boundary, the uniqueness of limit solutions follows in this case without further assumptions.

References

[1] J.D. Buckmaster and G.S.S. Ludford, *Theory of Laminar Flames*, Cambridge University Press, Cambridge, 1982. MR **84f:**80011

[2] L.A. Caffarelli, C. Lederman, and N. Wolanski, *Uniform estimates and limits for a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J. **46 (2)** (1997), 453–490. MR **98i:**35099

[3] ———, *Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J. **46 (3)** (1997), 719–740. MR **99c:**35116

[4] L.A. Caffarelli and J.L. Vazquez, *A free boundary problem for the heat equation arising in flame propagation*, Trans. Amer. Math. Soc. **347** (1995), 411–441. MR **95e:**35097

[5] J. Fernández Bonder and N. Wolanski, *A free boundary problem in combustion theory.* Interfaces Free Bound., **2** (2000), 381–411.

[6] A. Langlois, *Sur l'ètude asymptotique d'un système parabolique modèlisant des flames presque èquidiffusives.* PhD dissertation, Ecole Central de Lyon, December 2000.

[7] C. Lederman, J.L. Vazquez, and N. Wolanski, *Uniqueness of solution to a free boundary problem from combustion.* Trans. Amer. Math. Soc. **353** (2001), no. 2, 655–692.

[8] ———, *Uniqueness of solution to a two-phase free boundary problem from combustion*, to appear in Advances Differential Equations.

[9] A. Petrosyan, *On existence and uniqueness in a free boundary problem from combustion*, preprint.

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