MULTIPLE SOLUTIONS FOR THE $p-$LAPLACE EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this note we show the existence of at least three nontrivial solutions to the following quasilinear elliptic equation $-\Delta_p u + |u|^{p-2}u = f(x,u)$ in a smooth bounded domain Ω of $\mathbb{R}^N$ with nonlinear boundary conditions $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u)$ on $\partial\Omega$. The proof is based on variational arguments.

1. Introduction.

Let us consider the following nonlinear elliptic problem:

$$
\text{(P)} \qquad \begin{cases} -\Delta_p u + |u|^{p-2}u = f(x,u) & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases}
$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$laplacian and $\partial/\partial\nu$ is the outer unit normal derivative.

Problems like (P) appears naturally in several branches of pure and applied mathematics, such as the study of optimal constants for the Sobolev trace embedding (see [5, 10, 12, 11]); the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [7, 16]), non-Newtonian fluids, reaction diffusion problems, flow through porus media, nonlinear elasticity, glaciology, etc. (see [1, 2, 3, 6]).

The purpose of this note, is to prove the existence of at least three nontrivial solutions for (P) under adequate assumptions on the sources terms f and g. This result extends previous work by the author [8, 9].

Here, no oddness condition is imposed in f or g and a positive, a negative and a sign-changing solution are found. The proof relies on the Lusternik–Schnirelman method for non-compact manifolds (see [14]).

For a related result with Dirichlet boundary conditions, see [15] and more recently [4, 17]. The approach in this note follows the one in [15].

Throughout this work, by (weak) solutions of (P) we understand critical points of the associated energy functional acting on the Sobolev space $W^{1,p}(\Omega)$:

$$
\text{(1)} \qquad \Phi(v) = \frac{1}{p}\int_\Omega |\nabla v|^p + |v|^p\,dx - \int_\Omega F(x,v)\,dx - \int_{\partial\Omega} G(x,v)\,dS,
$$

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where $F(x,u) = \int_0^u f(x,z)\,dz$, $G(x,u) = \int_0^u g(x,z)\,dz$ and dS is the surface measure.

We will denote

$$\mathcal{F}(v) = \int_\Omega F(x,v)\,dx \qquad \text{and} \qquad \mathcal{G}(v) = \int_{\partial\Omega} G(x,v)\,dS, \tag{2}$$

so the functional Φ can be rewritten as

$$\Phi(v) = \frac{1}{p}\|v\|^p_{W^{1,p}(\Omega)} - \mathcal{F}(v) - \mathcal{G}(v).$$

2. Assumptions and statement of the results.

The precise assumptions on the source terms f and g are as follows:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x,0) = 0$ for every $x \in \Omega$.

(F2) There exist constants $p < q < p^* = Np/(N-p)$, $s > p^*/(p^*-q)$, $t = sq/(2+(q-2)s) > p^*/(p^*-2)$ and functions $a \in L^s(\Omega)$, $b \in L^t(\Omega)$, such that for $x \in \Omega$, $u, v \in \mathbb{R}$,

$$|f_u(x,u)| \le a(x)|u|^{q-2} + b(x),$$
$$|(f_u(x,u) - f_u(x,v))u| \le (a(x)(|u|^{q-2} + |v|^{q-2}) + b(x))|u-v|.$$

(F3) There exist constants $c_1 \in (0, 1/(p-1))$, $c_2 > p$, $0 < c_3 < c_4$, such that for any $u \in L^q(\Omega)$

$$c_3\|u\|^q_{L^q(\Omega)} \le c_2 \int_\Omega F(x,u)\,dx \le \int_\Omega f(x,u)u\,dx \le$$
$$c_1 \int_\Omega f_u(x,u)u^2\,dx \le c_4\|u\|^q_{L^q(\Omega)}.$$

(G1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $y \in \partial\Omega$. Moreover, $g(y,0) = 0$ for every $y \in \partial\Omega$.

(G2) There exist constants $p < r < p_* = (N-1)p/(N-p)$, $\sigma > p_*/(p_*-r)$, $\tau = \sigma r/(2+(r-2)\sigma) > p_*/(p_*-2)$ and functions $\alpha \in L^\sigma(\partial\Omega)$, $\beta \in L^\tau(\partial\Omega)$, such that for $y \in \partial\Omega$, $u, v \in \mathbb{R}$,

$$|g_u(y,u)| \le \alpha(y)|u|^{r-2} + \beta(y),$$
$$|(g_u(y,u) - g_u(y,v))u| \le (\alpha(y)(|u|^{r-2} + |v|^{r-2}) + \beta(y))|u-v|.$$

(G3) There exist constants $k_1 \in (0, 1/(p-1))$, $k_2 > p$, $0 < k_3 < k_4$, such that for any $u \in L^r(\partial\Omega)$

$$k_3\|u\|^r_{L^r(\partial\Omega)} \le k_2 \int_{\partial\Omega} G(x,u)\,dS \le \int_{\partial\Omega} g(x,u)u\,dS \le$$
$$k_1 \int_{\partial\Omega} g_u(x,u)u^2\,dx \le k_4\|u\|^r_{L^r(\partial\Omega)}.$$

Remark 1. Assumptions (F1)–(F3) implies, since the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $1 < q < p^*$ is compact, that $\mathcal{F}$ is C^1 with compact derivative. Analogously, (G1)–(G3) implies the same facts for $\mathcal{G}$ by the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ for $1 < r < p_*$.

So the main result of the paper reads:

Theorem 1. *Under assumptions* (F1)–(F3), (G1)–(G3)*, there exist three different, nontrivial, (weak) solutions of problem* (P). *Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

3. Proof of Theorem 1.

The proof uses the same approach as in [15]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that Φ has a critical point in K_i. These sets will be subsets of smooth manifolds $M_i \subset W^{1,p}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$
\begin{aligned}
M_1 &= \{u \in W^{1,p}(\Omega) \mid && \textstyle\int_{\partial\Omega} u_+ \, dS > 0 \text{ and} \\
& && \|u_+\|^p_{W^{1,p}(\Omega)} = \langle \mathcal{F}'(u), u_+\rangle + \langle \mathcal{G}'(u), u_+\rangle\},\\
M_2 &= \{u \in W^{1,p}(\Omega) \mid && \textstyle\int_{\partial\Omega} u_- \, dS > 0 \text{ and} \\
& && \|u_-\|^p_{W^{1,p}(\Omega)} = \langle \mathcal{F}'(u), u_-\rangle + \langle \mathcal{G}'(u), u_-\rangle\},\\
M_3 &= M_1 \cap M_2,
\end{aligned}
$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of u, and $\langle \cdot, \cdot\rangle$ is the duality pairing of $W^{1,p}(\Omega)$.

Finally we define

$$
\begin{aligned}
K_1 &= \{u \in M_1 \mid u \geq 0\},\\
K_2 &= \{u \in M_2 \mid u \leq 0\},\\
K_3 &= M_3.
\end{aligned}
$$

For the proof of the Theorem, we need the following Lemmas.

Lemma 1. *There exist $c_j > 0$ such that, for every $u \in K_i$, $i = 1, 2, 3$,*

$$\|u\|^p_{W^{1,p}(\Omega)} \leq c_1 \left(\int_\Omega f(x,u)u \, dx + \int_{\partial\Omega} g(x,u)u \, dS \right) \leq c_2 \Phi(u) \leq c_3 \|u\|^p_{W^{1,p}(\Omega)}.$$

Proof. As $u \in K_i$, we have that

$$\|u\|^p_{W^{1,p}(\Omega)} = \int_\Omega f(x,u)u \, dx + \int_{\partial\Omega} g(x,u)u \, dS.$$

This proves the first inequality.

Now, by (F3) and (G3)

$$\int_\Omega F(x,u) \, dx \leq \frac{1}{k_2} \int_\Omega f(x,u)u \, dx, \qquad \int_{\partial\Omega} G(x,u) \, dS \leq \frac{1}{c_2} \int_{\partial\Omega} g(x,u)u \, dS.$$

So, for $C = \max\{\frac{1}{k_2}; \frac{1}{c_2}\} < \frac{1}{p}$, we have

$$\Phi(u) \leq (\frac{1}{p} - C)\|u\|^p_{W^{1,p}(\Omega)}.$$

This proves the third inequality.

To prove the middle inequality we proceed as follows:

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \int_\Omega F(x,u)\,dx - \int_{\partial\Omega} G(x,u)\,dS\\ &= \frac{1}{p}\left(\int_\Omega f(x,u)u\,dx + \int_{\partial\Omega} g(x,u)u\,dS\right) - \left(\int_\Omega F(x,u)\,dx + \int_{\partial\Omega} G(x,u)\,dS\right)\\ &\ge (\frac{1}{p} - C)\left(\int_\Omega f(x,u)u\,dx + \int_{\partial\Omega} g(x,u)u\,dS\right).\end{aligned}$$

This finishes the proof. □

Lemma 2. *There exists $c > 0$ such that*

$$\begin{aligned}&\|u_+\|_{W^{1,p}(\Omega)} \ge c \quad \textit{for} \quad u \in K_1,\\ &\|u_-\|_{W^{1,p}(\Omega)} \ge c \quad \textit{for} \quad u \in K_2 \quad \textit{and}\\ &\|u_+\|_{W^{1,p}(\Omega)}, \|u_-\|_{W^{1,p}(\Omega)} \ge c \quad \textit{for} \quad u \in K_3.\end{aligned}$$

Proof. By the definition of K_i, by (F3) and (G3), we have that

$$\|u_\pm\|^p_{W^{1,p}(\Omega)} = \int_\Omega f(x,u)u_\pm\,dx + \int_{\partial\Omega} g(x,u)u_\pm\,dS \le c(\|u_\pm\|^q_{L^q(\Omega)} + \|u_\pm\|^r_{L^r(\partial\Omega)}).$$

Now the proof follows by the Sobolev immersion Theorem and by the Sobolev trace Theorem, as $p < q, r$. □

Lemma 3. *There exists $c > 0$ such that $\Phi(u) \ge c\|u\|^p_{W^{1,p}(\Omega)}$ for every $u \in W^{1,p}(\Omega)$ such that $\|u\|_{W^{1,p}(\Omega)} \le c$.*

Proof. By (F3), (G3) and the Sobolev immersions we have

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \mathcal{F}(u) - \mathcal{G}(u) \ge \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c(\|u\|^q_{L^q(\Omega)} + \|u\|^r_{L^r(\partial\Omega)})\\ &\ge \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c(\|u\|^q_{W^{1,p}(\Omega)} + \|u\|^r_{W^{1,p}(\Omega)}) \ge c\|u\|^p_{W^{1,p}(\Omega)},\end{aligned}$$

if $\|u\|_{W^{1,p}(\Omega)}$ is small enough, as $p < q, r$. □

The following lemma describes the properties of the manifolds M_i.

Lemma 4. *M_i is a $C^{1,1}$ sub-manifold of $W^{1,p}(\Omega)$ of co-dimension 1 ($i = 1, 2$), 2 ($i = 3$) respectively. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_u W^{1,p}(\Omega) = T_u M_i \oplus span\{u_+, u_-\},$$

where $T_u M$ is the tangent space at u of the Banach manifold M. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof. Let us denote

$$\begin{aligned}\bar{M}_1 &= \left\{u \in W^{1,p}(\Omega) \mid \int_{\partial\Omega} u_+\,dS > 0\right\}\\ \bar{M}_2 &= \left\{u \in W^{1,p}(\Omega) \mid \int_{\partial\Omega} u_-\,dS > 0\right\}\\ \bar{M}_3 &= \bar{M}_1 \cap \bar{M}_2.\end{aligned}$$

Observe that $M_i \subset \bar{M}_i$.

By the Sobolev trace Theorem, the set $\bar{M}_i$ is open in $W^{1,p}(\Omega)$, therefore it is enough to prove that M_i is a smooth sub-manifold of $\bar{M}_i$. In order to do this, we will construct a $C^{1,1}$ function $\varphi_i : \bar{M}_i \to \mathbb{R}^d$ with $d = 1$ $(i = 1, 2)$, $d = 2$ $(i = 3)$ respectively and M_i will be the inverse image of a regular value of φ_i.

In fact, we define: For $u \in \bar{M}_1$,

$$\varphi_1(u) = \|u_+\|^p_{W^{1,p}(\Omega)} - \langle \mathcal{F}'(u), u_+\rangle - \langle \mathcal{G}'(u), u_+\rangle.$$

For $u \in \bar{M}_2$,

$$\varphi_2(u) = \|u_-\|^p_{W^{1,p}(\Omega)} - \langle \mathcal{F}'(u), u_-\rangle - \langle \mathcal{G}'(u), u_-\rangle.$$

For $u \in \bar{M}_3$,

$$\varphi_3(u) = (k_1(u), k_2(u)).$$

Obviously, we have $M_i = \varphi_i^{-1}(0)$. We need to show that 0 is a regular value for φ_i. To this end we compute, for $u \in M_1$,

$$\begin{aligned}
\langle \nabla\varphi_1(u), u_+\rangle =& p\|u_+\|^p_{W^{1,p}(\Omega)} - \int_\Omega f_u(x,u)u_+^2 + f(x,u)u_+\,dx \\
& - \int_{\partial\Omega} g_u(x,u)u_+^2 + g(x,u)u_+\,dS \\
=& (p-1)\int_\Omega f(x,u)u_+\,dx - \int_\Omega f_u(x,u)u_+^2\,dx \\
& + (p-1)\int_{\partial\Omega} g(x,u)u_+\,dS - \int_{\partial\Omega} g_u(x,u)u_+^2\,dS.
\end{aligned}$$

By (F3) and (G3) the last term is bounded by

$$(p-1-c_1^{-1})\int_\Omega f(x,u)u_+\,dx + (p-1-k_1^{-1})\int_{\partial\Omega} g(x,u)u_+\,dS.$$

Recall that $c_1, k_1 < 1/(p-1)$. Now, by Lemma 1, this is bounded by

$$-c\|u_+\|^p_{W^{1,p}(\Omega)}$$

which is strictly negative by Lemma 2. Therefore, M_1 is a smooth sub-manifold of $W^{1,p}(\Omega)$. The exact same argument applies to M_2.

Since trivially

$$\langle \nabla\varphi_1(u), u_-\rangle = \langle \nabla\varphi_2(u), u_+\rangle = 0$$

for $u \in M_3$, the same conclusion holds for M_3.

To see that K_i is complete, let u_k be a Cauchy sequence in K_i, then $u_k \to u$ in $W^{1,p}(\Omega)$. Moreover, $(u_k)_\pm \to u_\pm$ in $W^{1,p}(\Omega)$. Now it is easy to see, by Lemma 2 and by continuity that $u \in K_i$.

Finally, by the first part of the proof we have the decomposition

$$T_uW^{1,p}(\Omega) = T_uM_i \oplus \operatorname{span}\{u_+, u_-\}.$$

Now let $v \in T_uW^{1,p}(\Omega)$ be a unit tangential vector, then $v = v_1 + v_2$ where v_i are given by

$$v_2 = (\nabla\varphi_i(u)|_{\operatorname{span}\{u_+,u_-\}})^{-1}\langle \nabla\varphi_i(u), v\rangle \in \operatorname{span}\{u_+, u_-\}, \quad v_1 = v - v_2 \in T_uM_i.$$

From these formulas and from the estimates given in the first part of the proof, the uniform continuity follows. □

Now, we need to check the Palais-Smale condition for the functional Φ restricted to the manifold M_i.

Lemma 5. *The functional $\Phi|_{K_i}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_k\} \subset K_i$ be a Palais-Smale sequence, that is $\Phi(u_k)$ is uniformly bounded and $\nabla\Phi|_{K_i}(u_k) \to 0$ strongly. We need to show that there exists a subsequence u_{k_j} that converges strongly in K_i.

Let $v_j \in T_{u_j}W^{1,p}(\Omega)$ be a unit tangential vector such that

$$\langle \nabla\Phi(u_j), v_j\rangle = \|\nabla\Phi(u_j)\|_{(W^{1,p}(\Omega))'}.$$

Now, by Lemma 4, $v_j = w_j + z_j$ with $w_j \in T_{u_j}M_i$ and $z_j \in \text{span}\{(u_j)_+, (u_j)_-\}$.

Since $\Phi(u_j)$ is uniformly bounded, by Lemma 1, u_j is uniformly bounded in $W^{1,p}(\Omega)$ and hence w_j is uniformly bounded in $W^{1,p}(\Omega)$. Therefore

$$\|\Phi(u_j)\|_{(W^{1,p}(\Omega))'} = \langle \nabla\Phi(u_j), v_j\rangle = \langle \nabla\Phi|_{K_i}(u_j), v_j\rangle \to 0.$$

As u_j is bounded in $W^{1,p}(\Omega)$, there exists $u \in W^{1,p}(\Omega)$ such that $u_j \rightharpoonup u$, weakly in $W^{1,p}(\Omega)$. As it is well known that the unrestricted functional Φ satisfies the Palais-Smale condition (cf. [9] and [13]), the lemma follows.

See [15] for the details. □

We now immediately obtain

Lemma 6. *Let $u \in K_i$ be a critical point of the restricted functional $\Phi|_{K_i}$. Then u is also a critical point of the unrestricted functional Φ and hence a weak solution to* (P).

With all this preparatives, the proof of the Theorem follows easily.

Proof of Theorem 1. The proof now is a standard application of the Lusternik–Schnirelman method for non-compact manifolds. See [14]. □

References

[1] D. Arcoya and J.I. Diaz. *S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology.* J. Differential Equations, **150** (1998), 215–225.

[2] C. Atkinson and K. El Kalli. *Some boundary value problems for the Bingham model.* J. Non-Newtonian Fluid Mech. **41** (1992), 339–363.

[3] C. Atkinson and C.R. Champion. *On some boundary value problems for the equation* $\nabla(F(|\nabla w|)\nabla w) = 0$. Proc. R. Soc. London A, **448** (1995), 269–279.

[4] T. Bartsch and Z. Liu. *On a superlinear elliptic p-Laplacian equation.* J. Differential Equations, **198** (2004), 149–175.

[5] M. del Pino and C. Flores. *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains.* Comm. Partial Differential Equations, **26** (11-12) (2001), 2189-2210.

[6] J.I. Diaz. *Nonlinear partial differential equations and free boundaries.* Pitman Publ. Program 1985.

[7] J. F. Escobar, *Uniqueness theorems on conformal deformations of metrics, Sobolev inequalities, and an eigenvalue estimate.* Comm. Pure Appl. Math., **43** (1990), 857–883.

[8] J. Fernández Bonder. *Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities.* Abstr. Appl. Anal., **2004** (2004), no. 12, 1047–1056

[9] J. Fernández Bonder and J.D. Rossi. *Existence results for the p-Laplacian with nonlinear boundary conditions.* J. Math. Anal. Appl., **263** (2001), 195–223.

[10] J. Fernández Bonder and J.D. Rossi. *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains.* Comm. Pure Appl. Anal. **1** (2002), no. 3, 359–378.

[11] J. Fernández Bonder, E. Lami-Dozo and J.D. Rossi. *Symmetry properties for the extremals of the Sobolev trace embedding.* Ann. Inst. H. Poincaré Anal. Non Linèaire, **21** (2004), no. 6, 795–805.

[12] J. Fernández Bonder, S. Martínez and J.D. Rossi. *The behavior of the best Sobolev trace constant and extremals in thin domains.* J. Differential Equations, **198** (2004), no. 1, 129–148.

[13] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., no. 65, Amer. Math. Soc., Providence, R.I. (1986).

[14] J.T. Schwartz. *Generalizing the Lusternik-Schnirelman theory of critical points.* Comm. Pure Appl. Math., **17** (1964), 307–315.

[15] M. Struwe. *Three nontrivial solutions of anticoercive boundary value problems for the Pseudo-Laplace operator.* J. Reine Angew. Math. **325** (1981), 68–74.

[16] P. Tolksdorf. *Regularity for a more general class of quasilinear elliptic equations.* J. Differential Equations, **51** (1984), 126–150.

[17] Z. Zhang, J. Chen and S. Li. *Construction of pseudo-gradient vector field and sign-changing multiple solutions involving p-Laplacian.* J. Differential Equations, **201** (2004), 287–303.

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