ASYMPTOTIC BEHAVIOR OF THE BEST SOBOLEV TRACE CONSTANT IN EXPANDING AND CONTRACTING DOMAINS

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Abstract. We study the asymptotic behavior for the best constant and extremals of the Sobolev trace embedding \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega) \) on expanding and contracting domains. We find that the behavior strongly depends on \( p \) and \( q \).

For contracting domains we prove that the behavior of the best Sobolev trace constant depends on the sign of \( qN - pN + p \) while for expanding domains it depends on the sign of \( q - p \). We also give some results regarding the behavior of the extremals, for contracting domains we prove that they converge to a constant when rescaled in a suitable way and for expanding domains we observe when a concentration phenomena takes place.

1. Introduction. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N, N \geq 2 \). Of importance in the study of boundary value problems for differential operators in \( \Omega \) are the Sobolev trace inequalities. For any \( 1 < p < N \), and \( 1 < q \leq p^* = p(N - 1)/(N - p) \) we have that \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega) \) and hence the following inequality holds:

\[
S_q \|u\|_{L^q(\partial\Omega)}^p \leq \|u\|_{W^{1,p}(\Omega)}^p,
\]

for all \( u \in W^{1,p}(\Omega) \). This is known as the Sobolev trace embedding Theorem. The best constant for this embedding is the largest \( S_q \) such that the above inequality holds, that is,

\[
S_q(\Omega) = \inf_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p + |u|^p \, dx}{\left( \int_{\partial\Omega} |u|^q \, d\sigma \right)^{p/q}}.
\]

Moreover, if \( 1 < q < p^* \) the embedding is compact and as a consequence we have the existence of extremals, i.e. functions where the infimum is attained, see

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These extremals are weak solutions of the following problem

\[
\begin{aligned}
&\Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\
&|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega,
\end{aligned}
\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) is the \(p\)-Laplacian and \(\frac{\partial}{\partial \nu}\) is the outer unit normal derivative.

Standard regularity theory and the strong maximum principle, [16], show that any extremal \(u\) belongs to the class \(C^{1,\alpha}_{\text{loc}}(\Omega) \cap C^\alpha(\Omega)\) and that is strictly one signed in \(\Omega\), so we can assume that \(u > 0\) in \(\Omega\). Let us fix \(p, q\) with \(1 < q < p^*\) and \(\Omega\) a bounded smooth domain in \(\mathbb{R}^N\), \(C^1\) is enough for our calculations. For \(\mu > 0\) we consider the family of domains

\[
\Omega_\mu = \mu \Omega = \{\mu x ; x \in \Omega\}.
\]

The purpose of this work is to describe the asymptotic behavior of the best Sobolev trace constants \(S_q(\Omega_\mu)\) as \(\mu \to 0^+\) and \(\mu \to +\infty\).

As a precedent, see [4] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding domains for \(p = 2\) and \(q > 2\). In that paper it is proved that the extremals develop a peak near the point where the curvature of the boundary attains a maximum. In [5] and [13] a related problem in the half-space \(\mathbb{R}^N_+\) for the critical exponent is studied. See also [6], [7] for other geometric problems that leads to nonlinear boundary conditions.

Let us call \(u_\mu\) an extremal corresponding to \(\Omega_\mu\). Making a change of variables, we go back to the original domain \(\Omega\). If we define \(v_\mu(x) = u_\mu(\mu x)\), we have that \(v_\mu \in W^{1,p}(\Omega)\) and

\[
S_q(\Omega_\mu) = \mu^{(Nq-Np+p)/q} \int_\Omega \frac{|\nabla v_\mu|^p + |v_\mu|^p}{\left( \int_{\partial\Omega} |v_\mu|^q \, d\sigma \right)^{p/q}} \, dx.
\]

We can assume, and we do so, that the functions \(u_\mu\) are chosen so that

\[
\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.
\]

We remark that the quantity (1) is not homogeneous under dilations or contractions of the domain. This is a remarkable difference with the study of the Sobolev embedding \(W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)\). First, we deal with the case \(\mu \to 0^+\). As we will see the behavior of the Sobolev constant and extremals is very different when the domain is contracted than when it is expanded. Our first result is the following:

**Theorem 1.1.** Let \(1 < q < p^*\), then

\[
\lim_{\mu \to 0^+} \frac{S_q(\Omega_\mu)}{\mu^{(Nq-Np+p)/q}} = \frac{|\Omega|}{|\partial\Omega|^{p/q}}
\]

and if we scale the extremals \(u_\mu\) to the original domain \(\Omega\) as \(v_\mu(x) = u_\mu(\mu x), x \in \Omega\), with \(\|v_\mu\|_{L^q(\partial\Omega)} = 1\), then \(v_\mu\) is nearly constant in the sense that \(v_\mu \to |\partial\Omega|^{-1/q}\) in \(W^{1,p}(\Omega)\).
Observe that the behavior of the Sobolev trace constant, strongly depends on $p$ and $q$. If we call $\beta_{pq} = (Nq - Np + p)/q$ then we have that, as $\mu \to 0+$,

\[
\begin{align*}
S_q & \to 0 \quad \text{if } \beta_{pq} > 0, \\
S_q & \to +\infty \quad \text{if } \beta_{pq} < 0, \\
S_q & \to C \neq 0 \quad \text{if } \beta_{pq} = 0.
\end{align*}
\]

Let us remark that the influence of the geometry of the domain appears in (4).

In the special case $p = q$, problem (2) becomes a nonlinear eigenvalue problem. For $p = 2$, this eigenvalue problem is known as the Steklov problem, [2]. In [8] it is proved, applying the Ljusternik-Schnirelman critical point Theory on $C^1$ manifolds, that there exists a sequence of variational eigenvalues $\lambda_k \not\to +\infty$ and it is easy to see that the first eigenvalue $\lambda_1(\Omega)$ verifies $\lambda_1(\Omega) = S_2(\Omega)$. So Theorem 1.1 shows a difference in the behavior of the eigenvalue problem of (2) with respect to the domain with the behavior of the first eigenvalue of the following Dirichlet problem

\[
\begin{align*}
-\Delta_p u &= \lambda|u|^{p-2}u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where it is a well known fact that $\lambda_1$ increases as the domain decreases, see [1], [10].

The variational eigenvalues $\lambda_k$ of (2) are characterized by

\[
\frac{1}{\lambda_k} = \sup_{C \subset C_k} \min_{u \in C} \frac{\|u\|_{L^p(\partial\Omega)}}{\|u\|_{W^{1,p}(\Omega)}},
\]

where $C_k = \{ C \subset W^{1,p}(\Omega); \ C \text{ compact, symmetric and } \gamma(C) \geq k \}$ and $\gamma$ is the Krasnoselski genus (see [11]). It is shown in [9] that there exists a second eigenvalue for (2) and that it coincides with the second variational eigenvalue $\lambda_2$. Moreover, the following characterization of the second eigenvalue $\lambda_2$ holds

\[
\lambda_2 = \inf_{u \in A} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \right\},
\]

where $A = \{ u \in W^{1,p}(\Omega); \|u\|_{L^p(\partial\Omega)} = 1 \text{ and } |\partial\Omega^+| \geq c \}$, $\partial\Omega^+ = \{ x \in \partial\Omega; \ u(x) > 0 \}$ and $\partial\Omega^−$ is defined analogously. Concerning the eigenvalue problem, we have the following result.

**Theorem 1.2.** There exists a constant $\tilde{\lambda}_2$ such that

\[
\lim_{\mu \to 0+} \mu^{p-1}\lambda_2(\Omega_\mu) = \tilde{\lambda}_2.
\]

This constant $\tilde{\lambda}_2$ is the first nonzero eigenvalue of the following problem

\[
\begin{align*}
\Delta_p u &= 0 \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial n} &= \tilde{\lambda}|u|^{p-2}u \quad \text{on } \partial\Omega.
\end{align*}
\]

Moreover, if we take an eigenfunction $u_{2,\mu}$ associated to $\lambda_2(\Omega_\mu)$ and scale it to $\Omega$ as in Theorem 1.1, we obtain that $v_{2,\mu} = \tilde{v}_2$ in $W^{1,p}(\Omega)$, where $\tilde{v}_2$ is an eigenfunction of (7) associated to $\tilde{\lambda}_2$. Also, every eigenvalue $\lambda_2(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu)$ of (2) (variational or not) behaves as $\lambda(\Omega_\mu) \sim \mu^{1-p}$ as $\mu \to 0+$. Finally, if $\mu_j \to 0$ and $\lambda_j = \lambda(\Omega_{\mu_j})$ is a sequence of eigenvalues such that there exists $\lambda$ with

\[
\lim_{j \to \infty} \mu_j^{p-1}\lambda_j = \lambda,
\]
let \((v_j)\) be the sequence of associated eigenfunctions rescaled as in Theorem 1.1, then \((v_j)\) has a convergent subsequence \((v_{j_k})\) and a limit \(v\), that is an eigenfunction of (7) with eigenvalue \(\lambda\).

Observe that the first eigenvalue of (7) is zero with associated eigenfunction a constant. Hence Theorem 1.1 says that the first eigenvalue and the first eigenfunction of our problem (2) converges to the ones of (7). Theorem 1.2 says that \(\lambda(\Omega_\mu) \to +\infty\) as \(\mu \to 0^+\) for the remaining eigenvalues and that problem (7) is a limit problem for (2) when \(\mu \to 0^+\). We believe that Theorem 1.2 is our main result.

Now, we deal with the case \(\mu \to +\infty\). In this case we find, as before, that the behavior strongly depends on \(p\) and \(q\). We prove,

**Theorem 1.3.** Let \(\beta_{pq} = (qN - pN + p)/q\). It holds

1. If \(1 < q < p\), \(0 < c_1 \mu^{\beta_{pq} - 1} \leq S_q(\Omega_\mu) \leq c_2 \mu^{\beta_{pq} - 1}\).
2. If \(p \leq q < p^*\), \(0 < c_1 \leq S_q(\Omega_\mu) \leq c_2 < \infty\).

For the lower bound in (2) in the case \(p < q < p^*\) we have to assume that the corresponding extremals \(v_\mu\) rescaled such that \(\max_{\Omega} v_\mu = 1\) verify \(|\nabla v_\mu| \leq C\mu\).

As before the behavior of the Sobolev trace constant depends on \(p\) and \(q\). We have that, as \(\mu \to +\infty\),

\[
S_q \to 0 \quad \text{if } \beta_{pq} - 1 < 0, \text{ i.e. } q < p,
\]

\[
0 < c_1 \leq S_q \leq c_2 < \infty \quad \text{if } \beta_{pq} - 1 \geq 0, \text{ i.e. } q \geq p.
\]

The hypothesis \(|\nabla v_\mu| \leq C\mu\) is a regularity assumption, see [15] for \(C^{1,\alpha}_{\text{loc}}\) regularity results. As a consequence of our arguments we have that the extremals do not develop a peak if \(1 < q < p\) as in this case we have that

\[
c_1 \leq \int_{\partial\Omega} |v_\mu|^p d\sigma \leq c_2,
\]

and

\[
\int_{\partial\Omega} |v_\mu|^q d\sigma = 1.
\]

For \(p = q\) it is proved in [12] that the first eigenvalue \(\lambda_1(\Omega_\mu) = S_p(\Omega_\mu)\) is isolated and simple. As a consequence of this if \(\Omega\) is a ball the extremal \(v_\mu\) is radial and hence it does not develop a peak. Finally, for \(q > p\) the extremals develop peaking concentration phenomena in the sense that, for every \(a > 0\),

\[
a^p|\partial\Omega \cap \{v_\mu > a\}| \to 0, \quad \text{as } \mu \to +\infty,
\]

with \(\max_{\partial\Omega} v_\mu = 1\). This is in concordance with the results of [4] where for \(p = 2, q > 2\) they find that the extremals concentrates, with the formation of a peak, near a point of the boundary where the curvature maximizes. We believe that for
q > p, extremals develop a single peak as in the case p = 2. Nevertheless that kind of analysis needs some fine knowledge of the limit problem in $\mathbb{R}^N_+$ that is not yet available for the p–Laplacian.

Let us give an idea of the proof of the lower bounds. In the case $p = q$ we can obtain the lower bound by an approximation procedure. We replace $W^{1,p}(\Omega)$ by an increasing sequence of subspaces in the minimization problem. Then we prove a convergence result and find a uniform bound from below for the approximating problems. We believe that this idea can be used in other contexts. For the case $q > p$ we use our assumption $|\nabla v_\mu| \leq C\mu$ to prove a reverse Hölder inequality for the extremals on the boundary that allows us to reduce to the case $p = q$.

Finally, for large $\mu$, in the case $p = q$ we can prove that every eigenvalue is bounded.

**Theorem 1.4.** Let $\lambda_1(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu)$ be an eigenvalue of (2) in $\Omega_\mu$ (variational or not). Then there exists two constants, $C_1, C_2 > 0$, independent of $\mu$ such that $0 < C_1 \leq \lambda(\Omega_\mu) \leq C_2 < +\infty$, for every $\mu$ large.

The rest of the paper is organized as follows. In Section 2, we deal with the case $\mu \to 0$ and in Section 3, we study the case $\mu \to +\infty$. Throughout the paper, by C we mean a constant that may vary from line to line but remains independent of the relevant quantities.

**2. Behavior as $\mu \to 0^+$.** In this section we focus on the case $\mu \to 0^+$. First we prove Theorem 1.1 and then study the case where $q = p$ (the eigenvalue problem).

Let us begin with the following Lemma.

**Lemma 2.1.** Under the assumptions of Theorem 1.1, it follows that

$$S_q(\Omega_\mu) \leq \mu^{(Nq-Np+p)/q} \frac{|\Omega|}{|\partial \Omega|^{p/q}}.$$  

**Proof.** Let us recall that

$$S_q(\Omega_\mu) = \inf_{u \in W^{1,p}(\Omega_\mu) \setminus \{0\}} \frac{\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dx}{\left(\int_{\partial \Omega_\mu} |u|^q \, d\sigma\right)^{p/q}}.$$  

Then, taking $u \equiv 1$ it follows that

$$S_q(\Omega_\mu) \leq \mu^{(Nq-Np+p)/q} \frac{|\Omega|}{|\partial \Omega|^{p/q}},$$

as we wanted to see.

This Lemma shows that the ratio $S_q(\Omega_\mu)/\mu^{(Nq-Np+p)/q}$ is bounded. So a natural question will be to determine if it converges to some value. This is answered in Theorem 1.1 that we prove next.

**Proof of Theorem 1.1.** Let $u_\mu \in W^{1,p}(\Omega_\mu)$ be a extremal for $S_q(\Omega_\mu)$ and define $v_\mu(x) = u_\mu(\mu x)$, we have that $v_\mu \in W^{1,p}(\Omega)$. We can assume that the functions $u_\mu$ are chosen so that

$$\int_{\partial \Omega} |v_\mu|^q \, d\sigma = 1.$$
Equation (3) and Lemma 2.1 give, for $\mu < 1$,

$$\|v_\mu\|_{W^{1,p}(\Omega)}^p \leq \int_\Omega \mu^{-p} |\nabla v_\mu|^p + |v_\mu|^p \, dx \leq \frac{|\Omega|}{|\partial \Omega|^{p/q}},$$

so there exists a function $v \in W^{1,p}(\Omega)$ and a sequence $\mu_j \to 0^+$ such that

$$v_{\mu_j} \to v \quad \text{weakly in } W^{1,p}(\Omega),$$
$$v_{\mu_j} \rightharpoonup v \quad \text{in } L^p(\Omega),$$
$$v_{\mu_j} \to v \quad \text{in } L^q(\partial \Omega).$$

Moreover,

$$\int_\Omega |\nabla v_{\mu_j}|^p \, dx \leq \frac{|\Omega|}{|\partial \Omega|^{p/q}} \mu_j^p.$$ 

Hence $\nabla v_{\mu_j} \to 0$ in $L^p(\Omega)$. It follows that the limit $v$ is a constant and must verify $\int_{|\partial \Omega|} |v|^q = 1$, hence $v = \text{constant} = |\partial \Omega|^{-1/q}$ and so the full sequence $v_\mu$ converges weakly in $W^{1,p}(\Omega)$ to $v$. From our previous bounds we have

$$v_\mu \to \frac{1}{|\partial \Omega|^{1/q}} \text{ in } L^p(\Omega) \quad \text{and} \quad \int_\Omega |\nabla v_\mu|^p \, dx \to 0.$$ 

Therefore, we have strong convergence, $v_\mu \to |\partial \Omega|^{-1/q}$ in $W^{1,p}(\Omega)$. The proof is finished.

Now we turn our attention to the case $p = q$ which is a nonlinear eigenvalue problem. We recall that Theorem 1.1 says that $\lambda_1(\Omega_\mu) = S_p(\Omega_\mu) \sim \mu \to 0$. First we focus on the behavior of the second eigenvalue $\lambda_2$. For the proof of Theorem 1.2 we need the following Lemmas. We believe that these results have independent interest.

**Lemma 2.2.** Let $h \in L^p(\partial \Omega)$. Then, problem

$$\begin{cases}
\Delta_p w = 0 & \text{in } \Omega, \\
|\nabla w|^{p-2} \frac{\partial w}{\partial \nu} = h(x) & \text{on } \partial \Omega,
\end{cases} \tag{8}$$

has a weak solution if and only if $\int_{\partial \Omega} h(x) \, d\sigma = 0$. Moreover, the solution is unique up to an additive constant.

**Proof.** It is straightforward to check that if there exists a weak solution to (8) then $\int_{\partial \Omega} h(x) \, d\sigma = 0$.

Now, let $X = \{ w \in W^{1,p}(\Omega); \int_\Omega w \, dx = 0 \}$. By a standard compactness argument, one can verify that the following Poincare inequality holds,

$$\|w\|_{L^p(\Omega)} \leq C \|\nabla w\|_{L^p(\Omega)}, \tag{9}$$

for every $w \in X$ and some constant $C$. Let us now define

$$\Phi(w) = \int_\Omega |\nabla w|^p \, dx - \int_{\partial \Omega} h(x)w \, d\sigma. \tag{10}$$

Critical points of $\Phi$ in $W^{1,p}(\Omega)$ are weak solutions of (8). By (9), $\Phi$ is a strictly convex, bounded below functional on $X$, and so there exists a unique function $w \in X$ such that $\Phi'(w)(v) = 0$ for every $v \in X$. Now, using the fact that $\int_{\partial \Omega} h(x) \, d\sigma = 0$, it is easy to see that $\Phi'(w)(v) = 0$ for every $v \in W^{1,p}(\Omega)$ and the proof is now complete.
Now we find a variational characterization of the first non-zero eigenvalue of the limit problem (7).

Lemma 2.3. Let \( \tilde{\lambda}_2 \) be defined by

\[
\tilde{\lambda}_2 = \inf_{u \in Y - \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\partial\Omega} |u|^p \, d\sigma},
\]

where \( Y = \{ u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^{p-2} u \, d\sigma = 0 \} \). Then the infimum is attained.

Proof. Let \( u_n \) be a minimizing sequence with \( \|u_n\|_{L^p(\partial\Omega)} = 1 \). By a compactness argument we can extract a subsequence, that we still call \( u_n \), such that

\[
u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega),
\]

\[
u_n \to u \text{ in } L^p(\Omega),
\]

\[
u_n \to u \text{ in } L^p(\partial\Omega)
\]

Hence \( u \in Y - \{0\}, \|u\|_{L^p(\partial\Omega)} = 1 \). Moreover, we have that

\[
\int_{\Omega} |\nabla u|^p \, dx \leq \liminf_n \int_{\Omega} |\nabla u_n|^p \, dx = \tilde{\lambda}_2.
\]

Therefore \( u \) is a minimizer.

Now we are ready to deal with the proof of Theorem 1.2 which is the main result of the paper.

Proof of Theorem 1.2. We can assume that \( 0 \in \Omega \) and then we can take \( u(x) = x_1 \) in the characterization of \( \lambda_2 \) given by (6) to obtain

\[
\lambda_2(\Omega_\mu) \leq \frac{\mu |\Omega_\mu| + \int_{\Omega_\mu} |x_1|^p \, dx}{\int_{\partial\Omega_\mu} |x_1|^p \, d\sigma} = \mu^{1-p} \frac{\int_\Omega |y_1|^p \, dy}{\int_{\partial\Omega} |y_1|^p \, d\sigma} \leq C \mu^{1-p}.
\]

Hence if we consider \( v_{2,\mu} \) any eigenfunction associated to \( \lambda_2(\Omega_\mu) \) normalized with \( \|v_{2,\mu}\|_{L^p(\partial\Omega)} = 1 \) we get

\[
C \mu^{1-p} \geq \lambda_2(\Omega_\mu) = \mu^{1-p} \left( \int_\Omega |\nabla v_{2,\mu}|^p \, dx + \mu^p \int_\Omega |v_{2,\mu}|^p \, dx \right).
\]

Therefore \( \|\nabla v_{2,\mu}\|_{L^p(\Omega)} \leq C \). As we have that \( \|v_{2,\mu}\|_{L^p(\partial\Omega)} = 1 \), it follows that \( \|v_{2,\mu}\|_{W^{1,p}(\Omega)} \leq C \), hence we can extract a subsequence \( \mu_j \to 0^+ \) such that

\[
v_{2,\mu_j} \rightharpoonup \tilde{v}_2 \text{ weakly in } W^{1,p}(\Omega),
\]

\[
v_{2,\mu_j} \to \tilde{v}_2 \text{ in } L^p(\Omega),
\]

\[
v_{2,\mu_j} \to \tilde{v}_2 \text{ in } L^p(\partial\Omega)
\]

Therefore we have that

\[
\int_{\partial\Omega} |\tilde{v}_2|^p \, d\sigma = 1.
\]

As it is proved in [9], \( |\{v_{2,\mu_j} > 0\} \cap \partial\Omega|, |\{v_{2,\mu_j} < 0\} \cap \partial\Omega| > c \) independent of \( \mu_j \), then \( \tilde{v}_2 \) changes sign. Hence, we get

\[
\int_\Omega |\nabla \tilde{v}_2|^p \, dx \neq 0.
\]
Taking a subsequence, if necessary, we can assume that
\[
\frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \to \lambda \quad \text{as } \mu \to 0+
\]
and, as
\[
\frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \int_\Omega |\nabla v_{2,\mu}|^p \, dx + \mu^p \int_\Omega |v_{2,\mu}|^p \, dx,
\]
passing to the limit
\[
0 \neq \int_\Omega |\nabla \tilde{v}_2|^p \, dx \leq \liminf_{\mu \to 0+} \int_\Omega |\nabla v_{2,\mu}|^p \, dx = \bar{\lambda},
\]
hence we obtain that \(\bar{\lambda} \neq 0\).

Taking \(\varphi \equiv 1\) in the weak form of the equation satisfied by \(v_{2,\mu}\), we get that
\[
\mu^p \int_\Omega |v_{2,\mu}|^{p-2} v_{2,\mu} \, dx = \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \int_\Omega |v_{2,\mu}|^{p-2} v_{2,\mu} \, d\sigma.
\]
Passing again to the limit we have that
\[
\tilde{v}_2 \in Y = \left\{ u \in W^{1,p}(\Omega); \int_{\partial \Omega} |u|^{p-2} u \, d\sigma = 0 \right\}.
\]
Let \(w\) be a function where the infimum (11) is attained with \(\|w\|_{L^p(\partial \Omega)} = 1\). As \(w \in A\) (see (6)), we have
\[
\int_\Omega |\nabla w|^p + \mu^p |w|^p \, dx \geq \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \int_\Omega |\nabla v_{2,\mu}|^p + \mu^p |v_{2,\mu}|^p \, d\sigma.
\]
Taking the limit as \(\mu \to 0+\) we get
\[
\bar{\lambda}_2 = \int_\Omega |\nabla w|^p \, dx \geq \lim_{\mu \to 0} \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \geq \int_\Omega |\nabla \tilde{v}_2|^p \, dx \geq \inf_{\|z\|_{L^p(\partial \Omega)} = 1, z \in Y} \int_\Omega |\nabla z|^p = \tilde{\lambda}_2.
\]
Therefore
\[
\lim_{\mu \to 0} \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \bar{\lambda}_2
\]
and
\[
\int_\Omega |\nabla v_{2,\mu}|^p \, dx \to \int_\Omega |\nabla \tilde{v}_2|^p \, dx,
\]
from where it follows that \(v_{2,\mu} \to \tilde{v}_2\) strongly in \(W^{1,p}(\Omega)\). Once again, we pass to the limit as \(\mu \to 0+\) in the weak formulation satisfied by \(v_{2,\mu}\) to get that \(\tilde{v}_2\) is an eigenfunction associated to \(\bar{\lambda}_2\). By the characterization of \(\bar{\lambda}_2\) given in Lemma 11 we get that this is the first non-zero eigenvalue for problem (7).

Now we find the behavior of the remaining eigenvalues. Let \(\lambda(\Omega_\mu)\) be an eigenvalue (variational or not). Then, as the variational eigenvalues \(\lambda_k(\Omega_\mu)\) form an unbounded sequence, there exists \(k\) such that \(\lambda_2(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu)\). Now, let \(x_1, \ldots, x_k \in \partial \Omega\) and \(r = r(k)\) be such that \(\text{dist}(x_i, x_j) > 2r\). Let \(\phi \in C^\infty(\Omega)\) be a nonnegative function with support \(B(0, r)\) and let \(\phi_j(x) = \phi(x - x_j)\).

Now, let us define \(S_k = \operatorname{span}\{\phi_1, \ldots, \phi_k\} \cap \{v \in W^{1,p}(\Omega); \|v\|_{W^{1,p}(\Omega)} = 1\}\) and \(S_{k,\mu} = \{v(x/\mu); v \in S_k\}\); then \(\gamma(S_k) = \gamma(S_{k,\mu}) = k\). Hence
\[
\frac{1}{\lambda_k(\Omega_\mu)} = \sup_{\gamma(S) \geq k} \inf_{u \in S} \frac{\int_{\partial \Omega} |u|^p \, d\sigma}{\int_{\Omega} |\nabla u|^p + |u|^p \, dx} \geq \inf_{u \in S_{k,\mu}} \frac{\int_{\partial \Omega} |u|^p \, d\sigma}{\int_{\Omega} |\nabla u|^p + |u|^p \, dx}.
\]
Changing variables we get,

\[
\frac{1}{\lambda_k(\Omega_\mu)} \geq \mu^{p-1} \inf_{v \in S_k} \frac{\int_{\partial \Omega} |v|^p d\sigma}{\int_\Omega |\nabla v|^p + \mu^p |v|^p dx}.
\]

(12)

As \( \phi_i \) have disjoint support,

\[
\|v\|_{L^p(\Omega)} = \left\| \sum_{i=1}^k a_i \phi_i \right\|_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|_{L^p(\Omega)} \leq \sum_{i=1}^k |a_i|^p \|\phi_i\|_{L^p(B(0,r))}.
\]

and

\[
\|\nabla v\|_{L^p(\Omega)} = \left\| \sum_{i=1}^k a_i \nabla \phi_i \right\|_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\nabla \phi_i\|_{L^p(\Omega)} \leq \sum_{i=1}^k |a_i|^p \|\nabla \phi\|_{L^p(B(0,r))}.
\]

As the boundary of \( \Omega \) is regular we have that there exists a constant \( C_k \) such that

\[
\|v\|_{L^p(\partial \Omega)} = \left\| \sum_{i=1}^k a_i \phi_i \right\|_{L^p(\partial \Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|_{L^p(\partial \Omega)} \geq C_k \sum_{i=1}^k |a_i|^p.
\]

Using these estimates in (12) we obtain

\[
0 < c \leq \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \leq \frac{\lambda(\Omega_\mu)}{\mu^{1-p}} \leq \frac{\lambda_k(\Omega_\mu)}{\mu^{1-p}} \leq C_k < +\infty
\]

and the result follows.

Finally we study the convergence of the eigenvalues and eigenfunctions corresponding to the rest of the spectrum. By our hypotheses we have that

\[
\lim_{j \to \infty} \frac{\lambda_j}{\mu_j^{1-p}} = \lambda.
\]

As \( v_j \) is bounded in \( W^{1,p}(\Omega) \) we can extract a subsequence (that we still call \( v_j \)) such that

\[
\begin{align*}
    v_j &\to v \text{ weakly in } W^{1,p}(\Omega), \\
    v_j &\to v \text{ in } L^p(\Omega), \\
    v_j &\to v \text{ in } L^p(\partial \Omega).
\end{align*}
\]

Using that \( v_j \) are solutions of (2), we obtain

\[
\int_\Omega |\nabla v_j|^{p-2} \nabla v_j \nabla \phi + \mu_j^p |v_j|^{p-2} v_j \phi \, dx = \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial \Omega} |v_j|^{p-2} v_j \phi \, d\sigma.
\]

(13)

Taking \( \phi \equiv 1 \) we get

\[
\int_\Omega \mu_j^p |v_j|^{p-2} v_j \, dx = \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial \Omega} |v_j|^{p-2} v_j \, d\sigma.
\]

The limit as \( j \to \infty \) gives us

\[
0 = \lambda \int_{\partial \Omega} |v|^{p-2} v \, d\sigma
\]
and, as $\lambda \neq 0$, we obtain that
\[
0 = \int_{\partial \Omega} |v|^{p-2} v \, d\sigma. \tag{14}
\]
By Lemma 2.2 and (14), there exists a unique $w \in W^{1,p}(\Omega)$ with
\[
\int_{\partial \Omega} |w|^{p-2} w \, d\sigma = 0
\]
that satisfies
\[
\begin{aligned}
\Delta_p w &= 0 \quad \text{in } \Omega, \\
|\nabla w|^{p-2} \frac{\partial w}{\partial \nu} &= \lambda |v|^{p-2} v \quad \text{on } \partial \Omega.
\end{aligned} \tag{15}
\]
Combining (13), the variational formulation of (15) with $\phi = v_j - w$ and the fact that we are dealing with a strongly monotone operator (see [3]), we get
\[
\alpha \|\nabla v_j - \nabla w\|_{L^p(\Omega)}^p \leq \int_{\Omega} (|\nabla v_j|^{p-2} \nabla v_j - |\nabla w|^{p-2} \nabla w)(\nabla v_j - \nabla w) \, dx
\]
\[
= -\mu_j^p \int_{\Omega} |v_j|^{p-2} v_j (v_j - w) \, dx + \frac{\lambda_j}{\mu_j^p} \int_{\partial \Omega} |v_j|^{p-2} v_j (v_j - w) \, d\sigma
\]
\[
-\lambda \int_{\partial \Omega} |v|^{p-2} v (v_j - w) \, d\sigma
\]
\[
\leq C \mu_j^p + \left( \frac{\lambda_j}{\mu_j^p} - \lambda \right) \int_{\partial \Omega} |v_j|^{p-2} v_j (v_j - w) \, d\sigma
\]
\[
+ \lambda \int_{\partial \Omega} (|v_j|^{p-2} v_j - |v|^{p-2} v) (v_j - w) \, d\sigma.
\]
The first two terms go to zero as $j \to \infty$. Concerning the last one, we have that it is bounded by
\[
(|v_j|_{L^p(\partial \Omega)} + \|v\|_{L^p(\partial \Omega)})^{p-2} \|v_j - v\|_{L^p(\partial \Omega)} \|v_j - w\|_{L^p(\partial \Omega)} \quad \text{if } p \geq 2,
\]
\[
M \|v_j - v\|_{L^p(\partial \Omega)}^{p-1} \|v_j - w\|_{L^p(\partial \Omega)} \quad \text{if } p < 2.
\]
Therefore, taking the limit $j \to \infty$, we get $\nabla v_j \to \nabla w$ in $L^p(\Omega)$ and as $\nabla v_j \to \nabla v$ weakly in $L^p(\Omega)$ we conclude that $\nabla v = \nabla w$ and so $v = w$ and $v_j \to v$ strongly in $W^{1,p}(\Omega)$. Finally, taking limits in (13) we obtain that $v$ is a weak solution of (7) as we wanted to prove. 

3. Behavior as $\mu \to +\infty$. In this section we study the behavior of the Sobolev constant in expanding domains, that is when $\mu \to +\infty$. To clarify the exposition we divide the proof of Theorem 1.3 in several Lemmas. Let us begin by the upper bounds.

**Lemma 3.1.** Let $p = q$, then there exists a constant $C > 0$ such that $S_p(\Omega_\mu) = \lambda_1(\Omega_\mu) \leq C$, for every $\mu$ large.
Proof. We have \( p = q \) and look for a bound on the first eigenvalue \( \lambda_1(\Omega_\mu) \). Changing variables as before we have that

\[
\lambda_1(\Omega_\mu) = \inf_{v \in W^{1,p}(\Omega)} \frac{\mu \left( \int_{\Omega} \mu^{-p} |\nabla v|^p + |v|^p \, dx \right)}{\int_{\partial \Omega} |v|^q \, d\sigma}.
\]

We choose \( v(x) \) such that \( v = a = \text{constant} \) on \( \partial \Omega \) and \( v = 0 \) in \( \Omega \); \( \text{dist}(x, \partial \Omega) \geq r \) with \( |\nabla v| \leq C/r \). We fix \( a \) such that

\[
\int_{\partial \Omega} |v|^q \, d\sigma = 1,
\]

that is \( a = |\partial \Omega|^{-1/q} \). As for \( r \) small we have that \( |\Omega \setminus \Omega_r| \sim r|\partial \Omega| \) we get

\[
\int_{\Omega} |v|^p \, d\sigma \leq Cr.
\]

Using that \( |\nabla v| \leq C/r \) we obtain

\[
\int_{\Omega} |\nabla v|^p \, d\sigma \leq \frac{C}{r^{p-1}},
\]

therefore

\[
\lambda_1(\Omega_\mu) \leq C \mu \left( C \frac{\mu^{-p}}{r^{p-1}} + Cr \right).
\]

Finally, choose \( r = \mu^{-1} \) to obtain the desired result.

\[ \square \]

Lemma 3.2. Let \( p < q < p^* \), then there exists a constant \( C > 0 \) such that \( S_q(\Omega_\mu) \leq C \), for every \( \mu \) large.

Proof. As we mentioned in the introduction, we have that

\[
S_q(\Omega_\mu) = \mu^{(N_q-N_p+p)/q} \inf_{v \in W^{1,p}(\Omega)} \int_{\Omega} \mu^{-p} |\nabla v|^p + |v|^p \, dx \left( \int_{\partial \Omega} |v|^q \, d\sigma \right)^{p/q}.
\]  \( \tag{16} \)

Now, let us choose a point \( x_0 \in \partial \Omega \) and let \( \phi \in C^\infty(\Omega) \) with support \( B(x_0, \mu^{-1}) \), and \( \|\phi\|_{L^q(\partial \Omega)} = 1 \).

Arguing as in Section 2, we have that

\[
\mu^{(N_q-N_p+p)/q} \int_{\Omega} |\phi|^p \, dx \leq C,
\]

and

\[
\mu^{(N_q-N_p+p)/q} \mu^{-p} \int_{\Omega} |\nabla \phi|^p \, dx \leq C.
\]

Therefore, taking \( \phi = v \) in (16), we get \( S_q(\Omega_\mu) \leq C \), and this ends the proof. \( \square \)

Lemma 3.3. Let \( 1 < q < p \), then we have \( S_q(\Omega_\mu) \leq C \mu^{(N-1)(q-p)/q} \), for some constant \( C > 0 \). Remark that this says that \( \lim_{\mu \to \infty} S_q(\Omega_\mu) = 0 \).
Proof. We observe that the same calculations of Lemma 3.2 show that $S_q$ is bounded independently of $\mu$ for $1 < q < p$. Now, as in the case $p = q$ (Lemma 3.1), let us take $v(x)$ such that $v = a = constant$ on $\partial \Omega$ and $v = 0$ in $\Omega_r = \{x \in \Omega : dist(x, \partial \Omega) \geq r\}$. We fix $a$ such that

$$
\int_{\partial \Omega} |v|^q \, d\sigma = 1.
$$

Using the same arguments as in Lemma 3.1 we get

$$
S_q(\Omega \mu) \leq C \mu^{(Nq - Np + p)/q} \left( C^{-p} + C r \right)
$$

and choosing $r = \mu^{-1}$ we obtain $S_q(\Omega \mu) \leq C \mu^{(Nq - Np + p - q)/q}$.

Now let us prove that the extremals concentrates at the boundary.

**Lemma 3.4.** Let $1 < q < p^\ast$. The extremals concentrate at the boundary in the sense that

$$
\int_{\Omega} |v_{\mu}|^p \, dx \to 0 \quad \text{as} \ \mu \to +\infty,
$$

while

$$
\int_{\partial \Omega} |v_{\mu}|^q \, d\sigma = 1.
$$

*Proof.* Let $v_{\mu}$ be an extremal such that $\|v_{\mu}\|_{L^q(\partial \Omega)} = 1$. From our previous bound we get, for $p = q$,

$$
\mu^{1-p} \int_{\Omega} |\nabla v_{\mu}|^p \, dx + \mu \int_{\Omega} |v_{\mu}|^p \, dx \leq C.
$$

Hence

$$
\int_{\Omega} |v_{\mu}|^p \, dx \leq \frac{C}{\mu} \to 0 \quad \text{as} \ \mu \to +\infty.
$$

Now we turn back to the case $1 < q < p$. We have, from our previous calculations,

$$
S_q(\Omega \mu) \leq C \mu^{(Nq - Np + p - q)/q}.
$$

Hence

$$
\int_{\Omega} |v_{\mu}|^p \, dx \leq C \mu^{(N-1)(q-p)/q} \to 0 \quad \mu \to +\infty.
$$

Finally, for $p < q < p^\ast$ we get that

$$
\mu^{(Nq - Np + p)/q} \int_{\Omega} |v_{\mu}|^p \, dx \leq C
$$

and therefore, as we are in the case $q > p$ and so $Nq > p(N - 1)$, we get

$$
\int_{\Omega} |v_{\mu}|^p \, dx \leq \frac{C}{\mu^{(Nq - Np + p)/q}} \to 0 \quad \text{as} \ \mu \to +\infty.
$$

The proof is now complete.

To get the bound from below for $\lambda_1$ in the case $p = q$ we use the following idea, first we replace the minimization problem in $W^{1,p}(\Omega)$ with a minimization problem in a sequence of increasing subspaces and next we find that for an adequate choice of the subspaces we get a uniform lower bound for the approximate problems. This idea combined with a convergence result for the approximations gives the desired result. So, let us first state and prove the convergence result. Since this procedure works for every $1 < q < p^\ast$ we prove it in full generality.
Now we want to describe a general approximation procedure for \( S_q \). These results are essentially contained in [14] but we reproduce the main arguments here in order to make the paper self-contained.

The Sobolev trace constant \( S_q \) can be characterized as

\[
S_q = \inf_{v \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v|^p + |v|^p \, dx; \quad \int_{\partial \Omega} |v|^q \, d\sigma = 1 \right\}.
\]  

(17)

As we have already mentioned, the idea is to replace the space \( W^{1,p}(\Omega) \) with a subspace \( V_h \) in the minimization problem (17). To this end, let \( V_h \) be an increasing sequence of closed subspaces of \( W^{1,p}(\Omega) \), such that

\[
\left\{ u_h \in V_h; \int_{\partial \Omega} |u_h|^q \, d\sigma = 1 \right\} \neq \emptyset
\]  

and

\[
\lim_{h \to 0} \inf_{u_h \in V_h} \|v - u_h\|_{W^{1,p}(\Omega)} = 0, \quad \forall \|v\|_{W^{1,p}(\Omega)} = 1.
\]  

(18)

We observe that the only requirement on the subspaces \( V_h \) is (18). This allows us to choose \( V_h \) as the usual finite elements spaces, for example.

With this sequence of subspaces \( V_h \) we define our approximation of \( S_q \) by

\[
S_{q,h} = \inf_{u_h \in V_h} \left\{ \int_\Omega |\nabla u_h|^p + |u_h|^p \, dx; \quad \int_{\partial \Omega} |u_h|^q \, d\sigma = 1 \right\}.
\]  

(19)

We have that, under hypothesis (18), \( S_{q,h} \) approximates \( S_q \) when \( h \to 0 \).

**Theorem 3.1.** Let \( v \) be an extremal for (17). Then, there exists a constant \( C \) independent of \( h \) such that,

\[
|S_q - S_{q,h}| \leq C \inf_{u_h \in V_h} \|u_h - v\|_{W^{1,p}(\Omega)},
\]

for every \( h \) small enough.

**Proof.** As \( V_h \subset W^{1,p}(\Omega) \) we have that

\[
S_q \leq S_{q,h}. \tag{20}
\]

Let us choose \( w \in V_h \) such that \( \|w - v\|_{W^{1,p}(\Omega)} \leq \inf_{V_h} \|v - u_h\|_{W^{1,p}(\Omega)} + \varepsilon \). We have

\[
S_{q,h}^{1/p} = \|u_h\|_{W^{1,p}(\Omega)} \leq \left( \frac{\|w\|_{L^q(\partial \Omega)}}{\|w\|_{L^q(\partial \Omega)}} \right) \leq \left( \frac{\|w - v\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,p}(\Omega)}}{\|w\|_{L^q(\partial \Omega)}} \right) = \left( \frac{\|w - v\|_{W^{1,p}(\Omega)} + S_q^{1/p}}{\|w\|_{L^q(\partial \Omega)}} \right).
\]

Now we use that

\[
\|w\|_{L^q(\partial \Omega)} \leq \|w\|_{L^q(\partial \Omega)} - 1 \leq \|w\|_{L^q(\partial \Omega)} - \|v\|_{L^q(\partial \Omega)} \leq \|w - v\|_{L^q(\partial \Omega)} \leq C\|w - v\|_{W^{1,p}(\Omega)}\]

and hypothesis (18) to obtain that for every \( h \) small enough,

\[
S_{q,h} \leq \left( \frac{\|w - v\|_{W^{1,p}(\Omega)} + S_q^{1/p}}{1 - C\|w - v\|_{W^{1,p}(\Omega)}} \right)^p \leq S_q + C\|w - v\|_{W^{1,p}(\Omega)}. \tag{21}
\]

The result follows from (20) and (21). □
Now we prove a result regarding the convergence of the approximate extremals. We will not use it but it completes the analysis of the approximations.

**Theorem 3.2.** Let $u_h$ be a function in $V_h$ where the infimum (19) is archived. Then from any sequence $h \to 0$ we can extract a subsequence $h_j \to 0$ such that $u_{h_j}$ converges strongly to an extremal in $W^{1,p}(\Omega)$. That is, there exists an extremal of (17), $v$, with

$$\lim_{h_j \to 0} \|u_{h_j} - v\|_{W^{1,p}(\Omega)} = 0.$$ 

**Proof.** Theorem 3.1 and hypothesis (18) gives that

$$\lim_{h \to 0} \|u_h\|_{W^{1,p}(\Omega)} = \lim_{h \to 0} S_{q,h} = S_q.$$ 

Hence there exists a constant $C$ such that for every $h$ small enough, $\|u_h\|_{W^{1,p}(\Omega)} \leq C$. Therefore we can extract a subsequence, that we denote by $u_{h_j}$, such that

$$u_{h_j} \rightharpoonup w \quad \text{weakly in } W^{1,p}(\Omega),$$

$$u_{h_j} \to w \quad \text{strongly in } L^p(\Omega),$$

$$u_{h_j} \to w \quad \text{strongly in } L^q(\partial\Omega).$$

Hence, from the $L^q(\partial\Omega)$ convergence we have,

$$1 = \lim_{h_j \to 0} \int_{\partial\Omega} |u_{h_j}|^q d\sigma = \int_{\partial\Omega} |w|^q d\sigma.$$ 

Therefore $w$ is an admissible function in the minimization problem (17). Now we observe that, if $v$ is an extremal,

$$\|v\|_{W^{1,p}(\Omega)}^p \leq \|w\|_{W^{1,p}(\Omega)}^p \leq \liminf_{h_j \to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)}^p \leq \lim_{h_j \to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)}^p = \lim_{h_j \to 0} S_{q,h} = S_q = \|v\|_{W^{1,p}(\Omega)}^p,$$

and therefore,

$$\lim_{h_j \to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)} = \|w\|_{W^{1,p}(\Omega)} = S_q^{1/p}. \quad (23)$$

The space $W^{1,p}(\Omega)$ being uniformly convex, the weak convergence, (22), and the convergence of the norms, (23), imply the convergence in norm. Therefore $u_{h_j} \to w$ in $W^{1,p}(\Omega)$. This limit $w$ verifies $\|w\|_{W^{1,p}(\Omega)} = S_q$ and $\|w\|_{L^q(\partial\Omega)} = 1$. Hence it is an extremal and we have that $\lim_{h_j \to 0} \|u_{h_j} - w\|_{W^{1,p}(\Omega)} = 0$. 

With these convergence results we can prove the lower bound in the case $p = q$.

**Lemma 3.5.** Let $p = q$, then $S_p(\Omega_{\mu}) = \lambda_1(\Omega_{\mu}) \geq C$, for every $\mu$ large.

**Proof.** Let us choose a particular subspace $V_\mu$ of $W^{1,p}(\Omega)$. As the boundary of $\Omega$ is smooth, we can define new coordinates near the boundary as follows. As before we denote by $\Omega_r = \{ x \in \Omega; \text{dist}(x, \partial\Omega) \geq r \}$ and by $\partial\Omega_r = \{ x \in \Omega; \text{dist}(x, \partial\Omega) = r \}$ and we use the following construction. We define $\Phi(\xi, r) = \xi - r\nu(\xi)$, where $\nu(\xi)$ is the exterior normal vector at $\xi \in \partial\Omega$. $\Phi : \partial\Omega \times (0, R) \to \Omega \setminus \bar{\Omega}_R$. We recall that $\Phi$ is a diffeomorphism if $R$ is small enough. With this application $\Phi$ we can define a triangulation as follows. First, choose a uniform regular triangulation of size $h$ of the set $\partial\Omega \times (0, R)$. Now, by the application $\Phi$ we can get a triangulation of the strip $\Omega \setminus \bar{\Omega}_R$. In fact, we can select as nodes $x_{ij}$ the points $\Phi(\xi_i, r_j)$, where
$(\xi_i, r_j)$ is a node of the uniform mesh of $\partial \Omega \times (0, R)$. Our space $V_h$ is defined by all the continuous functions in $W^{1,p}(\Omega)$ that are linear over each triangle of the strip $\Omega \setminus \Pi_R$. This space is the usual space of linear finite elements in special triangulations defined using the mapping $\Phi$, see [3] for detailed information on the finite elements method.

Let us call $u_h$ the functions in $V_h$. We have indexed the nodes $x_{ij}$ in a way such that $x_{i1} \in \partial \Omega$ and $x_{ij}$ is at distance $j-1$ (in nodes) from the boundary, $\partial \Omega$. We denote by $u_{ij}$ the value of $u_h$ at the node $x_{ij}$ and by $a_{ij}$ the value of the gradient of $u_h$ on the triangle $T_{ij}$. We assume that the index $i$ runs from 1 to $l$ and $j$ from 1 to $k_0$. Remark that $k_0 \sim R/h$ and $l \sim |\partial \Omega|/h^{N-1}$.

We want to find a lower bound (independent of $h$ and $\mu$) on the approximation of the first eigenvalue, $\lambda_{1,h}(\Omega, \mu) = \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_{\Omega} |\nabla u_h|^p \, dx + \mu \int_{\Omega} |u_h|^p \, dx; \quad \int_{\partial \Omega} |u_h|^p \, d\sigma = 1 \right\}$.

To this end we consider a function $u_h \in V_h$ such that

$$\int_{\partial \Omega} |u_h|^p \, d\sigma = 1,$$

that is

$$\sum_{i=1}^l |u_{i1}|^p h^{N-1} \geq C_1.$$

Let $k$ be the first integer in $[1, k_0]$ such that

$$\sum_{i=1}^l |u_{ik}|^p h^{N-1} \leq \frac{C_1}{2}.$$

First, let us observe that if $k = k_0$ (there are $k_0$ triangles between the two boundaries of $\Omega \setminus \Omega_r$), then we have

$$\mu \int_{\Omega} |u_h|^p \, dx \geq \mu \sum_{j=2}^{k_0} \int_{T_{ij}} |u_h|^p \, dx \geq C \mu \sum_{j=2}^{k_0} \int_{T_{ij}} |u_{ij}|^p h^N \geq Ch \mu k_0 \sum_{j=2}^{k_0} \sum_{i=1}^l |u_{ij}|^p h^{N-1} \geq Ch \mu C_1 \frac{k_0}{2}.$$

As $k_0 \sim R/h$ we get that

$$\lambda_{1,h}(\Omega, \mu) = \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_{\Omega} |\nabla u_h|^p \, dx + \mu \int_{\Omega} |u_h|^p \, dx; \quad \int_{\partial \Omega} |u_h|^p \, d\sigma = 1 \right\} \geq \inf_{u_h \in V_h} \left\{ \mu \int_{\Omega} |u_h|^p \, dx; \quad \int_{\partial \Omega} |u_h|^p \, d\sigma = 1 \right\} \geq C \mu > 1$$

and we are done. Hence let us assume that $k < k_0$. As before we can bound the term $\mu \int_{\Omega} |u_h|^p \, dx$ by

$$\mu \int_{\Omega} |u_h|^p \, dx \geq C \mu \sum_{j=2}^{k} \sum_{i=1}^l |u_{ij}|^p h^N = Ch \mu \sum_{j=2}^{k} \sum_{i=1}^l |u_{ij}|^p h^{N-1} \geq Ch \mu C_1 \frac{k}{2}. \quad (24)$$
Now we observe that
\[ u_{i1} - u_{ik} = \sum_{j=1}^{k} a_{ij} h. \]

Using this fact we get,
\[ C \leq \left( \frac{1}{7} \sum_{i=1}^{l} |u_{i1}|^p \right)^{1/p} - \left( \frac{1}{7} \sum_{i=1}^{l} |u_{ik}|^p \right)^{1/p} \]
\[ \leq \left( \frac{1}{7} \sum_{i=1}^{l} |u_{i1} - u_{ik}|^p \right)^{1/p} = \left( \frac{k^p}{l} \sum_{i=1}^{l} \left| \frac{1}{k} \sum_{j=1}^{k} a_{ij} h \right|^p \right)^{1/p}. \]

Hence we get
\[ \frac{C l}{k^p - 1 h^p} \leq \sum_{i=1}^{l} \frac{1}{k} \sum_{j=1}^{k} |a_{ij}|^p \]

and finally,
\[ \mu^{1-p} \int_{\Omega} |\nabla u_h|^p \, dx \geq \frac{C \mu^{1-p} h^{N-1}}{k^p - 1 h^p - 1} \geq C. \] (25)

Using (24) and (25) we obtain
\[ \lambda_{1,h}(\Omega_{\mu}) = \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_{\Omega} |\nabla u_h|^p \, dx + \mu \int_{\Omega} |u_h|^p \, dx; \int_{\partial \Omega} |u_h|^p \, d\sigma = 1 \right\} \]
\[ \geq C(\mu h) + \frac{C}{(\mu h)^{p-1}}. \]

Hence, if we call \( \tau = \mu h \) we get that
\[ \lambda_{1,h}(\Omega_{\mu}) \geq F(\tau) \equiv C \tau + \frac{C}{\tau^{p-1}} \geq C. \]

Since the subspaces that we have chosen verify hypotheses (18), we can use the convergence result, Theorem 3.1, to get that \( \lambda_1(\Omega_{\mu}) = \lim_{h \to 0} \lambda_{1,h}(\Omega_{\mu}) \geq C. \)

Let us look at the case \( 1 < q < p \) more carefully, and obtain a bound from below using the lower bound obtained for \( \lambda_1(\Omega_{\mu}) \).

**Lemma 3.6.** Let \( 1 < q < p \). Then, for every \( \mu \) large, \( S_q(\Omega_{\mu}) \geq C \mu^{\beta_{p,q} - 1} \). Moreover this shows that, if \( v \) is an extremal,
\[ c_1 \left( \int_{\partial \Omega} |v|^q \, d\sigma \right)^{1/q} \geq \left( \int_{\partial \Omega} |v|^p \, d\sigma \right)^{1/p} \geq c_2 \left( \int_{\partial \Omega} |v|^q \, d\sigma \right)^{1/q}. \]

**Hence there is no peaking formation in this case.**
Proof. As we mentioned in the introduction, we have that
\[ S_q(\Omega_\mu) = \mu^{(N_q-N_p+p)/q} \inf_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p \, dx}{\left( \int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}} \]
\[ = \mu^{\beta_{pq}-1} \inf_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p \, dx}{\left( \int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}} \]
\[ = \mu^{\beta_{pq}-1} \inf_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p \, dx}{\int_{\partial\Omega} |v|^p \, d\sigma} \frac{\int_{\partial\Omega} |v|^p \, d\sigma}{\left( \int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}}. \]

Using that \(1 < q < p\) we get that, by Hölder’s inequality
\[ \int_{\partial\Omega} |v|^p \, d\sigma \geq C. \]

Hence, using our previous lower bound for \(\lambda_1(\Omega_\mu)\) we get that there exists a constant \(C\) such that
\[ S_q(\Omega_\mu) = \mu^{\beta_{pq}-1} \min_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p \, dx}{\int_{\partial\Omega} |v|^q \, d\sigma} \frac{\int_{\partial\Omega} |v|^p \, d\sigma}{\left( \int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}} \]
\[ \geq C \mu^{\beta_{pq}-1} \left( \int_{\partial\Omega} |v_\mu|^q \, d\sigma \right)^{p/q}. \]

Hence
\[ \int_{\partial\Omega} |v_\mu|^p \, d\sigma \leq C \left( \int_{\partial\Omega} |v_\mu|^q \, d\sigma \right)^{p/q}. \]

This ends the proof.

To finish the proof of Theorem 1.3 we need the following Lemma.

**Lemma 3.7.** Let \(p < q < p^*\). Then, for large \(\mu\), \(S_q(\Omega_\mu) \geq C\). Moreover, the extremals concentrates in the sense that \(a^p|\partial\Omega \cap \{v_\mu > a\}| \rightarrow 0\), as \(\mu \rightarrow +\infty\), with \(\max_{\Omega} v_\mu = 1\).

**Proof.** First we prove that there exists a constant \(C\) such that \(S_q(\Omega_\mu) \geq C\). Let \(v_\mu\) be an extremal in \(\Omega\). By rescaling \(v_\mu\) we can obtain an extremal \(\tilde{v}_\mu\) such that
max_{\Omega} \tilde{v}_\mu = 1. That is, \(0 < \tilde{v}_\mu \leq 1\) and there exits a point \(x_0 \in \partial \Omega\) with \(\tilde{v}_\mu(x_0) = 1\).

Arguing as in Lemma 3.6 we have

\[ S_q(\Omega_\mu) = \mu^{\beta_{pq}-1} \int_{\Omega} |x|^{-p} |\nabla \tilde{v}_\mu|^p dx \geq \mu^{\beta_{pq}-1} \left( \int_{\partial \Omega} |\tilde{v}_\mu|^p d\sigma \right)^{p/q}. \]  (26)

As \(\tilde{v}_\mu\) satisfies (2), by our hypothesis, we have that \(|\nabla \tilde{v}_\mu| \leq C\mu\). Hence

\[ \{x \in \partial \Omega; \tilde{v}_\mu(x) \geq 1/2\} \supseteq B(x_0, c/\mu) \cap \partial \Omega. \]

As \(q > p\) and \(0 < \tilde{v}_\mu \leq 1\) we have that

\[ \int_{\partial \Omega} |\tilde{v}_\mu|^p d\sigma \geq \int_{\partial \Omega} |\tilde{v}_\mu|^q d\sigma. \]

Therefore

\[ \mu^{\beta_{pq}-1} \left( \int_{\partial \Omega} |\tilde{v}_\mu|^p d\sigma \right)^{p/q} \geq \mu^{\beta_{pq}-1} \left( \int_{\partial \Omega} |\tilde{v}_\mu|^p d\sigma \right)^{(q-p)/q} \geq C\mu^{\beta_{pq}-1} \frac{1}{2^p} \int_{\partial \Omega} \frac{|\tilde{v}_\mu|^{p-q}}{\mu^q} d\sigma \geq C. \]

Using this bound and the lower bound for \(S_\mu(\Omega_\mu)\) in (26) we get the desired lower bound. Next, we prove the concentration property for the extremals. Using the same arguments as before, we get

\[ a^p|\partial \Omega \cap \{\tilde{v}_\mu > a\}| \leq \int_{\partial \Omega} |\tilde{v}_\mu|^p d\sigma \leq \frac{C}{\mu^{\beta_{pq}-1}} \to 0, \quad \text{as } \mu \to +\infty, \]

with \(\max_{\Omega} \tilde{v}_\mu = 1\). This proves the concentration phenomena.

We end the article proving that every eigenvalue is bounded as \(\mu \to +\infty\).

**Proof of Theorem 1.4.** The idea is similar as the one used in the proof of Theorem 1.2, see Section 2. Let \(x_1, \ldots, x_k \in \partial \Omega\) such that \(\text{dist}(x_1, x_j) > 2\mu\) and let \(\phi_j \in C^\infty(\Omega)\) with support \(B(x_j, \mu)\) and \(\text{max} \phi_j = 1\). Now, let us define \(S_k = \text{span}\{\phi_1, \ldots, \phi_k\} \cap \{u \in W^{1, p}(\Omega); \|u\|_{W^{1, p}(\Omega)} = 1\}\) and \(S_{k, \mu} = \{v(x/\mu); v \in S_k\}\). Then, \(\gamma(S_k) = \gamma(S_{k, \mu}) = k\). Hence

\[ \frac{1}{\lambda_k(\Omega_\mu)} = \sup_{\gamma(S) \geq k, \mu \in N} \inf_{u \in S_k} \int_{\Omega_\mu} |u|^p d\sigma \geq \inf_{u \in S_{k, \mu}} \int_{\Omega_\mu} |\nabla u|^p + |u|^p dx \geq \inf_{u \in S_{k, \mu}} \int_{\Omega_\mu} |\nabla u|^p + |u|^p dx. \]

Changing variables we get,

\[ \frac{1}{\lambda_k(\Omega_\mu)} \geq \mu^{p-1} \inf_{v \in S_k} \int_{\Omega} |v|^p d\sigma \geq \int_{\Omega} |\nabla v|^p + |\mu^p v|^p dx. \]  (27)
As \( \phi_i \) have disjoint support,

\[
\| v \|_{L^p(\Omega)} = \left| \sum_{i=1}^{k} a_i \phi_i \right|_{L^p(\Omega)}^p = \sum_{i=1}^{k} |a_i|^p \| \phi_i \|_{L^p(\Omega)}^p \leq C \sum_{i=1}^{k} |a_i|^p \mu^{-N}
\]

and

\[
\| \nabla v \|_{L^p(\Omega)} = \left| \sum_{i=1}^{k} a_i \nabla \phi_i \right|_{L^p(\Omega)}^p = \sum_{i=1}^{k} |a_i|^p \| \nabla \phi_i \|_{L^p(\Omega)}^p \leq C \sum_{i=1}^{k} |a_i|^p \mu^{-N+p}.
\]

As the boundary of \( \Omega \) is regular we have that there exists a constant \( C \) such that

\[
\| v \|_{L^p(\partial \Omega)} = \left| \sum_{i=1}^{k} a_i \phi_i \right|_{L^p(\partial \Omega)}^p = \sum_{i=1}^{k} |a_i|^p \| \phi_i \|_{L^p(\partial \Omega)}^p \geq C \sum_{i=1}^{k} |a_i|^p \mu^{1-N}.
\]

Using these estimates we get \( 0 < c \leq \lambda_1(\mu) \leq \lambda(\mu) \leq \lambda_k(\mu) \leq C_k < +\infty \).

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