UNIQUENESS OF MINIMAL ENERGY SOLUTIONS FOR A SEMILINEAR PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN

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ABSTRACT. In this paper we study a semilinear problem for the fractional laplacian that are the counterpart of the Neumann problems in the classical setting. We show uniqueness of minimal energy solutions for small domains.

1. INTRODUCTION

In recent years there has been an increasing amount of attention to problems involving nonlocal diffusion operators. These problems are so vast that is impossible to give a comprehensive list of references, just to cite a few, see [10, 11, 13, 16, 20, 25] for some physical models, [2, 17, 22] for some applications in finances, [6] for appications in fluid dynamics, [15, 19, 21] for application in ecology and [14] for some applications in image processing.

Among these applications, one operator that is of particular importance is the *fractional laplacian* that is defined (up to some normalization constant) as

$$(-\Delta)^s u(x) = \text{p.v.} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

where p.v. stands for *in principal value* and $\Omega \subset \mathbb{R}^n$ is a bounded domain.

This operator is classical and have been studied by several authors. See for instance [1, 4, 5, 7, 9, 8, 18, 23], etc.

In this paper, we address the following semilinear problem associated to $(-\Delta)^s$

$$(1.1) \qquad (-\Delta)^s u + u = \lambda |u|^{q-2} u, \quad \text{in } \Omega.$$

This is the fractional counterpart of the classical Neumann problem

$$(1.2) \qquad \begin{cases} -\Delta u + u = \lambda |u|^{q-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, q is a *subcritical exponent* in the sense of the Sobolev embeddings. That is,

$$1 \le q < 2_s^* := \begin{cases} \frac{2n}{n-2s} & \text{if } 2s < n \\ \infty & \text{if } 2s \ge n. \end{cases}$$

The problem can be separated into three different types of behavior: sublinear $(1 \le q < 2)$; linear $(q = 2)$ and superlinear $(q > 2)$.

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The linear case is by now well understood as an eigenvalue problem and will not be considered here.

For the sublinear and superlinear cases, the parameter λ is superfluous since if one gets a solution for some particular value of λ, then taking a suitable multiple of the solution the value of λ can be taken to be 1.

It is fairly easy to see that any solution to (1.1) is a critical point of the functional

$$\mathcal{F}(u) := \frac{\frac{1}{2}[u]_{s;\Omega}^2 + \|u\|_{2;\Omega}^2}{\|u\|_{q;\Omega}^2},$$

where

$$[u]_{s;\Omega} := \left(\iint_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}}$$

is the so-called *Gagliardo seminorm* of u and, as usual, $\|u\|_{r;\Omega}$ denotes the $L^r(\Omega)$−norm.

By standard variational methods, one can see that there exists *minimal energy solutions* to (1.1). That is, functions $u \in H^s(\Omega)$ such that

$$\mathcal{F}(u) = \inf_{v \in H^s(\Omega)} \mathcal{F}(v).$$

Moreover, by a direct application of the Ljusternik – Schnirelmann method, one can construct a sequence $\lambda_k \uparrow \infty$ of critical energy levels and a sequence of critical points $\{u_k\}_{k \in \mathbb{N}}$ of $\mathcal{F}$ associated to $\{\lambda_k\}_{k \in \mathbb{N}}$.

Therefore, there exist infinitely many solutions to problem (1.1).

In this paper, we focus on *minimal energy solutions* to (1.1). In particular to the multiplicity problem of such solutions.

To this end, inspired by the results of [12], given a domain Ω we consider the family of contracted domains

(1.3) $\Omega_\mu := \mu \cdot \Omega = \{\mu x \colon x \in \Omega\}$ as $\mu \downarrow 0$

and look for the asymptotic behavior of minimal energy solutions in Ω_μ as $\mu \downarrow 0$.

We first show that the asymptotic behavior of every minimal energy solution is the same and, using this asymptotic behavior, we are able to conclude the uniqueness of minimal energy solution for contracted domains.

Finally, we give an estimate on the contraction parameter in order to obtain the uniqueness result.

To end this introduction, we want to remark that the same ideas can be used to deal with the Neumann problem (1.2). The changes needed are easy and are left to the interested reader.

2. Preliminaries.

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $0 < s < 1$. The fractional order Sobolev space is defined as

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \colon \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2}+s}} \in L^2(\Omega \times \Omega) \right\}.$$

This space is endowed with the norm

$$\|u\|_{s;\Omega} := ([u]_{s;\Omega}^2 + \|u\|_{2;\Omega}^2)^{\frac{1}{2}}.$$

It is well known (see, for instance, [9]) that there exists a critical exponent

$$2_s^* := \begin{cases} \frac{2n}{n-2s} & \text{if } 2s < n, \\ \infty & \text{otherwise} \end{cases}$$

such that for any $1 \le q < 2_s^*$ the embedding $H^s(\Omega) \subset L^q(\Omega)$ is compact.

We define the Sobolev constant as the number

$$S(\Omega) = S_{s,q}(\Omega) := \inf_{u \in H^s(\Omega)} \frac{\|u\|_{s;\Omega}^2}{\|u\|_{q;\Omega}^2}.$$

It is easy to see, as a consequence of the compactness of the embedding, that there exists an extremal for $S(\Omega)$. That is a function $u \in H^s(\Omega)$ where the above infimum is attained.

Also, any extremal for $S(\Omega)$ is a minimal energy (weak) solution to (1.1).

The constant λ in (1.1) depends on the normalization of the extremal. For instance, if the extremal u is normalized as $\|u\|_{q;\Omega} = 1$ then $\lambda = S(\Omega)$.

Recall that the operator $(-\Delta)^s$ is a bounded operator between the Sobolev space $H^s(\Omega)$ and its dual $(H^s(\Omega))'$ and can be computed by

$$\langle (-\Delta)^s u, v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx dy.$$

Therefore, for a solution to (1.1) we mean a function $u \in H^s(\Omega)$ such that

$$\frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx dy + \int_\Omega uv \, dx = \lambda \int_\Omega |u|^{q-2} uv \, dx,$$

for every $v \in H^s(\Omega)$.

3. Asymptotic behavior in thin domains

Throughout this section we fix the exponent $q \in [1, 2_s^*)$, $q \ne 2$.

Our objective in this section is to study the asymptotic behavior of the constant $S(\Omega_\mu)$ as $\mu \downarrow 0$, where the contracted domains Ω_μ are given by (1.3).

We begin with a simple estimate.

Lemma 3.1. *Under the above notations, we have that*

$$S(\Omega_\mu) \le |\Omega_\mu|^{1 - \frac{2}{q}} = \mu^{n(1 - \frac{2}{q})} |\Omega|^{1 - \frac{2}{q}}.$$

Proof. The lemma follows just by taking $u = 1$ as a test function in the definition of $S(\Omega_\mu)$. $\square$

Now we want to be more precise. We need to show that the asymptotic behavior of $S(\Omega_\mu)$ is given precisely by $\mu^{n(1 - \frac{2}{q})}$ and also to study the behavior of the associated extremals.

Lemma 3.2. *Under the above notations, we have that*

$$\lim_{\mu \downarrow 0} \frac{S(\Omega_\mu)}{\mu^{n(1 - \frac{2}{q})}} = |\Omega|^{1 - \frac{2}{q}}.$$

Moreover if $u_\mu \in H^s(\Omega_\mu)$ is an extremal for $S(\Omega_\mu)$, the rescaled extremals $\bar{u}_\mu(x) := u_\mu(\mu x)$ normalized such that $\|\bar{u}_\mu\|_{q;\Omega} = 1$ verify that

$$\bar{u}_\mu \to |\Omega|^{-\frac{1}{q}} \qquad \text{strongly in } H^s(\Omega).$$

Proof. First, observe that for $v \in H^s(\Omega_\mu)$, if we denote $\bar{v}(x) = v(\mu x)$, then $\bar{v} \in H^s(\Omega)$. Moreover, $[v]_{s;\Omega_\mu} = \mu^{\frac{n}{2}-s}[\bar{v}]_{s;\Omega}$ and $\|v\|_{r;\Omega_\mu} = \mu^{\frac{n}{r}}\|\bar{v}\|_{r;\Omega}$ for $1 \le r < 2_s^*$. Therefore

$$\frac{\|v\|_{s;\Omega_\mu}^2}{\|v\|_{q;\Omega_\mu}^2} = \mu^{n(1-\frac{2}{q})}\frac{\mu^{-2s}[\bar{v}]_{s;\Omega}^2 + \|\bar{v}\|_{2;\Omega}^2}{\|\bar{v}\|_{q;\Omega}^2}.$$

Now, let $u_\mu \in H^s(\Omega_\mu)$ be an extremal for $S(\Omega_\mu)$ and let $\bar{u}_\mu(x) = u_\mu(\mu x)$. Then,

$$S(\Omega_\mu) = \mu^{n(1-\frac{2}{q})}\frac{\mu^{-2s}[\bar{u}_\mu]_{s;\Omega}^2 + \|\bar{u}_\mu\|_{2;\Omega}^2}{\|\bar{u}_\mu\|_{q;\Omega}^2}.$$

Now, by Lemma 3.1, it follows that

(3.1)
$$\frac{\mu^{-2s}[\bar{u}_\mu]_{s;\Omega}^2 + \|\bar{u}_\mu\|_{2;\Omega}^2}{\|\bar{u}_\mu\|_{q;\Omega}^2} \le |\Omega|^{1-\frac{2}{q}}.$$

Let us now fix the normalization of the extremal u_μ such that $\|\bar{u}_\mu\|_{q;\Omega} = 1$, and by (3.1), we obtain that $\bar{u}_\mu$ is bounded in $H^s(\Omega)$ uniformly on $\mu > 0$. So, there exists $\bar{u} \in H^s(\Omega)$ such that (up to some sequence $\mu_k \to 0$),

(3.2) $\qquad\qquad \bar{u}_\mu \rightharpoonup \bar{u} \qquad$ weakly in $H^s(\Omega)$,

(3.3) $\qquad\qquad \bar{u}_\mu \to \bar{u} \qquad$ strongly in $L^r(\Omega)$ for any $1 \le r < 2_s^*$.

Also, from (3.1) and (3.2), we have that

$$[\bar{u}]_{s;\Omega} \le \liminf_{\mu \downarrow 0}[\bar{u}_\mu]_{s;\Omega} = 0,$$

therefore $\bar{u}$ is constant and, since $\|\bar{u}_\mu\|_{q;\Omega} = 1$, from (3.3) we obtain that $\|\bar{u}\|_{q;\Omega} = 1$.

All these together imply that $\bar{u} = |\Omega|^{-\frac{1}{q}}$.

From these estimates, one easily concludes that

$$|\Omega|^{1-\frac{2}{q}} \le \liminf_{\mu \downarrow 0} \mu^{-2s}[\bar{u}_\mu]_{s;\Omega}^2 + \|\bar{u}_\mu\|_{2;\Omega}^2$$

$$= \liminf_{\mu \downarrow 0} \frac{S(\Omega_\mu)}{\mu^{n(1-\frac{2}{q})}} \le \limsup_{\mu \downarrow 0} \frac{S(\Omega_\mu)}{\mu^{n(1-\frac{2}{q})}} \le |\Omega|^{1-\frac{2}{q}}.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. Uniqueness of extremals for small domains

In this section we show the uniqueness of extremals if the domain is contracted enough.

For that purpose, observe that if u_μ is an extremal for $S(\Omega_\mu)$ and $\bar{u}_\mu$ is the rescaled extremal normalized as $\|\bar{u}_\mu\|_{q;\Omega} = 1$, then $\bar{u}_\mu$ is a weak solution of the problem

(4.1) $\qquad\qquad (-\Delta)^s u + \mu^{2s} u = \mu^{2s}\lambda_\mu |u|^{q-2}u \qquad$ in Ω,

where $\lambda_\mu = S(\Omega_\mu)\mu^{-n(1-\frac{2}{q})}$. Recall also, that $0 \le \lambda_\mu \le |\Omega|^{1-\frac{2}{q}}$ (c.f. Lemma 3.1).

So, we define the space

$$\mathcal{E} := \{u \in H^s(\Omega) \colon \|u\|_{q;\Omega} = 1\}.$$

It is easy to see that $\mathcal{E}$ is a C^1 manifold.

We then define $F \colon \mathcal{E} \times [0,1) \to (H^s(\Omega))'$ as

$$\langle F(u,\mu), v \rangle := \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx dy$$
$$+ \mu^{2s} \int_\Omega uv \, dx - \mu^{2s} \lambda_\mu \int_\Omega |u|^{q-2} uv \, dx.$$

Denote $u_0 = |\Omega|^{-\frac{1}{q}} \in \mathcal{E}$ and observe that $F(u_0, 0) = 0$.

Following the ideas of [12] we want to use the Implicit Function Theorem (IFT) to show the existence of a small number $\delta > 0$ and a curve $\phi \colon [0, \delta) \to \mathcal{E}$ such that

$$\phi(0) = u_0 \quad \text{and} \quad F(\phi(\mu), \mu) = 0 \text{ for every } 0 \leq \mu < \delta,$$

and if $(u, \mu) \in \mathcal{E} \times [0, \delta)$ is such that $F(u, \mu) = 0$ and u is close to u_0 then $u = \phi(\mu)$.

Observe that if we can apply the IFT then, combining this with Lemma 3.2, automatically we obtain the uniqueness of extremals of $S(\Omega_\mu)$ for μ small.

In order to be able to apply the IFT we need to check that $d_u F|_{(u_0,0)}$ is invertible (see [3] or [24]). Recall that since F is define on a manifold, the derivative is defined on the tangent space of $\mathcal{E}$ at the point u_0.

Let us begin with a couple of lemmas.

Lemma 4.1. *The tangent space of $\mathcal{E}$ at u_0, that we denote by $T_{u_0}\mathcal{E}$ is given by*

$$T_{u_0}\mathcal{E} = \left\{ v \in H^s(\Omega) \colon \int_\Omega v \, dx = 0 \right\}.$$

Proof. Let $v \in T_{u_0}\mathcal{E}$. Then, there exists a differentiable curve, $\alpha \colon (-1, 1) \to \mathcal{E}$ such that $\alpha(0) = u_0$ and $\dot{\alpha}(0) = v$.

But, since $\alpha(t) \in \mathcal{E}$ for every $t \in (-1, 1)$ it follows that

$$\int_\Omega |\alpha(t)|^q \, dx = 1 \quad \text{for every } t \in (-1, 1).$$

Differentiating both sides of the equality gives

$$\int_\Omega q|\alpha(t)|^{q-2} \alpha(t) \dot{\alpha}(t) \, dx = 0.$$

So, if we evaluate at $t = 0$ and recall that u_0 is constant, we obtain that

(4.2) $$\int_\Omega v \, dx = 0.$$

On the other hand, if $v \in H^s(\Omega)$ verifies (4.2), we construct the curve $\alpha \colon (-1, 1) \to \mathcal{E}$ as

$$\alpha(t) = \frac{u_0 + tv}{\|u_0 + tv\|_q}.$$

Straightforward computations show that $\alpha(0) = u_0$ and $\dot{\alpha}(0) = v$. $\qquad \square$

Now, we denote $\mathcal{A} = (\text{span}\{1\})^\perp = \{f \in (H^s(\Omega))' \colon \langle f, 1 \rangle = 0\}$.

Lemma 4.2. *We have that*

$$d_u F|_{(u_0,0)} \colon T_{u_0}\mathcal{E} \to \mathcal{A}.$$

Moreover, the following expression holds

$$\langle d_u F|_{(u_0,0)}(u), v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy.$$

Proof. To prove the Lemma, first observe that

$$\langle F(u,0), v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy,$$

for every $u \in \mathcal{E}$.

From this expression the lemma follows. $\square$

It remains to see that $d_u F|_{(u_0,0)}$ has a continuous inverse.

Lemma 4.3. *The derivative $d_u F|_{(u_0,0)} \colon T_{u_0}\mathcal{E} \to \mathcal{A}$ has a continuous inverse.*

Proof. First observe that $T_{u_0}\mathcal{E}$ is a Hilbert space with inner product given by

$$(u,v) = \langle d_u F|_{(u_0,0)}(u), v \rangle = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy.$$

So, the Lemma follows from the Riesz representation Theorem. $\square$

Combining Lemmas 4.1, 4.2, 4.3 we are in position to apply the IFT and conclude the main result of the section

Theorem 4.4. *Given $\Omega \subset \mathbb{R}^n$ smooth and of finite measure and $1 < q < 2_s^*$, there exists $\delta > 0$ such that $S(\Omega_\mu)$ has a unique extremal for $0 < \mu < \delta$.*

Proof. At this point the proof is a direct consequence of the IFT and the remarks made at the beginning of the section. $\square$

5. Estimates for the contraction parameter

Let us first define

$$\mu_0 = \sup\{\delta > 0 \colon \exists! \text{ normalized extremal for } S(\Omega_\mu) \ \forall \mu < \delta\}.$$

From the results of the previous section, we know that $\mu_0 > 0$. We now want to find a lower bound for μ_0.

Lemma 5.1. *There exists $u_0 \in H^s(\Omega_{\mu_0})$ an extremal for $S(\Omega_{\mu_0})$, such that the rescaled function $\bar{u}_0(x) = u_0(\mu_0 x)$ normalized as $\|\bar{u}_0\|_{q;\Omega} = 1$ verifies that $d_u F|_{(\bar{u}_0,\mu_0)}$ is not invertible.*

Proof. Assume the oposite.

We first claim that there is a unique extremal for $S(\Omega_{\mu_0})$. Otherwise, if $u_0 \neq u_1$ are extremals such that the rescaled functions $\bar{u}_i(x) = u_i(\mu_0 x)$, $i = 0, 1$ normalized as $\|\bar{u}_i\|_{q;\Omega} = 1$, $i = 0, 1$, verify that $d_u F|_{(\bar{u}_i,\mu_0)}$ is invertible for $i = 0, 1$. But then, by the IFT, there exists $\delta > 0$ and two curves $\phi_i \colon (\mu_0 - \delta, \mu_0 + \delta) \to \mathcal{E}$ such that

$$F(\phi_i(\mu), \mu) = 0, \quad \text{for every } \mu \in (\mu_0 - \delta, \mu_0 + \delta), \ i = 0, 1.$$

But this contradicts the uniqueness of extremals for $\mu < \mu_0$.

Now, let $\mu_k > \mu_0$ be such that $\mu_k \to \mu_0$ as $k \to \infty$ and let u_k be an extremal for $S(\Omega_{\mu_k})$. We normalized these extremals so that the rescaled functions $\bar{u}_k(x) = u_k(\mu_k x)$ verify $\|\bar{u}_k\|_{q;\Omega} = 1$.

We want to see that $\{\bar{u}_k\}_{k\in\mathbb{N}}$ converges to $\bar{u}_0$ that is the rescaled function of the unique extremal u_0 for $S(\Omega_{\mu_0})$.

In fact, it is immediate to see that $\sup_{k\in\mathbb{N}} \|\bar{u}_k\|_{s;\Omega} < \infty$, and so, up to a subsequence, there exists $\bar{w} \in H^s(\Omega)$ such that

$$\bar{u}_k \rightharpoonup \bar{w} \quad \text{weakly in } H^s(\Omega)$$

$$\bar{u}_k \to \bar{w} \quad \text{strongly in } L^r(\Omega), \text{ for every } 1 \le r < 2_s^*.$$

From these convergence results it follows that

$$1 = \|\bar{u}_k\|_{q;\Omega} \to \|\bar{w}\|_{q;\Omega}$$

and

$$\mu_0^{-2s}[\bar{w}]_{s,2;\Omega}^2 + \|\bar{w}\|_{2;\Omega}^2 \le \liminf_{k\to\infty} \mu_k^{-2s}[\bar{u}_k]_{s,2;\Omega}^2 + \|\bar{u}_k\|_{2;\Omega}^2 = \liminf_{k\to\infty} \frac{S(\Omega_{\mu_k})}{\lambda_{\mu_k}}.$$

Now, let $u_0 \in H^s(\Omega_{\mu_0})$ be the unique extremal for $S(\Omega_{\mu_0})$ normalized such that the rescaled function $\bar{u}_0$ satisfies $\|\bar{u}_0\|_{q;\Omega} = 1$.

Then,

$$\limsup_{k\to\infty} \frac{S(\Omega_{\mu_k})}{\lambda_{\mu_k}} \le \limsup_{k\to\infty} \mu_k^{-2s}[\bar{u}_0]_{s,2;\Omega}^2 + \|\bar{u}_0\|_{2;\Omega}^2 = \mu_0^{-2s}[\bar{u}_0]_{s,2;\Omega}^2 + \|\bar{u}_0\|_{2;\Omega}^2 = \frac{S(\Omega_{\mu_0})}{\lambda_{\mu_0}}$$

These two inequalities combined imply that $w(x) = \bar{w}(\mu_0^{-1} x)$ is an extremal for $S(\Omega_{\mu_0})$ and so $w = u_0$.

Now, since we are assuming that $d_u F|_{(u_0,\mu_0)}$ is invertible, we can apply the IFT as in the proof of Theorem 4.4 to conclude that for some $\delta > 0$ there is a unique extremal for $S(\Omega_\mu)$ for $\mu < \mu_0 + \delta$. But this contradicts the definition of μ_0. □

Remark 5.2. By a simple application of the Fredholm's alternative, it follows that $d_u F|_{(u_0,\mu_0)}$ is not invertible if and only if it has a nontrivial kernel.

The following Poincaré-type inequality plays an important role in the bound of μ_0

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, smooth and of finite measure. Let $0 < s < 1$ and $1 \le q < 2_s^*$. Then, there exists $c > 0$, that depends on q, s and Ω, such that*

$$c\|w\|_{q;\Omega}^2 \le \frac{1}{2}[w]_{s;\Omega}^2,$$

for every $w \in H^s(\Omega)$ such that $\int_\Omega w\, dx = 0$.

Proof. The proof follows by a standard compactness argument and is omitted. □

We are ready to prove the main result of the section.

Theorem 5.4. *Under the notations and assumptions of the section, we have that*

$$\mu_0 \ge \left(\frac{c}{(q-1)|\Omega|^{1-\frac{2}{q}}} \right)^{\frac{1}{2s}},$$

where $c > 0$ is the constant in the Poincaré-type inequality of Lemma 5.3.

Proof. By Lemma 5.1, there exists u_0 an extremal for $S(\Omega_{\mu_0})$ such that the rescaled function $\bar{u}_0$ normalized such that $\|\bar{u}_0\|_{q;\Omega} = 1$ verifies that $d_u F|_{(\bar{u}_0,\mu_0)}$ is not invertible.

Moreover, by Remark 5.2, $d_u F|_{(\bar{u}_0,\mu_0)}$ has a nontrivial kernel.

Let $0 \neq z \in \ker(d_u F|_{(\bar{u}_0,\mu_0)})$, then z is a nontrivial weak solution to the problem

$$(5.1) \qquad \begin{cases} (-\Delta)^s z + \mu_0^{2s} z = \mu_0^{2s}\lambda_{\mu_0}(q-1)|\bar{u}_0|^{q-2}z & \text{in } \Omega \\ \int_\Omega z\,dx = 0. \end{cases}$$

Using z as a test function in the weak formulation of (5.1) gives

$$(5.2) \qquad \frac{1}{2}[z]_{s,2;\Omega}^2 + \mu_0^{2s}\|z\|_{2;\Omega}^2 = \mu_0^{2s}\lambda_{\mu_0}(q-1)\int_\Omega |\bar{u}_0|^{q-2}z^2\,dx.$$

Now, we use the Poincaré-type inequality of Lemma 5.3, Lemma 3.1 and Hölder's inequality to deduce from (5.2) that

$$c\|z\|_{q;\Omega}^2 \leq \mu_0^{2s}|\Omega|^{1-\frac{2}{q}}(q-1)\|\bar{u}_0\|_{q;\Omega}^{q-2}\|z\|_{q;\Omega}^2.$$

Finally, recalling that $\|\bar{u}_0\|_{q;\Omega} = 1$ and that $z \neq 0$ we arrive that the desired result. $\qquad\square$

Acknowledgements

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