GLOBAL HÖLDER REGULARITY FOR EIGENFUNCTIONS OF THE FRACTIONAL g-LAPLACIAN

JULIÁN FERNÁNDEZ BONDER, ARIEL SALORT AND HERNÁN VIVAS

Abstract. We establish global Hölder regularity for eigenfunctions of the fractional $g-$Laplacian with Dirichlet boundary conditions where $g = G'$ and G is a Young functions satisfying the so called Δ_2 condition. Our results apply to more general semilinear equations of the form $(-\Delta_g)^s u = f(u)$.

1. Introduction

Given an open and bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary we consider the problem

$$\begin{cases} (-\Delta_g)^s u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1}$$

for $\lambda \in \mathbb{R}$ with $g = G'$, G being a Young function, and $(-\Delta_g)^s$ the fractional $g-$Laplacian ia given by

$$(-\Delta_g)^s u(x) = \text{p.v.} \int g\left(D_s u(x,y)\right) \frac{dy}{|x-y|^{n+s}}.$$

Here $D_s u$ is the $s-$Hölder quotient defined by

$$D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^s}. \tag{2}$$

The fractional $g-$Laplacian operator is the natural generalization of the fractional $p-$Laplacian when a non-power behavior of the $s-$Hölder quotient is considered. These type of operators have received much attention in recent years, see for instance [1, 2, 3, 4, 7, 8, 10, 14, 18, 19, 20, 21] and the references in these articles. Observe that in the particular case that $G(t) = t^p$, $p > 1$, the eigenvalue problem for the fractional $p-$Laplacian is recovered.

The non-local, non-linear and non-homogeneous eigenvalue problem (1) was treated in [18, 20], where existence of eigenvalues was proved. Eigenvalues with other boundary conditions were studied in [4]. Recently, a homogeneous version of (1) was dealt with in [10]. The particular case of powers was studied in [5, 11, 16], where L^∞ bound of eigenfunctions was obtained.

The local version of (1) was addressed for instance in [12, 13, 17]. In [17], by appealing to the regularity theory of G. Lieberman, the authors prove $C^{1,\alpha}$ regularity of the first eigenfunction. In this setting, lower bounds of eigenvalues were also proved in [19].

The aim of this note is to prove global Hölder regularity for solutions of (1) for a class of operators where the Young function G satisfies that $G(t)$ is comparable with $tG'(t)$ (see condition (6)) and G is sub-critical in the sense of condition (11). This class of Young functions includes powers, powers multiplied by logarithms and sum of different powers, among others functions (see Section 2 for further examples).

2020 *Mathematics Subject Classification.* 35J62; 35B65, 35P30.

Key words and phrases. Fractional $g-$Laplacian; elliptic regularity; nonlinear eigenvalue problems.

The main regularity result will be achieved via the following a priori bound on the L^∞ norm of u (see below for conditions and definitions):

Theorem 1. *Let $s \in (0,1)$, G a Young function satisfying (6) with $sp^+ < n$ and $u \in W_0^{s,G}(\Omega)$ a weak solution of (1) such that $\int_\Omega G(u)\,dx = \mu$ for some $\mu > 0$. Then there exists a constant $C = C(n, s, p^\pm, \mu) < \infty$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

The proof of Theorem 1 is based in the well-known De Giorgi's iteration scheme. This technique is a powerful tool in regularity analysis of elliptic and parabolic PDEs and it has been shown to be very versatile and adaptable to different contexts. In our problem, one of the main drawbacks to be overtaken in order to apply the iteration scheme is the possible lack of homogeneity of the equation. An extra difficulty is added by the fact that weak solutions satisfy an equation in terms of modular but not in terms of norms, however, both embedding theorems and Hölder type inequalities hold for norm of functions and not for modulars.

The desired global regularity is a consequence of the previous theorem and the following result recently proved by the authors (see [9]):

Theorem 2. *Under the assumptions of Theorem 1, and assuming that $g = G'$ satisfies*

$$p^- - 1 \leq \frac{tg'(t)}{g(t)} \leq p^+ - 1, \tag{3}$$

there exists $\alpha > 0$ such that $u \in C^\alpha(\Omega)$ and

$$\|u\|_{C^\alpha(\Omega)} \leq C$$

for some constant C depending of n, s, μ and $p^\pm$.

A slight modification of our arguments gives regularity of nonlinear problems of the form

$$\begin{cases} (-\Delta_g)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{4}$$

where the nonlinearity satisfies $f = F'$ for F a Young function such that

$$\eta^- \leq \frac{tf(t)}{F(t)} \leq \eta^+.$$

together with the sub-critical constrain $F \prec\prec G^*$, that is

$$\lim_{t\to\infty} \frac{F(kt)}{G^*(t)} = 0; \tag{5}$$

this condition is enough to ensure that the embeddings hold. Here G^* is the critical Sobolev Young function defined in (12). Condition (5) holds in particular if $\eta^+ \leq (p^-)^*$, the Sobolev conjugate of p^- which is the lower bound

$$(p^-)^* \leq \frac{t(G^*)'(t)}{G^*(t)}.$$

Theorem 3. *Under the assumptions of Theorem 1, and assuming that F is a Young function satisfying (5), let $u \in W_0^{s,G}(\Omega)$ be a weak solution of (4). Then there exists $C = C(n, s, p^\pm, \eta^\pm) < \infty$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

If G additionaly satisfies (3), there exists $\alpha > 0$ such that $u \in C^\alpha(\Omega)$ and

$$\|u\|_{C^\alpha(\Omega)} \leq C$$

for some constant C.

We would like to highlight the recent work [6] in which regularity estimates for quasilinear equations driven by the $g-$Laplacian are addressed via a Moser type approach. These results bear some resemblance with the ones presented here. However, we point out that they assume a stronger assumptions on the Young function G, i.e., the submultiplicativity condition (condition known as the Δ' condition). Moreover, in [6] the eigenvalue problem is only covered for the case in which g is equivalent to a power. We also allow for a broader range of growth behaviors in the semilinear setting (4).

The paper is organized as follows: in Section 2 we give the necessary definitions and provide some examples of Young functions that fit our setting; in Section 3 we prove some technical results that will be used in the proof of Theorem 1 and in Section 4 we prove Theorem 1 (and as a consequence Theorem 2).

2. Preliminaries and some technical results

An application $G\colon [0,\infty) \longrightarrow [0,\infty)$ is said to be a *Young function* if it admits the integral representation

$$G(t) = \int_0^t g(s)\,ds,$$

where the right-continuous function g defined on $[0,\infty)$ has the following properties:

(i) $g(0) = 0, \quad g(t) > 0$ for $t > 0$,
(ii) g is nondecreasing on $(0,\infty)$,
(iii) $\lim_{t\to\infty} g(t) = \infty$.

From these properties it is easy to see that a Young function G is continuous, nonnegative, strictly increasing and convex on $[0,\infty)$. Without of loss generality we can assume $G(1) = 1$ and we extend G to negative values in an even fashion: $G(-t) = G(t)$.

We will assume throughout the paper that G satisfies

$$1 < p^- \le \frac{tg(t)}{G(t)} \le p^+ < \infty. \tag{6}$$

Condition (6) is equivalent to ask G and $\tilde G$ to satisfy the Δ_2 *condition or doubling condition*, i.e.,

$$G(2t) \le 2^{p^+} G(t), \qquad \tilde G(2t) \le 2^{(p^-)'} \tilde G(t), \tag{7}$$

(we usually denote $\mathbf{C} := 2^{p^+}$) where the *complementary* function of a Young function G is the Young function $\tilde G$ defined as

$$\tilde G(t) = \sup\{ta - G(a)\colon a > 0\}.$$

This condition allows to split sums as

$$G(a+b) \le \tfrac{\mathbf{C}}{2}(G(a) + G(b)). \tag{8}$$

The following lemma will be useful often; its proof is elementary so we omit it.

Lemma 4. *For $\alpha \in [0,1]$ and $t \ge 0$*

$$G(\alpha t) \le \alpha G(t),$$

and for $\alpha \ge 1$ and $t \ge 0$

$$G(\alpha t) \ge \alpha G(t)$$

More generally, for any, $\alpha, t \ge 0$

$$G(t)\min\{\alpha^{p^-}, \alpha^{p^+}\} \le G(\alpha t) \le G(t)\max\{\alpha^{p^-}, \alpha^{p^+}\}, \tag{9}$$

$$G^{-1}(t)\min\{\alpha^{\frac{1}{p^-}},\alpha^{\frac{1}{p^+}}\}\le G^{-1}(\alpha t)\le G^{-1}(t)\max\{\alpha^{\frac{1}{p^-}},\alpha^{\frac{1}{p^+}}\}. \tag{10}$$

Examples of Young functions satisfying the assumptions of Theorem 1 include:

- $G(t)=t^p$, $t\ge 0$, $p>1$;
- $G(t)=t^p(1+|\log t|)$, $t\ge 0$, $p>1$;
- $G(t)=t^p\chi_{(0,1]}(t)+t^q\chi_{(1,\infty)}(t)$, $t\ge 0$, $p,q>1$;
- $G(t)$ given by the complementary function to $\tilde G(t)=(1+t)^{\sqrt{\log(1+t)}}-1$, $t\ge 0$.
- $G_1\circ\ldots\circ G_m$, $\max\{G_1,\ldots,G_m\}$ and $\sum_{j=1}^m a_jG_j$ where G_j is a Young function and $a_j\ge 0$ for $j=1,\ldots,m$.

We will assume also that

$$\int_0^1 \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau<\infty \quad\text{ and }\quad \int_1^{+\infty}\frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau=\infty \tag{11}$$

which are the conditions necessary for the Orlicz-Sobolev embeddings to hold (see Proposition 5). We also consider the *critical Young function* G^* defined as

$$(G^*)^{-1}(t):=\int_0^t \frac{G^{-1}(\tau)}{\tau^{\frac{n+s}{n}}}\,d\tau. \tag{12}$$

Condition (11) is fulfilled in particular when

$$sp^+<n. \tag{13}$$

We point out that this is not the optimal function Young function under which the embeddings hold; indeed, in [1] they are shown to hold for

$$G_{\frac{s}{n}}(t):=G\left(F^{-1}(t)\right)\quad\text{with}\quad F(t):=\left(\int_0^t\left(\frac{\tau}{G(\tau)}\right)^{\frac{s}{s-n}}\,d\tau\right)^{\frac{n-s}{n}}.$$

Throughout the paper, given a Young function G we will denote $H:=G^*\circ G^{-1}$, where G^* is the critical Sobolev Young function defined in (12) and G^{-1} is the inverse of G. Observe that H defines a new Young function.

Weak solutions of (1) satisfy

$$\langle(-\Delta_g)^s u,v\rangle=\lambda\int_\Omega g(u)v\,dx \qquad\text{for any } v\in W_0^{s,G}(\Omega) \tag{14}$$

where

$$\langle(-\Delta_g)^s u,v\rangle:=\iint_{\mathbb{R}^n\times\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(v(x)-v(y))}{|x-y|^{n+s}}\,dxdy.$$

From now on, the modulars in $L^G(\Omega)$ and $W^{s,G}(\Omega)$ will be denoted as

$$\Phi_G(u):=\int_\Omega G(|u|)\,dx \qquad \Phi_{s,G}(u):=\iint_{\mathbb{R}^n\times\mathbb{R}^n} G(|D_su|)\,d\mu,$$

respectively where the notation

$$d\mu:=\frac{dxdy}{|x-y|^n} \tag{15}$$

and D_su defined in (2) will be used throughout. Over the space

$$W^{s,G}(\Omega):=\{u\colon\mathbb{R}^n\to\mathbb{R}\text{ measurable s.t. }\Phi_{s,G}(u)+\Phi_G(u)<\infty\}$$

we define the norm

$$\|u\|_{s,G}:=\|u\|_G+[u]_{s,G}, \tag{16}$$

where

$$\|u\|_G := \inf\left\{\lambda > 0\colon \Phi_G\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

and

$$[u]_{s,G} := \inf\left\{\lambda > 0\colon \Phi_{s,G}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

is the $(s,G)-$*Gagliardo seminorm.*

We also denote

$$W_0^{s,G}(\Omega) := \{u \in W^{s,G}(\Omega)\colon u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

In this space $[\cdot]_{s,G}$ turns out to be an equivalent norm.

For the proof of the aforementioned facts and an introduction to fractional Orlicz-Sobolev spaces we refer to [7, 8]. The following embedding is proved in [3].

Proposition 5 (Embedding). *Let G be a Young function satisfying (6) and (11), then there is a positive constant C such that*

$$\|u\|_{G^*} \le C\|u\|_{s,G}.$$

As mentioned before, Proposition 5 is not the most general embedding result for $W^{s,G}(\Omega)$, see [1]. However, the simplicity of the formula for G^*, (12), allows us to simplify a lot of our arguments.

3. Some technical results

The purpose of this section is to gather some technical and useful inequalities which are the key of our argument. Recall that $H := G^* \circ G^{-1}$, where G^* is the critical Sobolev Young function defined in (12) and G^{-1} is the inverse of G.

Lemma 6. *Let G be a Young function satisfying (6). Then for any $u \in L^G(\Omega)$ we have that*

$$\|G(u)\|_H \le \max\{\|u\|_{G^*}^{p^+}, \|u\|_{G^*}^{p^-}\}.$$

Proof. When $\|u\|_{G^*} \ge 1$, (10) and the definition of the Luxemburg norm yield that

$$\int_\Omega H\left(\frac{G(u)}{\|u\|_{G^*}^{p^+}}\right) dx = \int_\Omega G^* \circ G^{-1}\left(\frac{G(u)}{\|u\|_{G^*}^{p^+}}\right) dx \le \int_\Omega G^*\left(\frac{u}{\|u\|_{G^*}}\right) dx = 1$$

which gives $\|G(u)\|_H \le \|u\|_{G^*}^{p^+}$. The case $\|u\|_{G^*} < 1$ is analogous. □

Lemma 7. *Let G be a Young function satisfying (6) and (11) and G^* be defined by (12). If we define*

$$K(t) := t(G^* \circ G^{-1})^{-1}\left(\tfrac{1}{t}\right) = t(G \circ (G^*)^{-1})\left(\tfrac{1}{t}\right)$$

then there exists a some constant $\bar{C} > 0$ (depending only on $n, s, p^\pm, \mathbf{C}$) such that

$$K(t) \le \bar{C}\max\{t, t^{\frac{sq}{n}}\} \quad \text{for all } t > 0 \text{ and } q < p^-.$$

Proof. Using the expression of G^* and (11) we have that, for $t \ge 1$,

$$K(t) = tG\left(\int_0^{\frac{1}{t}} \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\, d\tau\right) \le tC_1 \qquad \text{with } C_1 := G\left(\int_0^1 \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\, d\tau\right).$$

Let us deal now with the case $t<1$. From (8) we get

$$K(t)=tG\left(\int_0^1 \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau+\int_1^{\frac{1}{t}} \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau\right)$$
$$\leq \frac{\mathbf{C}}{2}t\left(G\left(\int_0^1 \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau\right)+G\left(\int_1^{\frac{1}{t}} \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau\right)\right):=(a)+(b).$$

As before, $(a)\leq \frac{\mathbf{C}}{2}C_1 t$. In order to bound (b), observe that

$$G^{-1}(\tau)\tau^{-\frac{s}{n}} \text{ is an increasing function for all } \tau>0 \tag{17}$$

whenever $sp^+<n$. Indeed, by using the change of variable $w=G(t)$, the last assertion is equivalent to $wG(w)^{-\frac{s}{n}}$ to be increasing, assertion which is true since due to (6) we get

$$(wG(w)^{-\frac{s}{n}})'=G(w)^{-\frac{s}{n}}-\frac{sw}{n}G(w)^{-\frac{s}{n}-1}g(t)>0 \iff \frac{tg(t)}{G(t)}\leq p^+<\frac{n}{s}.$$

Therefore, from (17) we have that

$$tG\left(\int_1^{\frac{1}{t}} \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau\right)\leq tG\left(G^{-1}\left(\frac{1}{t}\right)t^{\frac{s}{n}}\int_1^{\frac{1}{t}}\tau^{-1}\,d\tau\right)$$
$$\leq tG\left(G^{-1}\left(\frac{1}{t}\right)t^{\frac{s(1-\varepsilon)}{n}}t^{\frac{s\varepsilon}{n}}\log\left(\frac{1}{t}\right)\right)$$

where $\varepsilon>0$ is arbitrary. Since $t^{\frac{s\varepsilon}{n}}\log\left(\frac{1}{t}\right)\leq C_\varepsilon$, from the last expression together with (9) and the fact that $t<1$ we get

$$tG\left(\int_1^{\frac{1}{t}} \frac{G^{-1}(\tau)}{\tau^{1+\frac{s}{n}}}\,d\tau\right)\leq \tilde{C}_\varepsilon t^{1+\frac{sp}{n}(1-\varepsilon)p^-}G\left(G^{-1}\left(\frac{1}{t}\right)\right)\leq \tilde{C}_\varepsilon t^{\frac{sq}{n}}$$

for any $q<p^-$.

Putting together the bounds for (a) and (b) we get the desired estimate. □

The following lemma relates norms with modulars of weak solutions to (1).

Lemma 8. *If $u\in W^{s,G}(\Omega)$ satisfies that*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} G(|D_s u|)\,d\mu\leq M$$

for some $M\geq 1$, then it holds that

$$[u]_{s,G}\leq M^{\frac{1}{p^-}}.$$

Proof. Since $M\geq 1$, from (9) we have

$$1\geq \frac{1}{M}\iint_{\mathbb{R}^n\times\mathbb{R}^n} G(|D_s u|)\,d\mu\geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} G\left(M^{-\frac{1}{p^-}}|D_s u|\right)\,d\mu.$$

Then, the definition of $[\cdot]_{s,G}$ implies that $[u]_{s,G}\leq M^{\frac{1}{p^-}}$ as desired. □

We also have the general version of the Chebyshev's inequality, which is proved as the usual one:

Lemma 9. *Let G be a real valued, measurable in Ω, nonnegative and nondecreasing function. For any u measurable in Ω and real valued and $t > 0$ we have*

$$|\{x \in \Omega \colon u(x) \geq t\}| \leq \frac{1}{G(t)} \int_\Omega G(u(x))\, dx$$

We close this section with a simple real analysis result regarding sequences that satisfy a nonlinear recurrence relationship, its proof is elementary:

Lemma 10. *Let $\{a_k\}_k$ be a sequence of nonnegative real numbers and assume that exist $C > 0$ and $\delta \in (0, 1)$ such that*

$$a_{k+1} \leq C a_k^{1+\delta}, \quad k \geq 0. \tag{18}$$

Then there exists $\varepsilon_0 > 0$ such that

$$a_0 \leq \varepsilon_0 \quad \Rightarrow \quad \lim_{k\to\infty} a_k = 0.$$

4. Proof of the main results

In this section we give the proof of our main result, namely Theorem 1.

Proof of Theorem 1. The proof follows De Giorgi's L^2 implies L^∞ scheme; we are going to show that, if

$$\iint G(D_s u)\, d\mu \leq c\lambda \int_\Omega G(u)\, dx \tag{19}$$

for some constant $c = c(p^+, p^-)$, then there exists $\varepsilon_0 > 0$ such that

$$\int_\Omega G(u) = \mu \leq \varepsilon_0 \quad \Rightarrow \quad \|u\|_{L^\infty(\Omega)} \leq 1. \tag{20}$$

Note that (19) is readily implied by (14) (taking $v = u$) and (11):

$$\begin{aligned}
\iint G(D_s u)\, d\mu &\leq \frac{1}{p^-} \iint g(D_s u) D_s u\, d\mu \\
&= \frac{1}{p^-}\lambda \int_\Omega g(u) u\, dx \\
&\leq \frac{p^+}{p^-}\lambda \int_\Omega G(u)\, dx
\end{aligned}$$

and that (20) implies the general result by scaling: if $\mu > \varepsilon_0$ we can rescale:

$$\int_\Omega G\left(\frac{u}{C}\right)\, dx \leq \max\left\{C^{-p^-}, C^{-p^+}\right\} \mu \leq \varepsilon_0$$

by taking C sufficiently large. Notice that u/C fulfills that

$$\begin{aligned}
\iint_{\mathbb{R}^n\times\mathbb{R}^n} G\left(\left|D_s\left(\frac{u}{C}\right)\right|\right)\, d\mu &\leq \max\left\{C^{-p^-}, C^{-p^+}\right\} \iint_{\mathbb{R}^n\times\mathbb{R}^n} G(|D_s u|)\, d\mu \\
&\leq \max\left\{C^{-p^-}, C^{-p^+}\right\} c\lambda \int_\Omega G(u)\, dx \\
&\leq \frac{\max\left\{C^{-p^-}, C^{-p^+}\right\}}{\min\left\{C^{-p^-}, C^{-p^+}\right\}} c\lambda \int_\Omega G\left(\frac{u}{C}\right)\, dx \\
&=: C_0 \lambda \int_\Omega G\left(\frac{u}{C}\right)\, dx.
\end{aligned}$$

Then, by (20) we have

$$\|u\|_{L^\infty(\Omega)} \le C$$

and we get the desired result.

Let us prove (20). For any $k \in \mathbb{N}$ consider the function $w_k \in W_0^{s,G}(\Omega)$ defined as

$$w_k := (u - (1 - 2^{-k}))_+.$$

It is easy to see that these functions fulfill the following properties

$$\begin{aligned} &w_{k+1}(x) \le w_k(x) \quad \text{a.e. in } \mathbb{R}^n, \\ &\{w_{k+1} > 0\} \subset \{w_k > 2^{-(k+1)}\}. \end{aligned} \tag{21}$$

We further claim that:

$$u \le (2^{k+1} - 1) w_k \quad \text{in } \{w_{k+1} > 0\}. \tag{22}$$

Indeed, notice that $w_{k+1}(x) > 0$ implies $u(x) > 1 - 2^{-(k+1)}$ and that

$$2^{k+1} - 1 = \frac{1 - 2^{-(k+1)}}{1 - 2^{-(k+1)} - (1 - 2^{-k})}$$

and compute

$$\begin{aligned} (2^{k+1} - 1) w_k(x) &= (2^{k+1} - 1)\left(u(x) - (1 - 2^{-k})\right) \\ &= \frac{1 - 2^{-(k+1)}}{1 - 2^{-(k+1)} - (1 - 2^{-k})} u(x) - \frac{(1 - 2^{-(k+1)})(1 - 2^{-k})}{1 - 2^{-(k+1)} - (1 - 2^{-k})} \\ &= u(x) + \frac{1 - 2^{-k}}{1 - 2^{-(k+1)} - (1 - 2^{-k})} u(x) - \frac{(1 - 2^{-(k+1)})(1 - 2^{-k})}{1 - 2^{-(k+1)} - (1 - 2^{-k})} \\ &= u(x) + 2^{k+1}(1 - 2^{-k})\left(u(x) - (1 - 2^{-(k+1)})\right) > u(x), \end{aligned}$$

so (22) holds.

Now, since $0 \le w_k \le |u| + 1 \in L^G(\Omega)$ and

$$\lim_{k\to\infty} w_k = (u - 1)_+,$$

by the Dominated Convergence Theorem one gets that

$$\lim_{k\to\infty} \int_\Omega G(w_k)\, dx = \int_\Omega G((u - 1)_+)\, dx. \tag{23}$$

We want to get a recursive bound of the form

$$\int_\Omega G(w_{k+1})\, dx \le C_k \left(\int_\Omega G(w_k)\, dx \right)^{1+\delta} \tag{24}$$

for some $\delta > 0$ and some (increasing) sequence of constants $C_k > 0$. Indeed, (24) is exactly condition (18) in Lemma 10 so its proof would imply

$$\int_\Omega G(u_+)\, dx = \int_\Omega G(w_0)\, dx \le \varepsilon_0 \quad \Rightarrow \quad \lim_{k\to\infty} \int_\Omega G(w_k)\, dx = 0.$$

Finally this combined with (23), implies

$$u \le 1 \quad \text{a.e. in } \Omega.$$

Replacing u by $-u$ we get the other bound.

To prove (24) we start with the following inequality:

$$g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{v_+(x)-v_+(y)}{|x-y|^s}\right)\geq p^-G\left(\frac{v_+(x)-v_+(y)}{|x-y|^s}\right). \tag{25}$$

Indeed, we may assume without loss of generality that $v(x)\geq v(y)$. If $x,y\in\{v>0\}$ then (25) is just (6). If $x\in\{v>0\}$ and $y\in\{v\leq 0\}$ then we use the fact that g is increasing and (6) to get

$$g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{v_+(x)-v_+(y)}{|x-y|^s}\right)\geq g\left(\frac{v(x)}{|x-y|^s}\right)\left(\frac{v(x)}{|x-y|^s}\right)\geq p^-G\left(\frac{v_+(x)-v_+(y)}{|x-y|^s}\right)$$

as desired.

Now we use (25) with $v=u-(1-2^{-k})$ as follows:

$$\begin{aligned}\iint_{\mathbb{R}^{2n}}G(D_sw_{k+1})\,d\mu&=\iint_{\mathbb{R}^{2n}}G\left(\frac{(w_{k+1})_+(x)-(w_{k+1})_+(y)}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n}\\&\leq\frac{1}{p^-}\iint_{\mathbb{R}^{2n}}g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\left(\frac{(w_{k+1})_+(x)-(w_{k+1})_+(y)}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n}\\&=\frac{\lambda}{p^-}\int_\Omega g(|u|)\frac{u}{|u|}w_{k+1}\,dx\end{aligned}$$

where the last equality comes from testing the equation with w_{k+1}. Using this together with (22), (21) and (6) gives

$$\begin{aligned}\iint_{\mathbb{R}^{2n}}G(D_sw_{k+1})\,d\mu&\leq\frac{\lambda}{p^-}\int_\Omega g((2^{k+1}-1)w_{k+1})\frac{(2^{k+1}-1)w_{k+1}}{2^{k+1}-1}\,dx\\&\leq\frac{p^+}{p^-}\lambda\int_\Omega G((2^{k+1}-1)w_{k+1})\frac{1}{2^{k+1}-1}\,dx\\&\leq\frac{p^+}{p^-}\lambda(2^{k+1}-1)^{p^+-1}\int_\Omega G(w_{k+1})\,dx.\end{aligned}\tag{26}$$

Next, by using Hölder inequality for Orlicz spaces

$$\int_\Omega G(w_{k+1})\,dx\leq 2\|G(w_{k+1})\|_H\|\chi_{\{w_{k+1}>0\}}\|_{\tilde H}\tag{27}$$

where $H=G^*\circ G$ and $\tilde H$ is its conjugate. To get a bound for the first factor we recall that

$$\|\chi_{\{w_{k+1}>0\}}\|_{\tilde H}\leq|\{w_{k+1}>0\}|H^{-1}\left(|\{w_{k+1}>0\}|^{-1}\right)=:K\left(|\{w_{k+1>0}\}|\right)$$

(see [15], page 149) to get, using Lemma 7, (21), Lemma 9 and (9)

$$\begin{aligned}\|\chi_{\{w_{k+1}>0\}}\|_{\tilde H}&\leq\bar C\kappa(|\{w_{k+1}>0\}|\\&\leq\bar C\kappa(|\{w_k>2^{-(k+1)}\}|)\\&\leq\bar C\kappa\left(\frac{1}{G(2^{-(k+1)})}\int_\Omega G(w_k)\,dx\right)\\&\leq\bar C\kappa\left(\frac{1}{G(1)2^{-(k+1)p^+}}\int_\Omega G(w_k)\,dx\right)\\&\leq C_0\tilde C^{k+1}\kappa\left(\int_\Omega G(w_k)\,dx\right),\end{aligned}\tag{28}$$

where κ denotes the increasing function $\kappa(t) = \max\{t, t^{\frac{sp^-}{n}}\}$, $t > 0$ and

$$C_0 := \bar{C}\kappa(1/G(1)), \quad \tilde{C} = \kappa(2^{\frac{sp^-}{n}}) > 1.$$

Now we need to bound the other term in (27). For this, we use Lemma 6, Proposition 5, and Lemma 8 applied to (26)

$$\begin{aligned}\|G(w_{k+1})\|_H &\le \max\{\|w_{k+1}\|_{G^*}^{p^+}, \|w_{k+1}\|_{G^*}^{p^-}\}\\ &\le C\max\{[w_{k+1}]_{s,G}^{p^+}, [w_{k+1}]_{s,G}^{p^-}\}.\end{aligned}$$

Applying Lemma 6 to (26) with $M = \frac{p^+}{p^-}\lambda(2^{k+1}-1)^{p^+-1}\int_\Omega G(w_{k+1})\,dx$ gives (observe that $M \ge 1$ for k big enough)

$$[w_{k+1}]_{s,G} \le C(\lambda, p^\pm) 2^{\frac{(k+1)(p^+-1)}{p^-}} \left(\int_\Omega G(w_k)\right)^{\frac{1}{p^-}}.$$

The last two inequalities together give

$$\|G(w_{k+1})\|_H \le \bar{C}^k \max\left\{\int_\Omega G(w_k)\,dx, \left(\int_\Omega G(w_k)\,dx\right)^{\frac{p^+}{p^-}}\right\} \tag{29}$$

for some constant $\bar{C}(\lambda, p^\pm) > 1$

Inserting (28) and (29) in (27) we finally get

$$\begin{aligned}\int_\Omega G(w_{k+1})\,dx &\le C\bar{C}^k \max\left\{\left(\int_\Omega G(w_k)\right)^{1+\frac{p^+}{p^-}}, \left(\int_\Omega G(w_k)\right)^{\frac{sp^-}{n}+\frac{p^+}{p^-}}\right\}\\ &\le C\bar{C}^{k+1}\left(\int_\Omega G(w_k)\right)^{1+\delta}\end{aligned}$$

for some $\delta > 0$ and some $\bar{C}(\lambda, p^\pm) > 1$. This proves (24).

□

References

[1] A. Alberico, A. Cianchi, L. Pick and L. Slavíková *Fractional Orlicz-Sobolev embeddings.* J. Math. Pures Appl. (9) 149 (2021), 216–253. 1, 4, 5

[2] A. Alberico, A. Cianchi, L. Pick and L. Slavíková *On fractional Orlicz-Sobolev spaces.* Anal. Math. Phys. 11 (2021), no. 2, Paper No. 84, 21 pp. 1

[3] S. Bahrouni and H. Ounaies, *Embedding theorems in the fractional Orlicz-Sobolev space and applications to non-local problems*, Discret and continuous Dynamical systems, 40, no. 5, 2917–2944. (2020). 1, 5

[4] S. Bahrouni and A. Salort, *Neumann and Robin type boundary conditions in Fractional Orlicz-Sobolev spaces.* ESAIM: Control, Optimisation and Calculus of Variations, 27, S15. (2021). 1

[5] L. Brasco and E. Parini, *The second eigenvalue of the fractional p-Laplacian.* Advances in Calculus of Variations, 9(4), 323-355. (2016). 1

[6] M. L. Carvalho, E. D. Silva, J. C. Albuquerque and S. Bashrouni, *Regularity results for quasilinear elliptic problems driven by the fractional $\Phi-$Laplacian operator.* arXiv preprint arXiv:2111.05405v1. (2021). 3

[7] J. Fernández Bonder and A. M. Salort, *Fractional order Orlicz-Sobolev spaces.* Journal of Functional Analysis, 277(2), 333-367. (2019). 1, 5

[8] J. Fernández Bonder, M. Pérez-Llanos and A. M. Salort, A Hölder Infinity Laplacian obtained as limit of Orlicz Fractional Laplacians, Rev Mat Complut (2021). https://doi.org/10.1007/s13163-021-00390-2 1, 5

[9] J. Fernández Bonder, A. Salort and H. Vivas, *Interior and up to the boundary regularity for the fractional g-Laplacian: the convex case.* arXiv preprint arXiv:2008.05543. (2020). 2

[10] J. Fernández Bonder, A. Salort and H. Vivas, *Homogeneous eigenvalue problems in Orlicz-Sobolev spaces* 1

[11] G. Franzina and G. Palatucci, Fractional p-eigenvalues. arXiv preprint arXiv:1307.1789. (2013). 1

[12] M. García-Huidobro, V. Le, R. Manásevich and K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*. Nonlinear Differential Equations and Applications NoDEA, 6(2), 207-225. (1999). 1

[13] J.P. Gossez and R. Manásevich, *On a nonlinear eigenvalue problem in Orlicz-Sobolev spaces*. Proceedings. Section A, Mathematics-The Royal Society of Edinburgh, 132(4), 891. (2002). 1

[14] T. Jung; Q-H Choi, *Embedding theorems on the fractional Orlicz-Sobolev spaces*. Korean J. Math. 29 (2021), no. 1, 57–64. 1

[15] A. Kufner, O. John and S. Fucik, Function spaces (Vol. 3). Springer Science & Business Media. (1977). 9

[16] E. Lindgren and P. Lindqvist, *Fractional eigenvalues*. Calculus of Variations and Partial Differential Equations, 49(1), 795-826. (2014). 1

[17] M. Montenegro and S. Lorca, *The eigenvalue problem for quasilinear elliptic operators with general growth*. Applied Mathematics Letters, 25(7), 1045-1049. (2012). 1

[18] A. Salort, *Eigenvalues and minimizers for a non-standard growth non-local operator*. Journal of Differential Equations, 268(9), 5413-5439. (2020). 1

[19] A. Salort, *Lower bounds for Orlicz eigenvalues*. Discrete Contin. Dyn. Syst., accepted. 1

[20] A. Salort and H. Vivas, *Fractional eigenvalues in Orlicz spaces with no Δ_2 condition*. arXiv preprint arXiv:2005.01847. (2020). 1

[21] E. Silva, M. Carvalho, J. de Albuquerque and B. Bahrouni,*Compact embedding theorems and a Lions' type lemma for fractional Orlicz-Sobolev spaces*. J. Differential Equations 300 (2021), 487–512. 1

(JFB and AS) Instituto de Cálculo (IC), CONICET, Departamento de Matemática, FCEN - Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, C1428EGA, Av. Cantilo s/n, Buenos Aires, Argentina

Email address, JFB: `jfbonder@dm.uba.ar`
Email address, AS: `asalort@dm.uba.ar`

(HV) Instituto de Cálculo (IC), CONICET, Centro Marplatense de Investigaciones matemáticas, Dean Funes 3350, 7600 Mar del Plata, Argentina

Email address: `havivas@mdp.edu.ar`