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EXISTENCE FOR AN ELLIPTIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS VIA FIXED-POINT METHODS *

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Abstract. In this paper we prove the existence of nonnegative nontrivial solutions of the system

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v, \end{cases}$$

with nonlinear coupling through the boundary given by

$$\begin{cases} \frac{\partial u}{\partial n} = f(x,u,v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = g(x,u,v), \end{cases}$$

under suitable assumptions on the nonlinear terms f and g. For the proof we use a fixed-point argument and the key ingredient is a Liouville type theorem for a system of Laplace equations with nonlinear coupling through the boundary of power type in the half space.

1. Introduction. In this paper we study the existence via topological methods of nonnegative solutions of the following elliptic system:

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v, \end{cases} \tag{1.1}$$

with nonlinear coupling at the boundary given by

$$\begin{cases} \frac{\partial u}{\partial n} = f(x,u,v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = g(x,u,v). \end{cases} \tag{1.2}$$

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Here Ω is a bounded smooth domain in $\mathbb{R}^N$, $\partial/\partial n$ denotes the outer normal derivative and $f, g : \partial\Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are smooth positive functions with $f(x, 0, 0) = g(x, 0, 0) = 0$. Moreover we deal with the "superlinear" case (see Section 3 for detailed assumptions, (H1), (H2), (H3), on f and g).

Existence results for nonlinear elliptic systems have deserved a great deal of interest in recent years, in particular when the nonlinear term appears as a source in the equation, complemented with Dirichlet boundary conditions. There are two main classes of systems that can be treated variationally: Hamiltonian and gradient systems. The system (1.1)-(1.2) is called Hamiltonian if there exists a function H such that $H_v = f$ and $H_u = g$, and is called gradient if there exists F with $\nabla F = (f, g)$. Other problems without variational structure can be treated via fixed-point arguments. For this type of results see, among others, [2], [4], [6], [8], [9], [13] and the survey [7].

Here we address the existence problem for (1.1)–(1.2) without a variational assumption on f and g. To our knowledge, no existence result prior to this work is available for the nonlinear boundary-condition case.

The topological method (a fixed-point argument) we apply here, has been used by several authors to deal with problems without variational structure (see for instance [5], [10], [23], [24]), and, as in our case, they were forced to impose some growth restrictions on f and g (see Theorem 3.2 in Section 3).

In the course of the proof, we will need some knowledge of the following eigenvalue problem:

$$\begin{cases} \Delta\varphi = \varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = \lambda\varphi & \text{on } \partial\Omega. \end{cases}$$

We collect the results that we need in Section 2, and we include the proofs, though they are straightforward, in order to make the paper self-contained.

The main difficulty in carrying out the fixed-point argument is to obtain L^∞ a priori bounds for (1.1)–(1.2). This difficulty is overcome by means of the blow-up technique introduced by Gidas-Spruck [17]. The key ingredient in making this technique work is a Liouville-type theorem for the system

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ \Delta v = 0, & \end{cases} \tag{1.3}$$

with boundary conditions

$$\begin{cases} \frac{\partial u}{\partial n} = v^p & \text{on } \partial\mathbb{R}^N_+, \\ \frac{\partial v}{\partial n} = u^q. & \end{cases} \tag{1.4}$$

In [20], Bei Hu studies the single equation,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial n} = u^p & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.5}$$

There he proves that if $1 < p < \frac{N}{N-2}$ there is no nontrivial nonnegative classical solution of (1.5). In [19] such nonexistence-type result is applied to compute the blow-up rate of a parabolic problem. In [14] the authors use a similar result to obtain the blow-up rate for a parabolic system.

Here, in Section 4, we adapt the moving plane technique to deal with the system (1.3)–(1.4), and we obtain the same type of result with similar restrictions on the exponents (see Theorems 3.5 and 3.6). This result seems to be of independent interest.

We remark that we can also deal with the semilinear case

$$\begin{cases} -\Delta u + u = r(x,u,v) & \text{in } \Omega, \\ -\Delta v + v = s(x,u,v), \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial n} = f(x,u,v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = g(x,u,v), \end{cases}$$

using the same ideas (see Remark 3.1). Since the main novelty here comes from the boundary terms, we present our results for (1.1)–(1.2).

The paper is organized as follows: in Section 2, we analyze the eigenvalue problem; in Section 3, we state and prove our main results (Theorems 3.2, 3.4 and 3.7). Finally, in Section 4 we prove the nonexistence results (Theorems 3.5 and 3.6).

2. The eigenvalue problem. In this section we analyze the following eigenvalue problem:

$$\begin{cases} \Delta \varphi = \varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = \lambda\varphi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The proof of the results are rather standard, so we only sketch them. In fact we prove

Theorem 2.1. *There exists a first positive eigenvalue λ_1 with positive eigenfunction φ_1 of (2.1). Moreover, if $\mu > \lambda_1$ there is no nonnegative nontrivial solution of*

$$\begin{cases} \Delta w = w & \text{in } \Omega, \\ \frac{\partial w}{\partial n} \geq \mu w & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Proof. First, we observe that the operator $A : L^2(\partial\Omega) \to L^2(\partial\Omega)$ given by $Af = u\,|_{\partial\Omega}$, where u is the solution of

$$\begin{cases} \Delta u &= u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= f \quad \text{on } \partial\Omega \end{cases}$$

is compact, self-adjoint and $\ker(A) = \{0\}$ (see [11], [12]). To see that it is positive, we observe that

$$\langle Af, f\rangle = \int_{\partial\Omega} uf = \int_{\partial\Omega} u\frac{\partial u}{\partial n} = \int_\Omega |\nabla u|^2 + \Delta u u = \int_\Omega |\nabla u|^2 + u^2.$$

Then there exists a nonincreasing sequence of positive eigenvalues of A, $0 < \mu_n$ with $\mu_n \to 0$. Then, if we take $0 < \lambda_1 = 1/\mu_1$ we only have to show that the corresponding eigenfunction φ_1 is positive. For that purpose we observe that φ_1 is a solution of the following minimization problem:

$$\lambda_1 = \min_{\int_{\partial\Omega} u^2 = 1} \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2 \right).$$

As $|u|$ is also a solution if u is a solution, we can choose φ_1 positive (see [11], [12] for the details).

To finish the proof of the theorem, assume that we have a nonnegative solution of (2.2). We multiply the equation by φ_1 and, after integration by parts, we get

$$\int_\Omega w\Delta\varphi_1 - \int_{\partial\Omega} w\frac{\partial\varphi_1}{\partial n} + \int_{\partial\Omega} \frac{\partial w}{\partial n}\varphi_1 = \int_\Omega w\varphi_1.$$

Hence we obtain

$$\mu \int_{\partial\Omega} w\varphi_1 \le \lambda_1 \int_{\partial\Omega} w\varphi_1,$$

a contradiction, unless $w \equiv 0$. □

3. Main results. As we have described in the introduction, we will use a topological argument in order to obtain our existence result. More precisely, we want to apply the following fixed-point theorem that can be found, for instance, in [6] (Theorem 3.1).

Theorem 3.1. *Let $\mathcal{C}$ be a cone in a Banach space X and $S : \mathcal{C} \to \mathcal{C}$ a compact mapping such that $S(0) = 0$. Assume that there are real numbers $0 < r < R$ and $t > 0$ such that*

1. *$x \neq tSx$ for $0 \leq t \leq 1$ and $x \in \mathcal{C}$, $\|x\| = r$, and*
2. *there exists a compact mapping $H : \overline{B}_R \times [0, \infty) \to \mathcal{C}$ (where $B_\rho = \{x \in \mathcal{C} : \ \|x\| < \rho\}$) such that*
   - (a) *$H(x, 0) = S(x)$ for $\|x\| = R$.*
   - (b) *$H(x, t) \neq x$ for $\|x\| = R$ and $t > 0$.*
   - (c) *$H(x, t) = x$ has no solution $x \in \overline{B}_R$ for $t \geq t_0$.*

Then S has a fixed point in $U = \{x \in \mathcal{C} : \ r < \|x\| < R\}$.

To apply this theorem, we proceed as follows. Consider the space

$$X = \{(u, v) \ : \ u, v \in C(\overline{\Omega})\},$$

with the norm $\|(u, v)\| = \|u\|_\infty + \|v\|_\infty$, which makes it a Banach space. Let $S : X \to X$ be the solution operator defined by $S(\phi, \psi) = (u, v)$, where (u, v) is the solution of

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = f(x, \phi, \psi) & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

$$\begin{cases} \Delta v = v & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = g(x, \phi, \psi) & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We observe that a fixed point of S is a solution of (1.1)–(1.2).

Now let us see that S satisfies the hypotheses of Theorem 3.1. By standard regularity theory, [18], as the normal derivatives of u and v are bounded in L^∞ it follows that u and v are C^α; hence S is a compact operator. As $f(x, 0, 0) = g(x, 0, 0) = 0$ we have, by Hopf's lemma, that $S(0) = 0$.

Let $\mathcal{C}$ be the cone $\mathcal{C} = \{(u, v) \in X : \ u \geq 0, \ v \geq 0\}$. It follows from the maximum principle that $S(\mathcal{C}) \subset \mathcal{C}$.

To verify (1) in Theorem 3.1 we argue by contradiction. Let us assume that for every $r > 0$ there exists a $0 \leq t \leq 1$ and a pair (U, V) such that

$$\begin{cases} \Delta U = U & \text{in } \Omega, \\ \Delta V = V, \end{cases} \tag{3.3}$$

$$\begin{cases} \frac{\partial U}{\partial n} = tf(x, U, V) & \text{on } \partial\Omega, \\ \frac{\partial V}{\partial n} = tg(x, U, V). \end{cases} \tag{3.4}$$

We multiply the first equation of (3.3) by φ_1, the first eigenfunction of (2.1), and we obtain

$$0 = \int_\Omega (\Delta U - U)\varphi_1 = \int_\Omega U(\Delta\varphi_1 - \varphi_1) + t\int_{\partial\Omega} f(x,U,V)\varphi_1 - \int_{\partial\Omega} U\frac{\partial\varphi_1}{\partial n}.$$

Hence

$$0 = t\int_{\partial\Omega} f(x,U,V)\varphi_1 - \lambda_1\int_{\partial\Omega} U\varphi_1.$$

We assume that f and g are "superlinear"; in fact, we make the following hypothesis, which we call (H1):

$$f(x,U,V) \le \varepsilon(U+V) \quad \text{and} \quad g(x,U,V) \le \varepsilon(U+V), \tag{H1}$$

for small $\|(U,V)\|$. Using (H1), we obtain

$$\lambda_1\int_{\partial\Omega} U\varphi_1 \le \varepsilon t\int_{\partial\Omega}(U+V)\varphi_1.$$

Analogously, for V we get

$$\lambda_1\int_{\partial\Omega} V\varphi_1 \le \varepsilon t\int_{\partial\Omega}(U+V)\varphi_1.$$

Adding both inequalities we conclude that $\lambda_1 \le 2\varepsilon$, a contradiction if ε satisfies $\varepsilon < \lambda_1/2$.

To see (2) we define H as follows: $H((\phi,\psi),t) = S(\phi+t,\psi+t)$. Clearly (a) holds.

To see (c) we have to impose any of the following conditions (we call this (H2)):

(H2.i) There exist real numbers $\mu > \lambda_1$ and $C > 0$ such that $f(x,u,v) \ge \mu u - C$ uniformly in $x \in \overline{\Omega}$ and $v \in \mathbb{R}_+$.

(H2.ii) There exist real numbers $\mu > \lambda_1$ and $C > 0$ such that $g(x,u,v) \ge \mu v - C$ uniformly in $x \in \overline{\Omega}$ and $u \in \mathbb{R}_+$.

(H2.iii) There exist real numbers $\mu > \lambda_1$ and $C > 0$ such that $f(x,u,v) + g(x,u,v) \ge \mu(u+v) - C$ uniformly in $x \in \overline{\Omega}$.

For instance, assume that (H2.i) holds. Then we observe that for t large enough we have $f(x,u+t,v+t) \ge \mu(u+t) - C \ge \mu u$, with $\mu > \lambda_1$, and hence, for t large, u is a nonnegative solution of

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} \ge \mu u & \text{on } \partial\Omega, \end{cases}$$

which contradicts Theorem 2.1.

The other cases can be handled in a similar fashion. Finally, condition (b) is an immediate consequence of an a priori bound for the system

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v, & \end{cases} \tag{3.5}$$

$$\begin{cases} \frac{\partial u}{\partial n} = f(x, u+t, v+t) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = g(x, u+t, v+t). & \end{cases} \tag{3.6}$$

Hence we have proved our main result, provided we have an a priori L^∞ bound for (3.5)-(3.6).

Theorem 3.2. *Let f and g satisfy* (H1)–(H2)*; if there exists a constant C such that for every solution (u, v) of* (1.1)-(1.2) *it holds that $\|u\|_\infty$, $\|v\|_\infty \le C$, then the system* (1.1)-(1.2) *has a nontrivial positive solution.*

Now our aim is to prove that, under further conditions on f and g, the nonnegative solutions of (1.1)-(1.2) are bounded in L^∞, so Theorem 3.2 applies. To do so, we apply the blow-up technique introduced by Gidas and Spruck [17]. We argue by contradiction. Assume that there is no such a priori bound; then there exists a sequence of positive solutions (u_n, v_n) such that $\max\{\|u_n\|_\infty,\ \|v_n\|_\infty\} \to \infty$ Let β_1, β_2 be two positive numbers to be fixed. We can assume that $\|u_n\|_\infty \to \infty$ and that $\|u_n\|_\infty^{\beta_2} \ge \|v_n\|_\infty^{\beta_1}$. As $\overline{\Omega}$ is compact and u_n is continuous, we can choose $x_n \in \overline{\Omega}$ such that $u_n(x_n) = \max_{\overline{\Omega}} u_n$. Moreover, it follows from the maximum principle that $x_n \in \partial\Omega$. Again, by the compactness of $\overline{\Omega}$, we can assume that $x_n \to x_0 \in \partial\Omega$.

We define γ_n such that $\gamma_n^{\beta_1}\|u_n\|_\infty = 1$. This sequence γ_n goes to 0 as $n \to \infty$. Let

$$w_n(y) = \gamma_n^{\beta_1} u_n(\gamma_n y + x_n), \qquad z_n(y) = \gamma_n^{\beta_2} v_n(\gamma_n y + x_n).$$

These functions are defined in $\Omega_n = \{y \in \mathbb{R}^N : \ \gamma_n y + x_n \in \Omega\}$. We observe that $0 \le w_n$, $z_n \le 1$ and $w_n(0) = 1$.

On f and g we impose the following condition (hypothesis (H3)):

$$\begin{aligned} f(x,u,v) &= a(x)u^{p_{11}} + b(x)v^{p_{12}} + h_1(x,u,v), \\ g(x,u,v) &= c(x)u^{p_{21}} + d(x)v^{p_{22}} + h_2(x,u,v), \end{aligned} \tag{H3}$$

where $0 < k \le a, b, c, d \le K < \infty$ and h_i are lower-order terms,

$$|h_i(x,u,v)| \le c_i(1 + |u|^{\alpha_{i1}} + |v|^{\alpha_{i2}}).$$

Here the exponents α_{ij} satisfy $0 \le \alpha_{ij} < p_{ij}$. Hence, w_n and z_n satisfy

$$\begin{cases} \Delta w_n = \gamma_n^2 w_n & \text{in } \Omega_n, \\ \Delta z_n = \gamma_n^2 z_n, \end{cases} \tag{3.7}$$

$$\begin{cases} \frac{\partial w_n}{\partial n} = \gamma_n^{\beta_1(1-p_{11})+1} a(*) w_n^{p_{11}} + \gamma_n^{\beta_1+1-\beta_2 p_{12}} b(*) z_n^{p_{12}} \\ \qquad + \gamma_n^{\beta_1+1} h_1(*, \gamma_n^{-\beta_1} w_n, \gamma_n^{-\beta_2} z_n) & \text{on } \partial\Omega_n, \\ \frac{\partial z_n}{\partial n} = \gamma_n^{\beta_2+1-\beta_1 p_{21}} c(*) w_n^{p_{21}} + \gamma_n^{\beta_2(1-p_{22})+1} d(*) z_n^{p_{22}} \\ \qquad + \gamma_n^{\beta_2+1} h_2(*, \gamma_n^{-\beta_1} w_n, \gamma_n^{-\beta_2} z_n). \end{cases} \tag{3.8}$$

Now we want to pass to the limit in (3.7)-(3.8), so we need to know what happens with the coefficients of the leading terms.

We distinguish two cases in terms of p_{ij}: the weakly coupled case and the strongly coupled case.

1) Weakly coupled case. We say that the system is weakly coupled if there exist $\beta_1,\ \beta_2$ such that

$$\begin{array}{ll} \beta_1(1-p_{11})+1=0, & \beta_1+1-\beta_2 p_{12}>0, \\ \beta_2(1-p_{22})+1=0, & \beta_2+1-\beta_1 p_{21}>0. \end{array} \tag{3.9}$$

Thus, in this case we choose $\beta_1 = \frac{1}{p_{11}-1}$, $\beta_2 = \frac{1}{p_{22}-1}$. These conditions impose

$$1 < p_{11}, p_{22}, \quad p_{12} < \frac{p_{11}(p_{22}-1)}{p_{11}-1}, \quad \text{and} \quad p_{21} < \frac{p_{22}(p_{11}-1)}{p_{22}-1}.$$

2) Strongly coupled case. We say that the system is strongly coupled if there exist $\beta_1,\ \beta_2$ such that

$$\begin{array}{ll} \beta_1(1-p_{11})+1>0, & \beta_1+1-\beta_2 p_{12}=0, \\ \beta_2(1-p_{22})+1>0, & \beta_2+1-\beta_1 p_{21}=0. \end{array} \tag{3.10}$$

Thus, in this case we choose $\beta_1 = \frac{p_{12}+1}{p_{12}p_{21}-1}$, $\beta_2 = \frac{p_{21}+1}{p_{12}p_{21}-1}$. These conditions impose

$$1 < p_{21}p_{12}, \quad p_{11} < 1 + \frac{p_{21}p_{12}-1}{p_{12}+1}, \quad \text{and} \quad p_{22} < 1 + \frac{p_{21}p_{12}-1}{p_{21}+1}.$$

First we deal with the weakly coupled case.

As w_n, z_n are C^α (see [18]) and f, g are smooth, we have (see [22], [15]) that w_n, z_n are uniformly bounded in $C^{1+\alpha}$. Hence, by standard Schauder theory, [18], we obtain that w_n, z_n are uniformly bounded in $C^{2+\alpha}$. Using a compactness argument we can assume that $(w_n, z_n) \to (w, z)$ in $C^{2+\beta} \times C^{2+\beta}$ with $\beta < \alpha$. We observe that the domains Ω_n approach $\mathbb{R}^N_+$. Therefore, passing to the limit we obtain a nontrivial nonnegative bounded solution w of

$$\left\{ \begin{array}{lcll} \Delta w & = & 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial w}{\partial n} & = & a(x_0) w^{p_{11}} & \text{on } \partial \mathbb{R}^N_+. \end{array} \right. \tag{3.11}$$

Bei Hu in [20] proved the following nonexistence theorem:

Theorem 3.3. *The only nonnegative classical solution of* (3.11) *is* $w \equiv 0$ *when* $1 < p_{11} < \frac{N}{N-2}$ *(*p_{11} *is subcritical) if* $N \geq 3$ *or* $0 < p_{11}$ *if* $N = 2$.

The proof of Theorem 3.3 relies on the *moving plane method*, introduced by Alexandroff and then used by several authors to study the symmetry properties of many elliptic equations ([1], [16], [21], etc.). We want to remark that in the critical case, $p_{11} = \frac{N}{N-2}$, there exist nontrivial nonnegative solutions of (3.11), [3].

Using Theorem 3.3 we get a contradiction, and this proves the a priori bound in the weakly coupled case. In summary, we have proved the following result:

Theorem 3.4. *Assume that the system* (1.1)–(1.2) *satisfies* (H3) *and is weakly coupled. If* $1 < p_{11}, p_{22} < \frac{N}{N-2}$ *(*$N \geq 3$*) or* $0 < p_{11}, p_{22}$ *(*$N = 2$*), then there exists a constant* C *such that every nonnegative solution* (u, v) *satisfies*

$$\|u\|_\infty, \|v\|_\infty \leq C.$$

Next, we deal with the strongly coupled case. Passing to the limit as in the previous case, we obtain a nontrivial, nonnegative solution of

$$\left\{ \begin{array}{lcll} \Delta w & = & 0 & \text{in } \mathbb{R}^N_+, \\ \Delta z & = & 0, & \end{array} \right. \tag{3.12}$$

with boundary conditions

$$\left\{ \begin{array}{lcll} \frac{\partial w}{\partial n} & = & b(x_0) z^{p_{12}} & \text{on } \partial \mathbb{R}^N_+, \\ \frac{\partial z}{\partial n} & = & c(x_0) w^{p_{21}}. & \end{array} \right. \tag{3.13}$$

For this problem, using the moving planes technique, in the next section we prove the following Liouville-type theorems:

Theorem 3.5. *Suppose $N \geq 3$, and $p_{12}, p_{21} \leq \frac{N}{N-2}$ but not both equal to $\frac{N}{N-2}$, with $p_{12}p_{21} > 1$. Let (w, z) be a classical nonnegative solution of* (3.12)–(3.13)*; then $w \equiv z \equiv 0$.*

For the case $N = 2$ we have to suppose that w or z is bounded, and we obtain the same conclusion with no restriction on the exponents p_{12}, p_{21}. More precisely, we have

Theorem 3.6. *Let $N = 2$, and $p_{12}, p_{21} > 0$. Let (w, z) be a classical nonnegative solution of* (3.12)–(3.13) *with w bounded; then $w \equiv z \equiv 0$.*

Again, applying Theorems 3.5 and 3.6 we get a contradiction in the strongly coupled case. In summary we have proved the following theorem:

Theorem 3.7. *Assume that the system* (1.1)–(1.2) *satisfies* (H3) *and is strongly coupled. If $1 < p_{12}p_{21}$ and $p_{12}, p_{21} \leq \frac{N}{N-2}$ but not both equal $(N \geq 3)$ or $0 < p_{12}, p_{21}$ $(N = 2)$, then there exists a constant C such that every nonnegative solution (u, v) satisfies $\|u\|_\infty, \|v\|_\infty \leq C$.*

Remark 3.1 We observe that the same techniques apply to the semilinear case,

$$\begin{cases} -\Delta u + u &= r(x, u, v) \quad \text{in } \Omega, \\ -\Delta v + v &= s(x, u, v), \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial n} &= f(x, u, v) \quad \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} &= g(x, u, v). \end{cases}$$

The differences arises in the blow-up argument. When we apply this technique, the points x_n need not lie in $\partial\Omega$, and hence we have to discriminate the case where $x_n \to x_0 \notin \partial\Omega$. In this case we can pass to the limit, but we lose the boundary condition, obtaining a semilinear problem in $\mathbb{R}^N$.

4. Nonexistence results. This section is devoted to the proofs of Theorems 3.5 and 3.6.

Throughout this section, for the sake of simplicity, we write p, q instead of p_{12}, p_{21} and u, v instead of w, z. Also we assume that $b(x_0) = c(x_0) = 1$.

Proof of Theorem 3.5. To apply the *moving plane method* we use the following notation: for $\lambda \in \mathbb{R}$, let

$$\Sigma_\lambda = \{(x_1, ..., x_N) : x_1 > 0,\ x_N < \lambda\},\ T_\lambda = \{(x_1, ..., x_N) : x_1 \geq 0,\ x_N = \lambda\},$$

$$\widetilde{\Sigma}_\lambda = \overline{\Sigma_\lambda} - \{(0, ..., 0, 2\lambda)\}, \qquad B_\mu^+(y_0) = B_\mu(y_0) \cap \{x_1 > 0\}.$$

Let (u, v) be a positive solution of (1.3)-(1.4) and $\alpha_1 = -\frac{p+1}{pq-1}$, $\alpha_2 = -\frac{q+1}{pq-1}$ (we observe that, as $pq > 1$, α_1 and α_2 are negative). Then define

$$\overline{u}(x) = \mu^{-\alpha_1} u(\mu x), \qquad \overline{v}(x) = \mu^{-\alpha_2} v(\mu x).$$

As (u, v) satisfies (1.3)-(1.4), $(\overline{u}, \overline{v})$ satisfies

$$\begin{cases} \Delta\overline{u} &= 0, & \Delta\overline{v} &= 0, \\ \frac{\partial\overline{u}}{\partial n} &= \overline{v}^p, & \frac{\partial\overline{v}}{\partial n} &= \overline{u}^q. \end{cases} \tag{4.1}$$

By (4.1), if $\overline{u} \equiv 0$, then $\overline{v} \equiv 0$; then we can suppose that $u \not\equiv 0$, $v \not\equiv 0$. Now we observe that if $\mu < 1$

$$\begin{aligned} &\sup\nolimits_{x\in B_1^+(0)} \overline{u}(x) \le \mu^{-\alpha_1} \sup\nolimits_{x\in B_\mu^+(0)} u(x) \le C\mu^{-\alpha_1}, \\ &\sup\nolimits_{x\in B_1^+(0)} \overline{v}(x) \le \mu^{-\alpha_2} \sup\nolimits_{x\in B_\mu^+(0)} v(x) \le C\mu^{-\alpha_2}. \end{aligned} \tag{4.2}$$

Also

$$\begin{aligned} &\inf\nolimits_{x\in B_1^+(0)} \overline{u}(x) \ge \mu^{-\alpha_1} \inf\nolimits_{x\in B_\mu^+(0)} u(x) \ge c\mu^{-\alpha_1}, \\ &\inf\nolimits_{x\in B_1^+(0)} \overline{v}(x) \ge \mu^{-\alpha_2} \inf\nolimits_{x\in B_\mu^+(0)} v(x) \ge c\mu^{-\alpha_2}. \end{aligned} \tag{4.3}$$

Let ε_1, ε_2 be the following numbers, which are positive by the maximum principle:

$$\varepsilon_1 = \min_{|x|=1,\, x_N\ge 0} \overline{u}(x) > 0, \qquad \varepsilon_2 = \min_{|x|=1,\, x_N\ge 0} \overline{v}(x) > 0.$$

Next we observe that if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then by a comparison argument,

$$\begin{cases} \overline{u}(x) &\ge & \frac{\varepsilon}{|x|^{N-2}} & |x| \ge 1 \quad x_1 > 0, \\ \overline{v}(x) &\ge & \frac{\varepsilon}{|x|^{N-2}}. \end{cases} \tag{4.4}$$

Now we use Kelvin's inversion to define

$$\varphi(x) = \frac{\overline{u}(\frac{x}{|x|^2})}{|x|^{N-2}}, \qquad \psi(x) = \frac{\overline{v}(\frac{x}{|x|^2})}{|x|^{N-2}}.$$

As $(\overline{u}, \overline{v})$ satisfies (4.1), this pair of functions, (φ, ψ), satisfies

$$\begin{cases} \Delta\varphi(x) &= 0, & \Delta\psi(x) &= 0, \\ \frac{\partial\varphi}{\partial n}(x) &= \frac{\psi^p(x)}{|x|^{N-(N-2)p}}, & \frac{\partial\psi}{\partial n}(x) &= \frac{\varphi^q(x)}{|x|^{N-(N-2)q}}. \end{cases}$$

As a consequence of (4.4), we obtain

$$\psi(x) = \frac{\overline{v}(\frac{x}{|x|^2})}{|x|^{N-2}} \geq \varepsilon, \quad \varphi(x) = \frac{\overline{u}(\frac{x}{|x|^2})}{|x|^{N-2}} \geq \varepsilon, \quad \text{in } |x| \leq 1, \quad x_1 > 0.$$

Also, by (4.2), we have

$$\begin{aligned} \varphi(x) &= \frac{\overline{u}(\frac{x}{|x|^2})}{|x|^{N-2}} \leq \frac{\sup_{y\in B_1^+(0)} \overline{u}(y)}{|x|^{N-2}} \leq \frac{C\mu^{-\alpha_1}}{|x|^{N-2}} \quad \text{if } |x| \geq 1, \quad x_1 > 0, \\ \psi(x) &= \frac{\overline{v}(\frac{x}{|x|^2})}{|x|^{N-2}} \leq \frac{\sup_{y\in B_1^+(0)} \overline{v}(y)}{|x|^{N-2}} \leq \frac{C\mu^{-\alpha_2}}{|x|^{N-2}} \quad \text{if } |x| \geq 1, \quad x_1 > 0. \end{aligned} \tag{4.5}$$

In order to prove symmetry properties of φ and ψ, we set

$$\Phi_\lambda(x) = \varphi_\lambda(x) - \varphi(x), \qquad \Psi_\lambda(x) = \psi_\lambda(x) - \psi(x),$$

where for $\lambda < 0$ we define

$$\begin{aligned} \varphi_\lambda(x_1, ..., x_N) &= \varphi(x_1, ..., x_{N-1}, 2\lambda - x_N) = \varphi(x_\lambda), \\ \psi_\lambda(x_1, ..., x_N) &= \psi(x_1, ..., x_{N-1}, 2\lambda - x_N) = \psi(x_\lambda). \end{aligned}$$

Now we can begin the *moving plane method*.

Lemma 4.1. *If $-\lambda$ is big enough, then*

$$\Phi_\lambda, \Psi_\lambda \geq 0 \qquad \textit{in } \widetilde{\Sigma}_\lambda.$$

Proof. Let us start by defining the following functions:

$$\overline{\Phi}_\lambda(x) = |z|^\beta \Phi_\lambda(x), \qquad \overline{\Psi}_\lambda(x) = |z|^\beta \Psi_\lambda(x),$$

where $z = x + e_1 = x + (1, 0, ..., 0)$. These functions satisfy

$$\left. \begin{aligned} -\Delta\overline{\Phi}_\lambda + \tfrac{2\beta}{|z|^2}\, z\cdot\nabla\overline{\Phi}_\lambda + \tfrac{\beta(N-2-\beta)}{|z|^2}\overline{\Phi}_\lambda = 0, \\ -\Delta\overline{\Psi}_\lambda + \tfrac{2\beta}{|z|^2}\, z\cdot\nabla\overline{\Psi}_\lambda + \tfrac{\beta(N-2-\beta)}{|z|^2}\overline{\Psi}_\lambda = 0. \end{aligned} \right. \quad \text{in } \Sigma_\lambda$$

We choose $\beta = \frac{N-2}{2}$ so that the coefficient of order zero in both equations is nonnegative.

At the boundary, these functions satisfy

$$\begin{aligned}
-\frac{\partial\overline{\Phi}_\lambda}{\partial x_1}\ |_{x_1=0} &= -\Big(\frac{\partial|z|^\beta}{\partial x_1}\Phi_\lambda(x)+|z|^\beta\frac{\partial\Phi_\lambda}{\partial x_1}(x)\Big)\ |_{x_1=0}\\
&= -\Big(\frac{\beta}{|z|^2}\overline{\Phi}_\lambda+|z|^\beta\frac{\partial}{\partial x_1}(\varphi_\lambda(x)-\varphi(x))\Big)\ |_{x_1=0}\\
&= -\frac{\beta}{|z|^2}\overline{\Phi}_\lambda+|z|^\beta\Big(\frac{1}{|x_\lambda|^{N-(N-2)p}}\psi_\lambda^p-\frac{1}{|x|^{N-(N-2)p}}\psi^p\Big).
\end{aligned}$$

Now, as $|x_\lambda|\le|x|$ in $\overline{\Sigma_\lambda}$ ($\lambda<0$), by the mean value theorem,

$$\begin{aligned}
\Big(\frac{1}{|x_\lambda|^{N-(N-2)p}}\psi_\lambda^p-\frac{1}{|x|^{N-(N-2)p}}\psi^p\Big) &\ge \frac{1}{|x|^{N-(N-2)p}}\left(\psi_\lambda^p-\psi^p\right)\\
&= \frac{1}{|x|^{N-(N-2)p}}\left(p\xi^{p-1}\Psi_\lambda\right),
\end{aligned}$$

where ξ lies between ψ_λ and ψ. Then

$$-\frac{\partial\overline{\Phi}_\lambda}{\partial x_1}\ |_{x_1=0}\ge -\frac{\beta}{|z|^2}\overline{\Phi}_\lambda+\overline{\Psi}_\lambda\frac{1}{|x|^{N-(N-2)p}}p\xi^{p-1}. \tag{4.6}$$

Analogously,

$$-\frac{\partial\overline{\Psi}_\lambda}{\partial x_1}\ |_{x_1=0}\ge -\frac{\beta}{|z|^2}\overline{\Psi}_\lambda+\overline{\Phi}_\lambda\frac{1}{|x|^{N-(N-2)q}}q\zeta^{q-1}, \tag{4.7}$$

where ζ lies between φ_λ and φ.

Now suppose that the statement of the lemma is false; that is,

$$\inf_{x\in\widetilde{\Sigma}_\lambda}\overline{\Phi}_\lambda=-\delta<0.$$

We have

$$\begin{aligned}
|\overline{\Phi}_\lambda(x)| &= |z|^\beta|\varphi_\lambda(x)-\varphi(x)|\le|z|^\beta\left(|\varphi_\lambda(x)|+|\varphi(x)|\right)\\
&\le\Big(\frac{C\mu^{-\alpha_1}}{|x_\lambda|^{N-2}}+\frac{C\mu^{-\alpha_1}}{|x|^{N-2}}\Big)|z|^\beta\le\frac{3C\mu^{-\alpha_1}}{|x|^{\frac{N-2}{2}}},\quad\text{if } |x| \text{ is big enough.}
\end{aligned}$$

Analogously,

$$|\overline{\Psi}_\lambda(x)|\le\frac{3C\mu^{-\alpha_2}}{|x|^{\frac{N-2}{2}}}.$$

Now, near the point $(0,...,0,2\lambda)$ (more precisely, for $|x-(0,...,0,2\lambda)|\le 1$), we have

$$\begin{aligned}\overline{\Phi}_\lambda(x) &\ge |z|^\beta\left(\varepsilon-\varphi(x)\right) \ge |z|^\beta\Big(\varepsilon-\frac{C\mu^{-\alpha_1}}{|x|^{N-2}}\Big)\\ &\ge |z|^\beta\Big(\varepsilon-\frac{C\mu^{-\alpha_1}}{|\lambda|^{N-2}}\Big)>0, \quad \text{if } -\lambda \text{ is big enough.}\end{aligned}$$

In a similar way we obtain, for $|x-(0,...,0,2\lambda)|\le 1$, $\overline{\Psi}_\lambda(x)>0$. Then the infimum must be located in $y\in\overline{\Sigma}_\lambda\backslash B_1(0,...,0,2\lambda)$.

By the maximum principle, $y\notin \mathrm{int}(\widetilde{\Sigma}_\lambda)$ and $y\notin \mathrm{T}_\lambda$ because $\overline{\Phi}_\lambda\equiv 0$ in T_λ; then y must be in $\{(x_1,...,x_N);x_1=0\}$.

If $\overline{\Psi}_\lambda(y)\ge 0$ we are done because by (4.6) the normal derivative of $\overline{\Phi}_\lambda$ must be positive at y, a fact that contradicts Hopf's Lemma.

If not, $\psi_\lambda(y)<\psi(y)$ and then $\inf\overline{\Psi}_\lambda(x)=\overline{\Psi}_\lambda(y^*)<0$, and by an analogous argument, $\varphi_\lambda(y^*)<\varphi(y^*)$.

Then we have, by (4.5),

$$\xi(y)\le\frac{C\mu^{-\alpha_2}}{|y|^{N-2}},\qquad \zeta(y^*)\le\frac{C\mu^{-\alpha_1}}{|y^*|^{N-2}}. \tag{4.8}$$

By Hopf's Lemma, we can suppose that the normal derivative of $\overline{\Phi}_\lambda$ is negative at y; that is, using (4.8)

$$\begin{aligned}0>-\frac{\partial\overline{\Phi}_\lambda}{\partial x_1}\,|_{x=y} &\ge -\frac{\beta}{|z|^2}\overline{\Phi}_\lambda(y)+\overline{\Psi}_\lambda(y)\frac{1}{|y|^{N-(N-2)p}}p\xi^{p-1}\\ &\ge -\frac{\beta}{1+|y|^2}\overline{\Phi}_\lambda(y)+\overline{\Psi}_\lambda(y)\frac{1}{|y|^2}pC\mu^{-\alpha_2(p-1)}.\end{aligned}$$

Then, we have

$$\frac{\beta}{1+|y|^2}\delta<-\frac{p}{|y|^2}C\mu^{-\alpha_2(p-1)}\overline{\Psi}_\lambda(y).$$

Replacing in (4.7), we get

$$\begin{aligned}-\frac{\partial\overline{\Psi}_\lambda}{\partial x_1}\,|_{x=y^*} &\ge -\frac{\beta}{1+|y^*|^2}\overline{\Psi}_\lambda(y)-\frac{q}{|y^*|^2}C\mu^{-\alpha_1(q-1)}\delta\\ &\ge \frac{\beta^2}{1+|y^*|^2}\delta\frac{|y|^2}{1+|y|^2}\frac{1}{pC\mu^{-\alpha_2(p-1)}}-\frac{q}{|y^*|^2}\delta C\mu^{-\alpha_1(q-1)}\\ &\ge\Big[\frac{\beta^2}{pC\mu^{-\alpha_2(p-1)}}-qC\mu^{-\alpha_1(q-1)}\Big]\frac{\delta}{|y^*|^2}.\end{aligned} \tag{4.9}$$

We observe that, as $pq > 1$, if we choose μ small enough, we get that the last term is positive, which is a contradiction, and the lemma is proved. $\square$

Let us now start to move the plane.

Lemma 4.2. *If* $\lambda_0 = \sup\{\lambda < 0 : \Phi_\gamma, \Psi_\gamma \geq 0$ *in* $\widetilde{\Sigma}_\gamma\ \forall\ \gamma < \lambda\}$, *then* $\lambda_0 = 0$.

Proof. Suppose that $\lambda_0 < 0$. By continuity, we have

$$\Phi_{\lambda_0}, \Psi_{\lambda_0} \geq 0 \qquad \text{in } \widetilde{\Sigma}_{\lambda_0}.$$

In the boundary $\{x_1 = 0\} \cap \overline{\Sigma}_{\lambda_0}$, by (4.6) and (4.7) these functions satisfy

$$\begin{aligned} \frac{\partial \Phi_{\lambda_0}}{\partial n} &= \frac{\psi_\lambda^p}{|x_\lambda|^{N-(N-2)p}} - \frac{\psi^p}{|x|^{N-(N-2)p}} \geq \frac{p}{|x|^{N-p(N-2)}} \xi^{p-1} \Psi_{\lambda_0} \geq 0, \\ \frac{\partial \Psi_{\lambda_0}}{\partial n} &= \frac{\varphi_\lambda^q}{|x_\lambda|^{N-(N-2)q}} - \frac{\varphi^q}{|x|^{N-(N-2)q}} \geq \frac{q}{|x|^{N-q(N-2)}} \zeta^{q-1} \Phi_{\lambda_0} \geq 0. \end{aligned} \tag{4.10}$$

Now, by (4.10) (as $N - p(N-2) \geq 0$, $N - q(N-2) > 0$ and $\lambda_0 < 0$), $\Phi_{\lambda_0}, \Psi_{\lambda_0} \not\equiv 0$ in $\widetilde{\Sigma}_{\lambda_0}$; then, by the maximum principle, we have

$$\Phi_{\lambda_0}, \Psi_{\lambda_0} > 0 \qquad \text{in } \overline{\Sigma}_{\lambda_0} - \{T_{\lambda_0} \cup \{(0, ..., 0, 2\lambda_0)\}\}. \tag{4.11}$$

Now, let us define the following numbers, which by (4.11) are positive:

$$\begin{aligned} \delta_1 &= \inf\{\Phi_{\lambda_0} : x_1 > 0,\ |x - (0, ..., 0, 2\lambda_0)| = \frac{|\lambda_0|}{2}\}, \\ \delta_2 &= \inf\{\Psi_{\lambda_0} : x_1 > 0,\ |x - (0, ..., 0, 2\lambda_0)| = \frac{|\lambda_0|}{2}\}, \end{aligned}$$

$\delta = \min\{\delta_1, \delta_2\}$. The point $(0, ..., 0, 2\lambda_0)$ might be a singularity point for Φ_{λ_0} and Ψ_{λ_0}; to control this fact, we define h_ε to be the solution of the following problem:

$$\begin{cases} \Delta h_\varepsilon = 0 & \text{in} \quad \varepsilon < |x - (0, ..., 0, 2\lambda_0)| < \frac{1}{2}|\lambda_0|,\ x_1 > 0, \\ h_\varepsilon = \delta & \text{on} \quad |x - (0, ..., 0, 2\lambda_0)| = \frac{1}{2}|\lambda_0|,\ x_1 \geq 0, \\ h_\varepsilon = 0 & \text{on} \quad |x - (0, ..., 0, 2\lambda_0)| = \varepsilon,\ x_1 \geq 0, \\ \frac{\partial h_\varepsilon}{\partial n} = 0 & \text{on} \quad \varepsilon < |x - (0, ..., 0, 2\lambda_0)| < \frac{1}{2}|\lambda_0|,\ x_1 = 0. \end{cases}$$

By the maximum principle, we have

$$\Phi_{\lambda_0}, \Psi_{\lambda_0} \geq h_\varepsilon \quad \text{in } \varepsilon \leq |x - (0, ..., 0, 2\lambda_0)| \leq \tfrac{1}{2}|\lambda_0|,\ |x_1| \geq 0.$$

Now, let $\varepsilon \to 0$, and as $\lim_{\varepsilon\to 0^+} h_\varepsilon(x) \equiv \delta$, we obtain

$$\Phi_{\lambda_0}, \Psi_{\lambda_0} \geq \delta \quad \text{in } 0 < |x-(0,...,0,2\lambda_0)| \leq \tfrac{1}{2}|\lambda_0|, \;\; |x_1| \geq 0.$$

As, in $\widetilde{\Sigma}_{\lambda_0}$, $\overline{\Phi}_{\lambda_0} \geq \Phi_{\lambda_0}$, $\overline{\Psi}_{\lambda_0} \geq \Psi_{\lambda_0}$, we obtain

$$\lim_{\lambda\searrow\lambda_0} \inf_{\substack{|x-(0,\dots,0,2\lambda_0)|\leq|\lambda_0|/2 \\ x_1\geq 0}} \overline{\Phi}_\lambda \geq \inf_{\substack{|x-(0,\dots,0,2\lambda_0)|\leq|\lambda_0|/2 \\ x_1\geq 0}} \Phi_{\lambda_0} \geq \delta,$$

and an analogous inequality holds for $\overline{\Psi}_\lambda$.

By the definition of λ_0, there exists a sequence (λ_k), $\lambda_k \searrow \lambda_0$, such that $\inf_{x\in\widetilde{\Sigma}_{\lambda_k}} \overline{\Phi}_{\lambda_k}(x) < 0$ or $\inf_{x\in\widetilde{\Sigma}_{\lambda_k}} \overline{\Psi}_{\lambda_k}(x) < 0$. Let us suppose that

$$\inf_{x\in\widetilde{\Sigma}_{\lambda_k}} \overline{\Phi}_{\lambda_k}(x) < 0. \tag{4.12}$$

Clearly, $\lim_{|x|\to\infty} \overline{\Phi}_{\lambda_k}(x) = 0$; then the infimum (4.12) must be located in some point $x^k \in \overline{\Sigma}_{\lambda_k} - B_{\frac{|\lambda_0|}{2}}(0,...,0,2\lambda_0)$ if $|\lambda_k - \lambda_0|$ is small enough.

From the equation for $\overline{\Phi}_{\lambda_k}$ one finds that x^k cannot be an interior point. Since $\overline{\Phi}_{\lambda_k} \equiv 0$ on T_{λ_k} it follows that x^k is located on the lateral wall

$$\big\{x : x_1 = 0,\; x_N < \lambda_k,\; |x-(0,...,0,2\lambda_0)| \geq \frac{|\lambda_0|}{2}\big\}.$$

Then the tangential derivative $\frac{\partial\overline{\Phi}_{\lambda_k}}{\partial x_N}(x^k) = 0$. Now, as $\overline{\Phi}_{\lambda_k}$, $\overline{\Psi}_{\lambda_k}$ satisfy (4.6) and (4.7), the infimum of $\overline{\Psi}_{\lambda_k}$ must also be less than 0, and by analogous considerations must be located in the lateral wall too.

By the boundary conditions (4.6), (4.7) and by (4.9) we have that $\overline{\Phi}_{\lambda_k}$ cannot take a negative minimum at a point on the boundary $\{x_1 = 0\}\cap\{|x| > 1\}$; then we must have $|x^k| \leq 1$. Therefore we can assume (via a subsequence) that $\lim_{k\to\infty} x^k = x_0$. Then we have

$$\overline{\Phi}_{\lambda_0}(x_0) = 0, \qquad \frac{\partial\overline{\Phi}_{\lambda_0}}{\partial x_N} = 0, \qquad x_0 \in T_{\lambda_0} \cap \{x_1 = 0\}, \tag{4.13}$$

and, as a consequence of (4.13), we get

$$\frac{\partial\Phi_{\lambda_0}}{\partial x_N}(x_0) = 0. \tag{4.14}$$

Let g be the solution of the following elliptic problem:

$$\begin{cases} \Delta g &= 0 \quad \text{in } \{3/2\lambda_0 < x_N < \lambda_0,\ x_1^2 + \cdots + x_{N-1}^2 < 1\}, \\ g(x) &= 0 \quad \text{on } \{x_N = \lambda_0\} \cap \{x_1^2 + \cdots + x_{N-1}^2 \leq 1\}, \\ g(x) &= 0 \quad \text{on } \{x_1^2 + \cdots + x_{N-1}^2 = 1\} \cap \{3/2\lambda_0 \leq x_N \leq \lambda_0\}, \\ g(x) &= \eta \quad \text{on } \{x_N = 3/2\lambda_0\} \cap \{x_1^2 + \cdots + x_{N-1}^2 \leq 1\}, \end{cases}$$

where $\eta = \inf\{\Phi_{\lambda_0}(x) : \ x_N = 3/2\lambda_0,\ x_1^2 + \cdots + x_{N-1}^2 \leq 1\} > 0$. By construction, we have $\Phi_{\lambda_0} \geq g$. Now, as g is symmetric with respect to $\{x_1 = 0\}$, we have

$$\frac{\partial g}{\partial n}(x) = -\frac{\partial g}{\partial x_1}(x) = 0 \ \text{ on } \{x_1 = 0\},$$

and as $\Phi_{\lambda_0}(x_0) = g(x_0) = 0$,

$$\frac{\partial \Phi_{\lambda_0}}{\partial x_N}(x_0) \leq \frac{\partial g}{\partial x_N}(x_0).$$

But, by Hopf's Lemma, $\frac{\partial g}{\partial x_N}(x_0)$ must be negative, which is a contradiction of (4.14) and proves our claim. $\square$

End of the proof of Theorem 3.5. From the last lemma we have that

$$\varphi(x_1, ..., -x_N) \geq \varphi(x_1, ..., x_N), \qquad x_N < 0.$$

As the same is valid for $x_N > 0$ we obtain that φ is symmetric with respect to the x_N axis.

The same argument shows that φ is symmetric with respect to every direction perpendicular to x_1, and hence $\varphi(x) = q(x_1, |(x_2, ..., x_N)|)$. We conclude that u and v depend also on x_1 and $|(x_2, ..., x_N)|$. As the origin is arbitrary we obtain that u and v are functions of x_1 only and we can easily see that this is not possible unless $u \equiv v \equiv 0$. $\square$

Proof of Theorem 3.6. As before, if $u \equiv 0$, then $v \equiv 0$; hence we can suppose that u and v are not identically zero. By the maximum principle, we have

$$c = \inf_{|x|=2R;\ x_1 \geq 0} v(x) > 0,$$

and by hypothesis $\|u\|_{L^\infty} \leq L$.

We now construct the auxiliary function $\psi(x) = c\frac{(2R)^\varepsilon}{|x|^\varepsilon}$. A direct calculation shows that

$$
\begin{aligned}
&-\Delta\psi < 0 \quad \text{for } x \neq 0 \text{ since } N = 2 \text{ and } \varepsilon > 0,\\
&\frac{\partial \psi}{\partial n} = 0 \leq \frac{\partial v}{\partial n} \quad \text{on } \{x_1 = 0\},\\
&\psi(x) = c \leq v(x) \quad \text{on } \{x : \ |x| = 2R \text{ and } x_1 \geq 0\},\\
&\lim_{M\to\infty} \inf_{|x|>M} (v(x) - \psi(x)) \geq 0.
\end{aligned}
$$

It follows from the maximum principle that $v(x) \geq \psi(x)$, for $|x| \geq 2R,\ x_1 \geq 0$. Now, letting $\varepsilon \to 0^+$, we obtain $v(x) \geq c$, for $|x| \geq 2R,\ x_1 \geq 0$. Next, let $K > 2R$ be a large positive number, and take a smooth cut-off function $\zeta(x)$ such that

$$
\begin{array}{ll}
\zeta(x) \equiv 0 & \text{on } \{|x| \leq K\} \cup \{|x| \geq 4K\},\\
\zeta(x) \equiv 1 & \text{on } \{2K \leq |x| \leq 3K\},\\
0 \leq \zeta(x) \leq 1, & |\nabla \zeta(x)| \leq \frac{C}{K}.
\end{array}
$$

Multiplying the equation $\Delta u = 0$ by $u^{-1}\zeta^2$ and integrating by parts, we obtain

$$
\begin{aligned}
\int_{\{x_1=0\}} \frac{\zeta^2}{u} v^p dS + \int\int_{\{x_1>0\}} \zeta^2 \frac{|\nabla u|^2}{u^2} dx &= \int\int_{\{x_1>0\}} 2\zeta\nabla\zeta \frac{\nabla u}{u} dx\\
&\leq \int\int_{\{x_1>0\}} |\nabla\zeta|^2 dx + \int\int_{\{x_1>0\}} \zeta^2 \frac{|\nabla u|^2}{u^2} dx.
\end{aligned}
$$

It follows that

$$
\int_{\{x_1=0\}} \frac{\zeta^2}{u} v^p dS \leq \int\int_{\{x_1>0\}} |\nabla\zeta|^2 \, dx,
$$

which implies that

$$
\frac{c^p}{L} K \leq \int_{2K}^{3K} \frac{v^p}{u}(0, x_2)\, dx_2 \leq \frac{C^2}{K^2} |B_{4K}(0)| \leq 16\pi^2 C^2.
$$

This is a contradiction if K is large enough

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