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BLOW-UP VS. SPURIOUS STEADY SOLUTIONS

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Abstract. In this paper, we study the blow-up problem for positive solutions of a semidiscretization in space of the heat equation in one space dimension with a nonlinear flux boundary condition and a nonlinear absorption term in the equation. We obtain that, for a certain range of parameters, the continuous problem has blow-up solutions but the semidiscretization does not and the reason for this is that a spurious attractive steady solution appears.

1. Introduction

For many differential equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow up). Typical examples where this happens are problems involving reaction terms in the equation (see [6] and the references therein).

In this paper we are interested in numerical approximations of problems with blow-up. In particular, we study the long time behaviour of solutions of the semidiscretization in space of the following parabolic problem:

$$\left\{ \begin{array}{rcll} u_t & = & u_{xx} - \lambda u^p & \text{in } (0,1)\times[0,T), \\ u_x(1,t) & = & u(1,t)^q & \text{on } [0,T), \\ u_x(0,t) & = & 0 & \text{on } [0,T), \\ u(x,0) & = & u_0(x) \geq 0 & \text{in } [0,1], \end{array} \right. \tag{1.1}$$

where $p, q > 1$ and $\lambda > 0$ are parameters.

For this type of problems, existence and regularity of solutions have been proved in [5], [2] for an initial data that satisfies a compatibility condition. In the general case one can obtain a solution in H^1 by a standard approximation procedure (see [2] for the details).

In our problem one has a reaction term at the boundary and an absorption term in the equation. These two terms compete and the blow up phenomenon occurs if and only if $p < 2q - 1$ or $p = 2q - 1$ with $\lambda < q$ (see [5], [2]).

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In fact there holds:

Theorem 1.1 ([2], Theorems 4.1, 4.2 and 4.7 and [5])**.**

1. *Suppose that* $p < 2q - 1$ *or* $p = 2q - 1$ *with* $\lambda < q$*, if* $u_0 > v$*, where* v *is any maximal stationary solution; then* u *blows up in finite time.*
2. *Suppose that* $p > 2q - 1$ *or* $p = 2q - 1$ *with* $\lambda \geq q$*; then every positive solution is global.*

The numerical semidiscrete version of (1.1) proposed here comes from a first order finite element approximation on the variable x with a uniform mesh (this is not an essential requirement) keeping t continuous (from a well known fact in this case this scheme coincides with a classical finite difference second order scheme). A further mass lumping technique simplifies the scheme and the subsequent proofs. For other approximations of problems with blow-up we refer to [3], [1] and references therein.

The equation reads as follows:

$$\begin{cases} MU' = -AU + u_N^q e_N - \lambda M U^p, \\ U(0) = u_0^I, \end{cases} \tag{1.2}$$

where $U = (u_1, ..., u_N)$, $e_N = (0, ..., 0, 1)$, M is the mass matrix, A is the stiffness matrix and u_0^I is the Lagrange interpolation of u_0. If we write this equation as a system, we obtain

$$\begin{cases} u_1' &= \frac{2}{h^2}(u_2 - u_1) - \lambda u_1^p, & \\ u_k' &= \frac{1}{h^2}(u_{k+1} - 2u_k + u_{k-1}) - \lambda u_k^p, & 2 \leq k \leq N-1, \\ u_N' &= \frac{2}{h^2}(u_{N-1} - u_N) - \lambda u_N^p + \frac{2}{h} u_N^q, & \\ u_i(0) &= u_0(x_i), & 1 \leq i \leq N, \end{cases} \tag{1.3}$$

where x_i is a uniform partition of the interval $[0, 1]$.

For this numerical scheme a straightforward adaptation of the proof of Theorem 2 of [3] gives the following convergence theorem for regular solutions.

Theorem 1.2. *Let* $u \in C^{2,1}([0, 1] \times [0, T_1])$ *be the solution of* (1.1) *and* u_h *its semidiscrete approximation. Then there exists a constant* C *depending on* T_1 *and* u *such that, for* h *small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T_1])} \leq C h^{\frac{3}{2}}.$$

We want to describe the cases in which the blow-up phenomenon occurs for (1.3). In section 2 we prove the following theorem:

Theorem 1.3. *Let* $U = (u_1, ..., u_N)$ *be a positive solution of* (1.3)*. Then*

1. *if* $p \leq q$ *and the initial datum is large enough,* U *has finite blow-up time;*
2. *if* $p > q$*,* U *is global.*

Therefore we want to give an explanation for the different behaviour of the solutions in the continuous and semidiscrete cases expressed by Theorems 1.1 and 1.3. For that purpose we prove in section 3 that there exists a spurious steady solution that is attractive and goes to infinity as h (the mesh parameter) goes to zero. In fact, we prove the following theorem:

Theorem 1.4. *Assume $q < p < 2q-1$. Then there exists a spurious steady solution $W = (w_1, ..., w_N)$ of (1.3), which is attractive and verifies that $|w_N| \sim \frac{1}{h^{1/(p-q)}}$. So if the initial datum is large enough, the solution is global and converges, as t goes to infinity, to a spurious stationary solution for which w_N is of order $h^{-1/(p-q)}$.*

2. Semidiscrete blow-up

In this section we describe when the blow-up phenomena occurs for the semidiscrete scheme (1.3) in terms of the parameters p, q.

Case $p \le q$. Let

$$\Phi(u) \equiv \int_0^1 \frac{(u_x)^2}{2} + \lambda \int_0^1 \frac{u^{p+1}}{p+1} - \frac{u^{q+1}(1,t)}{q+1};$$

then Φ is a Lyapunov functional for (1.1). We want to observe that if u_0 verifies $\Phi(u_0) < 0$, then u has finite blow-up time (see [2], Theorem 4.5). The discrete analogous of Φ is

$$\Phi_h(U) \equiv \frac{1}{2}\langle A^{1/2}U; A^{1/2}U\rangle + \frac{\lambda}{p+1}\langle MU^p; U\rangle - \frac{U_N^{q+1}}{q+1}.$$

As before, this Φ_h is a Lyapunov functional for (1.2). Now, let $W = (w_1, ..., w_N)$ be a stationary solution of (1.2). Then we have

$$0 = -AW - \lambda MW^p + w_N^q \cdot e_N; \tag{2.1}$$

multiplying (2.1) by W and using the fact that $1 < p \le q$, we obtain

$$\begin{aligned} 0 &= -\frac{1}{2}\langle A^{1/2}W; A^{1/2}W\rangle - \lambda \sum_{i=1}^N m_{ii}\frac{w_i^{p+1}}{2} + \frac{w_N^{q+1}}{2} \\ &\ge -\frac{1}{2}\langle A^{1/2}W; A^{1/2}W\rangle - \lambda \sum_{i=1}^N m_{ii}\frac{w_i^{p+1}}{p+1} + \frac{w_N^{q+1}}{q+1} = -\Phi_h(W). \end{aligned}$$

So every positive stationary solution of (1.2) has positive "energy" (i.e. $\Phi_h(W) \ge 0$) and then if U_0 satisfies $\Phi_h(U_0) < 0$, as every global solution must converge to a stationary one (see [4]), it must blow-up. Now, it is easy to check that $\Phi_h(u_0^I) \to \Phi(u_0)$ and therefore we conclude that, if $\Phi(u_0) < 0$, then u and u_h blow up for every small h. □

Remark. As an alternative proof of this fact, we observe that if the initial datum satisfies $u_N^{q-1}(0) > 2/h$, the solution blows up in finite time, because u_N satisfies

$$u_N' \ge -\frac{2u_N}{h^2} - \lambda u_N^p + \frac{2}{h}u_N^q \ge u_N^q$$

if h is small enough. Then, as $q > 1$, u_N blows-up, and so does U. Moreover, after integration, we have

$$T_h - t \le \int_{u_N(t)}^{\infty} \frac{1}{x^q}dx < \infty. \tag{2.2}$$

From this proof it seems that the size that the initial data needs to guarantee blow-up depends on h. From the former proof we can see that this is not the case.

Case $p > q$. In this case, we have that every solution is globally defined. Suppose not; then

$$\lim_{t \nearrow T} \max_{k=1,\dots,N} |u_k(t)| = \infty.$$

Hence there exist $t_0 < T$ and $1 \le j \le N$ such that

$$\max_{k=1,..,N} \sup_{t \in [0,t_1]} u_k(t) = u_j(t_0) > M;$$

therefore $u_j'(t_0) \ge 0$. Now we can assume that $j = N$, because if not from the equations (1.3) and $u_j'(t_0) \ge 0$, we deduce that $u_{j+1} = u_{j-1} = u_j$ so $U(t_0)$ is constant. But with $j = N$ (as $p \ge q$)

$$u_N'(t_0) = \frac{2(u_{N-1}(t_0) - u_N(t_0))}{h^2} - \lambda u_N^p(t_0) + \frac{2}{h} u_N^q(t_0) < 0,$$

which is a contradiction. This proves Theorem 1.3. □

The latter case shows that this semidiscrete scheme has substantial differences in the global behaviour of the solutions with the real equation, since for the case $q < p < 2q - 1$, we know that (1.1) has initial data that blows-up in finite time but all the solutions of (1.3) are global.

In order to explain this phenomenon, we will proceed to make an analysis of the steady solutions of (1.3).

3. Spurious steady solutions

In this section we will assume that $q < p < 2q - 1$.

We want to look at stationary solutions of (1.3), i.e., solutions of

(3.1)
$$\begin{cases} 0 = & \frac{2}{h^2}(w_2 - w_1) - \lambda w_1^p, \\ 0 = & \frac{1}{h^2}(w_{k+1} - 2w_k + w_{k-1}) - \lambda w_k^p, \qquad 2 \le k \le N-1, \\ 0 = & \frac{2}{h^2}(w_{N-1} - w_N) - \lambda w_N^p + \frac{2}{h} w_N^q. \end{cases}$$

From this equation we can obtain w_2 as an increasing function of w_1,

$$w_2 = w_1 + \lambda h^2 w_1^p \equiv F_2(w_1)$$

and from the second equation we can obtain w_3 as a function of w_1 and w_2 and, using the former equation, as a function of w_1

$$w_3 = 2w_2 - w_1 + \lambda h^2 w_2^p = 2F_2(w_1) - w_1 + \lambda h^2 (F_2(w_1))^p \equiv F_3(w_1).$$

We observe that w_3 is increasing as a function of w_1 and also that the differences $w_2 - w_1$ and $w_3 - w_2$ are increasing functions of w_1.

We can continue with this procedure and obtain an increasing sequence of $w_1 < w_2 < \dots . < w_k = F_k(w_1) < \dots < w_N$ (also each F_k is a increasing function of w_1) that satisfies

$$\begin{aligned} w_k &= 2w_{k-1} - w_{k-2} + \lambda h^2 w_{k-1}^p \\ &= 2F_{k-1}(w_1) - F_{k-2}(w_1) + \lambda h^2 (F_{k-1}(w_1))^p \equiv F_k(w_1). \end{aligned}$$

We have obtained that every solution of (3.1) is increasing.

Now, if $W = (w_1, ..., w_N)$ is a solution of (3.1), then the last two coordinates $w_{N-1} = F_{N-1}(w_1)$ and $w_N = F_N(w_1)$ have to satisfy the last condition

$$0 = 2\frac{F_{N-1} - F_N}{h^2} - \lambda(F_N)^p + \frac{2F_N^q}{h} \equiv G(w_1). \tag{3.2}$$

Then every positive solution of (3.1) gives a solution of (3.2). Conversely if we have a positive w_1 such that $G(w_1) = 0$, then we can obtain a solution of (3.1) by taking $w_k = F_k(w_1)$.

We observe that as $F_N(w_1)$ is a continuous increasing function of w_1 and has range $[0, +\infty)$, then there exists x such that

$$F_N(x) = \frac{a}{h^{\frac{1}{p-q}}}.$$

For that value of x,

$$G(x) \geq -\frac{2a}{h^{2+\frac{1}{p-q}}} + \frac{2a^q}{h^{1+\frac{q}{p-q}}} - \lambda\frac{a^p}{h^{\frac{p}{p-q}}} \geq 0$$

if $a < (\frac{2}{\lambda})^{\frac{1}{p-q}}$ and h is small enough (here we are using the fact that $p < 2q - 1$).

Moreover, there exists y such that

$$F_N(y) = \frac{b}{h^{\frac{1}{p-q}}},$$

and if $b \geq (\frac{2}{\lambda})^{\frac{1}{p-q}}$,

$$G(y) \leq \frac{2b^q}{h^{1+\frac{q}{p-q}}} - \lambda\frac{b^p}{h^{\frac{p}{p-q}}} \leq 0.$$

As $G(w_1)$ is continuous and satisfies $G(x) \geq 0$ and $G(y) \leq 0$, we obtain that there exists a solution of $G(w_1) = 0$ (and therefore a solution of (3.1)) that satisfies

$$F_N(w_1) = \frac{c}{h^{\frac{1}{p-q}}}$$

with $(\frac{2}{\lambda})^{\frac{1}{p-q}} - \varepsilon < c < (\frac{2}{\lambda})^{\frac{1}{p-q}}$ if h is small enough ($h = h(\varepsilon)$).

Now we want to show that this spurious steady solution W is attractive. For that purpose, we only have to observe that the linearization of (1.2) at W has all the eigenvalues with negative real part. The linearization has the form

$$Z' = (-M^{-1}A + B)Z$$

where A is the stiffness matrix (and hence positive semidefinite), M is the mass matrix (which is diagonal with positive entries) and B is a diagonal matrix that has the following coefficients:

$$b_{ii} = -p\lambda w_i^{p-1}, \quad 1 \leq i \leq N-1, \qquad \text{and} \qquad b_{NN} = -p\lambda w_N^{p-1} + \frac{2qw_N^{q-1}}{h}.$$

Now, if we take ε such that $(\frac{2}{\lambda})^{\frac{1}{p-q}} - \varepsilon > (\frac{2q}{\lambda p})^{\frac{1}{p-q}}$, we get $w_N > (\frac{2q}{p\lambda h})^{\frac{1}{p-q}}$; then the matrix B is negative definite and hence $-M^{-1}A + B$ is negative definite. This proves Theorem 1.4. □

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