# TIME–SPACE WHITE NOISE ELIMINATES GLOBAL SOLUTIONS IN REACTION DIFFUSION EQUATIONS

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ABSTRACT. We prove that perturbing the reaction–diffusion equation $u_t = u_{xx} + (u_+)^p$ $(p > 1)$, with time–space white noise produces that solutions explodes with probability one for every initial datum, opposite to the deterministic model where a positive stationary solution exists.

## 1. INTRODUCTION

In this paper we study the following parabolic SPDE with additive noise

$$u_t = u_{xx} + f(u) + \sigma \dot{W}(x,t), \tag{1.1}$$

in an interval $(0,1)$, complemented with homogeneous Dirichlet boundary conditions. Here $W$ is a $2-$dimensional Brownian sheet, $\sigma$ is a positive parameter and $f$ is a locally Lipschitz real function.

We restrict ourselves to one space dimension since for higher dimensions the solution to (1.1) (if it exists) it is not expected to be a function valued process and have to be understood in a distributional sense. But in this case there is no natural way to define $f(u)$, see [13] for more on this.

This type of problems appear naturally in several branches of pure and applied mathematics such as population dynamics, chemical reactions, chemotaxis in biological systems, etc.

This equation, in the deterministic case (i.e. $\sigma = 0$), have been widely studied in the literature. One problem that has drawn the attention to the PDE community is the appearance of singularities in finite time, no matter how smooth the initial data is. This phenomena is known as *blow-up*. What happens is that solutions go to infinity in finite time, that is, there exists a time $T < \infty$ such that

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_\infty = \infty.$$

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A well known condition on the nonlinear term $f$ that assures this phenomena is when $f$ is a nonnegative convex function with

$$\int^{\infty} \frac{1}{f} < \infty.$$

For a general reference of these facts and much more on blow-up problems, see the book [14] and the surveys [1, 5].

For a large class of nonlinearities $f$ (for instance, for the model problems $f(s) = (s_+)^p$, with $p > 1$), problem (1.1) with $\sigma = 0$ admits a stationary positive solution $v$ and hence, since the comparison principle holds for this equation, for every initial datum $u_0 \le v$ the solution to (1.1) is global in time.

Surprisingly, the situation changes drastically for $\sigma > 0$. We prove that, in this case, there is no global solution. In fact, for every initial nonnegative datum $u_0$, the solution to (1.1) blows up with probability one.

Stochastic partial differential equations with blow-up has been considered by C. Mueller in [10, 11] and C. Mueller and R. Sowers in [12]. In those papers, a linear drift with a nonlinear multiplicative noise is considered and the explosion is due to this latter term.

A similar result, but in some sense in the opposite direction, was proved by Mao, Marion and Renshaw in [9]. There, the authors prove for a system of ODEs that arise in population dynamics and that have blow-up solutions, that perturbing some coefficients of the system with a small Brownian noise, global solutions a.s. are obtained for every initial data.

In other problem, a common way to interpret the asymptotic behavior of $u$ is the following: consider first the deterministic case $\sigma = 0$. In this case there is some kind of competition between the diffusion, which diffuses the zero boundary condition to the interior of the domain and the nonlinear source $f(u)$ that induces $u$ to grow very fast.

Again in the deterministic case, it was proved in [3] that for small initial datum $u_0$, $u \to 0$ as $t \to +\infty$ while for $u_0$ large, there exist a finite time $T$ such that $\|u(\cdot, t)\|_\infty \nearrow +\infty$ as $t \nearrow T$. More precisely, it is proved that for every data $u_0$, there exists a critical parameter $\lambda^*$ such that if we solve the PDE with initial data $\lambda u_0$, for $\lambda < \lambda^*$ the solution converges to 0 uniformly, for $\lambda > \lambda^*$ the solution blows-up in finite time and for $\lambda = \lambda^*$ the solution converges uniformly to the unique positive steady state.

For small noise $\sigma \ll 1$ one could expect a similar behavior. Of course we can not expect convergence to the zero solution as $t \to \infty$ since in this case $v \equiv 0$ is not invariant for (1.1), but it is reasonable to suspect the existence of an invariant measure close to the stationary solution of the deterministic PDE and convergence to this invariant measure for small initial datum as $t \to \infty$.

However, that is not the case. We prove in Section 3 that for every initial datum $u_0$ solutions to (1.1) blow-up in finite time with probability one.

Numerical simulations, as well as heuristical arguments, suggest that, for small initial data $u_0$, metastability could be taking place in this case. Metastability appears here since, while the noise remains relatively small, the solution stays in the domain of attraction of the zero solution of the deterministic problem. But, as soon as the noise became large, the solution escapes this domain of attraction and hence the reaction term begins to dominate and pushes forward the solution until ultimately explosion cannot be prevented by the action of the noise.

**Organization of the paper.** The paper is organized as follows. In Section 2 we give the rigorous meaning of (1.1) and give the references where the foundations for the study of this kind of equation were laid. Section 3 deals with the proof of the main result of this paper: the explosion of the solutions of (1.1). Finally, in Section 4 we show some numerical simulations for this equation.

## 2. Formulation of the problem

We begin this section discussing the rigorous meaning of (1.1), the references for this being [2, 7, 13, 15]. There are two alternatives: the *integral* and the *weak* formulation as described in [2, 13, 15]. The last being more suitable for our purposes. Both formulations are equivalent as is shown in [15].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which is supposed to be right continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}-$null sets of $\mathcal{F}$. We are given a space-time white noise on $\mathbb{R}_+ \times [0,1]$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $u_0 \in C_0([0,1])$.

Assume for a moment that $f$ is globally Lipschitz, multiply (1.1) by a test function $\varphi \in C^2((0,1)) \cap C_0([0,1])$ and integrate to obtain

$$\begin{aligned} &\int_0^1 u(x,t)\varphi(x)\,dx - \int_0^1 u_0(x)\varphi(x)\,dx = \\ &\quad \int_0^t \int_0^1 u(s,x)\varphi_{xx}(x)\,dx\,ds + \int_0^t \int_0^1 f(u(s,x))\varphi(x)\,dx\,ds \\ &\quad + \sigma \int_0^t \int_0^1 \varphi(x)\,dW(x,s). \end{aligned} \tag{2.1}$$

Alternatively, the integral formulation of the problem is constructed by means of the function $G$, the fundamental solution of the heat equation for the domain $(0,1)$.

$$u(x,t) - \int_0^1 G_t(x,y) u_0(y)\,dy = \\ - \int_0^t \int_0^1 G_{t-s}(x,y) f(u(y,s))\,dyds + \sigma \int_0^t \int_0^1 G_{t-s}(x,y) dW(y,s).$$

As a solution to (1.1) we understand an $\mathcal{F}_t-$adapted process with values in $C_0([0,1])$ that verifies (2.1) for every $\varphi \in C^\infty((0,1)) \cap C_0([0,1])$.

In [2, 15] it is proved that there exists a unique solution to this problem and that the integral and weak formulations are equivalent.

For $f$ locally Lipschitz globally defined solutions do not exist in general. Nevertheless, existence of local in time solutions is proved by standard arguments: consider for each $n \in \mathbb{N}$ the globally Lipschitz function $f_n(x) = f(-n)\mathbf{1}_{(-\infty,-n]} + f(x)\mathbf{1}_{(-n,n)} + f(n)\mathbf{1}_{[n,+\infty)}$ and $u^n$, the unique solution of (1.1) with $f$ replaced by $f_n$. Let $T_n$ be the first time at which $\|u^n(\cdot,t)\|_\infty$ reaches the value $n$. Then $(T_n)_n$ is an increasing sequence of stopping times and we define the maximal existence time of (1.1) as $T := \lim T_n$. It is easy to see that $u^{n+1}\mathbf{1}_{\{t<T_n\}} = u^n\mathbf{1}_{\{t<T_n\}}$ a.s. and hence there exist the limit $u(x,t) = \lim u^n(x,t)$ for $t < T$ which verifies

$$\begin{aligned} &\int_0^1 u(x,t\wedge T)\varphi(x)\,dx - \int_0^1 u_0(x)\varphi(x)\,dx = \\ &\int_0^{t\wedge T}\int_0^1 u(s,x)\varphi_{xx}(x)\,dx\,ds + \int_0^{t\wedge T}\int_0^1 f(u(s,x))\varphi(x)\,dx\,ds \\ &+ \sigma \int_0^{t\wedge T}\int_0^1 \varphi(x)\,dW(x,s). \end{aligned} \tag{2.2}$$

So we say that $u$ solves (1.1) up to the explosion time $T$. We also say that $u$ blows up in finite time if $\mathbb{P}(T < \infty) > 0$. Observe that if $T(\omega) < \infty$ then

$$\lim_{t \nearrow T(\omega)} \|u(\cdot,t,\omega)\|_\infty = \infty.$$

## 3. Explosions

In this section, we show that equation (1.1) blows-up in finite time with probability one for every initial datum $u_0 \in C_0([0,1])$.

In order to prove the blow-up of $u$, we define the function

$$\Phi(t) := \int_0^1 \phi(x) u(x,t)\,dx.$$

Here $\phi(x) > 0$ is the normalized first eigenfunction of the Dirichlet Laplacian in $(0,1)$. That is, $\phi(x) = \frac{\pi}{2}\sin(\pi x)$ and hence we can use it as a test function in (2.1) to obtain

$$\Phi(t) = -\lambda_1 \int_0^t \Phi(s)\, ds + \int_0^t \int_0^1 \phi(x) f(u(x,s))\, dx ds + \sigma \int_0^t \int_0^1 \phi(x)\, dW(x,s).$$

We denote by $z_0 := \Phi(0) = \int_0^1 \phi(x) u_0(x)\, dx$.

Now, as $f$ is convex, by Jensen's inequality, we get

$$\int_0^1 \phi(x) f(u(x,s))\, dx \geq f\Big( \int_0^1 \phi(x) u(x,s)\, dx \Big) = f(\Phi(s)).$$

Moreover, since $\phi$ is a positive function with $L^1-$norm equal to 1, it is easy to see that

$$B(t) := \int_0^t \int_0^1 \phi(x)\, dW(x,s),$$

is a Brownian motion.

Combining all these facts, we obtain that $\Phi$ verifies the (one dimensional) stochastic differential inequality

$$d\Phi(t) \geq \big( -\lambda_1 \Phi(t) + f(\Phi(t)) \big)\, dt + \sigma dB(t).$$

Define $z(t)$ to be the one-dimensional process that verifies

$$dz = (-\lambda_1 z + f(z))\, dt + \sigma dB,$$

with initial condition $z(0) = z_0$. Then, $e(t) = \Phi(t) - z(t)$ verifies

$$de \geq \Big( -\lambda_1 e + \frac{f(\Phi) - f(z)}{\Phi - z} e \Big)\, dt.$$

Observe that $e$ verifies a deterministic differential inequality. Hence, as $e(0) = 0$ it is easy to check that $e(t) > 0$ as long as it is defined.

Therefore, $\phi(t) \geq z(t)$ as long as $\phi$ is defined.

The following lemma proves that $z$ explodes with probability one.

**Lemma 3.1.** *Let $z$ be the solution of*

$$dz = (-\lambda_1 z + f(z))\, dt + \sigma dB, \qquad z(0) = 0. \tag{3.1}$$

*Then $z$ explodes in finite time with probability one.*

*Proof.* The proof is just an application of the *Feller Test for explosions* ([8], Chapter 5). Using the same notation as in [8] we obtain the scale function for (3.1) to be

$$p(x) = \int_0^x \exp\left( -\frac{2}{\sigma^2} \int_0^s b(\xi)\, d\xi \right)\, ds$$

Here $b(\xi) = -\lambda_1 \xi + f(\xi)$.

It is easy to see that, as $\int^\infty 1/f < \infty$,

$$p(-\infty) = -\infty, \qquad p(+\infty) < +\infty,$$

and hence the Feller Test implies that, if $S$ is the explosion time of $z$, we get

$$\mathbb{P}\left(\lim_{t\nearrow S} z(t) = +\infty\right) = 1$$

To prove that $\mathbb{P}(S < +\infty) = 1$ we have to consider the function

$$v(x) = 2\int_0^x \frac{p(x) - p(y)}{\sigma^2 p(y)}\, dy.$$

The behavior of $v$ at $+\infty$ is given $1/f$ and hence $v(+\infty) < +\infty$, which implies that

$$\mathbb{P}(S < \infty) = 1.$$

This completes the proof. □

These facts all together, imply that there exists a (random) time $T = T(\omega) < \infty$ a.s. such that

$$\lim_{t\nearrow T} \|u(\cdot,t)\|_\infty = \infty \quad \text{a.s.}$$

So we have proved the following Theorem.

**Theorem 3.2.** *Let $f$ be a nonnegative, convex locally Lipschitz function such that*

$$\int^\infty \frac{1}{f} < \infty.$$

*Then, for every nonnegative initial datum $u_0 \geq 0$ the solution $u$ to* (1.1) *blows-up in finite (random) time $T$ with*

$$\mathbb{P}^{u_0}(T < \infty) = 1.$$

## 4. Numerical simulations

In this section we show some numerical simulations of (1.1). We perform all the simulations with the reaction $f(u) = (u_+)^2$, $\sigma = 6.36$ and initial datum $u_0 \equiv 0$.

To perform the simulations we discretize the space variables with second order finite differences in a uniform mesh of size $h = 0.02$ (i.e.: $n = 50$ nodes). With this discretization we obtain a system of SDE that reads

$$du_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1})dt + f(u_i)dt + \frac{\sigma}{\sqrt{h}} dw_i, \quad 2 \leq i \leq n-1.$$

accompanied with the boundary conditions $u_1 = u_n = 0$, $u_i(0) = u_0(ih)$, $1 \leq i \leq n$. The Brownian motions $w_i$ are obtained by space integration of the Brownian sheet in the interval $[(i - 1/2)h, (i + 1/2)h)$.

To integrate this system we use an adaptive procedure similar to the one developed in [4] for the one dimensional case. Here we adapt the time step as in that work replacing the value of the solution (which is a real number) by the $L^1-$norm of $u$, as is done in [6] for the deterministic case.

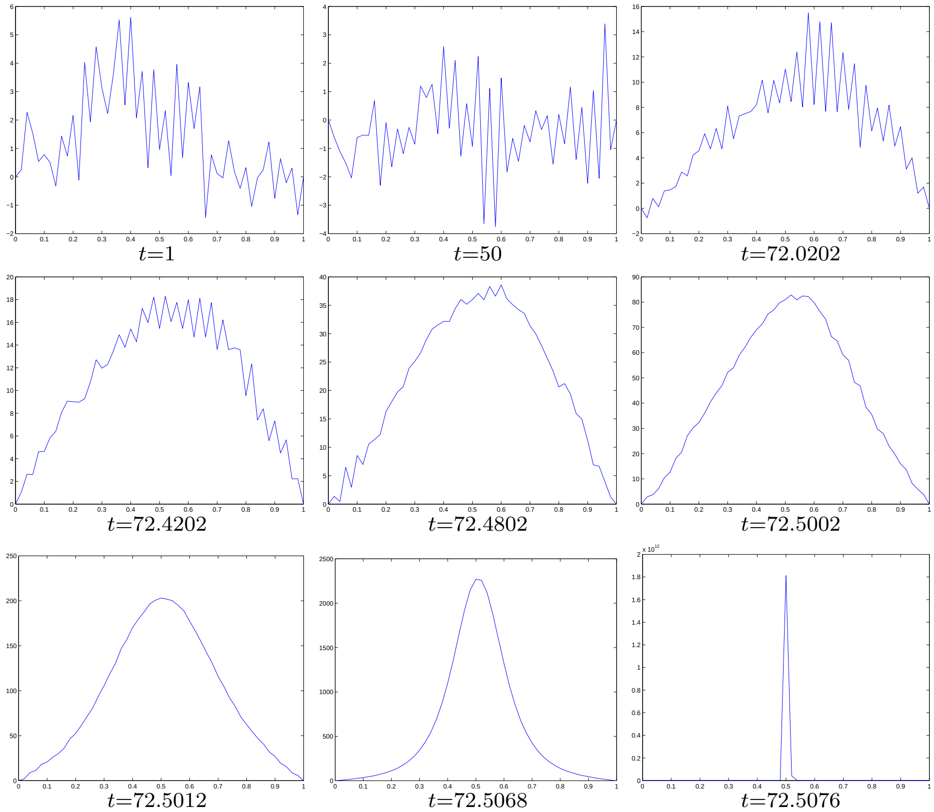


FIGURE 1. Profiles of a sample solution at different times.

| Snapshot | Time | $\|u(\cdot,t,\omega)\|_\infty$ |
|---|---|---|
| 1 | 1.0000 | 5.6159 |
| 2 | 50.0000 | 3.3863 |
| 3 | 72.0202 | 15.5104 |
| 4 | 72.4202 | 18.2885 |
| 5 | 72.4802 | 38.5848 |
| 6 | 72.5002 | 82.8705 |
| 7 | 72.5012 | 203.0799 |
| 8 | 72.5068 | $2.2695 \times 10^3$ |
| 9 | 72.5076 | $1.8128 \times 10^{12}$ |

TABLE 1. The maximum of the solution at differen times

We want to remark that adaptivity is esencial in this case since a fixed time step procedure gives rise to globally defined approximations.

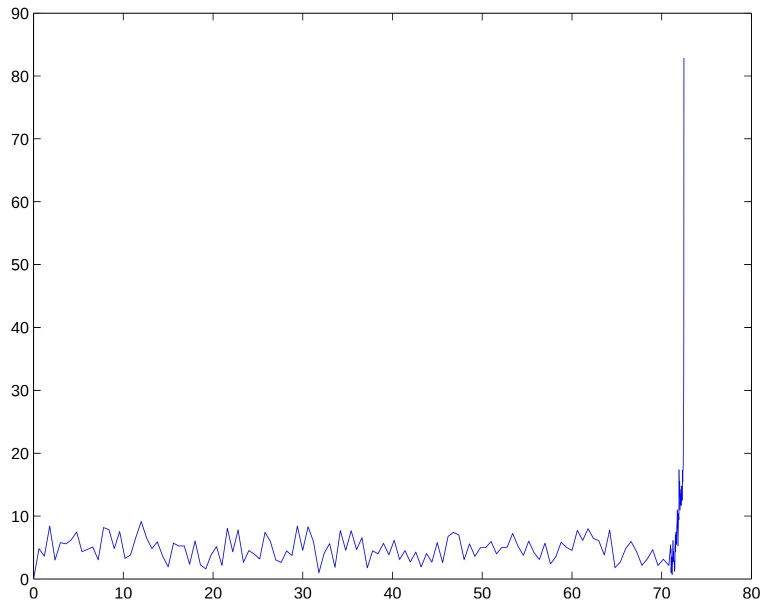


FIGURE 2. The evolution of the maximum of a sample solution with initial data $u_0 \equiv 0$

On the other hand, we want to remark that convergence of this numerical scheme and/or the explosion times is not proved. Neither that the asymptotic behavior of the numerical approximations agrees with that of the continuous solutions in some sense. That is the subject of a forthcoming paper.

In spite that in Theorem 3.2 we prove that solutions to (1.1) blow up with probability one for every $\sigma > 0$ and every initial data, we want to remark that it is not possible to observe that in numerical simulations since for small $\sigma$, the explosion time is exponentially large when the initial datum is small.

Essentially, in order to blow-up, the solution needs to be greater than the positive stationary solution of the deterministic problem (i.e. the solution of $v_{xx} = -f(v)$, which is of size 12 when $f(v) = (v_+)^2$) plus the order of the noise $\sigma$. Once the solution is in that range of values, the noise can not prevent the explosion.

The probability that such an event occurs in a finite fixed time interval depends on $\sigma$ and is exponentially small.

So, to show the explosive behavior we choose to do the simulations with $\sigma = 6.36$ and initial datum $u_0 \equiv 0$. We ran the code with $\sigma \leq 5$ until time $t = 1000$ and we did not observe explosions but a meta-stable behavior.

The features of a particular sample path are shown in Figure 1.

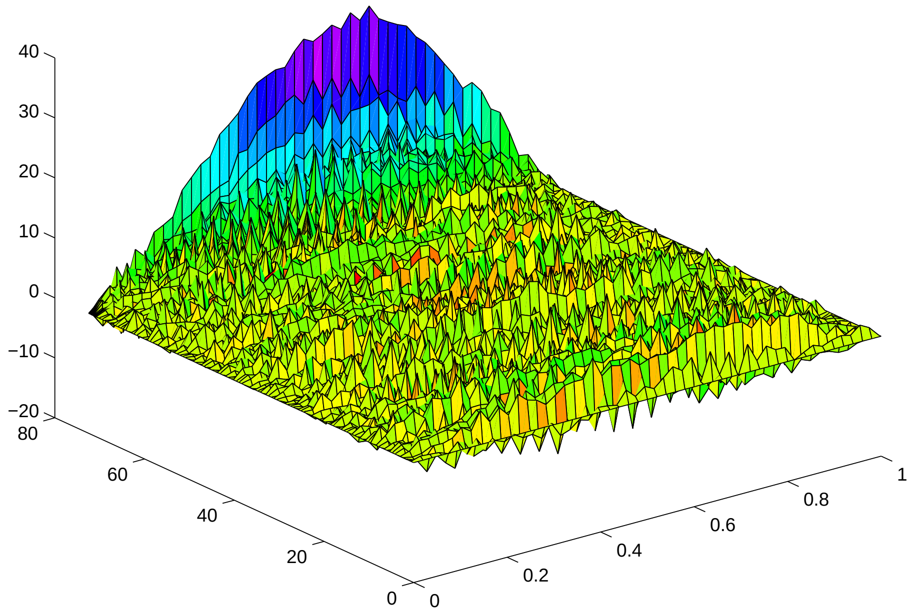


FIGURE 3. The graph of a sample solution with initial datum $u_0 \equiv 0$

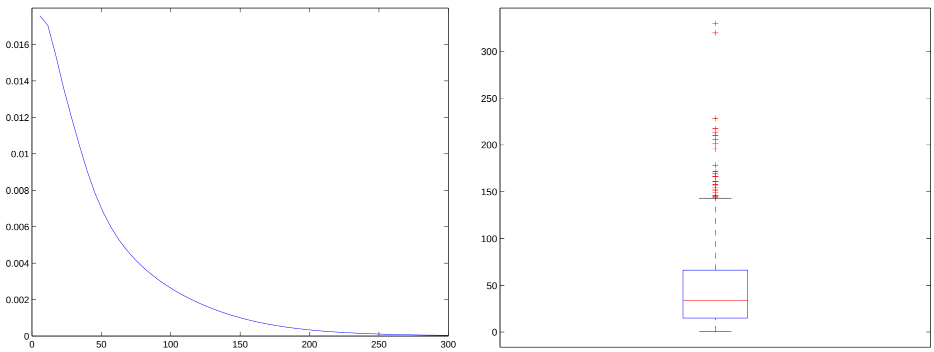


FIGURE 4. The kernel density estimator of the explosion time for $\sigma = 6.36$ and the corresponding box–plot.

Table 1 shows the times at where the solution is drawn and the $L^\infty$– norm of the solution at that time.

In Figure 2 we show the evolution of the $L^\infty$ norm and in Figure 3 is the whole picture as a function of $x$ and $t$ of a sample path.

Finally, Figure 4 shows some statistics: we perform 832 simulations of the solution with $\sigma = 6.36$ to obtain a sample of the explosion time. Actually, we stop the simulation when the maximum of the solution reaches the value $10^{13}$. The kernel density estimator of the data obtained by the simulation and the corresponding box–plot are shown. The sample mean is 46.8834 and the sample standard deviation 43.8857.

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