STABLE MANIFOLD APPROXIMATION FOR THE HEAT EQUATION WITH NONLINEAR BOUNDARY CONDITION

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ABSTRACT. In this paper, we study the dynamic behaviour of positive solutions of the heat equation in one space dimension with a nonlinear flux boundary condition of the type $u_x = u^p - u$ at $x = 1$. We analyze the behaviour of a semidiscrete numerical scheme in order to approximate the stable manifold of the only positive steady solution. We also obtain some stability properties of this positive steady solution and a description of its stable manifold.

1. INTRODUCTION.

In this paper, we study the long time behaviour of semidiscretizations in space of positive solutions of the following parabolic problem

$$\begin{cases} u_t = u_{xx} & \text{in } (0,1) \times [0,T), \\ u_x(1,t) = g(u(1,t)) & \text{on } [0,T), \\ u_x(0,t) = 0 & \text{on } [0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{on } [0,1], \end{cases} \quad (1.1)$$

where $g(s) = s^p - s$ with $p > 1$.

Parabolic reaction-diffusion problems like (1.1) or of a more general form, allowing for example source terms or with different boundary conditions, appear in several branches of applied mathematics. They have been used to model, for example, chemical reactions, heat transfer or population dynamics and have been studied by several authors. See [19] and the references therein.

For this type of problems, existence and regularity of solutions has been proved in [2], [3], [4], [13], [15] for an initial datum that satisfies a compatibility condition. In the general case one can obtain a solution in H^1 by a standard approximation procedure. Therefore in the rest

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of our work we will assume that our solution lies in H^1 and by a result of [4], (1.1) defines a compact local semiflow (Lemma 2.1).

First, we want to describe the behaviour of positive solutions of (1.1).

For many differential equations or systems the solutions can become unbounded in finite time (a phenomenum that is known as blow up). Typical examples where this happens are problems involving reaction terms in the equation (see for example [1], [6], [7], [10], [11], [19]). When the nonlinear term that appears at the boundary is of power type then it is well known that if the power involved is greater than 1, one has this blow up phenomenum (see [15], [17], [18], [20]), so in our case we have some solutions that blow up in finite time.

It is easy to see that problem (1.1) has two solutions that do not depend on t, $u \equiv 1$ and $u \equiv 0$. Also, it was shown in [12] that the following alternative holds: the solution has finite blow up time or it is bounded in H^1 norm (Lemma 2.3). If one takes the initial datum u_0 greater than 1 the solution must blow up in finite time, but if we take $0 \leq u_0 < 1$ then the solution must go to zero uniformly (in fact in H^1). By a simple linearization we can see that the fixed point $u \equiv 1$ is hyperbolic and has a stable manifold of codimension one. Therefore there exists initial data (different from $u_0 \equiv 1$) such that the corresponding solution goes to $u \equiv 1$. Then we want to describe this stable manifold.

Using ideas from [6] and [7], in section 2 we prove an interesting property of the stable manifold of the point $u \equiv 1$: for a fixed u_0 if one considers the family of solutions u_λ that have initial data of the form λu_0 then there exists only one critical value λ_c such that $\lambda_c u_0$ lies in the stable manifold of $u \equiv 1$ (Theorem 2.2)(see [8] for a similar result). We also show that $\lambda_c = \lambda_c(u_0)$ is continuous with respect to initial data (Theorem 2.3).

On the other hand, there exist several papers which deal with the numerical approximation of the blow up time an the blow up profile (see for example, [5] for the semilinear case and [9] for a problem similar to (1.1)).

After all this, we arrive at the main point of this paper: one can try to estimate this value λ_c numerically. Our second result solves this problem, we prove that if one considers a semidiscretization of (1.1) and replace the problem by a system of ordinary differential equations, then the new semidiscrete problem also shows the existence and uniqueness of a critical value $\lambda_{c,h}$ (Theorem 3.2) and moreover this critical value $\lambda_{c,h}$ goes to λ_c as the mesh parameter h goes to zero (Theorem 3.4).

We want to remark that the existence and uniqueness of the critical value, $\lambda_{c,h}$ is easier and shorter than its continuous counterpart.

The numerical semidiscrete version of (1.1) proposed here comes from a first degree finite element approximation on the variable x keeping t continuous (from a well known fact this case coincides with a classical finite difference second order scheme on x). A further mass lumping technique simplifies the scheme and preserves the maximum principle, allowing us to use comparison arguments (Lemma 3.1).

The paper is organized as follows, in section 2 we analyze the continuous problem, in section 3 the semidiscrete approximations and in the last section we present some numerical experiments.

2. The continuous equation.

We begin by stating a Lemma that says that (1.1) defines a compact local semiflow.

Lemma 2.1. *The problem* (1.1) *defines a local semiflow in* $C^+ = \{v \in H^1 \,:\, v \geq 0\}$ *and bounded orbits are relatively compact. If, in addition,*

$$\sup_{0\leq t<t_{max}(u_0)} \|u(\cdot,t)\|_{H^1} < \infty$$

then the orbit through u_0 *exist globally, that is,* $t_{max}(u_0) = \infty$.

Proof. As the maximum principle applies, if $u_0 \in C^+$ then $u(\cdot,t) \in C^+$ for all $t \in (0, t_{max}(u_0))$. For the rest of the proof, we observe that we fall into the hypotheses of Theorem 14.5 of [4] □

The following Lemma shows that (1.1) has a Lyapunov functional.

Lemma 2.2. *Let*

$$V(u(\cdot,t)) = \frac{1}{2}\int_0^1 (u_x(x,t))^2 dx - G(u(1,t)), \tag{2.1}$$

where $G(s) = \frac{s^{p+1}}{p+1} - \frac{s^2}{2}$ *is a primitive of* g. *Then* V *is a Lyapunov functional for* (1.1).

Proof. The Lemma is an immediate consequence of

$$\frac{dV(u(\cdot,t))}{dt} = -\int_0^1 (u_t(x,t))^2 dx. \tag{2.2}$$

□

Next, we state a Lemma, that can be found in [12] that says that global solutions of (1.1) are bounded in H^1.

Lemma 2.3. *Let $u_0 \in C^+$, then if $u(\cdot, t)$ is the solution of (1.1) with u_0 as initial datum, the following alternative holds: $u(\cdot, t)$ blows up in finite time (i.e. $t_{max}(u_0) < \infty$), or it remains bounded in H^1.*

As a consequence of these results we have, by a well known fact (see [14]), the following Corollary.

Corollary 2.1. *The possible ω-limits of a global solution of (1.1) are $u \equiv 0$ or $u \equiv 1$ (those are the only steady states of (1.1)).*

Proof. This is a consequence of Lemma 2.1, Lemma 2.2 and Lemma 2.3 (see [14]) □

Remark . From the well known energy identity

$$\int_0^t \int_0^1 u_t^2 dx dt + V(u(\cdot, t)) = V(u_0),$$

and Lemma 2.3 it follows that if $t_{max}(u_0) < \infty$ then $u(1, t) \to \infty$ when $t \nearrow t_{max}(u_0)$.

Lemma 2.4. *If u is a solution of (1.1) different from 0, such that there exists t_0 with*

$$V(u(\cdot, t_0)) \leq 0,$$

then u blows up in finite time.

Proof. First, we observe that by (2.2) we can assume that $V(u(\cdot, t_0)) < 0$.

Suppose that u is globally defined, then by Corollary 2.1, $u \to 0$ in H^1 or $u \to 1$ in H^1, and by continuity of V, we must have $V(u(\cdot, t)) \searrow V(0) = 0$ or $V(u(\cdot, t)) \searrow V(1) = \frac{1}{2} - \frac{1}{p+1} > 0$, which is a contradiction and the Lemma is proved □

For a fixed u_0 we consider the family of solutions u_λ that has initial data λu_0. Let us define

$$\lambda_+ \equiv \inf\{\lambda \ : \ u_\lambda \text{ blows up } \}$$

and

$$\lambda_- \equiv \sup\{\lambda \ : \ u_\lambda \to 0\}$$

It is immediate to see that these sets are not empty so these numbers are well defined.

The following result shows that the steady solution $u \equiv 1$ is unstable and that $\lambda_+ = \lambda_-$.

Theorem 2.1. *Let u_0 be an initial datum for* (1.1) *such that* $u(\cdot,t)\to 1$ *in* H^1

(1) *If* $v_0 \le u_0,\ v_0 \neq u_0$, *then* $v(\cdot,t)\to 0$ *in* H^1.
(2) *If* $v_0 \ge u_0,\ v_0 \neq u_0$, *then* $v(\cdot,t)$ *blows up in finite time.*

Proof. We prove (1), (2) follows in a similar way.

We start by making the following remark, which is an immediate consequence of the maximum principle: If $v \le u$, and u is globally defined, then v is globally defined.

Now, by the preceding remark, v is globally defined, and then, v must converge to a stationary positive solution of (1.1).

Then $v \to 1$ or $v \to 0$ in H^1. Suppose then that $v \to 1$.

Now, let $w = u - v$, then w is a solution of

$$\begin{cases} w_t &= w_{xx}, \\ w_x(0,t) &= 0, \\ w_x(1,t) &= g(u(1,t)) - g(v(1,t)), \\ w(x,0) &= u_0 - v_0 \equiv w_0 \ge 0 \qquad w_0 \neq 0. \end{cases}$$

Now, $g(u(1,t)) - g(v(1,t)) = g'(\xi(t))w(1,t)$, with $v(1,t) \le \xi(t) \le u(1,t)$, then $\xi(t) \to 1$ as $t \to \infty$.

Then there exists $\alpha > 0$ and t_0 such that $g'(\xi(t)) > \alpha$ if $t \ge t_0$, and then w verifies

$$\begin{cases} w_t &= w_{xx} & \text{in } (0,1)\times(t_0,+\infty), \\ w_x(0,t) &= 0 & \text{on } (t_0,+\infty), \\ w_x(1,t) &\ge \alpha w(1,t) & \text{on } (t_0,+\infty), \\ w(x,t_0) &= u(x,t_0) - v(x,t_0) > 0 & \text{on } (0,1), \end{cases}$$

which contradicts the fact that $w \to 0$ and proves our claim. □

By Theorem 2.1 we have that $\lambda_- = \lambda_+ \equiv \lambda_c$ so we want to prove that u_{λ_c} lies on the stable manifold of $u \equiv 1$.

To prove this fact, we need some auxiliary results to get bounds on the L^∞ norms of global solutions of (1.1).

First we will control a global solution by its Lyapunov functional (Proposition 2.1), and then we will observe that if the initial datum is uniformly bounded, so is the Lyapunov functional (Lemma 2.8).

Lemma 2.5. *Let u be a global solution of* (1.1), *then*

$$\int_0^1 u^2(x,t_2)dx - \int_0^1 u^2(x,t_1)dx \ge -4V(u(\cdot,t_1))(t_2-t_1) + 2\int_{t_1}^{t_2} \left(g(u(1,t))u(1,t) - 2G(u(1,t))\right)dt$$

Proof. If we multiply (1.1) by u and integrate, we get

$$\frac{1}{2}\int_0^1 \Big(u^2(x,t_2) - u^2(x,t_1)\Big)\,dx = -\int_{t_1}^{t_2}\int_0^1 (u_x(x,t))^2 dxdt + \int_{t_1}^{t_2} g(u(1,t))u(1,t)dt. \quad (2.3)$$

Then, by (2.1) and by Lemma 2.2

$$\begin{aligned}\int_0^1 \Big(u^2(x,t_2) - u^2(x,t_1)\Big)\,dx &= \\ &= -4\int_{t_1}^{t_2} V(u(\cdot,t))dt + 2\int_{t_1}^{t_2} \left(g(u(1,t))u(1,t) - 2G(u(1,t))\right) dt \\ &\geq -4V(u(\cdot,t_1))(t_2-t_1) + 2\int_{t_1}^{t_2} \left(g(u(1,t))u(1,t) - 2G(u(1,t))\right) dt\end{aligned}$$

as we wanted to show □

Lemma 2.6. *Let u_0 be an initial datum such that $\|u_0\|_\infty \leq \mathcal{A}$ and that the corresponding solution of* (1.1) *is globally defined. Then, for any $t_0 > 0$ there exists $C > 0$ depending only on p and $\mathcal{A}$ such that*

$$\int_0^1 u^2(x,t)dx \leq C\left(V(u(\cdot,t_0))^{\frac{2}{p+1}} + 1\right), \qquad \forall t \geq t_0.$$

Proof. First, let us observe that, by the maximum principle and by Hopf's Lemma, $\max_{x\in[0,1]} u(x,t) \leq \max\{\sup_{0<s<t} u(1,s), \max_{x\in[0,1]} u_0(x)\}$, then if

$$\max_{x\in[0,1]} u(x,t) \leq \max_{x\in[0,1]} u_0(x) \ \forall t \geq t_0,$$

it is enough to take $C = \mathcal{A}^2$ to prove the Lemma (we observe that by Lemma 2.4 $V(u(\cdot,t_0)) > 0$).

Then, we can suppose that $\max_{x\in[0,1]} u(x,t) \leq \sup_{0<s<t} u(1,s)$.

Again, by the maximum principle, we can assume that

$$u(1,t) \geq u(x,t) - C$$

with C depending only on p and $\mathcal{A}$, therefore, we get

$$u^2(1,t) \geq \frac{1}{2}u^2(x,t) - C$$

and thus

$$u^2(1,t) \geq \frac{1}{2}\int_0^1 u^2(x,t)dx - C. \quad (2.4)$$

Let $h(t) = \int_0^1 u^2(x,t)dx$. Then we have, by Lemma 2.5

$$h(t_2) - h(t_1) \geq -4V(u(\cdot,t_1))(t_2-t_1) + 2\int_{t_1}^{t_2}(g(u(1,t))u(1,t) - 2G(u(1,t)))dt$$

but $g(u)u - 2G(u) = \frac{p-1}{p+1}u^{p+1}$, then replacing in the latter inequality we get, by (2.4),

$$\begin{aligned} h(t_2) - h(t_1) &\geq -4V(u(\cdot,t_1))(t_2-t_1) + C\int_{t_1}^{t_2}\left(u^2(1,t)\right)^{\frac{p+1}{2}}dt \\ &\geq -4V(u(\cdot,t_1))(t_2-t_1) + C\int_{t_1}^{t_2} h(t)^{\frac{p+1}{2}}dt - C(t_2-t_1). \end{aligned}$$

Now, if we divide by $(t_2 - t_1)$ and take $t_2 \to t_1$, we get

$$h'(t_1) \geq -4V(u(\cdot,t_1)) - C + Ch^{\frac{p+1}{2}}(t_1).$$

Now, if the statement of the Lemma is false, then the latter inequality implies that $h(t)$ blows up in finite time, which contradicts the fact that u is global. □

Lemma 2.7. *Let u be a global solution of* (1.1) *with $\|u_0\|_\infty \leq \mathcal{A}$, then for any $t_0 > 0$ there exists a constant $C = C(p,\mathcal{A}) > 0$ such that*

(1) $\int_0^1 u^2(x,t_2)dx - \int_0^1 u^2(x,t_1)dx \leq C(t_2-t_1)^{1/2}(V(u(\cdot,t_1))^{\frac{p+3}{2p+2}} + V(u(\cdot,t_1))^{\frac{1}{2}})$

(2) $\int_{t_1}^{t_2}(g(u(1,t))u(1,t) - 2G(u(1,t)))dt \leq$
$C\left[(t_2-t_1)V(u(\cdot,t_1)) + (t_2-t_1)^{1/2}\left(V(u(\cdot,t_1))^{\frac{p+3}{2p+2}} + V(u(\cdot,t_1))^{\frac{1}{2}}\right)\right]$

for any $t_2 > t_1 > t_0$.

Proof. By Hölder's inequality, we have

$$\begin{aligned} &\int_0^1 u^2(x,t_2)dx - \int_0^1 u^2(x,t_1)dx = \\ &= \int_0^1\int_{t_1}^{t_2}\frac{\partial u^2(x,t)}{\partial t}dtdx = 2\int_0^1\int_{t_1}^{t_2}u(x,t)u_t(x,t)dtdx \\ &\leq 2\left(\int_0^1\int_{t_1}^{t_2}u^2(x,t)dtdx\right)^{1/2}\left(\int_0^1\int_{t_1}^{t_2}u_t^2(x,t)dtdx\right)^{1/2}. \end{aligned} \tag{2.5}$$

Now, by Lemma 2.6

$$\begin{aligned} \left(\int_0^1\int_{t_1}^{t_2}u^2(x,t)dtdx\right)^{1/2} &\leq C\left(\int_{t_1}^{t_2}(V(u(\cdot,t_1))^{\frac{2}{p+1}}+1)dt\right)^{1/2} \\ &\leq C(t_2-t_1)^{1/2}(V(u(\cdot,t_1))^{\frac{1}{p+1}}+1) \end{aligned}$$

and, by (2.2)

$$\begin{aligned} \left(\int_0^1\int_{t_1}^{t_2}u_t^2(x,t)dtdx\right)^{1/2} &= \left(\int_{t_1}^{t_2}-\frac{dV(u(\cdot,t))}{dt}dt\right)^{1/2} \\ &= (V(u(\cdot,t_1)) - V(u(\cdot,t_2)))^{1/2} \leq V(u(\cdot,t_1))^{1/2}. \end{aligned}$$

Then, (2.5) is bounded by

$$CV(u(\cdot,t_1))^{1/2}(t_2-t_1)^{1/2}(V(u(\cdot,t_1))^{\frac{1}{p+1}}+1) = C(t_2-t_1)^{1/2}(V(u(\cdot,t_1))^{\frac{p+3}{2p+2}} + V(u(\cdot,t_1))^{\frac{1}{2}})$$

which proves (1).

Now

$$\begin{aligned}\int_0^1 u^2(x,t_2)dx - \int_0^1 u^2(x,t_1)dx &= 2\int_0^1\int_{t_1}^{t_2} u(x,t)u_t(x,t)dtdx \\ &= 2\int_0^1\int_{t_1}^{t_2} u(x,t)u_{xx}(x,t)dtdx \\ &= -2\int_{t_1}^{t_2}\int_0^1 (u_x(x,t))^2 dtdx + 2\int_{t_1}^{t_2} g(u(1,t))u(1,t)dt \\ &= -4\int_{t_1}^{t_2} V(u(\cdot,t))dt + 2\int_{t_1}^{t_2}(g(u(1,t))u(1,t) - 2G(u(1,t)))dt\end{aligned}$$

and (2) follows from (1) and Lemma 2.2. □

The following Corollary is an immediate consequence of Lemma 2.7

Corollary 2.2. *Let u be a global solution of* (1.1) *with $\|u_0\|_\infty \le \mathcal{A}$, then for any $t_0 > 0$ it holds*

$$\int_{t_1}^{t_2} u(1,t)^{p+1}dt \le C\left[(t_2-t_1)V(u(\cdot,t_1)) + (t_2-t_1)^{1/2}\left(V(u(\cdot,t_1))^{\frac{p+3}{2p+2}} + V(u(\cdot,t_1))^{\frac{1}{2}}\right)\right].$$

for any $t_2 > t_1 > t_0$ and $C = C(p,\mathcal{A})$.

Now we are ready to prove the key Proposition.

Proposition 2.1. *Let u be a global solution of* (1.1) *with $\|u_0\|_\infty \le \mathcal{A}$, then for any $t_0 > 0$ there exists $C = C(p,\mathcal{A}) > 0$ such that*

$$\|u(\cdot,t)\|_\infty \le C\left(V(u(\cdot,t_0))^\alpha + V(u(\cdot,t_0))^\beta + V(u(\cdot,t_0))^\gamma\right) \qquad \forall\ t \ge t_0$$

where α, β and γ are positive constants which depend only on p.

Proof. By the maximum principle, we have that

$$\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty + \sup_{0<s<t} u(1,s) \le \mathcal{A} + \sup_{0<s<t} u(1,s),$$

so to finish the proof, we only need to control $u(1,t)$.

Now, by Corollary 2.2, we have (for $t_2 > t_1 > t_0 > 0$)

$$\int_{t_1}^{t_2} u(1,t)^{p+1}dt \leq C\left[(t_2-t_1)V(u(\cdot,t_0)) + (t_2-t_1)^{1/2}\left(V(u(\cdot,t_0))^{\frac{p+3}{2p+2}} + V(u(\cdot,t_0))^{\frac{1}{2}}\right)\right].$$

Then, if $\delta = |t_2 - t_1|$ is small (say $\delta \leq V(u(\cdot,t_0))^{\frac{p+3}{p+1}-2} \equiv \delta_0$)

$$\int_{t_1}^{t_2} u(1,t)^{p+1}dt \leq C\delta^{1/2}\left(V(u(\cdot,t_0))^{\frac{p+3}{2p+2}} + V(u(\cdot,t_0))^{\frac{1}{2}}\right).$$

Therefore, for any interval $[t_1,t_2]$ of length $\delta \leq \delta_0$, there exists a point $t_3 \in [t_1,t_2]$ such that

$$u(1,t_3) \leq C\frac{V(u(\cdot,t_0))^{\frac{p+3}{2(p+1)^2}} + V(u(\cdot,t_0))^{\frac{1}{2(p+1)}}}{\delta^{\frac{1}{2(p+1)}}} \equiv \frac{D}{\delta^{\frac{1}{2(p+1)}}}. \tag{2.6}$$

To finish the proof of the proposition, we need to control the growth of $u(1,t)$ in intervals of the form $[t_3, t_3+2\delta]$, with t_3 and δ as in (2.6).

Let $\overline{u}(x,t) = \phi(a(x)+b(t))$ where

$$\phi'(s) = \phi^p(s), \qquad \text{i.e. } \phi(s) = \frac{M(p)}{(s_0-s)^{1/(p-1)}},$$

with $M(p) = (\frac{1}{p-1})^{\frac{1}{p-1}}$

$$b(t) = \delta^{-1/2}t,$$

$$a(x) = \begin{cases} s_0 - \delta^{\frac{p}{2(p+1)}} & 0 \leq x \leq 1-\delta^{\frac{p}{2(p+1)}} \\ (s_0 - \delta^{\frac{p}{2(p+1)}}) + \frac{1}{3\delta^{\frac{p}{p+1}}}(x-1+\delta^{\frac{p}{2(p+1)}})^3 & 1-\delta^{\frac{p}{2(p+1)}} \leq x \leq 1. \end{cases}$$

Now it is easy to check that we can choose $\delta_1 = \delta_1(p)$, such that if $\delta < \delta_1$, then $\overline{u}$ is a super solution of (1.1) which is well defined $\forall t \in [0,2\delta]$, then

$$u(x,t_3+t) \leq \overline{u}(x,t) \qquad \forall\ t \in [0,2\delta]$$

and by simple computation, we get

$$u(1,t) \leq CD\delta^{-\frac{3p+2}{2(p+1)}} \qquad \forall\ t \in [t_3, t_3+2\delta]$$

and the Proposition is proved □

Lemma 2.8. *Let u be a global solution of* (1.1) *with $\|u_0\|_\infty \leq \mathcal{A}$. Then for any $t_0 > 0$ there exists $C = C(t_0,p,\mathcal{A})$ such that $V(u(\cdot,t_0)) \leq C$.*

Proof. To prove this Lemma, let v be the solution of (1.1) with $v_0 \equiv \mathcal{A}$. We observe that if t is small ($t < T/2$ where v is defined in $[0, T)$) then

$$\|u(\cdot, t)\|_\infty \leq \|v(\cdot, t)\|_\infty \leq C \tag{2.7}$$

where C only depends on T which only depends on p and $\mathcal{A}$.

The standard energy estimate (2.3) gives us

$$\frac{1}{2}(\|u(\cdot, t_0)\|_{L^2}^2 - \|u(\cdot, 0)\|_{L^2}^2) = -\int_0^{t_0} \|u_x(\cdot, s)\|_{L^2}^2 ds + \int_0^{t_0} g(u)u(1, s) ds.$$

Using (2.7) we get

$$\int_0^{t_0} \|u_x(\cdot, s)\|_{L^2}^2 ds \leq C.$$

Then by the mean value theorem, there exist $t_1 \leq t_0$ such that

$$t_0 \|u_x(\cdot, t_1)\|_{L^2}^2 \leq C.$$

Finally we observe that

$$V(u(\cdot, t_0)) \leq V(u(\cdot, t_1)) \leq C$$

□

Now we are ready to prove the main result of the Section.

Theorem 2.2. *Let u_0 be a positive function in $[0, 1]$, and let u_λ be the solution of* (1.1) *with $u_\lambda(x, 0) = \lambda u_0(x)$.*

Then, there exists a critical value λ_c such that

1. *if $\lambda < \lambda_c$ then $u_\lambda \to 0$ in H^1.*
2. *if $\lambda > \lambda_c$ then u_λ blows up in finite time.*
3. *if $\lambda = \lambda_c$ then $u_\lambda \to 1$ in H^1.*

Proof. Let us start by defining the following sets

$$\begin{aligned} I_0 &= \{\lambda > 0 : \ u_\lambda \to 0\}, \\ I_\infty &= \{\lambda > 0 : \ u_\lambda \text{ blows up in finite time}\}, \\ I_g &= \{\lambda > 0 : \ u_\lambda \text{ is globally defined}\}. \end{aligned}$$

Then $\mathbb{R}_{>\nvdash} = \mathbb{I}_{\eth} \cup \mathbb{I}_\infty$

In order to prove the Theorem, it is enough to show that I_0 and I_∞ are open, or equivalently, that I_0 is open and I_g is closed.

Let us start by showing that I_0 is open. If $\lambda \in I_0$, then there exists t_0 such that $\|u_\lambda(\cdot, t)\|_\infty < 1/2$ if $t \geq t_0$.

Now, by continuity with respect to the initial data, there exists $\varepsilon > 0$ such that

$$\forall \ \mu > 0 : \ |\lambda - \mu| < \varepsilon, \ \|u_\mu(\cdot, t_0)\|_\infty < 3/4 < 1.$$

Therefore, by Theorem 2.1, $u_\mu \to 0$ in H^1, and hence $\mu \in I_0$.

On the other hand, let $(\lambda_k)_{k\geq 1} \subset I_g$ be a sequence such that $\lambda_k \nearrow \lambda_0$. Then, there exists $T > 0$ such that u_{λ_0} is defined in $[0, T]$. In order to show that $\lambda_0 \in I_g$, by Lemma 2.1 and Lemma 2.2, it is enough to show that for every $t_1 < T$ there exists a constant C independent of t_1 such that, $\|u_{\lambda_0}(\cdot, t)\|_\infty \leq C \quad \forall\ t \leq t_1$. By Proposition 2.1 and by Lemma 2.8, taking $t_0 < t_1$ there exists $C = C(t_0, p, \lambda_0\|u_0\|_\infty)$ such that

$$\|u_{\lambda_k}(\cdot, t)\|_\infty \leq C \qquad \forall\ t > t_0\ \forall\ k \geq 1.$$

The fact that $\|u_{\lambda_0}(\cdot, t)\|_\infty \leq C \quad \forall\ t_0 < t < t_1$ follows from the continuous dependence on the initial data. □

We end this section by showing that λ_c is continuous with respect to the initial data. First we will need the following Lemma

Lemma 2.9. *Let u be a solution of* (1.1)*. Then, for every $M > 1$ there exists $L > 0$ such that if $u(1, t) > M \quad \forall t \in [0, L]$, then there exists $T_0 < L$ such that $u(x, T_0) > 1 \ \ \forall x \in [0, 1]$ and hence u blows up.*

Proof. Let v the solution of the following auxiliary problem

$$\begin{cases} v_t &= v_{xx}, \\ v_x(1,t) &= \sigma(p), \\ v_x(0,t) &= 0, \\ v(x,0) &= 0, \end{cases} \tag{2.8}$$

where $\sigma(p) = M^p - M > 0$.

Now the proof follows by a comparison argument. In fact if $u(x, t)$ is a solution of (1.1) for which $u(1, t) > M \quad \forall t \in [0, L]$, then we claim that

$$v(x, t) \leq u(x, t) \qquad \forall t \in [0, L]\ \forall x \in [0, 1].$$

To prove the claim we observe that $w = u - v$ verifies

$$\begin{aligned} w_t &= w_{xx}, \\ w_x(0,t) &= 0, \\ w_x(1,t) &= g(u_\lambda(1,t)) - \sigma(p), \\ w(x,0) &\geq 0. \end{aligned}$$

But $g(u(1, t)) - \sigma(p) > 0$ implies $w(x, t) > 0$, giving $v(x, t) \leq u(x, t) \ \ \forall t \in [0, L]\ \forall x \in [0, 1]$. At this point we just observe that the solution of (2.8) overcome 1 in finite time T_0, depending only on p and M, so taking $L > T_0$, $v(x, T_0) > 1$ and the same holds for $u(x, t)$. □

Theorem 2.3. *$\lambda_c = \lambda_c(u_0)$ is continuous with respect to the initial data (with the L^∞ norm).*

Proof. Let $\varepsilon > 0$ and let $\lambda \equiv \lambda_c(u_0) - \varepsilon$. Now, u_λ must go to zero uniformly, so there exists a time T such that $\|u_\lambda(\cdot, T)\|_\infty \leq 1/2$. If $\|u_0 - v_0\|_\infty < \delta$, by continuity of (1.1) with respect to the initial data we must have that $\|v_\lambda(\cdot, T)\|_\infty \leq 3/4$, and then, by Theorem 2.1, $v_\lambda \to 0$ uniformly. So $\lambda_c(v_0) > \lambda = \lambda_c(u_0) - \varepsilon$.

For the other inequality, let now $\lambda \equiv \lambda_c(u_0) + \varepsilon$. Now, as u_λ blows up in finite time, for $M > 1$ there exists $T > 0$ and $\eta > 0$ such that

$$2M \leq u_\lambda(1, t) < \infty \qquad \forall t \in [T, T + \eta].$$

η can be taken to be half of the blow up time of the solution of (1.1) with initial datum $\|u_\lambda(\cdot, T)\|_\infty$.

By continuity of (1.1) with respect to the initial data we have $v_\lambda(1, t) > M$ for $t \in [T, T + \eta]$ if $\|u_0 - v_0\|_\infty \leq \delta_1$.

Let now $\lambda_c(u_0) < \lambda_2 < \lambda$ such that

$$2M \leq u_{\lambda_2}(1, t) < \infty \qquad \forall t \in [T + \eta, T + 2\eta].$$

Again, by continuity of (1.1) with respect to the initial data we have $v_{\lambda_2}(1, t) > M$ for $t \in [T + \eta, T + 2\eta]$ if $\|u_0 - v_0\|_\infty \leq \delta_2$ and by the maximum principle, $v_\lambda \geq v_{\lambda_2}$, so

$$v_\lambda(1, t) > M \qquad \forall t \in [T, T + 2\eta].$$

Iterating this argument, we can find a sequence $\delta_k > 0$, such that

$$v_\lambda(1, t) > M \qquad \forall t \in [T, T + k\eta]$$

if $\|u_0 - v_0\|_\infty < \delta_k$.

So, by Lemma 2.9, if $k_0\eta > L$ we have that v_λ blows up in finite time if $\|u_0 - v_0\|_\infty \leq \delta_0 \equiv \delta_{k_0}$ so $\lambda_c(v_0) < \lambda = \lambda_c(u_0) + \varepsilon$. ☐

3. The semidiscrete case

We begin by a description of our numerical scheme.

Let $x_i = \frac{i}{N}$ $0 \leq i \leq N$ be a partition of the interval $(0, 1)$ into subintervals $I_i = (x_i, x_{i+1})$, of length $h = \frac{1}{N}$. Let V_h the set of continuous functions which are affine on each I_i. We consider the basis functions of V_h taking as usual φ_i, with $\varphi_i(x_j) = \delta_i^j$. Now let

$$u_h(x, t) = \sum_{i=0}^{N} u_i(t)\varphi_i(x). \tag{3.1}$$

For a fixed t, u_h belongs to $H^1(0, 1)$ so in order to construct an approximate solution of (1.1) we proceed as follows: replacing (3.1) in the weak formulation of (1.1) we get a system of ordinary differential equations for $U = (u_0(t), ..., u_N(t))$,

$$MU' = -AU + g(U), \tag{3.2}$$

where M is the mass matrix, A is the stiffness matrix, and $g(U) = g(u_N)e_N$ with $e_N = (0, ..., 1)$.

Mass lumping upon the matrix M in (3.2) gives

$$\begin{array}{rcl} u_0' &=& 2h^{-2}(-u_0 + u_1), \\ u_i' &=& h^{-2}(u_{i+1} - 2u_i + u_{i-1}), \\ u_N' &=& 2h^{-2}(-u_N + u_{N-1} + hg(u_N)). \end{array} \tag{3.3}$$

So, from now on, we will suppose that M is diagonal and $M_{ii} = h$ if $1 \le i \le N-1$ and $M_{00} = M_{NN} = h/2$.

In order to study the asymptotic behaviour of (3.3) we need the following results.

Lemma 3.1. *(Maximum principle) Let $h > 0$ be fixed, and let $U = (u_0, ..., u_N)$ be a solution of*

$$\begin{array}{rcl} u_0' &\le& 2h^{-2}(-u_0 + u_1), \\ u_i' &\le& h^{-2}(u_{i+1} - 2u_i + u_{i-1}), \\ u_N' &\le& 2h^{-2}(-u_N + u_{N-1} + hg(u_N)). \end{array} \tag{3.4}$$

Then

$$\max_{k=0,\dots,N} u_k(t) \le \max\{\max_{k=0,\dots,N} u_k(0); \sup_{0<\tau<t} u_N(\tau)\}$$

Proof. Let us first suppose that $U = (u_0, ..., u_N)$ verifies

$$\begin{array}{rcl} u_0' &<& 2h^{-2}(-u_0 + u_1), \\ u_i' &<& h^{-2}(u_{i+1} - 2u_i + u_{i-1}), \\ u_N' &<& 2h^{-2}(-u_N + u_{N-1} + hg(u_N)). \end{array} \tag{3.5}$$

Now, if the maximum is attained in an interior node, say $0 < j < N$, let t_0 be the first time when this happens, then we have

$$u_j'(t_0) \ge 0 \quad \text{and} \quad u_j(t_0) \ge u_k(t_0) \ 0 \le k \le N.$$

On the other hand, by (3.5) we get $u_j'(t_0) < 0$ which leads to a contradiction.

Now, it is easy to see from (3.5) that the maximum cannot be reached at u_0 and so the "maximum principle" follows.

To complete the proof, we just observe that if $Z = h^2(0, ..., k^2, ..., N^2)$, then $U_\varepsilon(t) \equiv U(t) + \varepsilon Z$ verifies (3.5) whenever U verifies (3.4). As $\varepsilon > 0$ is arbitrary, the Lemma follows □

Lemma 3.2. *Let $h > 0$ be fixed, U a global solution of* (3.3)*. Then, U is bounded in $\mathbb{R}^{\mathbb{N}+1}$.*

Proof. If a global solution of (3.3) is not bounded, then, by Lemma 3.1, necesarilly $u_N(t) \to \infty$ as $t \to \infty$, but u_N satisfies

$$u'_N = 2h^{-2}(-u_N + u_{N-1} + hg(u_N)) \geq C_1(h)u_N^p - C_2(h)u_N \geq C_3(h)u_N^p$$

so u_N cannot be globally defined □

Next, we show that the semidiscrete version of problem (1.1) also has a Lyapunov functional and hence, by Lemma 3.2, every global solution must converge to a stationary one.

Lemma 3.3. *Let $h > 0$ be fixed, U a solution of* (3.3)*, and $G(s) = \frac{s^{p+1}}{p+1} - \frac{s^2}{2}$ a primitive of g. Then*

$$V_h(U(t)) = \frac{1}{2} < A^{1/2}U(t), A^{1/2}U(t) > -G(u_N(t)) \qquad (3.6)$$

is a Lyapunov functional for the system (3.3).

Proof. This is just a discrete analogous of Lemma 2.2. □

In the same spirit of the continuous case (see Theorem 2.1), we have

Theorem 3.1. *Let $h > 0$ be fixed, $U_0 = (u_0, u_1, ..., u_N)$ be a positive initial datum, and let $U(t)$ be the solution of* (3.3) *with $U(0) = U_0$, suppose that $U(t) \to (1, ..., 1)$*

(1) *If $V_0 \leq U_0$, $V_0 \neq U_0$, then $V(t) \to 0$.*
(2) *If $V_0 \geq U_0$, $V_0 \neq U_0$, then $V(t)$ blows up in finite time.*

Here $V(t)$ denotes the solution such that $V(0) = V_0$.

Proof. We will only sketch the proof of (1). By comparison with U we have that $U \geq V$ so we obtain global existence for V, and then Lemma 3.2 and the Lyapunov functional given in Lemma 3.3 implies that $V(t) \to 0$ or $V(t) \to (1, ..., 1)$, but the latter can not occur, otherwise $Z = U - V$ should tend to zero and satisfies

$$MZ' = -AZ + g'(\eta)z_N e_N$$

with $g'(\eta) > \delta > 0$ which contradicts the fact $Z \to 0$. □

The proof of the existence and uniqueness of the critical value $\lambda_{c,h}$ is much easier than its continuous counterpart.

Theorem 3.2. *Let $h > 0$ be fixed, $U_0 = (u_0, u_1, ..., u_N)$ be a positive initial datum, and let $U_\lambda(t)$ be the solution of* (3.3) *with $U_\lambda(0) = \lambda U_0$.*

Then, there exists a critical value $\lambda_{c,h}$, depending on h such that

(1) *if* $\lambda < \lambda_{c,h}$ *then* $U_\lambda(t) \to 0$.
(2) *if* $\lambda > \lambda_{c,h}$ *then* $U_\lambda(t)$ *blows up in finite time.*
(3) *if* $\lambda = \lambda_{c,h}$ *then* $U_\lambda(t) \to (1, ..., 1)$.

Proof. As in Theorem 2.2 we define the following sets

$$\begin{aligned} I_{0,h} &= \{\lambda > 0 : \ U_\lambda \to 0\}, \\ I_{\infty,h} &= \{\lambda > 0 : \ U_\lambda \text{ blows up in finite time}\}, \\ I_{g,h} &= \{\lambda > 0 : \ U_\lambda \text{ is globally defined}\}. \end{aligned}$$

Then $\mathbb{R}_{>\nvdash} = \mathbb{I}_{\eth,\eqsim} \cup \mathbb{I}_{\infty,\eqsim}$.

Now, it is enough to show that $I_{0,h}$ and $I_{\infty,h}$ are open, so let $\lambda_0 \in I_{0,h}$, then $U_{\lambda_0} \to 0$ and for t_0 large enough $U_{\lambda_0}(t_0) < 1/2$. The continuous dependence on the initial condition allows us to conclude. In fact there exist an $\varepsilon > 0$ such that if $|\lambda - \lambda_0| < \varepsilon$ then $\|U_\lambda(t_0)\|_\infty < 3/4 < 1$, and $I_{0,h}$ is open. To prove the same for $I_{\infty,h}$ we use similar arguments. From (3.3) we obtain

$$u'_N \geq 2h^{-2}(-u_N + hg(u_N)),$$

then $\|U_\lambda(t)\|_\infty$ cannot be globally bounded if $(U_\lambda)_N(t_0) > K(h)$, with $K(h)$ large enough (it is enough to choose the last positive root of $-s + hg(s)$ as $K(h)$). Let $\lambda_0 \in I_{\infty,h}$ then there exists t_0 such that $(U_{\lambda_0})_N(t_0) > K(h) + 1$, and by continuity respect to the initial condition, $(U_\lambda)_N(t) > K(h)$ provided λ is closed to λ_0, this implies the desired result. □

In order to obtain the convergence of this critical values $\lambda_{c,h}$ to the critical value λ_c we need the following convergence Theorem for regular solutions whose proof can be found in [9].

Theorem 3.3. *Let* $u \in C^{2,1}([0,1] \times [0,T])$, $T < t_{max}(u_0)$, *be a solution of* (1.1) *and* u_h *its semidiscrete approximation given by* (3.1). *Then there exists a constant* C *depending on* T *and* u *such that, for* h *small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T])} \leq Ch^{\frac{3}{2}}.$$

Remark . The required regularity of u can be obtained by taking initial data that are compatible with the boundary conditions (see [15]), namely

$$u'_0(0) = 0 \qquad u'_0(1) = g(u_0(1)).$$

We observe that this type of initial data are dense in H^1.

Now we prove the discrete version of Lemma 2.9

Lemma 3.4. *Let $u_h(x,t)$ be the semidiscrete approximation of u given by (3.1). Then for every $M > 1$ there exists $h_0 > 0$ and $L > 0$ such that if $u_h(1,t) > M \quad \forall t \in [0,L]$, then there exists $T_0 < L$ such that if $h < h_0$, then $u_h(x,T_0) > 1$ and hence u_h blows up, $\forall h < h_0$.*

Proof. By Lemma 2.9, there exists $M > 0$, $L > 0$ and $T_0 < L$ such that if $u(1,t) > M \quad \forall t \in [0,L]$ then $u(x,T_0) > 1 \quad \forall x \in [0,1]$. Now we choose a compatible initial datum v_0, such that $v_0 \le u_0$ and $v(x,T_0) > 1 \quad \forall x \in [0,1]$. Now by Theorem 3.3 and a comparison argument, if h is small enough, $u_h(x,T_0) > v_h(x,T_0) > 1 \quad \forall x \in [0,1]$ which proves the Lemma. □

Now we are ready to prove the main result of this section.

Theorem 3.4. *Let λ_c and $\lambda_{c,h}$ as in Theorems 2.2 and 3.2 respectively, then $\lambda_{c,h} \to \lambda_c$ as $h \to 0$.*

Proof. Let $\varepsilon > 0$, if we take $\lambda \equiv \lambda_c - \varepsilon$, we have that $u_\lambda \to 0$, then taking t_0 large enough $u_\lambda(x,t_0) < 1/2$.

Now we pick a compatible initial datum v_0 such that $v_0 \ge \lambda u_0$ and $v(x,t_0) < 1/2$.

As $v_h \to v$ in $L^\infty([0,1] \times [0,t_0])$ (by Theorem 3.3) it holds by a comparison argument, $u_{\lambda,h}(x,t_0) < v_h(x,t_0) < 3/4$ if $h < h_0$, then Theorem 3.1 implies that $u_{\lambda,h}(x,t) \to 0$, and then $\lambda_c - \varepsilon = \lambda \le \lambda_{c,h}$.

On the other hand let $\lambda \equiv \lambda_c + \varepsilon$. If $\overline{\lambda} < \lambda$ the blow up time $\overline{T}$ of $u_{\overline{\lambda}}$ is uniformly bounded (below) by the blow up time T of u_λ. Now we can choose $u_{\overline{\lambda}}$ in order to obtain $u_{\overline{\lambda}}(1,T) = M + 2$ and we can take a fixed value τ (independent of h) such that

$$2M > u_{\overline{\lambda}}(1, T+t) > M + 1 \qquad \forall t \in [0,\tau].$$

Now, as in the previous case, we pick a compatible initial datum v_0 such that $v_0 < \lambda u_0$ and

$$2M > v(1, T+t) > M + 1 \qquad \forall t \in [0,\tau].$$

Choosing h_1 small enough, for all $h < h_1$,

$$\|v_h - v\|_{L^\infty([0,T+\tau]\times[0,1])} < 1$$

in particular

$$|v_h(1,T+t) - v(1,T+t)| < 1 \qquad \forall t \in [0,\tau],$$

which gives

$$2M + 1 > v_h(1,T+t) > M \qquad \forall t \in [0,\tau].$$

Taking another $\overline{\lambda}$ and repeating inductively the last argument we obtain a finite number of decreasing $h = h_k$, $\lambda_k < \lambda_c + \varepsilon$ and compatible initial data $v_{0,k}$ such that $v_{0,k} < \lambda u_0$ and

$$2M + 1 > v_{k,h}(1, T + t) > M \qquad \forall t \in [0, k\tau] \qquad \forall h \leq h_k.$$

This implies, using a comparison argument between $v_{k,h}$ and $u_{\lambda,h}$

$$u_{\lambda,h}(1, T + t) > M \qquad \forall t \in [0, k\tau]$$

and if $k\tau$ in big enough ($k\tau > L$) we fall into the hypothesis of Lemma 3.4 then we can conclude that $u_{\lambda,h}$ has finite blow up time and then $\lambda_{c,h} < \lambda = \lambda_c + \varepsilon$ for all $h < h_k$. □

We end this section by showing that the steady solution $U = (1, ..., 1)$ is hyperbolic and its stable manifold has dimension $N - 1$ as can be expected.

Theorem 3.5. *Let $U = (1, ..., 1)$ be the only positive steady solution of* (3.3). *Then U is hyperbolic and its stable manifold is an hypersurface.*

Proof. The linearized problem of (1.1) at 1, has the form

$$\begin{cases} v_t &= v_{xx}, \\ v_x(0,t) &= 0, \\ v_x(1,t) &= g'(1)v(1,t). \end{cases}$$

Now, it is enough to observe that all the eigenvalues but one of the linearization of (1.1) at 1 are negatives, and the last one is positive.

On the other hand, using standard techniques, we have an uniform bound in H^1 norm for the error of the discrete approximation of the following problem

$$\begin{cases} v_{xx} &= f, \\ v_x(0,t) &= 0, \\ v_x(1,t) &= g'(1)v(1,t), \end{cases}$$

i.e. we have an inequality of the form,

$$\|v - v_h\|_{H^1} \leq Ch\|f\|_{H^1}.$$

Then we only have to observe that, from a well known fact (see [16]), the discrete eigenvalues converge to the continuous ones. □

N	$\lambda_{c,h}$
10	0.9755
20	0.9766
50	0.9770
100	0.9770

Table 1

4. Numerical experiments

In this section we show some numerical experiments for the semidiscrete version of (1.1) with $g(s) = s^2 - s$ and

$$u_0(x) = \frac{1}{2}\cos(2\pi x) + 1$$

The results were obtained integrating the ODE system (3.3) by using an adaptive Runge-Kutta method.

Table 1 gives the approximate $\lambda_{c,h}$ with four exact decimals. The values obtained suggest that the convergence rate is of order h^2 however we were not able to prove it.

In Fig. 1, we show the evolution of the solution with $h = 1/100$, $\lambda = 0.9771 > \lambda_{c,h}$.

It is worth remarking that in Fig. 1 we observe that the solution stays close to 1 for a long period of time, after which it rapidly goes to infinity (this happens because we have chosen a value of lambda greater but close to the critical one).

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