ESTIMATES FOR EIGENVALUES OF QUASILINEAR ELLIPTIC SYSTEMS. PART II

JULIÁN FERNÁNDEZ BONDER AND JUAN P. PINASCO

ABSTRACT. In this paper we find explicit lower bounds for Dirichlet eigenvalues of a weighted quasilinear elliptic system of resonant type in terms of the eigenvalues of a single p-Laplace equation. Also we obtain asymptotic bounds by studying the spectral counting function which is defined as the number of eigenvalues smaller than a given value.

1. Introduction

In this work we will study the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda r(x)\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -\Delta_q v = \lambda r(x)\beta|u|^{\alpha}|v|^{\beta-2}v \end{cases} \tag{1.1}$$

in Ω with zero Dirichlet boundary conditions, $u = v = 0$ on $\partial\Omega$. Here, $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$, $r \in L^\infty(\Omega)$ is a strictly positive function, $r(x) \geq m > 0$ (less regularity conditions on r and $\partial\Omega$ are enough, see the remarks at the end of the paper), $\lambda \in \mathbb{R}$ is the eigenvalue parameter, $1 < q \leq p < +\infty$, and α, β are positive constants satisfying

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1.$$

The eigenvalue problem for (1.1) was studied in several works, let us mention among them Boccardo and de Figueiredo [4], Fleckinger, Manásevich, Stravrakakis, and De Thélin [18], Manasevich and Mawhin [22], and the references therein.

In particular, the first or principal eigenvalue has deserved a great deal of attention, and several properties were analyzed like existence, unicity, positivity, and isolation in bounded or unbounded domains, with different boundary conditions and with or without weights (and for the weighted problem, indefinite and singular weights were considered). Also, the positivity of both associated eigenfunctions can be found in the literature. We refer the interested reader to [1], [9], [13], [21], [27], [30], and [31] among others.

The existence of a sequence of eigenvalues $\{\lambda_k\}$ of problem (1.1) was proved in [7] by using the abstract theory developed by Amann in [2], and the existence of generalized eigenvalues was obtained in [8]. Moreover, an upper bound of the first eigenvalue was obtained in terms of the first eigenvalue of the p-laplacian and, for the one dimensional problem, upper bounds for all the variational eigenvalues were obtained, namely

Key words and phrases. elliptic system, p laplacian, eigenvalue bounds,
2000 *Mathematics Subject Classification.* 34L30, 35P30, 34L15, 35P15 .

$$\lambda_k \le \frac{\Lambda_{(p),k}}{p}\Big[1+\left(\frac{p}{q}\right)^{q+1}\big(m\Lambda_{(p),k}\big)^{(q-p)/p}\Big], \tag{1.2}$$

Here, $\Lambda_{(p),k}$ stands for the k^{th} eigenvalue of the one dimensional p-laplacian, and m is the lower bound of the weight. By using that $\Lambda_{(p),k} \sim \left(\pi_p/\int_\Omega r^{1/p}\right)^p k^p$ when $k\to\infty$ (see [14]), we obtain the asymptotic upper bound

$$\lambda_k \le \left(\frac{\pi_p}{\int_\Omega r^{1/p}}\right)^p \frac{k^p}{p} + ck^q \sim \left(\frac{\pi_p}{\int_\Omega r^{1/p}}\right)^p \frac{k^p}{p} \tag{1.3}$$

(throughout this work, we will write $f\sim g$ to denote that $\lim_{k\to\infty} f/g = 1$). Let us note that inequality (1.2) is an explicit upper bound of λ_k, whereas (1.3) is an asymptotic bound.

It would be desirable to obtain also lower bounds due to several applications to bifurcation problems, anti-maximum principles, and existence or non-existence of solutions (see for example [3], [11], [12], [16], [27], [28], [29], [30]). However, the results in [8] and [13] only gives lower bounds of the first eigenvalue.

Hence, in this paper we give explicit and asymptotic lower bounds for the k^{th} eigenvalue of a system in $\Omega\subset\mathbb{R}^N$. The asymptotic bounds depend on the smaller exponent of the system, q, instead of p:

$$ck^q \le \lambda_k,$$

when $k\to\infty$, and explicit lower bounds depends on a combination of the eigenvalues of both the p and q laplacians.

Also, when $p=q$, we obtain the correct order of growth of the k^{th} eigenvalue of a system in any dimension $N\ge 1$. We have

$$ck^{p/N} \le \lambda_k \le Ck^{p/N},$$

where the constants c, C depends only on p, r, N and the measure of Ω.

In the one dimensional case we have a better result when $\alpha=\beta$,

$$\lambda_k \sim Ck^p.$$

For a single p-laplacian equation without weight the order of growth of the eigenvalues was given in [20], $ck^{p/N} \le \Lambda_{(p),k} \le Ck^{p/N}$, and better asymptotic constants c, C were computed in [19]. A better order of growth was conjectured in this paper, namely $\Lambda_{(p),k}\sim Ck^{p/N}$. This was achieved for weighted problems only for $N=1$ with different techniques in [14], [23], [24].

In order to prove the asymptotic bounds, we will study the spectral counting function $N(\lambda)$ defined as

$$N(\lambda) = \#\{k : \lambda_k \le \lambda\},$$

and we will find asymptotic bounds for its growth. Let us note that inequalities like $ck^b \le \lambda_k \le Ck^a$, for certain constants c, C and exponents a, b could be stated equivalently in terms of $N(\lambda)$ as

$$(C^{-1}\lambda)^{1/a} \le N(\lambda) \le (c^{-1}\lambda)^{1/b}.$$

The main tool used in this work is a generalization of the Dirichlet-Neumann bracketing together with comparison and variational arguments.

We will consider first the special case $N = 1$. In that case, equation (1.1) reads

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda r(x)\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -(|v'|^{q-2}v')' = \lambda r(x)\beta|u|^{\alpha}|v|^{\beta-2}v \end{cases} \tag{1.4}$$

in $[0,1]$ and the main result is the following theorem:

Theorem 1.1. *Let $N(\lambda)$ be the eigenvalue counting function of problem* (1.4) *with Dirichlet boundary conditions.*

(1) *If $q < p$, then*

$$c_1\lambda^{1/p} \le N(\lambda) \le C_1\lambda^{1/q} + C_2\lambda^{1/p} \qquad \text{as } \lambda \to \infty.$$

(2) *If $q = p$, then*

$$c_2\lambda^{1/p} \le N(\lambda) \le (C_1 + C_2)\lambda^{1/p} \qquad \text{as } \lambda \to \infty.$$

(3) *If $q = p$, and $\alpha = \beta$, then*

$$N(\lambda) \sim c_2\lambda^{1/p} \qquad \text{as } \lambda \to \infty.$$

Remark 1.2. The constants c_1, c_2, C_1, C_2 in Theorem 1.1 can be computed explicitly. In fact, from the proof of the Theorem, it follows that

$$c_1 := \frac{p^{1/p}\|r^{1/p}\|_{L^1}}{\pi_p}, \qquad c_2 := 2^{1-1/p}c_1$$

$$C_1 := \frac{\alpha^{1/p}\|r^{1/p}\|_{L^1}}{\pi_p}, \qquad C_2 := \frac{\beta^{1/q}\|r^{1/q}\|_{L^1}}{\pi_q}.$$

Of independent interest is the upper bound for $N(\lambda)$ in Theorem 1.1. We derive it for a different eigenvalue problem which gives explicit lower bounds for the eigenvalues of the system. We state it separately here, and Subsection 3.1 is devoted to its proof:

Theorem 1.3. *Let $S_p = \{\Lambda_{(p),k}/\alpha\}$ and $S_q = \{\Lambda_{(q),k}/\beta\}$ be the sets of variational eigenvalues of each p and q laplacian equations respectively. Let us introduce the set $S = S_p \cup S_q$, ordered as a sequence $\{\mu_k\}$ with $\mu_1 \le \mu_2 \le \ldots \le \mu_k \le \ldots$. Then, $\mu_k \le \lambda_k$ for every $k \in \mathbb{N}$, where $\{\lambda_k\}$ is the set of variational eigenvalues of problem* (1.4).

Let us note that Theorem 1.3 gives explicit lower bounds, which holds for every $k \in \mathbb{N}$.

Similar results are valid for the N-dimensional case. In Section 4 we consider problem (1.1). The results are slightly worse than the previous ones, and for brevity, we will consider only the case $r \equiv 1$. The general case follows by using the Sturm theory and the bounds $m \le r \le M$ [19].

Theorem 1.4. *Let $N(\lambda)$ be the eigenvalue counting function of problem* (1.1).

(1) *If $q \le p$, then*

$$\bar{c}_1\lambda^{N/p} \le N(\lambda) \le \bar{C}_1\lambda^{N/q} + \bar{C}_2\lambda^{N/p} \qquad \text{as } \lambda \to \infty.$$

(2) *If $q = p$, then*

$$\bar{c}_2\lambda^{N/p} \le N(\lambda) \le (\bar{C}_1 + \bar{C}_2)\lambda^{N/p} \qquad \text{as } \lambda \to \infty.$$

Remark 1.5. As in Theorem 1.1 the constants $\bar c_1$ and $\bar c_2$ can be computed explicitly. In fact,

$$\bar c_1 := \frac{p^{N/p}|\Omega|}{(\pi_p^p N)^{N/p}}, \qquad \bar c_2 := 2^{1-N/p}\bar c_1.$$

However, the constants $\bar C_1$ and $\bar C_2$ depend on the lower bound for the $k-$th eigenvalue of the $p-$Laplacian given in [20] (see also [19]) which is not known explicitly. In particular, the dependence of these constants on p (or q) is not well understood.

The missing item could be proved only when $p = q = 2$ and $\alpha = \beta = 1$, which is related to the bilaplacian with Navier's boundary conditions:

$$\begin{cases} \Delta\Delta u = \lambda^2 u \\ u = \Delta u = 0. \end{cases}$$

However, a subtle detail concerning the signs of solutions must be considered. See Remark 3.6 at the end of the proof of Theorem 1.1.

We close the paper with section 5, where some generalizations and open problems will be briefly discussed.

2. Some Previous Results

In this Section we recall some previous results which will be needed in the rest of the paper. The only new result is Proposition 2.7, which has some interest since it provides an explicit lower bound for the first eigenvalue of the N dimensional p-laplacian.

2.1. **Variational setting.** The variational characterization of eigenvalues follows from the abstract theory developed by Amman (see [2]).

A proof of the existence of infinitely many eigenpairs for problem (1.1) could be found in [10]. By an eigenpair of problem (1.1), we mean a pair $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + |\nabla u|^{q-2}\nabla u\cdot\nabla\psi = \lambda\int_\Omega r(x)(\alpha|u|^{\alpha-2}u\phi|v|^\beta + \beta|u|^\alpha|v|^{\beta-2}v\psi)$$

for any test-function pair $(\phi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

It is convenient to work with the variational characterization of the eigenvalues, defined through the Rayleigh quotient,

$$\lambda_k = \inf_{C\in\mathcal{C}_k}\sup_{(u,v)\in C}\frac{\frac{1}{p}\int_\Omega|\nabla u|^p + \frac{1}{q}\int_\Omega|\nabla v|^q}{\int_\Omega r(x)|u|^\alpha|v|^\beta}, \tag{2.1}$$

where $\mathcal{C}_k$ is the class of compact symmetric ($C = -C$) subsets of $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ of (Krasnoselskii) genus greater or equal that k.

This approach is due to Browder [6], and following Riddell [26] it is easy to prove the equivalence between (2.1) and the characterization of the eigenvalues given by Amman's theory, see for example [8].

For the one dimensional p-laplace equation

$$-(|u'|^{p-2}u')' = \lambda r(x)|u|^{p-2}u \tag{2.2}$$

in $\Omega = [a, b]$ we have:

$$\Lambda_{(p),k} = \inf_{C\in\mathcal{C}_k} \sup_{u\in C} \frac{\int_a^b |u'|^p}{\int_a^b r(x)|u|^p}, \tag{2.3}$$

where now we work on the space $W_0^{1,p}(a, b)$.

2.2. **One Dimensional Case.** For the one dimensional case and constant weight $r \equiv 1$, all the eigenvalues and eigenfunctions could be find explicitly as in [10]. We state this result in the next lemma:

Lemma 2.1 ([10], Theorem 3.1)**.** *The eigenvalues $\lambda_{(p),k}$ and eigenfunctions $u_{(p),k}$ of equation* (2.2) *with $r \equiv 1$ on an interval of lenght L are given by*

$$\Lambda_{(p),k} = \frac{\pi_p^p k^p}{L^p},$$
$$u_{(p),k} = \sin_p(\pi_p x/L).$$

The function $\sin_p(x)$ is obtained by integrating equation (2.2), its first zero is π_p, given by

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}.$$

Moreover, they coincide with the variational eigenvalues. We have:

Lemma 2.2. *All the eigenvalues of equation* (2.2) *are given by* (2.3)*.*

See [14] and the references therein for a proof.

2.3. **The Spectral Counting Function.** We will study the spectral counting function $N(\lambda)$ of problem (1.1) defined as

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\}.$$

Sometimes we will use $N_{sys}(\lambda)$, $N_p(\lambda)$, and $N_q(\lambda)$ to denote the eigenvalue counting functions of the system, the p-laplacian and the q-laplacian, respectively. If confusion could arise, we will write $N(\lambda, \Omega)$ to denote explicitly the set Ω where the eigenvalue problem is considered, and also $N^D(\lambda)$ and $N^N(\lambda)$ to indicate the Dirichlet and Neumann boudary conditions, although in this work we will avoid the Neumann boundary condition.

The main tool in order to obtain the asymptotic expansion of $N(\lambda)$ is the Dirichlet-Neumann bracketing. The following proposition could be find in [14]:

Proposition 2.3. *Let $U_1, U_2 \in \mathbb{R}^N$ be disjoint open sets such that $(\overline{U_1 \cup U_2})^{int} = U$ and $|U \setminus U_1 \cup U_2|_N = 0$, where $|A|_N$ stands for the N-dimensional Lebesgue measure of the set A. Then,*

$$\begin{aligned} N^D(\lambda, U_1) + N^D(\lambda, U_2) = N^D(\lambda, U_1 \cup U_2) \leq N^D(\lambda, U) \\ \leq N^N(\lambda, U) \leq N^N(\lambda, U_1 \cup U_2) = N^N(\lambda, U_1) + N^N(\lambda, U_2). \end{aligned}$$

The explicit expression of eigenvalues together with Proposition 2.3 gives the following asymptotic expansion for the eigenvalue counting function $N(\lambda)$:

Lemma 2.4. *Let $r(x)$ be a bounded continuous function in Ω. Then, when $\lambda \to \infty$,*

$$N(\lambda, \Omega) = \frac{\lambda^{1/p}}{\pi_p} \int_\Omega r^{1/p} + o(\lambda^{1/p}).$$

That is,

$$\Lambda_{(p),k} \sim \frac{\pi_p^p k^p}{(\int_\Omega r^{1/p})^p},$$

For a proof, see [14], [24]. The error term $o(\lambda^{1/p})$ denotes that

$$\frac{N(\lambda,\Omega) - \lambda^{1/p}\int_\Omega r^{1/p}/\pi_p}{\lambda^{1/p}} \to 0$$

when $\lambda \to \infty$. The error term $N(\lambda,\Omega) - \lambda^{1/p}\int_\Omega r^{1/p}/\pi_p$ could be improved as $O(\lambda^{d/p})$ for regular weights r, where d is the Minkowski dimension of $\partial\Omega$, see [15] for details. However, in this work we are not interested in error terms, and whenever we write

$$c\lambda^a \le N(\lambda), \qquad N(\lambda) \le c\lambda^a,$$

it must be understood that

$$1 \le \liminf_{\lambda\to\infty} \frac{N(\lambda)}{c\lambda^a}, \qquad 0 \le \limsup_{\lambda\to\infty} \frac{N(\lambda)}{c\lambda^a} \le 1.$$

2.4. **The N-dimensional case.** In order to find a lower bound for $N(\lambda)$ we need to find an upper bound of $\Lambda_{(p),1}$ which enable us to bound the number of eigenvalues of the system less than a given λ. We will follow the ideas in [19], by fixing a value of λ and by covering Ω by a grid of squares of side L such that the number of eigenvalues of the p-laplacian in each square would be equal to one. For example, we may use the bounds in [19], [20]:

Lemma 2.5. *Let $\Lambda_{(p),k}$ be the k^{th} eigenvalue of the p-laplacian. Then, there exists $c_p, C_p \in \mathbb{R}$ such that*

$$c_p k^{p/N} \le \Lambda_{(p),k} \le C_p k^{p/N}.$$

However, the main drawback of this approach is the fact that we ignore the precise values of the constants c_p, C_p.

Hence, we will compute explicit upper and lower bounds of $\Lambda_{(p),1}$ by using the first eigenvalue $\nu_{(p),1}$ of the pseudo p-laplacian on a square Q_L of side of length L with a constant coefficient $r \equiv 1$:

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = \nu |u|^{p-2} u, \tag{2.4}$$

that is,

$$\nu_{(p),1} = \inf_{u\in W_0^{1,p}} \frac{\int_{Q_L}\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^p}{\int_{Q_L} |u|^p},$$

As in [20], given Ω we consider two squares $Q_1 \subset \Omega \subset Q_2$, and the bounds follows from the following Propositions:

Proposition 2.6. *Let* $\Omega_1 \subset \Omega_2$. *Then, the eigenvalues of Problem* (1.1) *satisfy*

$$\Lambda_{(p),k}(\Omega_2) \leq \Lambda_{(p),k}(\Omega_1).$$

The proof follows easily from the variational characterization of eigenvalues and the fact that $W_0^{1,p}(\Omega_1) \subset W_0^{1,p}(\Omega_2)$.

Proposition 2.7. *Let* $Q_L \subset \mathbb{R}^N$, *and* $\Lambda_{(p),1}$, *be the first eigenvalues of the p-laplacian in* Q_L. *Then,*

$$\begin{array}{lll} \frac{\pi_p^p N}{L^p} \leq \Lambda_{(p),1} \leq \frac{\pi_p^p N^{p/2}}{L^p} & \text{if } 2 < p, \\ \frac{\pi_p^p N^{p/2}}{L^p} \leq \Lambda_{(p),1} \leq \frac{\pi_p^p N}{L^p} & \text{if } p < 2. \end{array}$$

Proof. Due to the equivalence of norms in $\mathbb{R}^N$, we have

$$|x|_q \leq C_p |x|_p$$

for any $x \in \mathbb{R}^N$, where $C_p = 1$ if $p \leq q$, and $C_p = N^{(p-q)/2q}$ if $p \geq q$ (see, for instance, [17]).

We fix the set $B = \{u \in W_0^{1,p} : \int_{Q_L} |u|^p\}$, and we have the following characterization of the first eigenvalues $\Lambda_{(p),1}$, $\nu_{(p),1}$ of the p-laplacian and the pseudo p-laplacian in $\mathbb{R}^N$ respectively:

$$\nu_{(p),1} = \inf_{u \in B} \| |\nabla u|_p \|_p^p; \quad \Lambda_{(p),1} = \inf_{u \in B} \| |\nabla u|_2 \|_p^p.$$

Clearly, the previous norm inequality gives

$$\begin{array}{lll} \nu_{(p),1} \leq \Lambda_{(p),1} \leq N^{(p-2)/2} \nu_{(p),1} & \text{if } 2 < p, \\ N^{(p-2)/2} \nu_{(p),1} \leq \Lambda_{(p),1} \leq \nu_{(p),1} & \text{if } p < 2. \end{array}$$

Now, we have that

$$u_{(p),1} = \sin_p(\pi_p x_1 / L) \cdots \sin_p(\pi_p x_N / L), \quad \nu_{(p),1} = \frac{\pi_p^p N}{L^p}$$

is the first eigenpair of the pseudo p-laplacian on Q_L. This result follows by separation of variables, and $u_{(p),1}$ is the first eigenfunction since there exists only one positive eigenfunction of the pseudo p-laplacian (see [5]).

The proof is complete. □

3. One Dimensional Case

We will divide the proof of Theorem 1.1 in several lemmas, finding lower and upper bounds for $N(\lambda)$.

3.1. Upper Bounds for the Spectral Counting Function.

The most difficult problem is to find an upper bound for $N(\lambda)$, since this is equivalent to give lower bounds for the eigenvalues. Hence, we begin by studying the following system in $[a, b] \subset \mathbb{R}$:

$$\begin{cases} -(|u'|^{p-2} u')' = \mu r(x) \alpha |u|^{p-2} u \\ -(|v'|^{q-2} v')' = \mu r(x) \beta |v|^{q-2} v \end{cases} \tag{3.1}$$

with zero Dirichlet boundary conditions, coupled only on the eigenvalue parameter μ.

Lemma 3.1. *Let us consider the following variational problem*

$$\mu^k = \inf_{C\in\mathcal{C}_k} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q}\int_a^b |v'|^q}{\frac{\alpha}{p}\int_{\partial\Omega} r(x)|u|^p + \frac{\beta}{q}\int_a^b r(x)|v|^q},$$

with $C \subset W = W_0^{1,p}\times W_0^{1,q}([a,b])$, $\mathcal{C}_k$ as in Section 2. Then, μ^k correspond to an eigenvalue of system (3.1)*.*

Proof. The proof follows as usual, by noting that the equations in system (3.1) are the Euler Lagrange equations of the functional

$$\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q}\int_a^b |v'|^q - \mu\frac{\alpha}{p}\int_a^b r(x)|u|^p - \mu\frac{\beta}{q}\int_a^b r(x)|v|^q.$$

□

It is clear that any eigenvalue of the p and q laplacians corresponds to an eigenvalue of this system, and reciprocally. However, it remains to prove that they are all the *variational eigenvalues.*

Let us rename the sequences of eigenvalues of each equation, $S_p = \{\Lambda_{(p),k}/\alpha\}$ and $S_q = \{\Lambda_{(q),k}/\beta\}$, as $\{\mu_k\}$, where $\mu_k \in S_p\cup S_q$, and $\mu_1 \le \mu_2 \le \ldots \le \mu_k \le \ldots$; and let us call $S = \{\mu^k\}$. Our next task is to prove that $\mu_k = \mu^k$.

Theorem 3.2. *Let $\mu^k \in S$ be a variational eigenvalue. Then, $\mu^k = \mu_k$.*

Proof. Suppose that μ^k is a variational eigenvalue with an associate eigenfunction (u^k, v^k). If one of them is identically zero, say v^k, then (μ^k, u^k) is an eigenpair of the p-laplacian, and μ^k must coincide with one of the eigenvalues μ_j. Hence, $\mu^k \ge \mu_k$.

Let us note that if both u^k, v^k are not identically zero in $[a,b]$, then $(u^k,0)$ and $(0,v^k)$ are also eigenfunctions corresponding to μ^k, and $span[(u^k,0),(0,v^k)]\cap B_1(W)$ where $B_1(W)$ is the unit ball in W has genus 2 and then μ^k has multiplicity at least two, $\mu^k = \mu^{k+1}$ where $\mu^k = \mu_j \in S_p$, $\mu^{k+1} = \mu_i \in S_q$ for certain i, j.

Let us show that any eigenvalue μ_k of one equation is a *variational* eigenvalue of the system. Since the first part showed that $S \subset S_p \cup S_q$, i. e., $\mu^k \ge \mu_k$, we only need to show that there exists a compact symmetric set C of genus greater or equal than k such that

$$\sup_{(u,v)\in C} \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q}\int_a^b |v'|^q}{\frac{\alpha}{p}\int_a^b r(x)|u|^p + \frac{\beta}{q}\int_a^b r(x)|v|^q} \le \mu_k + \varepsilon$$

We can assume without loss of generality that $\mu_k = \lambda_{(p),j}$. Hence, there are at least $k-j$ eigenvalues corresponding to the q-laplacian lower or equal than μ_k, and we can take two sets $C_{(p),j} \in W_0^{1,p}([a,b])$ and $C_{(q),k-j} \in W_0^{1,q}([a,b])$ of genus greater or equal than j and $k-j$, respectively, such that

$$\sup_{u\in C_{(p),j}} \frac{\frac{1}{p}\int_a^b |u'|^p}{\frac{\alpha}{p}\int_a^b r(x)|u|^p} \le \mu_k + \varepsilon,$$

$$\sup_{v\in C_{(q),k-j}} \frac{\frac{1}{q}\int_a^b |v'|^q}{\frac{\beta}{q}\int_a^b r(x)|v|^q} \le \mu_k + \varepsilon.$$

Now, the product set $C = C_{(p),j} \times C_{(q),k-j}$ has genus greater or equal than k, and for any $(u,v) \in C$ we have

$$\sup_{(u,v)\in C} \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q}\int_a^b |v'|^q}{\frac{\alpha}{p}\int_a^b r(x)|u|^p + \frac{\beta}{q}\int_a^b r(x)|v|^q} \le \mu_k + \varepsilon,$$

since

$$\frac{a_1}{b_1} = \frac{\frac{1}{p}\int_a^b |u'|^p}{\frac{\alpha}{p}\int_a^b r(x)|u|^p} \le \mu_k + \varepsilon,$$

$$\frac{a_2}{b_2} = \frac{\frac{1}{q}\int_a^b |v'|^q}{\frac{\beta}{q}\int_a^b r(x)|v|^q} \le \mu_k + \varepsilon,$$

and

$$\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\} \le \frac{a_1+a_2}{b_1+b_2} \le \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\}.$$

Hence, we have $\mu^k \le \mu_k$, and the proof is finished. □

Our next Lemma proves Theorem 1.3.

Lemma 3.3. *Let λ_k be the eigenvalues of system* (1.4). *Then, $\mu_k \le \lambda_k$.*

Proof. Let us note that, for any $(u,v) \in W$, Young's inequality gives

$$\int_a^b r(x)|u|^\alpha|v|^\beta = \int_a^b r^{\alpha/p}(x)r^{\beta/q}(x)|u|^\alpha|v|^\beta \le \frac{\alpha}{p}\int_a^b r(x)|u|^p + \frac{\beta}{q}\int_a^b r(x)|v|^q,$$

and the result follows by the variational characterization of eigenvalues of each system. □

We are ready to prove the upper bounds of $N(\lambda)$ in Theorem 1.1.

Proposition 3.4. *If $q \le p$, then*

$$N_{sys}(\lambda) \le N_p(\alpha\lambda) + N_q(\beta\lambda).$$

Proof. From Lemmas 3.1 and 3.3, and Theorem 3.2, we have:

$$\begin{aligned} N_{sys}(\lambda) &= \#\{k : \lambda_k \le \lambda\} \\ &\le \#\{k : \mu^k \le \lambda\} \\ &= \#\{k : \mu_k \le \lambda\} \\ &= \#\{k : \Lambda_{(p),k} \le \alpha\lambda\} + \#\{k : \Lambda_{(q),k} \le \beta\lambda\} \\ &= N_p(\alpha\lambda) + N_q(\beta\lambda) \end{aligned}$$

□

3.2. **Lower Bounds of the Spectral Counting Function.** A lower bound for $N(\lambda)$ depends on upper bounds of eigenvalues, which usually are simpler to find, by using appropriate test functions.

From [8] we have the following upper bound

$$\lambda_k \leq \frac{\Lambda_{(p),k}}{p}\Big[1 + \big(\frac{p}{q}\big)^{q+1}\big(m\Lambda_{(p),k}\big)^{(q-p)/p}\Big].$$

The formulas in Lemmas 2.1 and 2.4 show that the second term is negligible as $k \to \infty$, which gives the asymptotic formula

Proposition 3.5. *If $q \leq p$, then*

$$N_{sys}(\lambda) \geq N_p(p\lambda).$$

Proof. From the previous bound we have:

$$\begin{aligned} N_{sys}(\lambda) &= \#\{k : \lambda_k \leq \lambda\} \\ &\geq \#\{k : \Lambda_{(p),k}/p \leq \lambda\} \\ &= N_p(p\lambda) \end{aligned}$$

□

3.3. **Proof of Theorem 1.1.** The proof of part (1) follows immediately from Propositions 3.4, 3.5, and the asymptotic expansion of Lemma 2.4, which give the bounds

$$ck^q \leq \lambda_k \leq Ck^p.$$

Clearly, this proves also part (2). We refine the constants c and C by using that

$$\lambda_k \leq \frac{2}{p}\Lambda_{(p),k}$$

by inequality (1.2).

In order to prove part (3), let us note that

$$N_{sys}(\lambda) \leq N_p(p\lambda/2) + N_p(p\lambda/2) = 2N_p(p\lambda/2).$$

On the other hand, inequality (1.2) gives only

$$\lambda_k \leq \frac{2}{p}\Lambda_{(p),k},$$

that is,

$$N_{sys}(\lambda) \geq N_p(p\lambda/2).$$

The factor 2 to achieve the equality follows from the fact that each eigenpair (λ, u) of a p-laplacian equation gives two eigenpairs of the system: $(p\lambda/2, u, u)$ and $(p\lambda/2, u, -u)$, and hence we have at least twice the number of eigenvalues of only one equation.

Remark 3.6. Let us observe that part (3) seems to contradict the well known case of the bilaplacian with Navier's boundary conditions. For example, on the interval $[0, 1]$ we have:

$$\begin{cases} (u'')'' = \lambda^2 u \\ u(0) = u''(0) = 0 \\ u(1) = u''(1) = 0 \end{cases}$$

and its number of eigenvalues is $N(\lambda) \sim \left(\frac{1}{2}\lambda\right)^{1/2}$. However, $p = 2$ and $r \equiv 1$ correspond to the following system

$$\begin{cases} -u'' = \lambda sign(u)|v| \\ -v'' = \lambda sign(v)|u| \end{cases}$$

where the signs of u and v are unrelated, and now the factor 2 could be easily recovered.

To our knowledge, this is a new derivation of the number of eigenvalues of the bilaplacian, and we may estimate the eigenvalues of a $2m^{th}$-order problem by estimating the ones of a system of lower order in much the same way.

4. Proof of Theorem 1.4

First, let us note that the upper bound for $N(\lambda)$ follows as in the one dimensional case, by considering the N-dimensional version of the eigenvalue problem (3.1), defining

$$\mu^k = \inf_{C \in \mathcal{C}_k} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q}{\frac{\alpha}{p}\int_\Omega |u|^p + \frac{\beta}{q}\int_\Omega |v|^q},$$

with $C \subset W = W_0^{1,p} \times W_0^{1,q}(\Omega)$, $\mathcal{C}_k$ as in Section 2.

Renaming the sequences of eigenvalues of each equation, $S_p = \{\Lambda_{(p),k}/\alpha\}$ and $S_q = \{\Lambda_{(q),k}/\beta\}$, as $\{\mu_k\}$, where $\mu_k \in S_p \cup S_q$, and $\mu_1 \le \mu_2 \le \ldots \le \mu_k \le \ldots$, we have $\mu_k = \mu^k$.

Now, we can use the lower bounds for $\Lambda_{(p),k}$, $\Lambda_{(q),k}$ given in Lemma 2.5, which gives the upper bounds (for the eigenvalue counting functions of only one equation) $N_p(\lambda) \le (\lambda/c_p)^{N/p}$, $N_q(\lambda) \le (\lambda/c_q)^{N/q}$, and hence

$$N(\lambda) \le (\lambda/c_p)^{N/p} + (\lambda/c_q)^{N/q}.$$

On the other hand, inequality (1.2) is valid only for the first eigenvalue in the N-dimensional case. We have, since $r \equiv 1$,

$$\lambda_1 \le \frac{\Lambda_{(p),1}}{p}\Big[1 + \left(\frac{p}{q}\right)^{q+1} \Lambda_{(p),1}^{(q-p)/p}\Big].$$

In order to find a lower bound of $N(\lambda)$, we will cover the set Ω with squares Q_j of side $L = \pi_p(N/\lambda)^{1/p}$, and by applying the Dirichlet Neumann bracketing (see Proposition 2.3), we have

$$N_{sys}(\lambda) \ge \#\{j : \Omega \subset \cup_{1\le j\le J} Q_j\} \equiv J$$

due to Propositions 2.6 and 2.7. When $\lambda \to \infty$, the covering approximates the volume of Ω, and we get

$$J \sim \frac{|\Omega|\lambda^{N/p}}{(\pi_p^p N)^{N/p}}.$$

The upper bound for $N(\lambda)$ follows from inequality

$$Ck^{p/N} \le \Lambda_{(p),k},$$

that is proved in [20]. The rest of the proof of Theorem 1.4 follows in much the same way than in the one dimensional case, by using the lower bounds for $\Lambda_{(q),1}$ given in Proposition 2.7. □

5. Final Remarks

Clearly, the main open problem in this subject is a complete characterization of the spectrum of system (1.1). It is not clear that the variational sequence $\{\lambda_k\}_k$ exhausts the spectrum even in the one dimensional case. A partial step in this direction -i. e., when $N = 1$- could be a good description of the nodal sets of eigenfunctions. However, this approach cannot be used for $N > 1$.

Still, there are many interesting problems about the sequence of variational eigenvalues. It would be desirable to find better bounds that the ones in Part (1) of both Theorems 1.1 and 1.4, since for a given q the bounds becomes useless when both p or k growths.

Another question is the role played by α and β. Let us observe that they are not involved in the upper bounds except in Part (3) of Theorem 1.1, and in that case they are very particular values related to p. Hence, we may ask how the asymptotic expansion of $N(\lambda)$ reflects the general case $\alpha \neq \beta$. Also, when p or q is equal to 2, we can improve the constants and bounds since in this case the eigenvalues of only one equation are well known.

It is possible to impose less regularity conditions on r and $\partial\Omega$. In the one dimensional case, Theorem 1.1 could be extended when $\alpha = \beta = p/2$ to more general sets following the methods in [14] and [15], and also second term of $N_{sys}(\lambda)$ could be found. We omit here this extension. On the N-dimensional case, the regularity of $\partial\Omega$ is not involved, since always we can bound the eigenvalues by considering interior and exterior sets with regular boundary, and using the monotonicity of eigenvalues. Also, indefinite weights could be studied as in [15], and the coefficients involved depend on $\|r^{\pm}\|_1$, where r^+ (respectively, r^-) is the positive (respectively, negative) part of r.

A different problem arise if we remove the conditions $0 < m \leq r \leq M$. If m is allowed to be zero, we may consider a set $\Omega_\varepsilon \subset \Omega$ where $r \geq \varepsilon$ and we obtain uppers bounds for the eigenvalues on Ω by considering the ones of Ω_ε. To obtain lower bounds, it is enough to define a weight $r + \varepsilon$. In the one dimensional case, this could be improved following the arguments in [8] for a system, or in [25] for the eigenvalues of only one equation, since it is possible to find lower bounds of eigenvalues in terms of $\|r\|_1$.

Also, it is possible to find lower bounds for unbounded r in much the same way for the one dimensional case provided that $\|r\|_1$ is finite. However, the previous approach are not valid for the N-dimensional case. It would be interesting to find lower bounds for unbounded weights even for the single equation.

Acknowledgements Supported by grant TX066 from Universidad de Buenos Aires, grants 03-05009 and 03-10608 from ANPCyT PICT, project 13900-5 from Fundacion Antorchas and CONICET (Argentina). The authors wishes to thank Professor Pablo De Nápoli for many interesting conversations.

References

[1] W. Allegretto and Y-X. Huang. *A Piccone's identity for the $p-$Laplacian and applications*, Nonlinear Anal. TMA, 32 nr. 7 (1996), 143-175.

[2] H. Amann. *Lusternik-Schnirelmann theory and nonlinear eigenvalue problems*, Math. Ann., 199 (1972), 55-72

[3] C. Azizieh and Ph. Clément. *A-priori estimates and continuations methods for positive solutions of p-Laplace equations.* J. Differential Equations, 179, nr. 1 (2002), 213-245.
[4] L. Boccardo and D. G. de Figueiredo. *Some Remarks on a System of Quasilinear Elliptic Equations*, NoDEA Nonlinear Differential Equations Appl., 9 (2002), 309-323.
[5] G. Bognar and O. Dosly. *The Application of Picone-type Identity for some Nonlinear Elliptic Differential Equations*, Acta Math. Univ. Comenian.(NS) Vol. LXXII, nr. 1 (2003), 4557.
[6] F. Browder. *Infinite Dimensional Manifolds and Non-Linear Elliptic Eigenvalue Problems*, Ann. of Math., 82 nr. 3 (1965), 459-477.
[7] P. de Napoli and C. Marianni. *Quasilinear Elliptic Systems of Resonant Type and Nonlinear Eigenvalue Problems*, Abstr. Appl. Anal., 7 nr. 3 (2002), 155-167
[8] P. de Napoli and J. P. Pinasco, *Estimates for Eigenvalues of Quasilinear Elliptic Systems*, J. Differential Equations, 227 nr. 1 (2006), 102-115.
[9] de Thélin, F. *Première valeur propre d'un système elliptique non linéaire*, Rev. Mat. Apl. 13, nr. 1 (1992), 1-8. See also C. R. Acad. Sci., Paris, Ser. I 311, nr. 10 (1990), 603-606.
[10] P. Drabek and R. Manásevich, *On the Closed Solutions to some nonhomegeneous eigenvalue problemes with p-laplacian*, Differential Integral Equations, 12 nr. 6 (1999), 773-788.
[11] P. Drabek, N. M. Stavrakakis and N. B. Zographopoulos, *Multiple Nonsemitrivial Solutions for Quasilinear Elliptic Systems*, Differential Integral Equations, 16, nr. 12 (2003), 15191531.
[12] P. Felmer, R. Manásevich, and F.de Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, Commun. Partial Differential Equations, 17 (1992), 2013-2029.
[13] J. Fernández Bonder. *Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities*, Abstr. Appl. Anal. 2004, nr. 12 (2004), 1047-1056.
[14] J. Fernández Bonder and J. P. Pinasco. *Asymptotic Behavior of the Eigenvalues of the One Dimensional Weighted p- Laplace Operator*, Ark. Mat., 41 (2003), 267-280.
[15] J. Fernández Bonder and J.P. Pinasco. *Eigenvalues of the p-laplacian in fractal strings with indefinite weights*, J. Math. Anal. Appl., 308, no. 2 (2005), 764-774.
[16] J. Fleckinger-Pellé, J.-P. Gossez, P. Takác, and F. de Thélin. *Nonexistence of solutions and an anti-maximum principle for cooperative systems with the p-Laplacian*, Math. Nachr. 194 (1998), 49-78.
[17] J. Fleckinger, E. M. Harrell II, and F. de Thélin. *Boundary behavior and estimates for solutions of equations containing the p-laplacian*, Electron. J. Differential Equations, Vol. 1999, nr. 38 (1999), 1-20.
[18] J. Fleckinger, R.F. Manásevich, N.M. Stravrakakis, and F. de Thélin. *Principal Eigenvalues for Some Quasilinear Elliptic Equations on* $\mathbb{R}^n$, Adv. Differential Equations, 2, nr. 6 (1997), 981-1003.
[19] L. Friedlander. *Asymptotic behaviour of eigenvalues of the p-Laplacian*, Commun. Partial Differential Equations, 14, nr. 8/9 (1989), 1059-1069.
[20] J. Garcia Azorero and I. Peral Alonso. *Comportement asymptotique des valeurs propres du p-laplacien* , C. R. Acad. Sci. Paris, Ser. I, 307 (1988), 75-78.
[21] D. A. Kandilakis, M. Magiropoulos, and N. B. Zographopoulos. *The first eigenvalue of p-Laplacian systems with nonlinear boundary conditions*, Boundary Value Problems, 2005, nr. 3 (2005), 307-321.
[22] R. Manásevich and J. Mawhin. *The spectrum of p-Laplacian Systems with various boundary Conditions and Applications*, Adv. Differential Equations, 5, nr. 10-12 (2000), 1289-1318.
[23] J. P. Pinasco, *On the Asymptotic Behavior of Eigenvalues of the Radial p-laplacian*, Manuscripta Math., 117, nr. 3 (2005), 363-371.
[24] J. P. Pinasco. *The Asymptotic behavior of Nonlinear Eigenvalues*, to appear in Rocky Mountain J. Math.
[25] J. P. Pinasco. *Comparison of Eigenvalues for the p-Laplacian with Integral Inequalities*, to appear in Appl. Math. Comput.
[26] R. C. Riddell. *Nonlinear Eigenvalue Problems and Spherical Fibrations of Banach Spaces*, J. Functional Anal., 18 (1975), 213-270.
[27] N. M. Stavrakakis and N. B. Zographopoulos. *Existence results for quasilinear elliptic systems in* $\mathbb{R}^N$, Electron. J. Differential Equations, 1999 nr. 39 (1999), 1-15.

[28] N. M. Stavrakakis and N. B. Zographopoulos. *Bifurcation Results for some Quasilinear Elliptic Systems on* $\mathbb{R}^N$, Adv. Differential Equations, 8, nr. 3 (2003), 315-336.
[29] J. Vélin, F. de Thélin. *Existence and nonexistence of nontrivial solutions for some nonlinear elliptic systems*, Rev. Mat. Complut., 6, nr. 1 (1993), 153194.
[30] N. Zographopoulos. *p-Laplacian Systems on Resonance*, Appl. Anal., 83, nr. 5 (2004), 509-519.
[31] N. Zographopoulos. *On the isolation of the principal eigenvalue for a p-laplacian systems*, to appear in Appl. Math. Lett.

Julián Fernández Bonder
FCEyN - Departamento de Matemática,
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I
(1428) Buenos Aires, Argentina.
e-mail: `jfbonder@dm.uba.ar`

Juan P. Pinasco
Instituto de Ciencias,
Universidad Nacional de General Sarmiento
J.M. Gutierrez 1150
Los Polvorines (1613)
Buenos Aires, Argentina.
e-mail: `jpinasco@ungs.edu.ar`