PRECISE ASYMPTOTIC OF EIGENVALUES OF RESONANT QUASILINEAR SYSTEMS

JULIÁN FERNÁNDEZ BONDER AND JUAN P. PINASCO

Abstract. In this work we study the sequence of variational eigenvalues of a system of resonant type involving $p-$ and $q-$laplacians on $\Omega \subset \mathbb{R}^N$, with a coupling term depending on two parameters α and β satisfying $\alpha/p + \beta/q = 1$. We show that the order of growth of the k^{th} eigenvalue depends on $\alpha + \beta$, $\lambda_k = O(k^{\frac{\alpha+\beta}{N}})$.

1. Introduction

This paper is devoted to the study of the asymptotic behavior of eigenvalues of resonant quasilinear systems

$$\begin{cases} -\Delta_p u = \lambda\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -\Delta_q v = \lambda\beta|u|^{\alpha}|v|^{\beta-2}v, \end{cases} \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u(x) = v(x) = 0 \qquad \text{on } \partial\Omega. \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, the $s-$laplacian operator is $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$, the exponents satisfy $1 < p, q < +\infty$, and the positive parameters α, β satisfy

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1. \tag{1.3}$$

The study of resonant systems has deserved a great deal of attention in the last years, and we may cite the works of Boccardo and de Figueiredo [9], Manasevich and Mawhin [26], Felmer, Manasevich and de Thèlin [17], Stavrakakis and Zographopoulos [31], among several others.

In several applications, such as bifurcation problems, anti-maximum principles, and existence or non-existence of solutions (see for example [4, 16, 17, 22, 31, 32, 33]) it is desirable to have precise bounds on the eigenvalues. In general this information is not well understood for elliptic systems, except for the first or principal eigenvalue. Several properties of this first eigenvalue were analyzed (existence, uniqueness, positivity, and isolation in bounded or unbounded domains, with different boundary conditions and with or without weights) and we refer the interested reader to [1, 13, 18, 25, 33] among others.

Key words and phrases. elliptic system, p laplacian, eigenvalue bounds,
2000 *Mathematics Subject Classification.* 35P30, 35P15, 34L30, 34L15.

Let us recall briefly that the existence of a sequence of variational eigenvalues for problem (1.1)-(1.2) was proved in [11], and the values λ_k are defined as

$$\lambda_k := \inf_{C\in\mathcal{C}_k} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx}{\int_\Omega |u|^\alpha |v|^\beta\,dx},$$

where $\mathcal{C}_k$ is the class of compact symmetric ($C=-C$) subsets of $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ of (Krasnoselskii) genus greater or equal than k.

Throughout this work, the eigenvalues are counted repeated according to their multiplicity. We say that λ_k has multiplicity r if $\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r-1} < \lambda_{k+r}$. In this case, it is a well know fact that the set of eigenfunctions corresponding to λ_k has genus greater or equal than r (see, for instance, [24]).

In the case of a single equation the existence of a sequence of variational eigenvalues together with the correct order of growth for these eigenvalues was first obtained by Garcia-Azorero and Peral in [24]. The constants in the asymptotic behavior were improved by Friedlander in [23]. Let us note that for the one-dimensional problem, these bounds can be refined, see [19, 20].

As for elliptic systems, the asymptotic growth of the eigenvalues is less understood even in the linear case. We may cite here the work of Protter [30], and also the works of Cantrell and Cosner [5, 6, 7] were lower bounds for the first eigenvalue were obtained. The exception for the lack of results in this direction comes from linear and nonlinear elasticity theory (see the survey of Antman [3]).

For nonlinear elliptic systems, up to our knowledge, the first work where this problem was addressed was [12] where we obtained a generalization of the Lyapunov inequality together with an upper bound of the variational eigenvalues in terms of the ones of a single p-laplacian equation for the one dimensional case. Later, in [21] we obtained lower and upper bounds for the eigenvalues of problem (1.1)-(1.2) in terms of the eigenvalues of a single p-laplacian and q-laplacian equations in any dimension $N \ge 1$. More precisely, for each k we prove that

$$c\left(\frac{k}{|\Omega|}\right)^{\frac{q}{N}} \le \lambda_k \le C\left(\frac{k}{|\Omega|}\right)^{\frac{p}{N}}$$

for suitable positive constants c, C depending on p and q.

We refer the interested reader to the introductions of [12, 21] for more information and references about the eigenvalues of quasilinear elliptic equations and resonant systems.

However, our previous bounds fail to reflect the coupling strength of the system which is given by the parameters α and β. Formally, by taking $\alpha = p$ and $\beta = 0$, we obtain a single p-laplacian equation replacing the system,

$$-\Delta_p u = \lambda p |u|^{p-2} u,$$

and similarly, for $\alpha = 0$ and $\beta = q$ we have

$$-\Delta_q v = \lambda q |v|^{q-2} v.$$

Hence, the order of growth of our upper (resp., lower) bound given in [21] is sharp for the case $\alpha = p$ (resp., $\beta = q$), since coincides with the true upper (resp., lower) order of growth of the eigenvalues (see [24]). On the other hand, both orders does not hold simultaneously even for those limit cases.

We can suspect that there exists a smooth transition for the order of growth of the eigenvalues between both limiting cases, and the main result of this work is to prove it. Namely, our main theorem is

Theorem 1.1. *Let $\{\lambda_k\}_k$ be the sequence of variational eigenvalues of problem (1.1)-(1.2). Then, there exist positive constants $c < C$ depending on p, q and $\Omega \subset \mathbb{R}^N$, such that*

$$c\left(\frac{k}{|\Omega|}\right)^{\frac{\alpha+\beta}{N}} \leq \lambda_k \leq C\left(\frac{k}{|\Omega|}\right)^{\frac{\alpha+\beta}{N}}.$$

Observe that if one consider linear operators, or even the same $p-$laplace operator in both equations (i.e. $p = q$), the coupling parameters α and β are not reflected in the asymptotics of the eigenvalues since $\alpha + \beta = p$ in this case. We believe that this fact may be the reason why this phenomenum was not discovered earlier.

The proof of Theorem 1.1 follows directly from Weyl-type bounds for the spectral counting function $N(\lambda)$ which gives the number of eigenvalues less than a given value, that is

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\}.$$

Theorem 1.1 is equivalent to the following asymptotic bound for $N(\lambda)$:

$$C^{-1}\lambda^{\frac{N}{\alpha+\beta}} \leq N(\lambda) \leq c^{-1}\lambda^{\frac{N}{\alpha+\beta}}.$$

Up to our knowledge, this is the first case in the literature where the coupling parameters of an elliptic system appear explicitly modifying the power on the asymptotic order of growth of the eigenvalues. For example, in linear second order problems on domains with parts of different dimensions, $\Omega = \Omega^{(N)} \cup \Omega^{(n)}$ with $n < N$ like the ones considered in [10], we always have

$$c(\Omega^{(N)})\lambda^{N/2} \leq N(\lambda) \leq C(\Omega^{(N)})\lambda^{N/2},$$

and the influence of the domain $\Omega^{(n)}$ appears only as a correction factor of lower order (see [28] for details). Also, in Steklov-like eigenvalue problems where the eigenvalue parameter appears both in the equation and the boundary condition in a domain $\Omega \subset \mathbb{R}^N$ (see for example [27] for $p = 2$, and [29] for $1 < p < \infty$),

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u \\ \quad |\frac{\partial u}{\partial \nu}|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u, \end{cases}$$

which can be thought as a system involving the laplacian and the Dirichlet-to-Neumann map, we have $N(\lambda) = O(\lambda^m)$, where the order of growth of $N(\lambda)$ is given by

$$m = \max\left\{\frac{N-1}{p-1}, \frac{N}{p}\right\}.$$

1.1. **Organization of the paper.** The paper is organized as follows. In Section §2 we review some facts about the eigenvalue problem for the single s-laplacian equation

$$-(|u'|^{s-2}u')' = |u|^{s-2}u,$$

which can be found for example in [14], or [15].

Also, we will consider the eigenvalue problem involving the pseudo $p-$laplacian operator

$$\hat{\Delta}_p = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\right),$$

and we recall some bounds proved in [21].

In Section §3 we prove Theorem 1.1 by using a scaling argument as in [23]. The main drawback of this approach is the fact that the constant on the asymptotic expansion remains unknown since they depend on the first Dirichlet eigenvalue and the second Neumann eigenvalue of problem (1.1)-(1.2) when Ω is the unit cube Q_1.

So, we are left with the problem of find lower and upper bounds for those eigenvalues, which is a problem of independent interest. Thus, we consider the one-dimensional problem

$$\left\{\begin{array}{rcl} -(|u'(x)|^{p-2}u'(x))' &=& \lambda\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -(|v'(x)|^{q-2}v'(x))' &=& \lambda\beta|u|^{\alpha}|v|^{\beta-2}v \end{array}\right. \tag{1.4}$$

on the interval (a, b), and we will focus on the Dirichlet boundary condition.

In Section §4 we prove the following lower bound for the first Dirichlet eigenvalue $\lambda_1(p, q)$ of the one dimensional problem (1.4):

Theorem 1.2. *Let $\Lambda_1(\alpha+\beta)$ be the first Dirichlet eigenvalue of*

$$-(|\varphi'|^{\alpha+\beta-2}\varphi')' = \Lambda|\varphi|^{\alpha+\beta-2}\varphi$$

on (a, b). Then,

$$\left(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{q}}\right)^{-1}\left(\frac{2}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta}\frac{\Lambda_1(\alpha+\beta)}{\alpha+\beta-1} \le \lambda_1(p, q).$$

The proof follows by using the Lyapunov inequality obtained in [12].

In [12] we obtained an upper bound of the first eigenvalue of the one dimensional problem (1.4) in terms of the first eigenvalue of the single p-laplacian equation. Moreover, upper bounds were obtained for all the variational eigenvalues, namely

$$\lambda_k \le \frac{\Lambda_k(p)}{p}\Big[1 + \left(\frac{p}{q}\right)^{q+1}\left(\Lambda_k(p)\right)^{(q-p)/p}\Big]. \tag{1.5}$$

Here, $\Lambda_k(p)$ stands for the k^{th} eigenvalue of the p-laplacian. Let us note that (1.5) holds for the one dimensional problem. In the $N-$dimensional case, (1.5) holds only for the first eigenvalue.

In this paper we improve this explicit upper bound for the first eigenvalue, and we use it to obtain asymptotic bounds for the k^{th} eigenvalue of a system in $\Omega \subset \mathbb{R}^N$ depending on the eigenvalues of the $\alpha+\beta-$laplacian.

Section §5 is devoted to the proof of the following upper bound of the first Dirichlet eigenvalue of problem (1.4):

Theorem 1.3. *Let $\Lambda_1(\alpha+\beta)$ be the first Dirichlet eigenvalue of equation*

$$-(|\varphi'|^{\alpha+\beta-2}\varphi')' = \Lambda|\varphi|^{\alpha+\beta-2}\varphi$$

on (a,b). Then,

$$\lambda_1(p,q) \leq \frac{\Lambda_1(\alpha+\beta)}{\alpha+\beta-1}\left[\frac{1}{p}+\frac{1}{q}\left(\frac{\pi_{\alpha+\beta}}{\int_0^{\pi_{\alpha+\beta}}\sin_{\alpha+\beta}^{\alpha+\beta}(t)dt}\right)^{1-\frac{q}{\alpha+\beta}}\right].$$

Here $\sin_s(x)$ is defined implicitely as

$$x = \int_0^{\sin_s(x)} \frac{dt}{(1-t^s)^{1/s}}$$

and π_s is given by

$$\pi_s = 2\int_0^1 \frac{dt}{(1-t^s)^{1/s}}.$$

See section §2 for more con these and see also the paper [14].

Finally, in Section §6 we close the paper with asymptotic upper bounds for the higher eigenvalues of system (1.1) by using the bounds obtained in Theorems 1.1 and 1.3, and some facts about eigenvalue problems involving the pseudo $p-$laplacian. We show that

$$\lambda_k(p,q) \leq c\lambda_k(\alpha+\beta),$$

for a fixed constant c depending only on α, β, and Ω. Also, we discuss the possibility of finding better estimates.

2. Some known facts

In this Section we recall some previous results which will be needed in the rest of the paper.

2.1. **Variational setting.** The variational characterization of eigenvalues follows from the abstract theory developed by Amman (see [2]). In [15] the authors showed the existence of infinitely many eigenpairs, i. e. $(u,v) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi + |\nabla v|^{q-2}\nabla v\nabla\psi\,dx = \lambda\int_\Omega(\alpha|u|^{\alpha-2}u\phi|v|^\beta + \beta|u|^\alpha|v|^{\beta-2}v\psi)\,dx$$

for any test-function pair $(\phi,\psi) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

It is convenient to work with the variational characterization of the eigenvalues, defined through the Rayleigh quotient,

$$\lambda_k = \inf_{C\in\mathcal{C}_k}\sup_{(u,v)\in C}\frac{\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{q}\int_\Omega|\nabla v|^q\,dx}{\int_\Omega|u|^\alpha|v|^\beta\,dx}, \tag{2.1}$$

where $\mathcal{C}_k$ is the class of compact symmetric ($C=-C$) subsets of $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ of (Krasnoselskii) genus greater or equal that k.

2.2. **One dimensional case.** For the one dimensional $s-$laplacian in $\Omega = [a, b]$

$$-(|u'|^{s-2}u')' = \Lambda |u|^{s-2}u \tag{2.2}$$

with Dirichlet boundary conditions, we have

$$\Lambda_k(s) = \inf_{C\in\mathcal{C}_k} \sup_{u\in C} \frac{\displaystyle\int_a^b |u'|^s\, dx}{\displaystyle\int_a^b |u|^s\, dx}, \tag{2.3}$$

with $u \in W_0^{1,s}(a, b)$.

Here, all the eigenvalues and eigenfunctions can be found explicitly (see [15]):

Theorem 2.1 (Del Pino, Drabek and Manasevich, [14])**.** *The eigenvalues $\Lambda_k(s)$ and eigenfunctions $u_{s,k}$ of equation* (2.2) *on the interval* $[0, L]$ *are given by*

$$\Lambda_k(s) = (s-1)\frac{\pi_s^s k^s}{L^s},$$

$$u_{s,k}(x) = \sin_s(\pi_s kx/L).$$

Remark 2.2. It was proved in [15] that they coincide with the variational eigenvalues given by equation (2.3). However, let us observe that the notation is different in both papers.

The function $\sin_s(x)$ is the solution of the initial value problem

$$-(|u'|^{s-2}u')' = (s-1)|u|^{s-2}u$$

$$u(0) = 0, \qquad u'(0) = 1,$$

and is defined implicitly as

$$x = \int_0^{\sin_s(x)} \frac{dt}{(1-t^s)^{1/s}}.$$

Moreover, its first zero is π_s, given by

$$\pi_s = 2\int_0^1 \frac{dt}{(1-t^s)^{1/s}}.$$

Let us note that both $\sin_s$ and $\sin_s'$ satisfy

$$|\sin_s| \le 1, \quad |\sin_s'| \le 1,$$

due to the Pythagorean like identity

$$|\sin_s|^s + |\sin_s'|^s = 1. \tag{2.4}$$

Finally, let us observe that the following integral is a constant depending only on s:

$$\int_0^{\pi_s} \sin_s^s(t)\, dt = K(s).$$

2.3. **The Spectral Counting Function.** Given the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$, we introduce the spectral counting function $N(\lambda)$ defined as

$$N(\lambda) = \#\{k\colon \lambda_k \leq \lambda\}.$$

To avoid confusion, we will use $N_{sys}(\lambda)$ or $N_s(\lambda)$ to denote the eigenvalue counting functions of the system and the $s-$laplacian respectively. If necessary, we will write $N(\lambda,\Omega)$ to denote explicitly the set Ω where the eigenvalue problem is considered, and even $N^D(\lambda)$ or $N^N(\lambda)$ to indicate the Dirichlet and Neumann boundary conditions.

The main tool in order to obtain the asymptotic expansion of $N(\lambda)$ is the classical Dirichlet-Neumann bracketing introduced by Courant [8]. The following proposition can be found in [19]:

Proposition 2.3 ([19], Theorem 2.1)**.** *Let $U_1, U_2 \in \mathbb{R}^N$ be disjoint open sets such that $(\overline{U_1\cup U_2})^{int} = U$ and $|U\setminus U_1\cup U_2| = 0$. Then,*

$$\begin{aligned} N^D(\lambda,U_1) + N^D(\lambda,U_2) &= N^D(\lambda,U_1\cup U_2) \\ &\leq N^D(\lambda,U) \\ &\leq N^N(\lambda,U) \\ &\leq N^N(\lambda,U_1\cup U_2) \\ &= N^N(\lambda,U_1) + N^N(\lambda,U_2). \end{aligned}$$

2.4. **The pseudo $p-$laplacian.** We will use the first eigenvalue $\nu_{p,1}$ of the pseudo $p-$laplacian on a cube Q_L of side of length L in order to bound the first eigenvalue of the $p-$laplacian on the same domain.

Proposition 2.4. *Let $Q_L \subset \mathbb{R}^N$, and $\Lambda_1(p)$, be the first eigenvalues of the p-laplacian in Q_L. Then,*

$$\begin{aligned} \frac{\pi_p^p N}{L^p} &\leq \Lambda_1(p) \leq \frac{\pi_p^p N^{p/2}}{L^p} &&\text{if } 2<p, \\ \frac{\pi_p^p N^{p/2}}{L^p} &\leq \Lambda_1(p) \leq \frac{\pi_p^p N}{L^p} &&\text{if } p<2. \end{aligned}$$

Proof. We only sketch the proof here, see [21], Proposition 2.7 for the details. For any $x\in\mathbb{R}^N$, we have $|x|_q \leq C_p|x|_p$ where $C_p = 1$ if $p\leq q$, and $C_p = N^{(p-q)/2q}$ if $p \geq q$. Hence,

$$\nu_{p,1} = \inf_{u\in W_0^{1,p}} \frac{\|\,|\nabla u|_p\|_p^p}{\|u\|_{L^p}}; \qquad \Lambda_1(p) = \inf_{u\in W_0^{1,p}} \frac{\|\,|\nabla u|_2\|_p^p}{\|u\|_{L^p}}.$$

The previous norm inequality gives

$$\begin{aligned} \nu_{p,1} &\leq \Lambda_1(p) \leq N^{(p-2)/2}\nu_{p,1} &&\text{if } 2<p, \\ N^{(p-2)/2}\nu_{p,1} &\leq \Lambda_1(p) \leq \nu_{p,1} &&\text{if } p<2, \end{aligned}$$

and the result follows since the first eigenpair of the pseudo $p-$laplacian on Q_L is

$$\nu_{p,1} = \frac{\pi_p^p N}{L^p},$$
$$u_{p,1} = \sin_p(\pi_p x_1/L)\cdots\sin_p(\pi_p x_N/L).$$

That is, the first eigenvalue is N times the one dimensional eigenvalue. □

3. Estimates for the Spectral Counting Function

Let us begin with the following scaling argument. We denote Q_1 the unit cube in $\mathbb{R}^N$, and Q_t, the scaled cube of side of length t.

Lemma 3.1. *Let $\lambda_1^1(p,q)$ (resp., $\lambda_1^t(p,q)$) be the first Dirichlet eigenvalue of problem 1.1 when $\Omega = Q_1$ (resp., $\Omega = Q_t$). Then,*

$$t^{\alpha+\beta}\lambda_1^t(p,q) = \lambda_1^1(p,q).$$

Proof. We have

$$\lambda_1^1(p,q) = \frac{\displaystyle\frac{1}{p}\int_{Q_1} |\nabla u|^p\,dx + \frac{1}{q}\int_{Q_1} |\nabla v|^q\,dx}{\displaystyle\int_{Q_1} |u|^\alpha |v|^\beta\,dx},$$

where (u,v) is an eigenfunction corresponding to $\lambda_1^1(p,q)$. Now, we choose the functions

$$u_t = tu_1\left(\frac{x}{t}\right), \qquad v_t = tv_1\left(\frac{x}{t}\right).$$

Clearly, $(u_t,v_t) \in W_0^{1,p}(Q_t)\times W_0^{1,q}(Q_t)$, and can be used as test functions in the variational characterization (2.1) for $\lambda_1^t(p,q)$. Then,

$$\lambda_1^t(p,q) \le \frac{\displaystyle\frac{1}{p}\int_{Q_t} |\nabla u_t|^p\,dx + \frac{1}{q}\int_{Q_t} |\nabla v_t|^q\,dx}{\displaystyle\int_{Q_t} |u_t|^\alpha |v_t|^\beta\,dx}.$$

Changing variables $s = x/t$ we obtain

$$\begin{aligned}
\lambda_1^t(p,q) \quad &\le \frac{\displaystyle\frac{1}{p}\int_{Q_t} |\nabla u_t|^p\,dx + \frac{1}{q}\int_{Q_t} |\nabla v_t|^q\,dx}{\displaystyle\int_{Q_t} |u_t|^\alpha |v_t|^\beta\,dx} \\
&= \frac{\displaystyle\frac{1}{p}\int_{Q_1} |\nabla u|^p\,tds + \frac{1}{q}\int_{Q_1} |\nabla v|^q\,tds}{\displaystyle\int_{Q_1} t^{\alpha+\beta}|u|^\alpha |v|^\beta\,tds} \qquad (3.1)\\
&= \frac{\lambda_1^1(p,q)}{t^{\alpha+\beta}}
\end{aligned}$$

To obtain the other inequality we repeat the same argument by choosing an eigenfunction (u_t,v_t) corresponding to $\lambda_1^t(p,q)$. □

Remark 3.2. In [1] the authors proved that the first eigenvalue of Problem (1.1) has an eigenfunction (u,v) satisfying $u>0$, $v>0$ in Ω. For any other eigenvalue, at least one of the eigenfunctions changes sign. However, let us note that both (u,v) and $(u,-v)$ are eigenfunctions corresponding to the first eigenvalue. By using this argument it is possible to show that the scaled functions (u_t,v_t) are the first eigenfunctions on Q_t.

The following lemma is similar to the previous one. However, several differences arise. The main one is that we cannot choose the second Neumann eigenfunction and use it as a test function, due to the minimax characterization. Let us note also that the first eigenvalue is $\mu_1 = 0$, with constant associated eigenfunctions.

Lemma 3.3. *Let $\mu_2^1(p,q)$ (resp., $\mu_2^t(p,q)$) be the second Neumann eigenvalue of problem* (1.1) *when $\Omega = Q_1$ (resp., $\Omega = Q_t$). Then,*

$$t^{\alpha+\beta}\mu_2^t(p,q) = \mu_2^1(p,q).$$

Proof. Instead of the second eigenfunction, we choose a compact symmetric set C of genus greater than or equal to 2 in $W_0^{1,p}(Q_1) \times W_0^{1,q}(Q_1)$. Now, we scale the functions as before, and then we obtain a set $C_t \subset W_0^{1,p}(Q_t) \times W_0^{1,q}(Q_t)$ of genus greater than or equal to 2. Since

$$\sup_{(u,v)\in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx}{\int_\Omega |u|^\alpha |v|^\beta\,dx} = \sup_{(u_t,v_t)\in C_t} \frac{\frac{1}{p}\int_\Omega |\nabla u_t|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v_t|^q\,dx}{\int_\Omega |u_t|^\alpha |v_t|^\beta\,dx},$$

changing variables again we get,

$$\begin{aligned} \mu_2^t(p,q) &\le \inf_{C_t\in\mathcal{C}_2} \sup_{(u_t,v_t)\in C_t} \frac{\frac{1}{p}\int_\Omega |\nabla u_t|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v_t|^q\,dx}{\int_\Omega |u_t|^\alpha |v_t|^\beta\,dx} \\ &= \inf_{C\in\mathcal{C}_2} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx}{\int_\Omega |u|^\alpha |v|^\beta\,dx} \qquad (3.2) \\ &= \mu_2^1(p,q). \end{aligned}$$

In order to obtain the reverse inequality, we take a compact symmetric set $C \subset W_0^{1,p}(Q_t) \times W_0^{1,q}(Q_t)$ of genus greater than or equal to 2, and we apply the same argument. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. For any λ fixed, we take a lattice of cubes of side length $t \ll 1$ in $\mathbb{R}^N$ with t depending on λ.

First we derive a lower bound for $N(\lambda)$ (equivalently, an upper bound for λ_k). From Lemma 3.1, $\lambda_1^t(p,q) = t^{-(\alpha+\beta)}\lambda_1^1(p,q)$, and taking

$$t = \left(\frac{\lambda}{\lambda_1^1(p,q)}\right)^{-\frac{1}{\alpha+\beta}},$$

we have

$$\lambda_1^t(p,q) = \lambda$$

Hence, each cube has at least two eigenvalues lower than or equal to λ. By using the Dirichlet Neumann bracketing, a lower bound for $N(\lambda)$ is given by $2K$, where

K is the number of cubes of the lattice contained in Ω. Since

$$t^N K \to |\Omega|$$

when $t \to 0$, we have

$$N(\lambda) \geq 2K \sim \frac{2|\Omega|}{t^N} = 2|\Omega| \left(\frac{\lambda}{\lambda_1^1(p,q)} \right)^{\frac{N}{\alpha+\beta}}.$$

The upper bound for $\lambda_k(p,q)$ follows since

$$k = N(\lambda_k(p,q)) \geq \frac{2|\Omega|}{\lambda_1^1(p,q)^{\frac{N}{\alpha+\beta}}} \lambda_k(p,q)^{\frac{N}{\alpha+\beta}}.$$

Let us find an upper bound for $N(\lambda)$. We use the bound for the second Neumann eigenvalue proved in Lemma 3.3, $\mu_2^t(p,q) = t^{-(\alpha+\beta)}\mu_2^1(p,q)$. Hence, taking

$$t = \left(\frac{c\lambda}{\mu_2^1(p,q)} \right)^{-\frac{1}{\alpha+\beta}},$$

for any $c > 1$, we have

$$\mu_2^t(p,q) = c\lambda > \lambda.$$

Therefore, each cube has at most two eigenvalues lower than or equal to λ. By using again the Dirichlet Neumann bracketing, an upper bound for $N(\lambda)$ is given by $2K$, where K is the number of cubes covering Ω. Since

$$t^N K \to |\Omega|$$

when $t \to 0$, we have

$$N(\lambda) \leq 2K \sim \frac{2|\Omega|}{t^N} = 2|\Omega| \left(\frac{c\lambda}{\mu_2^1(p,q)} \right)^{\frac{N}{\alpha+\beta}}.$$

The lower bound for $\lambda_k(p,q)$ follows since

$$k = N(\lambda_k(p,q)) \leq \frac{2|\Omega|}{\mu_2^1(p,q)^{\frac{N}{\alpha+\beta}}} (c\lambda_k(p,q))^{\frac{N}{\alpha+\beta}}.$$

The Theorem is proved. □

4. A Lower Bound for the First Eigenvalue

Let us prove now Theorem 1.2. We use the following Lyapunov inequality for systems proved in Part I:

Theorem 4.1 ([12], Theorem 1.5)**.** *Let us assume that there exists a positive solution of the system*

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' &= f(x)|u|^{\alpha-2}u|v|^{\beta} \\ -(|v'(x)|^{q-2}v'(x))' &= g(x)|u|^{\alpha}|v|^{\beta-2}v \end{cases}$$

on the interval (a,b), with Dirichlet boundary conditions. Then, we have that:

$$2^{\alpha+\beta} \leq (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}} \left(\int_a^b f(x)\,dx \right)^{\frac{\alpha}{p}} \left(\int_a^b g(x)\,dx \right)^{\frac{\beta}{q}} \tag{4.1}$$

This result gives Theorem 1.2 after replacing $f(x) = \alpha\lambda_1(p,q)$ and $g(x) = \beta\lambda_1(p,q)$. We have

$$2^{\alpha+\beta} \leq \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{q}} (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}+\frac{\alpha}{p}+\frac{\beta}{q}} \lambda_1(p,q), \tag{4.2}$$

and let us note that

$$\frac{\alpha}{p'} + \frac{\beta}{q'} + \frac{\alpha}{p} + \frac{\beta}{q} = \alpha\left(\frac{1}{p}+\frac{1}{p'}\right) + \beta\left(\frac{1}{q}+\frac{1}{q'}\right) = \alpha+\beta.$$

So,

$$2^{\alpha+\beta} \leq \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{q}} (b-a)^{\alpha+\beta} \lambda_1(p,q).$$

The desired result follows from this inequality and the explicit formula for $\Lambda_1(\alpha+\beta)$ in Theorem 2.1, since:

$$\frac{2^{\alpha+\beta}}{(b-a)^{\alpha+\beta}} \frac{\alpha+\beta-1}{\alpha+\beta-1} \left(\frac{\pi_{\alpha+\beta}}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta} = \frac{1}{\alpha+\beta-1}\left(\frac{2}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta} \Lambda_1(\alpha+\beta),$$

and therefore

$$\frac{1}{\alpha+\beta-1}\left(\frac{2}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta} \Lambda_1(\alpha+\beta) \leq \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{q}} \lambda_1(p,q).$$

This completes the proof of Theorem 1.2. □

5. An Upper Bound for the First Eigenvalue

Let us prove now Theorem 1.3. For the upper bound of the first eigenvalue, we need to improve the bound given in [12].

Given the variational characterization of the first eigenvalue,

$$\lambda_1(p,q) \leq \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q}\int_a^b |v'|^q}{\int_a^b |u|^\alpha |v|^\beta}$$

we choose $u = v = \varphi_1$, which is a multiple of the first Dirichlet eigenfunction of the single equation

$$-(|w'|^{\alpha+\beta-2}w')' = \Lambda |w|^{\alpha+\beta-2} w,$$

that is

$$\varphi_1(x) = \frac{b-a}{\pi_{\alpha+\beta}} \sin_{\alpha+\beta}\left(\pi_{\alpha+\beta}\frac{x-a}{b-a}\right).$$

Due to the Pythagorean-like identity (2.4) we have

$$|\varphi_1'| \leq 1.$$

Replacing u and v in the Rayleigh quotient we get

$$\lambda_1(p,q) \leq \frac{\frac{1}{p}\int_a^b |\varphi_1'|^p + \frac{1}{q}\int_a^b |\varphi_1'|^q}{\int_a^b |\varphi_1|^{\alpha+\beta}}$$

Now, since $\alpha+\beta < p$, and $|\varphi_1'| \leq 1$, the inequality

$$|\varphi_1'|^p \leq |\varphi_1'|^{\alpha+\beta}$$

holds. On the other hand, by using Hölder inequality we obtain

$$\int_a^b |\varphi_1'|^q \le \left(\int_a^b |\varphi_1'|^{\alpha+\beta} \right)^{\frac{q}{\alpha+\beta}} (b-a)^{1-\frac{q}{\alpha+\beta}}$$

We have

$$\begin{aligned} \lambda_1(p,q) &\le \frac{1}{p} \frac{\int_a^b |\varphi_1'|^{\alpha+\beta}}{\int_a^b |\varphi_1|^{\alpha+\beta}} + \frac{1}{q} \frac{\left(\int_a^b |\varphi_1'|^{\alpha+\beta}\right)^{\frac{q}{\alpha+\beta}} (b-a)^{1-\frac{q}{\alpha+\beta}}}{\int_a^b |\varphi_1|^{\alpha+\beta}} \\ &\le \frac{1}{p}\Lambda_1(\alpha+\beta) + \frac{1}{q}\Lambda_1(\alpha+\beta)^{\frac{q}{\alpha+\beta}} \frac{(b-a)^{1-\frac{q}{\alpha+\beta}}}{\left(\int_a^b |\varphi_1|^{\alpha+\beta}\right)^{1-\frac{q}{\alpha+\beta}}} \end{aligned}$$

Now we need an upper bound for

$$\left(\frac{b-a}{\int_a^b |\varphi_1|^{\alpha+\beta}} \right)^{1-\frac{q}{\alpha+\beta}}.$$

Indeed, we can compute the integral explicitly, obtaining

$$\begin{aligned} \int_a^b |\varphi_1|^{\alpha+\beta}\,dx &= \left(\frac{b-a}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta} \int_a^b \sin_{\alpha+\beta}^{\alpha+\beta}\left(\pi_{\alpha+\beta}\frac{x-a}{b-a}\right)dx \\ &= \left(\frac{b-a}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta+1} \int_0^{\pi_{\alpha+\beta}} \sin_{\alpha+\beta}^{\alpha+\beta}(t)\,dt \\ &= \left(\frac{b-a}{\pi_{\alpha+\beta}}\right)^{\alpha+\beta+1} K(\alpha+\beta). \end{aligned}$$

For brevity, let us call

$$\hat\Lambda_1(\alpha+\beta) = \frac{\Lambda_1(\alpha+\beta)}{\alpha+\beta-1}.$$

Then,

$$\lambda_1(p,q) \le \frac{1}{p}\hat\Lambda_1(\alpha+\beta) + \frac{1}{q}\hat\Lambda_1(\alpha+\beta)^{\frac{q}{\alpha+\beta}} \left(\frac{\pi_{\alpha+\beta}\pi_{\alpha+\beta}^{\alpha+\beta}}{(b-a)^{\alpha+\beta}K(\alpha+\beta)} \right)^{1-\frac{q}{\alpha+\beta}}.$$

Now, by using the explicit formula for $\lambda_1(\alpha+\beta)$, we have

$$\lambda_1(p,q) \le \frac{1}{p}\hat\Lambda_1(\alpha+\beta) + \frac{1}{q}\left(\frac{\pi_{\alpha+\beta}^{\alpha+\beta}}{(b-a)^{\alpha+\beta}}\right)^{\frac{q}{\alpha+\beta}} \left(\frac{\pi_{\alpha+\beta}^{\alpha+\beta}\pi_{\alpha+\beta}}{(b-a)^{\alpha+\beta}K(\alpha+\beta)} \right)^{1-\frac{q}{\alpha+\beta}}.$$

Expanding and collecting terms,

$$\lambda_1(p,q) \le \frac{1}{p}\hat\Lambda_1(\alpha+\beta) + \frac{1}{q}\frac{\pi_{\alpha+\beta}^{\alpha+\beta}}{(b-a)^{\alpha+\beta}} \left(\frac{\pi_{\alpha+\beta}}{K(\alpha+\beta)} \right)^{1-\frac{q}{\alpha+\beta}},$$

and finally we obtain

$$\lambda_1(p,q) \le \hat\Lambda_1(\alpha+\beta)\left[\frac{1}{p} + \frac{1}{q}\left(\frac{\pi_{\alpha+\beta}}{K(\alpha+\beta)}\right)^{1-\frac{q}{\alpha+\beta}}\right].$$

The proof of Theorem 1.3 is finished. □

6. An explicit lower bound on the Main Theorem

In Section §3 we obtained the following asymtotic bounds for the spectral counting function:

$$\frac{2|\Omega|}{\lambda_1^1(p,q)^{\frac{N}{\alpha+\beta}}}\,\lambda^{\frac{N}{\alpha+\beta}} \leq N(\lambda) \leq \frac{2|\Omega|}{\mu_2^1(p,q)^{\frac{N}{\alpha+\beta}}}\,(c\lambda)^{\frac{N}{\alpha+\beta}}.$$

An explicit lower bound can be given by using our previous upper bound, namely

$$\lambda_1(p,q) \leq \frac{\Lambda_1(\alpha+\beta)}{\alpha+\beta-1}\left[\frac{1}{p}+\frac{1}{q}\left(\frac{\pi_{\alpha+\beta}}{K(\alpha+\beta)}\right)^{1-\frac{q}{\alpha+\beta}}\right]$$

for one-dimensional problems.

Moreoever, for the $N-$dimensional case we need Proposition 2.4 together with the bound of Part I, equation (1.5). Let us recall that we can bound the first eigenvalue $\Lambda_1(p)$ by above by using the one of the pseudo $p-$laplacian,

$$\begin{array}{lll} \Lambda_1(p) \leq & N^{(p-2)/2}\nu_{p,1} & \text{if } 2<p, \\ \Lambda_1(p) \leq & \nu_{p,1} & \text{if } p<2, \end{array}$$

Since $\nu_{p,1} = N\pi_p^p$ for the cube Q_1, we obtain an explicit lower bound for $N(\lambda)$ in the $N-$dimensional case.

Remark 6.1. Although the previous formula holds for any constant $c>1$, it is convenient to take $c = 1+\lambda^{-1}$, since in this case we can rewrite the upper bound as

$$\frac{2|\Omega|}{\mu_2^1(p,q)^{\frac{N}{\alpha+\beta}}}\,\lambda^{\frac{N}{\alpha+\beta}} + O(1).$$

We conjecture that a stronger result holds, namely,

$$N(\lambda) = c(\Omega,\alpha,\beta)\lambda^{\frac{N}{\alpha+\beta}} + o(\lambda^{\frac{N}{\alpha+\beta}}).$$

Indeed, the proof follows immediately if it is true that

$$\lambda_1^1(p,q) = \mu_2^1(p,q).$$

The equality holds for a single equation and $p=2$ in any dimension, and also for the one dimensional $p-$laplace equation. Up to our knowledge, it is not known for systems, nor in the case $N>1$ even for a single $p-$laplacian equation.

Acknowledgements Supported by grant X078 from Universidad de Buenos Aires, grants 06-835, 06-290 from ANPCyT PICT. Both authors are members of CONICET (Argentina).

References

[1] W. Allegretto and Y-X. Huang. *A Piccone's identity for the p−Laplacian and applications*, Nonlinear Anal. TMA, 32 nr. 7 (1996), 143-175.

[2] H. Amann. *Lusternik-Schnirelmann theory and nonlinear eigenvalue problems*, Math. Ann., 199 (1972), 55-72.

[3] S. Antman. *The influence of elasticity on analysis: modern developments*, Bull. Amer. Math. Soc. (N.S.) 9 (1983), no. 3, 267–291.

[4] C. Azizieh and Ph. Clément. *A-priori estimates and continuations methods for positive solutions of p-Laplace equations.* J. Differential Equations, 179, nr. 1 (2002), 213-245.

[5] R. Cantrell. *On Coupled Multiparameter Nonlinear Elliptic System*, Trans. Amer. Math. Soc., 294 (1) (1986) 263-285.

[6] R. Cantrell and C. Cosner. *On The Generalized Spectrum for Second Order Elliptic Systems*, Trans. Amer. Math. Soc., 303 (1) (1987) 345-363.

[7] C. Cosner. *Estimates for Eigenfunctions and Eigenvalues of Nonlinear Elliptic Problems*, Trans. Amer. Math. Soc., 282 (1) (1984) 59-75.

[8] Courant, R. and Hilbert, D. *Methods of Mathematical Physics,* Vol 1, Interscience Publishers, Inc. New York (1953).

[9] L. Boccardo and D. G. de Figueiredo. *Some Remarks on a System of Quasilinear Elliptic Equations*, NoDEA Nonlinear Differential Equations Appl., 9 (2002), 309-323.

[10] C. Conca and E. Zuazua, *Asymptotic analysis of a multidimensional vibrating structure*, SIAM J. Math. Anal. 25 (1994), 836-858.

[11] P. de Napoli and C. Marianni. *Quasilinear Elliptic Systems of Resonant Type and Nonlinear Eigenvalue Problems*, Abstr. Appl. Anal., 7 nr. 3 (2002), 155-167.

[12] P. de Napoli and J. P. Pinasco, *Estimates for Eigenvalues of Quasilinear Elliptic Systems*, J. Differential Equations, 227 nr. 1 (2006), 102-115.

[13] F. de Thélin, *Première valeur propre d'un système elliptique non linéaire*, Rev. Mat. Apl. 13, nr. 1 (1992), 1-8. See also C. R. Acad. Sci., Paris, Ser. I 311, nr. 10 (1990), 603-606.

[14] M. Del Pino, P. Drabek and R. Manásevich, *The Fredholm Alternative at the First Eigenvalue for the One Dimensional p−laplacian*, Journal of Differential Equations, 151 (1999), 386-419.

[15] P. Drabek and R. Manásevich, *On the Closed Solutions to some nonhomegeneous eigenvalue problemes with p-laplacian*, Differential Integral Equations, 12 nr. 6 (1999), 773-788.

[16] P. Drabek, N. M. Stavrakakis and N. B. Zographopoulos, *Multiple Nonsemitrivial Solutions for Quasilinear Elliptic Systems*, Differential Integral Equations, 16, nr. 12 (2003), 15191531.

[17] P. Felmer, R. Manásevich, and F.de Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, Commun. Partial Differential Equations, 17 (1992), 2013-2029.

[18] J. Fernández Bonder. *Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities*, Abstr. Appl. Anal. 2004, nr. 12 (2004), 1047-1056

[19] J. Fernández Bonder and J. P. Pinasco. *Asymptotic Behavior of the Eigenvalues of the One Dimensional Weighted p- Laplace Operator*, Ark. Mat., 41 (2003), 267-280.

[20] J. Fernández Bonder and J.P. Pinasco. *Eigenvalues of the p-laplacian in fractal strings with indefinite weights*, J. Math. Anal. Appl., 308, no. 2 (2005), 764-774.

[21] J. Fernández Bonder and J.P. Pinasco. *Estimates for Eigenvalues of Quasilinear Elliptic Systems, Part II*, J. Differential Equations, 245, no. 4 (2008), 875-891.

[22] J. Fleckinger-Pellé, J.-P. Gossez, P. Takác, and F. de Thélin. *Nonexistence of solutions and an anti-maximum principle for cooperative systems with the p-Laplacian*, Math. Nachr. 194 (1998), 49-78.

[23] L. Friedlander. *Asymptotic behaviour of eigenvalues of the p-Laplacian*, Commun. Partial Differential Equations, 14, nr. 8/9 (1989), 1059-1069.

[24] J. Garcia Azorero and I. Peral Alonso. *Comportement asymptotique des valeurs propres du p-laplacien* , C. R. Acad. Sci. Paris, Ser. I, 307 (1988), 75-78.

[25] D. A. Kandilakis, M. Magiropoulos, and N. B. Zographopoulos. *The first eigenvalue of p-Laplacian systems with nonlinear boundary conditions*, Boundary Value Problems, 2005, nr. 3 (2005), 307-321.

[26] R. Manásevich and J. Mawhin. *The spectrum of p-Laplacian Systems with various boundary Conditions and Applications*, Adv. Differential Equations, 5, nr. 10-12 (2000), 1289-1318.

[27] J. Odhnoff, *Operators generated by differential problems with eigenvalue parameter in equation and boundary condition*, Thesis, Meddel. Lunds Univ. Mat. Semin. 14, 1-80 (1959).
[28] J. P. Pinasco. *Estimates for Eigenvalues of Elliptic Problems on domains with subdomains of different dimensions*, preprint.
[29] J. P. Pinasco. *Asymptotics for problems where the eigenvalue parameter appears on both the equation and the boundary condition*, preprint.
[30] M. Protter. *The Generalized Spectrum of Second Order Elliptic Systems*, Rocky Mountain J. of Math., 9 (3) (1979) 503-518.
[31] N. M. Stavrakakis and N. B. Zographopoulos. *Bifurcation Results for some Quasilinear Elliptic Systems on* $\mathbb{R}^N$, Adv. Differential Equations, 8, nr. 3 (2003), 315-336.
[32] J. Vélin, F. de Thélin. *Existence and nonexistence of nontrivial solutions for some nonlinear elliptic systems*, Rev. Mat. Complut., 6, nr. 1 (1993), 153194.
[33] N. Zographopoulos. *p-Laplacian Systems on Resonance*, Appl. Anal., 83, nr. 5 (2004), 509-519.

Julián Fernández Bonder
FCEyN - Departamento de Matemática,
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I
(1428) Buenos Aires, Argentina.
e-mail: jfbonder@dm.uba.ar

Juan P. Pinasco
FCEyN - Departamento de Matemática,
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I
(1428) Buenos Aires, Argentina.
e-mail: jpinasco@dm.uba.ar
On leave from Instituto de Ciencias,
Univ. Nacional de General Sarmiento