Uniform Bounds for the Best Sobolev Trace Constant

Julián Fernández Bonder
Departamento de Matemática, Universidad de Buenos Aires
Ciudad Universitaria (1428), Buenos Aires, Argentina
e-mail: jbonder@dm.uba.ar

Julio D. Rossi
Departamento de Matemática Universidad de Buenos Aires
Ciudad Universitaria (1428), Buenos Aires, Argentina
Current address: Departamento de Matemática, Universidad Católica,
Casilla 306, correo 22, Santiago, Chile
e-mail: jrossi@riemann.mat.puc.cl

Raúl Ferreira
Departamento de Matemáticas,
Universidad Autónoma de Madrid, Madrid, España
Current address: Departamento de Matemáticas, U. Carlos III de Madrid,
28911 Leganés, Spain
e-mail: rferreir@math.uc3m.es

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Abstract

We study the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$, looking at the dependence of the best constant and the extremals on $p$ and $q$. We prove that there exists a uniform bound (independent of $(p, q)$) for the best constant if and only if $(p, q)$ lies far from $(N, \infty)$. Also we study some limit cases, $q = \infty$ with $p > N$ or $p = \infty$ with $1 \leq q < \infty$.

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1 Introduction

Sobolev inequalities are very popular in the study of partial differential equations or in the calculus of variations and have been investigated by a great number of authors. Among them are the Sobolev trace inequalities. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. For any $1 \leq p \leq \infty$, we define the Sobolev trace conjugate as

$$p^* = \begin{cases} \frac{p(N-1)}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

If $1 \leq q \leq p^*$ (with strict second inequality if $p = N$), we have the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$ and hence the following inequality holds:

$$S \|u\|_{L^q(\partial \Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. This is known as the Sobolev trace embedding Theorem. The best constant for this embedding is the largest $S$ such that the above inequality holds, that is,

$$S_{p,q} = \inf_{u \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)} \left( \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left( \int_{\partial \Omega} |u|^q \, d\sigma \right)^{1/q}} \right)^{1/p} = \inf_{u \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)} Q_{p,q}(u).$$

Moreover, if $1 \leq q < p^*$ the embedding is compact and as a consequence we have the existence of extremals, i.e. functions where the infimum is attained, see [8]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{q-2}u & \text{on } \partial \Omega, \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian and $\frac{\partial}{\partial n}$ is the outer unit normal derivative. Using [13] and [14] we can assume that the extremals are positive, $u > 0$, ...
in \( \Omega \). In the special case \( p = q \), problem (1.2) becomes a nonlinear eigenvalue problem, that was studied in [8], [12]. For \( p = 2 \), this eigenvalue problem is known as the Steklov problem, [1]. From now on, let us call \( u_{p,q} \) an extremal corresponding to the exponents \((p, q)\).

The main purposes of this work are to study the possibility of a uniform bound (independent of \((p, q)\)) on \( S_{p,q} \) and to study the limit behavior of the best Sobolev trace constants \( S_{p,q} \) as \( p \to +\infty \) and as \( q \to +\infty \) and look at the limit cases \( p = \infty \), \( 1 \leq q \leq \infty \), and \( N < p < \infty \), \( q = \infty \). Our main result is the following.

**Theorem 1.1** Given \( A \) a set of admissible \((p, q)\),

\[
A \subset \{ (p, q) : 1 \leq p \leq \infty, 1 \leq q \leq p^* \}
\]

there exist constants \( C_1 \) and \( C_2 \) independent of \((p, q) \in A\) such that

\[
C_1 \leq S_{p,q} \leq C_2
\]

if and only if \( A \) verifies the following property, there is no sequence \((p_n, q_n) \in A\) with \( p_n \to N \) and \( q_n \to \infty \).

Notice that Theorem 1.1 says that we can obtain a uniform bound for \( S_{p,q} \) on \( A \) as long as \((p, q) \in A\) stays away from the point \((N, \infty)\). Observe that the upper bound, \( S_{p,q} \leq C_2 \), follows easily by taking \( u \equiv 1 \) in (1.1) and holds even if we are close to \((N, \infty)\). The main difficulty arises in the proof of the lower bound. As we will explain below, this is due to the fact that there exist functions in \( W^{1,N}(\Omega) \) that do not belong to \( L^\infty(\partial \Omega) \).

As we mentioned before, one of our concerns is to analyze the case \( p = \infty \) with \( 1 \leq q \leq \infty \), i.e., the immersion \( W^{1,\infty}(\Omega) \hookrightarrow L^q(\partial \Omega) \). The best constant is given by

\[
S_{\infty,q} = \inf_{u \in W^{1,\infty}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^q(\partial \Omega)}}.
\]

From this expression it is easy to see that \( S_{\infty,q} = 1/|\partial \Omega|^{1/q} \) and \( S_{\infty,\infty} = 1 \), with extremal \( u_{\infty,q} = u_{\infty,\infty} \equiv 1 \) in both cases (we normalize the extremals according to \( \|u_{\infty,q}\|_{L^\infty(\partial \Omega)} = \|u_{\infty,\infty}\|_{L^\infty(\partial \Omega)} = 1 \)). We prove that \( S_{\infty,\infty} = 1 \) is the limit of \( S_{p,q} \) as \( p, q \to \infty \) and also \( S_{\infty,q} \) is the limit of \( S_{p,q} \) when \( p \to \infty \).

**Theorem 1.2** Let \( S_{p,q} \) be the best Sobolev trace constant and \( u_{p,q} \) be any extremal normalized such that \( \|u_{p,q}\|_{L^\infty(\partial \Omega)} = 1 \). Then

\[
\lim_{p,q \to \infty} S_{p,q} = S_{\infty,\infty} = 1,
\]

and, for any \( 1 < r < \infty \), as \( p, q \to \infty \),

\[
\begin{align*}
u_{p,q} &\rightharpoonup u_{\infty,\infty} \equiv 1, & \text{weakly in } W^{1,r}(\Omega), \\
u_{p,q} &\to u_{\infty,\infty} \equiv 1, & \text{strongly in } C^\alpha(\overline{\Omega}).
\end{align*}
\]
Moreover, for fixed $1 \leq q < \infty$,

$$
\lim_{p \to \infty} S_{p,q} = S_{\infty,q} = \frac{1}{|\partial\Omega|^{1/q}};
$$

and, for any $1 < r < \infty$, as $p \to \infty$,

- $u_{p,q} \rightharpoonup u_{\infty,q} \equiv 1$, weakly in $W^{1,r}(\Omega)$,
- $u_{p,q} \rightarrow u_{\infty,q} \equiv 1$, strongly in $C^\alpha(\Omega)$.

The limit $q \to \infty$ with $p > N$ fixed is more subtle since we do not know a priori which is the extremal for the limit case. However we find an equation for the limit extremal.

**Theorem 1.3** Let $p > N$, then

$$
\lim_{q \to \infty} S_{p,q} = S_{p,\infty},
$$

and, up to subsequences, as $q \to \infty$,

- $u_{p,q} \rightharpoonup u_{p,\infty}$ weakly in $W^{1,p}(\Omega)$,
- $u_{p,q} \rightarrow u_{p,\infty}$ strongly in $C^\alpha(\bar{\Omega})$.

Moreover, there exists a measure $\mu \in C(\partial\Omega)^*$ with $\mu(\{u_{p,\infty} = 1\}) = 1$ such that $u_{p,\infty}$ is a weak solution of

$$
\begin{cases}
\Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S_{p,\infty}^p \mu \chi_{\{u \equiv 1\}} & \text{on } \partial\Omega.
\end{cases}
$$

We observe that $W^{1,N}(\Omega) \not\hookrightarrow L^\infty(\partial\Omega)$. Hence we expect that the best constant $S_{p,q}$ goes to zero as $(p,q) \to (N,\infty)$. This is the content of our next result.

**Theorem 1.4** The best constant $S_{p,q}$ goes to zero as $(p,q) \to (N,\infty)$ and moreover for any $\alpha < (N-1)/N$, there exists a constant $C$ such that

$$
S_{p,q} \leq C \max \left\{ \left( (p-N)_+ , \frac{1}{q} \right)^\alpha \right\}.
$$

For the dependence of $S_{p,q}(\Omega)$ with respect to the domain, see [4] and [9] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding and contracting domains. In [5] a related problem in the half-space $\mathbb{R}^N_+$ for the critical exponent is studied. See also [6], [7] for other geometric problems that lead to nonlinear boundary conditions, like the ones that appear in (1.2). The best constant in the Sobolev immersion, $W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega)$, has been studied by many authors, see for example [10]. More recently in [11] the authors analyze the limit as $p \to \infty$ of the related Dirichlet eigenvalue problem for the $p$-Laplacian.

The paper is organized as follows: first we deal with the limit cases. In sections 2 and 3 we prove Theorem 1.2 and Theorem 1.3 respectively, in section 4 we find estimates for $S_{p,q}$ near $(N,\infty)$, Theorem 1.4, and finally in section 5 we deal with the proof of our main result, Theorem 1.1.
2 Limit as $p \to +\infty$

In this section we prove Theorem 1.2.

Proof. First, we study the limit $p, q \to \infty$. In this case the natural limit problem is

$$S_{\infty, \infty} = \inf_{u \in W^{1,\infty}(\Omega)} \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial \Omega)}}{\|u\|_{L^\infty(\Omega)}}.$$  \hfill (1.1)

As we mentioned in the introduction $S_{\infty, \infty} = 1$ and the extremal is $u_{\infty, \infty} \equiv 1$ (normalized such that $\|u\|_{L^\infty(\partial \Omega)} = 1$). Now, taking $u = 1$ in (1.1), we get

$$S_{p,q} = \inf_{u \in W^{1,p}(\Omega)} Q_{p,q}(u) \leq \frac{|\Omega|^{1/p}}{|\partial \Omega|^{1/q}},$$  \hfill (2.1)

from where it follows that

$$\limsup_{p,q \to \infty} S_{p,q} \leq 1.$$  \hfill (2.2)

For $p > N$, let us denote by $u_{p,q}$ one extremal for (1.1) normalized such that $\|u_{p,q}\|_{L^\infty(\partial \Omega)} = 1$. Hence

$$\|u_{p,q}\|_{W^{1,p}(\Omega)} = S_{p,q} \|u_{p,q}\|_{L^p(\partial \Omega)} \leq S_{p,q} |\partial \Omega|^{1/q} \leq C,$n with $C$ independent of $p, q$. On the other hand, if $N < r < p$,

$$\|u_{p,q}\|_{W^{1,r}(\Omega)} \leq |\Omega|^{(p-r)/pr} \|u_{p,q}\|_{W^{1,p}(\Omega)} \leq C.$$n Hence, there exists $u \in W^{1,r}(\Omega)$ such that, up to a subsequence,

$$u_{p,q} \to u \text{ weakly in } W^{1,r}(\Omega),$$n $$u_{p,q} \to u \text{ strongly in } C^\alpha(\Omega).$$n

Observe that we can assume that the limit $u$ does not depend on $r$. In fact, we can choose a sequence $r_j \to \infty$ and in each $W^{1,r_j}$ we can extract a subsequence of $u_{p,q}$ that converges weakly. By a standard diagonal argument we obtain a subsequence that converges strongly in $C^\alpha$ and weakly in $W^{1,r_j}$ for every $j$ (and hence in $W^{1,r}$ for every $r$) to a limit function $u$.

In particular, $\|u\|_{L^\infty(\partial \Omega)} = 1$ and

$$S_{p,q} = Q_{p,q}(u_{p,q}) \geq \frac{|\Omega|^{-(p-r)/pr} \|u_{p,q}\|_{W^{1,r}(\Omega)}}{\|u_{p,q}\|_{L^p(\partial \Omega)}} \geq \frac{|\Omega|^{-(p-r)/pr} \|u_{p,q}\|_{W^{1,r}(\Omega)}}{|\partial \Omega|^{1/q}}.$$n

Hence

$$1 \geq \limsup_{p,q \to \infty} \frac{|\Omega|^{-(p-r)/pr} \|u_{p,q}\|_{W^{1,r}(\Omega)}}{|\partial \Omega|^{1/q}} \geq |\Omega|^{-1/r} \|u\|_{W^{1,r}(\Omega)},$$n

and therefore, taking the limit as $r \to \infty$, we get

$$1 \geq \|u\|_{W^{1,\infty}(\Omega)}.$$
We conclude that \( u \in W^{1,\infty}(\Omega) \) and that \( u \) is an extremal for \( S_{\infty,\infty} \) that satisfies \( \|u\|_{L^\infty(\partial\Omega)} = 1 \), and hence \( u \equiv 1 \).

Next, we focus on the case \( p \to +\infty \) with fixed \( 1 \leq q < \infty \). We consider the natural limit problem

\[
S_{\infty,q} = \inf_{u \in W^{1,\infty}(\Omega)} \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^q(\partial\Omega)}},
\]

and we note that the extremal is \( u_{\infty,q} \equiv 1 \) (normalized such that \( \|u_{\infty,q}\|_{L^\infty(\partial\Omega)} = 1 \)) and then the best constant is given by \( S_{\infty,q} = 1/|\partial\Omega|^{1/q} \).

Following the same argument given above we get that there exists \( u \in W^{1,r}(\Omega) \) such that, up to a subsequence,

\[
u_{p,q} \to u \text{ weakly in } W^{1,r}(\Omega),
\]

\[
u_{p,q} \to u \text{ strongly in } C^\alpha(\Omega).
\]

Moreover, we have the following inequalities,

\[
\frac{|\Omega|^{1/p}}{|\partial\Omega|^{1/q}} \geq S_{p,q} = Q_{p,q}(u_{p,q}) \geq \frac{|\Omega|^{- (p-r)/pr} \|u_{p,q}\|_{W^{1,r}(\Omega)}}{\|u_{p,q}\|_{L^q(\partial\Omega)}}.
\]

First we take the limit as \( p \to \infty \), and then the limit as \( r \to \infty \), to obtain

\[
\frac{1}{|\partial\Omega|^{1/q}} \geq S_{\infty,q} \geq \frac{\|u\|_{W^{1,\infty}(\Omega)}}{\|u\|_{L^q(\partial\Omega)}}.
\]

Therefore, we can conclude that \( u \in W^{1,\infty}(\Omega) \) and that it is an extremal for \( S_{\infty,q} \) which satisfies \( \|u\|_{L^\infty(\partial\Omega)} = 1 \). Hence \( u = u_{\infty,q} \equiv 1 \) and \( S_{\infty,q} = 1/|\partial\Omega|^{1/q} \). \( \square \)

3 Limit as \( q \to +\infty \) for fixed \( p > N \)

In this section we fix \( p > N \) and consider the limit of \( S_{p,q} \) and \( u_{p,q} \) when \( q \to \infty \). In order to clarify the exposition we divide the proof of Theorem 1.3 in two lemmas.

**Lemma 3.1** Let \( p > N \) be fixed. Then

\[
\lim_{q \to \infty} S_{p,q} = S_{p,\infty},
\]

and, up to subsequences, as \( q \to \infty \),

\[
u_{p,q} \to u_{p,\infty} \text{ weakly in } W^{1,p}(\Omega),
\]

\[
u_{p,q} \to u_{p,\infty} \text{ strongly in } C^\alpha(\Omega).
\]

**Proof.** Let \( u_{p,q} \) be an extremal for (1.1) normalized such that \( \|u_{p,q}\|_{L^\infty(\partial\Omega)} = 1 \). Then we have

\[
S_{p,q} = \frac{\|u_{p,q}\|_{W^{1,p}(\Omega)}}{\|u_{p,q}\|_{L^q(\partial\Omega)}} \geq \frac{\|u_{p,q}\|_{W^{1,p}(\Omega)}}{|\partial\Omega|^{1/q}}.
\]

(3.1)
Therefore, using (2.1), we have \( \| u_{p,q} \|_{W^{1,p}(\Omega)} \leq |\Omega|^{1/p} \). Hence, there exists a function \( u \in W^{1,p}(\Omega) \) such that, up to a subsequence,
\[
\begin{align*}
&u_{p,q} \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \\
&u_{p,q} \to u \quad \text{strongly in } L^{\infty}(\partial \Omega).
\end{align*}
\]
Hence \( \| u \|_{L^{\infty}(\partial \Omega)} = 1 \), and from (3.1) we get
\[
\liminf_{q \to \infty} S_{p,q} \geq \liminf_{q \to \infty} \| u_{p,q} \|_{W^{1,p}(\Omega)} \geq \| u \|_{W^{1,p}(\Omega)} \geq S_{p,\infty}.
\]
Now, let us see that \( u \) is an extremal for \( S_{p,\infty} \). We argue by contradiction. Assume that there exists \( v \in W^{1,p}(\Omega) \) such that
\[
Q_{p,\infty}(v) < Q_{p,\infty}(u).
\]
Then, for large \( q \) we have,
\[
Q_{p,q}(v) < Q_{p,q}(u),
\]
but as
\[
S_{p,q} \geq \frac{\| u_{p,q} \|_{W^{1,p}(\Omega)}}{|\partial \Omega|^{1/q}} \geq \frac{\| u \|_{W^{1,p}(\Omega)} - \varepsilon_q}{|\partial \Omega|^{1/q}} 
\geq \left( \frac{\| u \|_{L^p(\partial \Omega)}}{|\partial \Omega|^{1/q}} \right) \frac{\| u \|_{W^{1,p}(\Omega)} - \varepsilon_q}{\| u \|_{L^p(\partial \Omega)}} \frac{\| u \|_{W^{1,p}(\Omega)}}{\| u \|_{L^p(\partial \Omega)}}
\]
for some \( \varepsilon_q \) that goes to zero as \( q \to \infty \), we arrive to a contradiction.

To finish the proof of the Lemma, we observe that
\[
S_{p,q} \leq Q_{p,q}(u) \to Q_{p,\infty}(u) = S_{p,\infty}.
\]
Therefore, \( \limsup_{q \to \infty} S_{p,q} \leq S_{p,\infty} \).

**Lemma 3.2** Let \( p > N \) be fixed and let \( u_{p,\infty} \) be an extremal for (1.1) obtained as limit of a sequence of extremals \( u_{p,q} \), as \( q \to \infty \). Then there exists a measure \( \mu \in C(\partial \Omega)^* \), with \( \mu(\{ u_{p,\infty} \equiv 1 \}) = 1 \), such that \( u_{p,\infty} \) is a weak solution of
\[
\begin{align*}
\Delta_p u &= |u|^{p-2} u \quad \text{in } \Omega, \\
\frac{\partial |u|^{p-2} u}{\partial \nu} &= S_{p,\infty}^p \mu 1_{\{ u = 1 \}} \quad \text{on } \partial \Omega.
\end{align*}
\]

**Proof.** Let \( u_{p,q} \) be as in Lemma 3.1. As \( u_{p,q} \) is a weak solution of (1.2), we have that for every \( \phi \in W^{1,p}(\Omega) \),
\[
\int_{\Omega} (|\nabla u_{p,q}|^{p-2} \nabla u_{p,q} \nabla \phi + |u_{p,q}|^{p-2} u_{p,q} \phi) \, dx = S_{p,q}^p \left( \int_{\partial \Omega} |u_{p,q}|^q \, d\sigma \right)^{(p-q)/q} \int_{\partial \Omega} |u_{p,q}|^{q-2} u_{p,q} \phi \, d\sigma.
\]
Let us define $\Psi_q \in L^\infty(\partial \Omega)^*$ as

$$\Psi_q(\phi) = \left( \int_{\partial \Omega} |u_{p,q}|^q \, d\sigma \right)^{(p-q)/q} \int_{\partial \Omega} |u_{p,q}|^{q-2} u_{p,q} \phi \, d\sigma.$$  

By Hölder inequality, we get

$$|\Psi_q(\phi)| \leq \|u_{p,q}\|_{L^1(\partial \Omega)}^{p-1} \|\phi\|_{L^q(\partial \Omega)} \leq |\partial \Omega|^{p/q} \|u_{p,q}\|_{L^\infty(\partial \Omega)}^{p-1} \|\phi\|_{L^\infty(\partial \Omega)} \leq C \|\phi\|_{L^\infty(\partial \Omega)}$$

with $C$ independent of $q$. Therefore, $\|\Psi_q\| \leq C$ and hence if we call

$$v_q = \left( \int_{\partial \Omega} |u_{p,q}|^q \, d\sigma \right)^{(p-q)/q} |u_{p,q}|^{q-2} u_{p,q},$$

we have that $v_q$ is uniformly bounded in $L^1(\partial \Omega)$ and then, up to a subsequence, $v_q \rightharpoonup \mu$ weakly-* in the sense of measures.

In order to finish the proof, we will see that $\text{supp}(\mu) \subseteq \{u_{p,\infty} = 1\}$. To prove this, we consider a point $x_0 \in \partial \Omega$ such that $u_{p,\infty}(x_0) < 1 - 2\delta$ for some $\delta$ small enough. Hence, for $q$ large enough we have that $u_{p,q}(x_0) < 1 - \delta$. On the other hand, as $\|u_{p,\infty}\|_{L^\infty(\partial \Omega)} = 1$, and by the $C^\alpha$ convergence of $u_{p,q}$ to $u_{p,\infty}$ there exists a point $x_1 \in \partial \Omega$ and $r$ independent of $q$ such that $B_r(x_1) \cap \partial \Omega \subset \{x \in \partial \Omega : u_{p,q}(x) > 1 - \delta/2\}$. Therefore

$$|\partial \Omega|^{1/q} \geq \left( \int_{\partial \Omega} |u_{p,q}|^q \, d\sigma \right)^{1/q} \geq (1 - \delta/2) |B_r(x_1) \cap \partial \Omega|^{1/q},$$

where the first inequality follows from the fact that $\|u_{p,q}\|_{L^\infty(\partial \Omega)} = 1$. Now, we rewrite $v_q$ as follows,

$$v_q(x_0) = \left( \frac{u_{p,q}(x_0)}{\|u_{p,q}\|_{L^\infty(\partial \Omega)}} \right)^{q-1} \|u_{p,q}\|_{L^q(\partial \Omega)}^{p-1} \leq \left( \frac{1 - \delta}{(1 - \delta/2) |B_r(x_0) \cap \partial \Omega|^{1/q}} \right)^{q-1} |\partial \Omega|^{(p-1)/q}.$$  

Hence, we conclude that $v_q(x_0) \to 0$, and we get that the measure is supported in $\{x \in \partial \Omega : u(x) = 1\}$. Moreover, if we take $u_{p,\infty}$ as test function in the weak form of (3.2), we get

$$\int_{\Omega} (|\nabla u_{p,\infty}|^p + |u_{p,\infty}|^p) \, dx = S_{p,\infty}^p \int_{\partial \Omega \cap \{u_{p,\infty} = 1\}} \, d\mu.$$  

As $u_{p,\infty}$ is an extremal and verifies $\|u_{p,\infty}\|_{L^\infty(\partial \Omega)} = 1$ we have that

$$\int_{\Omega} (|\nabla u_{p,\infty}|^p + |u_{p,\infty}|^p) \, dx = S_{p,\infty}^p.$$  

Therefore $\mu(\partial \Omega \cap \{u_{p,\infty} = 1\}) = 1$. This completes the proof. ∎
4 Estimates for \((p, q)\) near \((N, \infty)\)

In this section we find an upper bound for the vanishing rate of \(S_{p,q}\) as \((p, q)\) approaches \((N, \infty)\), that is we prove Theorem 1.4.

Proof. If \(p < N\), using Holder inequality we have that there exist a constant \(C\) such that

\[ S_{p,q} \leq C S_{N,q}, \quad \text{for } p < N. \]

Hence, we can assume that \(p \geq N\). In order to obtain an upper bound on the decay rate, we suppose that \(0 \in \partial \Omega\), \(\alpha < (N - 1)/N\), and we consider the function

\[ u_{\varepsilon}(x) = \left( \ln(1 + \frac{1}{|x| + \varepsilon}) \right)^{\alpha} \in W^{1,p}(\Omega). \]

Then we obtain a bound for \(\|u_{\varepsilon}\|_{L^q(\partial \Omega)}\) as follows, given \(M < \|u_{\varepsilon}\|_{L^\infty(\partial \Omega)}\),

\[ \|u_{\varepsilon}\|_{L^q(\partial \Omega)} \geq \left( \int_{\{x \in \partial \Omega : u_{\varepsilon}(x) \geq M\}} |u_{\varepsilon}|^q \right)^{1/q} \geq M |\{x \in \partial \Omega : u_{\varepsilon}(x) \geq M\}|^{1/q}. \]

On the other hand, let us compute

\[ |\nabla u_{\varepsilon}|^p \leq \alpha^p \left( \ln(1 + \frac{1}{|x| + \varepsilon}) \right)^{(\alpha - 1)p} \left( \frac{1}{|x| + \varepsilon} \right)^p. \]

Hence,

\[ \int_{\Omega} |\nabla u_{\varepsilon}|^p \leq \frac{C}{\varepsilon} \int_0^C \frac{r^{N-1}}{(r + \varepsilon)^p} \left( \ln(1 + \frac{1}{r + \varepsilon}) \right)^{(\alpha - 1)p} dr \leq \frac{C}{\varepsilon^{p-N}}. \]

Moreover,

\[ \int_{\Omega} |u_{\varepsilon}|^p \leq C. \]

Summing up, we obtain that

\[ S_{p,\infty} \leq \frac{C}{\varepsilon^{p-N} M |\{x \in \partial \Omega : u_{\varepsilon}(x) \geq M\}|^{1/q}}. \]

If \(q(p - N) \geq 1\), we take \(M \sim 1/(p - N)^{\alpha}\) and \(\varepsilon \sim e^{-1/(p-N)}\) and if \(q(p - N) \leq 1\), \(M \sim q^\alpha\) and \(\varepsilon \sim e^{-q}\). With this choice, we obtain

\[ S_{p,q} \leq C \max \left\{ (p - N)_+, \frac{1}{q} \right\}^\alpha \to 0, \quad \text{as } (p, q) \to (N, \infty). \]

This ends the proof. \(\square\)
5 Uniform bounds for $S_{p,q}$

In this section we prove our main result, Theorem 1.1.

Proof. From Theorem 1.4 we get that the best constant $S_{p,q}$ degenerates as $(p, q) \to (N, \infty)$, hence to obtain uniform bounds we have to stay far from that point.

A uniform upper bound for $S_{p,q}$ follows from (2.1), namely,

$$S_{p,q} \leq \frac{|\Omega|^{1/p}}{|\partial \Omega|^{1/q}} \leq C_2,$$

for $1 \leq p, q \leq \infty$. The lower bound is more subtle. First we observe that, by Hölder’s inequality, we have

$$\|u\|_{L^{q_1}(\partial \Omega)} \leq |\partial \Omega|^{1 - \frac{1}{q_2}} \|u\|_{L^{q_2}(\partial \Omega)}$$

for $1 \leq q_1 \leq q_2$, and

$$\|u\|_{W^{1,q_2}(\Omega)} \leq |\Omega|^{\frac{1}{q_2} - \frac{1}{p_1}} \|u\|_{W^{1,p_1}(\Omega)}$$

for $1 \leq p_2 \leq p_1$. Therefore, there exists a constant $C$ independent of $1 \leq p \leq \infty$ and $1 \leq q \leq p^*$ such that

$$S_{p_{1,q_1}} \geq C S_{p_{2,q_2}},$$

for any $1 \leq q_1 \leq q_2$ and $1 \leq p_2 \leq p_1$. Inequality (5.2) says that in order to obtain lower bounds for $S_{p,q}$ we can enlarge $q$ and decrease $p$. Therefore, in order to get uniform bounds for $S_{p,q}$ in sets $A$ that are far from the point $(N, \infty)$ we can proceed as follows. From our assumptions on $A$ we have that there exists $s < N < r$ such that

$$A \subset \{(p, q) : p > r\} \cup \{(p, q) : 1 \leq p \leq r \text{ and } 1 \leq q \leq \min\{p^*, s^*\}\} = A_1 \cup A_2,$$

see Figure 1 below.

Figure 1.
From our previous estimate (5.2) we get that
\[ S_{p,q} \geq CS_{r,\infty} \tag{5.3} \]
for \((p, q) \in A_1\), and
\[ S_{p,q} \geq C \min_{1 \leq p \leq s} S_{p,p^*} \]
for \((p, q) \in A_2\). To estimate the value of the best Sobolev trace constant along the critical curve \((p, p^*)\) with \(1 \leq p \leq s\), we use interpolation theory, see [2], [3]. We have, for the trace operator \(T\):
\[ T : W^{1,1}(\Omega) \to L^1(\partial\Omega), \quad S_{1,1} \|Tu\|_{L^1(\partial\Omega)} \leq \|u\|_{W^{1,1}(\Omega)}, \]
and
\[ T : W^{1,s}(\Omega) \to L^{s^*}(\partial\Omega), \quad S_{s,s^*} \|Tu\|_{L^{s^*}(\partial\Omega)} \leq \|u\|_{W^{1,s}(\Omega)}. \]
Therefore,
\[ T : W^{1,p}(\Omega) \to L^q(\partial\Omega), \quad S_{p,q} \|Tu\|_{L^q(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}, \]
with
\[ \frac{1}{p} = \theta + \frac{1 - \theta}{s}, \quad \frac{1}{q} = \theta + \frac{1 - \theta}{s^*}, \tag{5.4} \]
and
\[ S_{p,q} \geq S_{1,1}^{1-\theta} S_{s,s^*}^\theta, \]
for any \(0 < \theta < 1\). We observe that if \((p, q)\) are given by (5.4) we have \(q = p^*\) hence there exists a constant \(C\) that only depends on \(s\) such that
\[ \min_{1 \leq p \leq s} S_{p,p^*} \geq \min\{S_{1,1}, S_{s,s^*}\} \geq C. \]
Hence we have a uniform lower bound
\[ S_{p,q} \geq C, \tag{5.5} \]
for \((p, q) \in A_2\). From (5.1), (5.3) and (5.5) we conclude the desired result. \(\square\)

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**References**


