Advanced Nonlinear Studies **3** (2003), 181-192

Uniform Bounds for the Best Sobolev Trace Constant

Julián Fernández Bonder*
Departamento de Matemática, Universidad de Buenos Aires
Ciudad Universitaria (1428), Buenos Aires, Argentina
e-mail: jfbonder@dm.uba.ar

Julio D. Rossi†
Departamento de Matemática Universidad de Buenos Aires
Ciudad Universitaria (1428), Buenos Aires, Argentina
Current address:*Departamento de Matemática, Universidad Católica,*
Casilla 306, correo 22, Santiago, Chile
e-mail: jrossi@riemann.mat.puc.cl

Raúl Ferreira‡
Departamento de Matemáticas,
Universidad Autónoma de Madrid, Madrid, España
Current address:*Departamento de Matemticas, U. Carlos III de Madrid,*
28911 Legans, Spain
e-mail: rferreir@math.uc3m.es

Received 29 August 2002
Communicated by Jean Mawhin

*The author acknowledges the support of ANPCyT PICT No. 03-05009
†The author acknowledges the support of ANPCyT PICT No. 03-05009, Fundación Antorchas and CONICET
‡The author acknowledges the support of AECI

Abstract

We study the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, looking at the dependence of the best constant and the extremals on p and q. We prove that there exists a uniform bound (independent of (p,q)) for the best constant if and only if (p,q) lies far from (N,∞). Also we study some limit cases, $q=\infty$ with $p>N$ or $p=\infty$ with $1 \le q \le \infty$.

2000 Mathematics Subject Classification. 35J65, 35J20, 35P30, 35P15.

Key words. Sobolev trace constants, p-Laplacian, nonlinear boundary conditions

1 Introduction

Sobolev inequalities are very popular in the study of partial differential equations or in the calculus of variations and have been investigated by a great number of authors. Among them are the Sobolev trace inequalities. Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \ge 2$. For any $1 \le p \le \infty$, we define the Sobolev trace conjugate as

$$p^* = \begin{cases} \dfrac{p(N-1)}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \ge N. \end{cases}$$

If $1 \le q \le p^*$ (with strict second inequality if $p = N$), we have the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and hence the following inequality holds:

$$S\|u\|_{L^q(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. This is known as the Sobolev trace embedding Theorem. The best constant for this embedding is the largest S such that the above inequality holds, that is,

$$S_{p,q} = \inf_{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\left(\displaystyle\int_\Omega |\nabla u|^p + |u|^p\,dx\right)^{1/p}}{\left(\displaystyle\int_{\partial\Omega} |u|^q\,d\sigma\right)^{1/q}} = \inf_{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} Q_{p,q}(u). \tag{1.1}$$

Moreover, if $1 \le q < p^*$ the embedding is compact and as a consequence we have the existence of extremals, i.e. functions where the infimum is attained, see [8]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian and $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative. Using [13] and [14] we can assume that the extremals are positive, $u > 0$, in Ω. In the special case $p = q$, problem (1.2) becomes a nonlinear eigenvalue problem, that was studied in [8], [12]. For $p = 2$, this eigenvalue problem is known as the *Steklov* problem, [1]. From now on, let us call $u_{p,q}$ an extremal corresponding to the exponents (p, q).

The main purposes of this work are to study the possibility of a uniform bound (independent of (p, q)) on $S_{p,q}$ and to study the limit behavior of the best Sobolev trace constants $S_{p,q}$ as $p \to +\infty$ and as $q \to +\infty$ and look at the limit cases $p = \infty$, $1 \le q \le \infty$ and $N < p < \infty$, $q = \infty$. Our main result is the following.

Theorem 1.1 *Given A a set of admissible (p, q),*

$$A \subset \{(p, q) \, : \, 1 \le p \le \infty, \, 1 \le q \le p^*\}$$

there exist constants C_1 and C_2 independent of $(p, q) \in A$ such that

$$C_1 \le S_{p,q} \le C_2$$

if and only if A verifies the following property, there is no sequence $(p_n, q_n) \in A$ with $p_n \to N$ and $q_n \to \infty$.

Notice that Theorem 1.1 says that we can obtain a uniform bound for $S_{p,q}$ on A as long as $(p, q) \in A$ stays away from the point (N, ∞). Observe that the upper bound, $S_{p,q} \le C_2$, follows easily by taking $u \equiv 1$ in (1.1) and holds even if we are close to (N, ∞). The main difficulty arises in the proof of the lower bound. As we will explain below, this is due to the fact that there exist functions in $W^{1,N}(\Omega)$ that do not belong to $L^\infty(\partial\Omega)$.

As we mentioned before, one of our concerns is to analyze the case $p = \infty$ with $1 \le q \le \infty$, i.e., the immersion $W^{1,\infty}(\Omega) \hookrightarrow L^q(\partial\Omega)$. The best constant is given by

$$S_{\infty,q} = \inf_{u \in W^{1,\infty}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^q(\partial\Omega)}}.$$

From this expression it is easy to see that $S_{\infty,q} = 1/|\partial\Omega|^{1/q}$ and $S_{\infty,\infty} = 1$, with extremal $u_{\infty,q} = u_{\infty,\infty} \equiv 1$ in both cases (we normalize the extremals according to $\|u_{\infty,q}\|_{L^\infty(\partial\Omega)} = \|u_{\infty,\infty}\|_{L^\infty(\partial\Omega)} = 1$). We prove that $S_{\infty,\infty} = 1$ is the limit of $S_{p,q}$ as $p, q \to \infty$ and also $S_{\infty,q}$ is the limit of $S_{p,q}$ when $p \to \infty$.

Theorem 1.2 *Let $S_{p,q}$ be the best Sobolev trace constant and $u_{p,q}$ be any extremal normalized such that $\|u_{p,q}\|_{L^\infty(\partial\Omega)} = 1$. Then*

$$\lim_{p,q \to \infty} S_{p,q} = S_{\infty,\infty} = 1,$$

and, for any $1 < r < \infty$, as $p, q \to \infty$,

$$\begin{aligned} u_{p,q} &\rightharpoonup u_{\infty,\infty} \equiv 1, && \textit{weakly in } W^{1,r}(\Omega), \\ u_{p,q} &\to u_{\infty,\infty} \equiv 1, && \textit{strongly in } C^\alpha(\overline{\Omega}). \end{aligned}$$

Moreover, for fixed $1 \le q < \infty$,

$$\lim_{p\to\infty} S_{p,q} = S_{\infty,q} = \frac{1}{|\partial\Omega|^{1/q}},$$

and, for any $1 < r < \infty$, *as* $p \to \infty$,

$$\begin{aligned} u_{p,q} &\rightharpoonup u_{\infty,q} \equiv 1, && \text{weakly in } W^{1,r}(\Omega), \\ u_{p,q} &\to u_{\infty,q} \equiv 1, && \text{strongly in } C^{\alpha}(\overline{\Omega}). \end{aligned}$$

The limit $q \to \infty$ with $p > N$ fixed is more subtle since we do not know a priori which is the extremal for the limit case. However we find an equation for the limit extremal.

Theorem 1.3 *Let* $p > N$, *then*

$$\lim_{q\to\infty} S_{p,q} = S_{p,\infty},$$

and, up to subsequences, as $q \to \infty$,

$$\begin{aligned} u_{p,q} &\rightharpoonup u_{p,\infty} && \text{weakly in } W^{1,p}(\Omega), \\ u_{p,q} &\to u_{p,\infty} && \text{strongly in } C^{\alpha}(\overline{\Omega}). \end{aligned}$$

Moreover, there exists a measure $\mu \in C(\partial\Omega)^*$ *with* $\mu(\{u_{p,\infty} = 1\}) = 1$ *such that* $u_{p,\infty}$ *is a weak solution of*

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S_{p,\infty}^p \mu \chi_{\{u\equiv 1\}} & \text{on } \partial\Omega. \end{cases}$$

We observe that $W^{1,N}(\Omega) \not\hookrightarrow L^\infty(\partial\Omega)$. Hence we expect that the best constant $S_{p,q}$ goes to zero as $(p,q) \to (N,\infty)$. This is the content of our next result.

Theorem 1.4 *The best constant* $S_{p,q}$ *goes to zero as* $(p,q) \to (N,\infty)$ *and moreover for any* $\alpha < (N-1)/N$, *there exists a constant* C *such that*

$$S_{p,q} \le C \max\left\{(p-N)_+,\ \frac{1}{q}\right\}^{\alpha}.$$

For the dependence of $S_{p,q}(\Omega)$ with respect to the domain, see [4] and [9] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding and contracting domains. In [5] a related problem in the half-space $\mathbb{R}^N_+$ for the critical exponent is studied. See also [6], [7] for other geometric problems that lead to nonlinear boundary conditions, like the ones that appear in (1.2). The best constant in the Sobolev immersion, $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, has been studied by many authors, see for example [10]. More recently in [11] the authors analyze the limit as $p \to \infty$ of the related Dirichlet eigenvalue problem for the $p-$Laplacian.

The paper is organized as follows: first we deal with the limit cases. In sections 2 and 3 we prove Theorem 1.2 and Theorem 1.3 respectively, in section 4 we find estimates for $S_{p,q}$ near (N,∞), Theorem 1.4, and finally in section 5 we deal with the proof of our main result, Theorem 1.1.

2 Limit as $p \to +\infty$

In this section we prove Theorem 1.2.

Proof. First, we study the limit $p, q \to \infty$. In this case the natural limit problem is

$$S_{\infty,\infty} = \inf_{u \in W^{1,\infty}(\Omega)} \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}.$$

As we mentioned in the introduction $S_{\infty,\infty} = 1$ and the extremal is $u_{\infty,\infty} \equiv 1$ (normalized such that $\|u\|_{L^\infty(\partial\Omega)} = 1$). Now, taking $u = 1$ in (1.1), we get

$$S_{p,q} = \inf_{u \in W^{1,p}(\Omega)} Q_{p,q}(u) \le \frac{|\Omega|^{1/p}}{|\partial\Omega|^{1/q}}, \tag{2.1}$$

from where it follows that

$$\limsup_{p,q\to\infty} S_{p,q} \le 1. \tag{2.2}$$

For $p > N$, let us denote by $u_{p,q}$ one extremal for (1.1) normalized such that $\|u_{p,q}\|_{L^\infty(\partial\Omega)} = 1$. Hence

$$\|u_{p,q}\|_{W^{1,p}(\Omega)} = S_{p,q}\|u_{p,q}\|_{L^q(\partial\Omega)} \le S_{p,q}|\partial\Omega|^{1/q} \le C,$$

with C independent of p, q. On the other hand, if $N < r < p$,

$$\|u_{p,q}\|_{W^{1,r}(\Omega)} \le |\Omega|^{(p-r)/pr}\|u_{p,q}\|_{W^{1,p}(\Omega)} \le C.$$

Hence, there exists $u \in W^{1,r}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} &u_{p,q} \rightharpoonup u \text{ weakly in } W^{1,r}(\Omega), \\ &u_{p,q} \to u \text{ strongly in } C^\alpha(\overline{\Omega}). \end{aligned}$$

Observe that we can assume that the limit u does not depend on r. In fact, we can choose a sequence $r_j \to \infty$ and in each W^{1,r_j} we can extract a subsequence of $u_{p,q}$ that converges weakly. By a standard diagonal argument we obtain a subsequence that converges strongly in C^α and weakly in W^{1,r_j} for every j (and hence in $W^{1,r}$ for every r) to a limit function u.

In particular, $\|u\|_{L^\infty(\partial\Omega)} = 1$ and

$$S_{p,q} = Q_{p,q}(u_{p,q}) \ge \frac{|\Omega|^{-(p-r)/pr}\|u_{p,q}\|_{W^{1,r}(\Omega)}}{\|u_{p,q}\|_{L^q(\partial\Omega)}} \ge \frac{|\Omega|^{-(p-r)/pr}\|u_{p,q}\|_{W^{1,r}(\Omega)}}{|\partial\Omega|^{1/q}}.$$

Hence

$$1 \ge \limsup_{p,q\to\infty} \frac{|\Omega|^{-(p-r)/pr}\|u_{p,q}\|_{W^{1,r}(\Omega)}}{|\partial\Omega|^{1/q}} \ge |\Omega|^{-1/r}\|u\|_{W^{1,r}(\Omega)},$$

and therefore, taking the limit as $r \to \infty$, we get

$$1 \ge \|u\|_{W^{1,\infty}(\Omega)}.$$

We conclude that $u \in W^{1,\infty}(\Omega)$ and that u is an extremal for $S_{\infty,\infty}$ that satisfies $\|u\|_{L^\infty(\partial\Omega)} = 1$, and hence $u \equiv 1$.

Next, we focus on the case $p \to +\infty$ with fixed $1 \le q < \infty$. We consider the natural limit problem

$$S_{\infty,q} = \inf_{u \in W^{1,\infty}(\Omega)} \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^q(\partial\Omega)}},$$

and we note that the extremal is $u_{\infty,q} \equiv 1$ (normalized such that $\|u_{\infty,q}\|_{L^\infty(\partial\Omega)} = 1$) and then the best constant is given by $S_{\infty,q} = 1/|\partial\Omega|^{1/q}$.

Following the same argument given above we get that there exists $u \in W^{1,r}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} &u_{p,q} \rightharpoonup u \text{ weakly in } W^{1,r}(\Omega), \\ &u_{p,q} \to u \text{ strongly in } C^\alpha(\overline{\Omega}). \end{aligned}$$

Moreover, we have the following inequalities,

$$\frac{|\Omega|^{1/p}}{|\partial\Omega|^{1/q}} \ge S_{p,q} = Q_{p,q}(u_{p,q}) \ge \frac{|\Omega|^{-(p-r)/pr}\|u_{p,q}\|_{W^{1,r}(\Omega)}}{\|u_{p,q}\|_{L^q(\partial\Omega)}}.$$

First we take the limit as $p \to \infty$, and then the limit as $r \to \infty$, to obtain

$$\frac{1}{|\partial\Omega|^{1/q}} \ge S_{\infty,q} \ge \frac{\|u\|_{W^{1,\infty}(\Omega)}}{\|u\|_{L^q(\partial\Omega)}}.$$

Therefore, we can conclude that $u \in W^{1,\infty}(\Omega)$ and that it is an extremal for $S_{\infty,q}$ which satisfies $\|u\|_{L^\infty(\partial\Omega)} = 1$. Hence $u = u_{\infty,q} \equiv 1$ and $S_{\infty,q} = 1/|\partial\Omega|^{1/q}$. □

3 Limit as $q \to +\infty$ for fixed $p > N$

In this section we fix $p > N$ and consider the limit of $S_{p,q}$ and $u_{p,q}$ when $q \to \infty$. In order to clarify the exposition we divide the proof of Theorem 1.3 in two lemmas.

Lemma 3.1 *Let $p > N$ be fixed. Then*

$$\lim_{q \to \infty} S_{p,q} = S_{p,\infty},$$

and, up to subsequences, as $q \to \infty$,

$$\begin{aligned} &u_{p,q} \rightharpoonup u_{p,\infty} && \textit{weakly in } W^{1,p}(\Omega), \\ &u_{p,q} \to u_{p,\infty} && \textit{strongly in } C^\alpha(\overline{\Omega}). \end{aligned}$$

Proof. Let $u_{p,q}$ be an extremal for (1.1) normalized such that $\|u_{p,q}\|_{L^\infty(\partial\Omega)} = 1$. Then we have

$$S_{p,q} = \frac{\|u_{p,q}\|_{W^{1,p}(\Omega)}}{\|u_{p,q}\|_{L^q(\partial\Omega)}} \ge \frac{\|u_{p,q}\|_{W^{1,p}(\Omega)}}{|\partial\Omega|^{1/q}}. \tag{3.1}$$

Therefore, using (2.1), we have $\|u_{p,q}\|_{W^{1,p}(\Omega)} \leq |\Omega|^{1/p}$. Hence, there exists a function $u \in W^{1,p}(\Omega)$ such that, up to a subsequence,

$$\begin{array}{ll} u_{p,q} \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega), \\ u_{p,q} \to u & \text{strongly in } L^{\infty}(\partial\Omega). \end{array}$$

Hence $\|u\|_{L^{\infty}(\partial\Omega)} = 1$, and from (3.1) we get

$$\liminf_{q\to\infty} S_{p,q} \geq \liminf_{q\to\infty} \|u_{p,q}\|_{W^{1,p}(\Omega)} \geq \|u\|_{W^{1,p}(\Omega)} \geq S_{p,\infty}.$$

Now, let us see that u is an extremal for $S_{p,\infty}$. We argue by contradiction. Assume that there exists $v \in W^{1,p}(\Omega)$ such that

$$Q_{p,\infty}(v) < Q_{p,\infty}(u).$$

Then, for large q we have,

$$Q_{p,q}(v) < Q_{p,q}(u),$$

but as

$$\begin{aligned} S_{p,q} \geq & \quad \frac{\|u_{p,q}\|_{W^{1,p}(\Omega)}}{|\partial\Omega|^{1/q}} \geq \frac{\|u\|_{W^{1,p}(\Omega)} - \varepsilon_q}{|\partial\Omega|^{1/q}} \\ \geq & \quad \left(\frac{\|u\|_{L^q(\partial\Omega)}}{|\partial\Omega|^{1/q}}\right) \frac{\|u\|_{W^{1,p}(\Omega)} - \varepsilon_q}{\|u\|_{L^q(\partial\Omega)}} > \frac{\|v\|_{W^{1,p}(\Omega)}}{\|v\|_{L^q(\partial\Omega)}} \end{aligned}$$

for some ε_q that goes to zero as $q \to \infty$, we arrive to a contradiction.

To finish the proof of the Lemma, we observe that

$$S_{p,q} \leq Q_{p,q}(u) \to Q_{p,\infty}(u) = S_{p,\infty}.$$

Therefore, $\limsup_{q\to\infty} S_{p,q} \leq S_{p,\infty}$. □

Lemma 3.2 *Let $p > N$ be fixed and let $u_{p,\infty}$ be an extremal for (1.1) obtained as limit of a sequence of extremals $u_{p,q}$, as $q \to \infty$. Then there exists a measure $\mu \in C(\partial\Omega)^*$, with $\mu(\{u_{p,\infty} \equiv 1\}) = 1$, such that $u_{p,\infty}$ is a weak solution of*

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S_{p,\infty}^p \mu \chi_{\{u\equiv 1\}} & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Proof. Let $u_{p,q}$ be as in Lemma 3.1. As $u_{p,q}$ is a weak solution of (1.2), we have that for every $\phi \in W^{1,p}(\Omega)$,

$$\int_{\Omega} (|\nabla u_{p,q}|^{p-2}\nabla u_{p,q}\nabla\phi + |u_{p,q}|^{p-2}u_{p,q}\phi)\,dx =$$

$$S_{p,q}^p \left(\int_{\partial\Omega} |u_{p,q}|^q\,d\sigma\right)^{(p-q)/q} \int_{\partial\Omega} |u_{p,q}|^{q-2}u_{p,q}\phi\,d\sigma.$$

Let us define $\Psi_q \in L^\infty(\partial\Omega)^*$ as

$$\Psi_q(\phi) = \left(\int_{\partial\Omega} |u_{p,q}|^q \, d\sigma \right)^{(p-q)/q} \int_{\partial\Omega} |u_{p,q}|^{q-2} u_{p,q} \phi \, d\sigma.$$

By Hölder inequality, we get

$$|\Psi_q(\phi)| \leq \|u_{p,q}\|^{p-1}_{L^q(\partial\Omega)} \|\phi\|_{L^q(\partial\Omega)} \leq |\partial\Omega|^{p/q} \|u_{p,q}\|^{p-1}_{L^\infty(\partial\Omega)} \|\phi\|_{L^\infty(\partial\Omega)} \leq C \|\phi\|_{L^\infty(\partial\Omega)}$$

with C independent of q. Therefore, $\|\Psi_q\| \leq C$ and hence if we call

$$v_q = \left(\int_{\partial\Omega} |u_{p,q}|^q \, d\sigma \right)^{(p-q)/q} |u_{p,q}|^{q-2} u_{p,q},$$

we have that v_q is uniformly bounded in $L^1(\partial\Omega)$ and then, up to a subsequence, $v_q \stackrel{*}{\rightharpoonup} \mu$ weakly-* in the sense of measures.

In order to finish the proof, we will see that $\mathrm{supp}(\mu) \subset \{u_{p,\infty} = 1\}$. To prove this, we consider a point $x_0 \in \partial\Omega$ such that $u_{p,\infty}(x_0) < 1 - 2\delta$ for some δ small enough. Hence, for q large enough we have that $u_{p,q}(x_0) < 1 - \delta$. On the other hand, as $\|u_{p,\infty}\|_{L^\infty(\partial\Omega)} = 1$, and by the C^α convergence of $u_{p,q}$ to $u_{p,\infty}$ there exists a point $x_1 \in \partial\Omega$ and r independent of q such that $B_r(x_1) \cap \partial\Omega \subset \{x \in \partial\Omega : u_{p,q}(x) > 1 - \delta/2\}$. Therefore

$$|\partial\Omega|^{1/q} \geq \left(\int_{\partial\Omega} |u_{p,q}|^q \, d\sigma \right)^{1/q} \geq (1 - \delta/2) |B_r(x_1) \cap \partial\Omega|^{1/q},$$

where the first inequality follows from the fact that $\|u_{p,q}\|_{L^\infty(\partial\Omega)} = 1$. Now, we rewrite v_q as follows,

$$\begin{aligned} v_q(x_0) &= \left(\frac{u_{p,q}(x_0)}{\|u_{p,q}\|_{L^q(\partial\Omega)}} \right)^{q-1} \|u_{p,q}\|^{p-1}_{L^q(\partial\Omega)} \\ &\leq \left(\frac{1-\delta}{(1-\delta/2)|B_r(x_0) \cap \partial\Omega|^{1/q}} \right)^{q-1} |\partial\Omega|^{(p-1)/q}. \end{aligned}$$

Hence, we conclude that $v_q(x_0) \to 0$, and we get that the measure is supported in $\{x \in \partial\Omega : \ u(x) = 1\}$. Moreover, if we take $u_{p,\infty}$ as test function in the weak form of (3.2), we get

$$\int_\Omega (|\nabla u_{p,\infty}|^p + |u_{p,\infty}|^p) \, dx = S^p_{p,\infty} \int_{\partial\Omega \cap \{u_{p,\infty} = 1\}} d\mu.$$

As $u_{p,\infty}$ is an extremal and verifies $\|u_{p,\infty}\|_{L^\infty(\partial\Omega)} = 1$ we have that

$$\int_\Omega (|\nabla u_{p,\infty}|^p + |u_{p,\infty}|^p) \, dx = S^p_{p,\infty}.$$

Therefore $\mu(\partial\Omega \cap \{u_{p,\infty} = 1\}) = 1$. This completes the proof. □

4 Estimates for (p, q) near (N, ∞)

In this section we find an upper bound for the vanishing rate of $S_{p,q}$ as (p, q) approaches (N, ∞), that is we prove Theorem 1.4.

Proof. If $p < N$, using Holder inequality we have that there exist a constant C such that

$$S_{p,q} \le C S_{N,q}, \qquad \text{for } p < N.$$

Hence, we can assume that $p \ge N$. In order to obtain a upper bound on the decay rate, we suppose that $0 \in \partial\Omega$, $\alpha < (N-1)/N$, and we consider the function

$$u_\varepsilon(x) = \left(\ln(1 + \frac{1}{|x| + \varepsilon}) \right)^\alpha \in W^{1,p}(\Omega).$$

Then we obtain a bound for $\|u_\varepsilon\|_{L^q(\partial\Omega)}$ as follows, given $M < \|u_\varepsilon\|_{L^\infty(\partial\Omega)}$,

$$\|u_\varepsilon\|_{L^q(\partial\Omega)} \ge \quad \left(\int_{\{x \in \partial\Omega : u_\varepsilon(x) \ge M\}} |u_\varepsilon|^q \right)^{1/q} \ge M |\{x \in \partial\Omega : u_\varepsilon(x) \ge M\}|^{1/q}.$$

On the other hand, let us compute

$$|\nabla u_\varepsilon|^p \le \alpha^p \left(\ln(1 + \frac{1}{|x| + \varepsilon}) \right)^{(\alpha-1)p} \left(\frac{1}{|x| + \varepsilon} \right)^p.$$

Hence,

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^p \le \quad & C \int_0^C \frac{r^{N-1}}{(r+\varepsilon)^p} \left(\ln(1 + \frac{1}{r + \varepsilon}) \right)^{(\alpha-1)p} dr \\ \le \quad & C \int_\varepsilon^C w^{N-p-1} (\ln w)^{(\alpha-1)p} \, dw \le \frac{C}{\varepsilon^{p-N}}. \end{aligned}$$

Moreover,

$$\int_\Omega |u_\varepsilon|^p \le C.$$

Summing up, we obtain that

$$S_{p,\infty} \le \frac{C}{\varepsilon^{p-N} M |\{x \in \partial\Omega : u_\varepsilon(x) \ge M\}|^{1/q}}.$$

If $q(p-N) \ge 1$, we take $M \sim 1/(p-N)^\alpha$ and $\varepsilon \sim e^{-1/(p-N)}$ and if $q(p-N) \le 1$, $M \sim q^\alpha$ and $\varepsilon \sim e^{-q}$. With this choice, we obtain

$$S_{p,q} \le C \max \left\{ (p-N)_+, \ \frac{1}{q} \right\}^\alpha \to 0, \qquad \text{as } (p, q) \to (N, \infty).$$

This ends the proof. $\square$

5 Uniform bounds for $S_{p,q}$

In this section we prove our main result, Theorem 1.1.

Proof. From Theorem 1.4 we get that the best constant $S_{p,q}$ degenerates as $(p,q) \to (N,\infty)$, hence to obtain uniform bounds we have to stay far from that point.

A uniform upper bound for $S_{p,q}$ follows from (2.1), namely,

$$S_{p,q} \leq \frac{|\Omega|^{1/p}}{|\partial\Omega|^{1/q}} \leq C_2, \tag{5.1}$$

for $1 \leq p,q \leq \infty$. The lower bound is more subtle. First we observe that, by Hölder's inequality, we have

$$\|u\|_{L^{q_1}(\partial\Omega)} \leq |\partial\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}} \|u\|_{L^{q_2}(\partial\Omega)}$$

for $1 \leq q_1 \leq q_2$, and

$$\|u\|_{W^{1,p_2}(\Omega)} \leq |\Omega|^{\frac{1}{p_2}-\frac{1}{p_1}} \|u\|_{W^{1,p_1}(\Omega)}$$

for $1 \leq p_2 \leq p_1$. Therefore, there exists a constant C independent of $1 \leq p \leq \infty$ and $1 \leq q \leq p^*$ such that

$$S_{p_1,q_1} \geq C S_{p_2,q_2}, \tag{5.2}$$

for any $1 \leq q_1 \leq q_2$ and $1 \leq p_2 \leq p_1$. Inequality (5.2) says that in order to obtain lower bounds for $S_{p,q}$ we can enlarge q and decrease p. Therefore, in order to get uniform bounds for $S_{p,q}$ in sets A that are far from the point (N,∞) we can proceed as follows. From our assumptions on A we have that there exists $s < N < r$ such that

$$A \subset \big\{(p,q) \,:\, p > r\big\} \cup \big\{(p,q) \,:\, 1 \leq p \leq r \text{ and } 1 \leq q \leq \min\{p^*, s^*\}\big\} = A_1 \cup A_2,$$

see Figure 1 below.

Figure 1.

From our previous estimate (5.2) we get that

$$S_{p,q} \geq C S_{r,\infty} \tag{5.3}$$

for $(p, q) \in A_1$, and

$$S_{p,q} \geq C \min_{1 \leq p \leq s} S_{p,p^*}$$

for $(p, q) \in A_2$. To estimate the value of the best Sobolev trace constant along the critical curve (p, p^*) with $1 \leq p \leq s$, we use interpolation theory, see [2], [3]. We have, for the trace operator T

$$T : W^{1,1}(\Omega) \to L^1(\partial\Omega), \qquad S_{1,1} \|Tu\|_{L^1(\partial\Omega)} \leq \|u\|_{W^{1,1}(\Omega)},$$

and

$$T : W^{1,s}(\Omega) \to L^{s^*}(\partial\Omega), \qquad S_{s,s^*} \|Tu\|_{L^{s^*}(\partial\Omega)} \leq \|u\|_{W^{1,s}(\Omega)}.$$

Therefore,

$$T : W^{1,p}(\Omega) \to L^q(\partial\Omega), \qquad S_{p,q} \|Tu\|_{L^q(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)},$$

with

$$\frac{1}{p} = \theta + \frac{1-\theta}{s}, \qquad \frac{1}{q} = \theta + \frac{1-\theta}{s^*}, \tag{5.4}$$

and

$$S_{p,q} \geq S_{1,1}^{\theta} S_{s,s^*}^{1-\theta},$$

for any $0 < \theta < 1$. We observe that if (p, q) are given by (5.4) we have $q = p^*$ hence there exists a constant C that only depends on s such that

$$\min_{1 \leq p \leq s} S_{p,p^*} \geq \min\{S_{1,1}, S_{s,s^*}\} \geq C.$$

Hence we have a uniform lower bound

$$S_{p,q} \geq C, \tag{5.5}$$

for $(p, q) \in A_2$. From (5.1), (5.3) and (5.5) we conclude the desired result. □

Acknowledgements. We want to thank R. Duran and P. Groisman for their valuable help.

References

[1] I. Babuska and J. Osborn, Eigenvalue Problems, Handbook of Numer. Anal., Vol. II (1991). North-Holland.

[2] A. P. Calderon, *Intermediate sapces and interpolation*, Stud. Math. **24** (1964), 113-190.

[3] C. P. Calderon and M. Milman, *Interpolation of Sobolev spaces. The real method*, Indiana Univ. Math. J. **32** (1983), 801-809.

[4] C. Flores and M. del Pino, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Comm. Partial Differential Equations **26**, no.11-12 (2001), 2189-2210.

[5] J. F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988), 687-698.

[6] J. F. Escobar, *Uniqueness theorems on conformal deformations of metrics, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math. **43** (1990), 857-883.

[7] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature*, Ann. of Math. **136** (1992), 1-50.

[8] J. Fernández Bonder and J.D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions*, J. Math. Anal., Appl. **263** (2001), 195-223.

[9] J. Fernández Bonder and J.D. Rossi, *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains*, Comm. Pure Appl. Anal. **1**, no.3 (2002), 359-378.

[10] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, NY (1983).

[11] P. Juutinen, P. Lindqvist and J. J. Manfredi, *The ∞-eigenvalue problem*, Arch. Rat. Mech. Anal. **148** (1999), 89-105.

[12] S. Martinez and J.D. Rossi, *Isolation and simplicity for the first eigenvalue of the $p-$Laplacian with a nonlinear boundary condition*, Abst. Appl. Anal. **7 (5)** (2002), 287-293.

[13] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, **51** (1984), 126-150.

[14] J.L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12 (3)** (1984), 191-202.