REGULARITY OF THE FREE BOUNDARY IN AN OPTIMIZATION PROBLEM RELATED TO THE BEST SOBOLEV TRACE CONSTANT

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ABSTRACT. In this paper we study the regularity properties of a free boundary problem arising in the optimization of the best Sobolev trace constant in the immersion $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ for functions that vanish in a subset of Ω. This problem is also related to a minimization problem for Steklov eigenvalues.

1. INTRODUCTION.

The study of Sobolev inequalities and of optimal constants is a subject of interest in the analysis of PDE's and related topics. It has been widely studied in the past by many authors and is still an area of intensive research. See for instance the book [1], and, for recent developments in this field, see the articles [6, 9, 10, 17] and the survey [7] among others.

The optimal Sobolev constant and its corresponding extremals (if they exist) are related to eigenvalue problems. In the case of the best Sobolev trace embedding $H^1(\Omega) \to L^q(\partial\Omega)$ where Ω is a bounded smooth domain in $\mathbb{R}^N$, the best constant and the extremal (that exists for $1 \le q < 2_* = 2(N-1)/(N-2)$ since the immersion is compact) give rise to the following elliptic problem with nonlinear boundary conditions

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u^{q-1} & \text{on } \partial\Omega. \end{cases}$$

The constant λ depends on the normalization of the extremal u. For instance if u is chosen so that $\|u\|_{L^q(\partial\Omega)} = 1$, then $\lambda = S$ the best Sobolev trace constant. In the linear case, $q = 2$, this problem becomes an eigenvalue problem that is known as the *Steklov eigenvalue problem* [19].

In this paper we are interested in the best Sobolev trace constant among functions that vanish in a subset of Ω. We try to optimize this best constant when varying the subset in the class of measurable sets with prescribed positive measure α. In a previous article [11], we proved that there exists an optimal set. In this paper we focus our attention on regularity properties of these optimal sets.

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More precisely, in [11] we studied the following problem. Let

$$\mathcal{J}(v) = \int_\Omega |\nabla v|^2 + v^2\,dx,$$
$$\mathcal{A}_\alpha = \{v \in H^1(\Omega) \ / \ \|v\|_{L^q(\partial\Omega)} = 1 \text{ and } |\{v > 0\}| = \alpha\}.$$

Then the problem is:

$$(P_\alpha) \qquad \text{Find } \phi_0 \in \mathcal{A}_\alpha \text{ such that } \quad S(\alpha) := \inf_{v\in\mathcal{A}_\alpha} \mathcal{J}(v) = \mathcal{J}(\phi_0).$$

In [11] we proved that there exists a solution ϕ_0 to (P_α) but the approach in [11] does not give any regularity properties of ϕ_0 nor of the *hole* $\{\phi_0 = 0\}$.

In this paper we consider a different approach. Instead of minimizing $\mathcal{J}(v)$ over $\mathcal{A}_\alpha$ we penalize the functional and minimize without the measure restriction. This approach has been used with great success by many authors starting with the work [2] (see also [3, 15, 16, 20], etc.). So, let

$$\mathcal{J}_\varepsilon(v) = \int_\Omega |\nabla v|^2 + v^2\,dx + F_\varepsilon(|\{v > 0\}|), \tag{1.1}$$

where

$$F_\varepsilon(s) = \begin{cases} \frac{1}{\varepsilon}(s-\alpha) & \text{if } s \ge \alpha \\ \varepsilon(s-\alpha) & \text{if } s < \alpha. \end{cases}$$

The penalized problem is to minimize $\mathcal{J}_\varepsilon$ over the class

$$\mathcal{K}_1 = \{v \in H^1(\Omega)\,/\,\|v\|_{L^q(\partial\Omega)} = 1\}.$$

For technical reasons, it is better to minimize in the class

$$\mathcal{K} = \{v \in H^1(\Omega)\,/\,\|v\|_{L^q(\Gamma_N)} = 1,\ v = \varphi_0 \text{ on } \Gamma_D\},$$

where $\emptyset \neq \Gamma_N \subset \partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_N$ is the closure of a relatively open set of the boundary and $\varphi_0 \in H^1(\Omega)$, $\varphi_0 \ge c_0 > 0$ on Γ_D. We will only need to assume that $\Gamma_D \neq \emptyset$ at the end of our arguments. See Section 4, Lemma 4.3.

So the penalized problem is:

$$(P_\varepsilon) \qquad \text{Find } u_\varepsilon \in \mathcal{K} \quad \text{such that} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v\in\mathcal{K}} \mathcal{J}_\varepsilon(v).$$

Observe that minimizing $\mathcal{J}_\varepsilon$ over $\mathcal{K}$ gives a problem with mixed boundary conditions. We believe that this problem has independent interest.

The main idea is to prove that for ε small any minimizer u_ε of $\mathcal{J}_\varepsilon$ in $\mathcal{K}$ satisfies $|\{u_\varepsilon > 0\}| = \alpha$, therefore the penalization term F_ε vanishes and hence we have a minimizer of our original problem. This allows us to avoid the passage to the limit (as $\varepsilon \to 0$) where uniform bounds are needed. To prove regularity of the minimizers of $\mathcal{J}_\varepsilon$ and their free boundaries, $\partial\{u_\varepsilon > 0\}$, is easier than the original problem, thanks to the results of [4].

The main theorem in this article is:

Theorem 1.1. *For every $\varepsilon > 0$ there exists a solution $u_\varepsilon \in \mathcal{K}$ to (P_ε). Moreover, any such solution is a locally Lipschitz continuous function and the free boundary $\partial\{u_\varepsilon > 0\}$ is locally a $C^{1,\beta}$ surface up to a set of $\mathcal{H}^{N-1}-$measure zero. In the case $N = 2$ the free boundary is locally a $C^{1,\beta}$ surface. Moreover, if $\Gamma_D \neq \emptyset$, for ε small we have $|\{u_\varepsilon > 0\}| = \alpha$.*

Outline of the paper. In Section 2 we begin our analysis of problem (P_ε) for fixed ε. First we prove the existence of a minimizer, local Lipschitz regularity and nondegeneracy near the free boundary (Theorem 2.1). Then we prove that a minimizer u_ε of (P_ε) is a weak solution to the following free boundary problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \{u > 0\} \cap \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda_\varepsilon & \text{on } \partial\{u > 0\} \cap \Omega, \end{cases}$$

where λ_ε is a positive constant (Theorem 2.6).

In Section 3, again for fixed ε, we analyze the regularity of the free boundary and show that, up to a set of $\mathcal{H}^{N-1}-$measure zero, $\partial\{u_\varepsilon > 0\}$ is locally a $C^{1,\beta}$ surface and, in the case $N = 2$, the free boundary has no exceptional points (Theorem 3.1). The proof of this result follows almost exactly the lines in [4], so we only remark the significant differences and refer to [4] for further details.

In Section 4 we analyze the behavior of the solutions to (P_ε) for small ε. We prove that, if $\Gamma_D \neq \emptyset$, the positivity set of the minimizer u_ε has measure α (Theorem 4.1).

Finally, in Section 5, we go back to our original problem and show, under some mild assumptions on the solutions ϕ_0 to (P_α), that they are also solutions to (P_ε) for small ε, so they inherit the properties of the solutions to (P_ε) (Theorem 5.1). These extra assumptions are satisfied, for instance, if Ω is a ball (Corollary 5.1). In the general case, without the assumption that $\Gamma_D \neq \emptyset$, we prove that the set of α's for which there is a solution to (P_α) with smooth free boundary is dense in $(0, |\Omega|)$ (Theorem 5.2). Then, we show that the minimizers of (P_ε) converge (up to a subsequence) to a solution to (P_α) (Theorem 5.3). We believe that this last result might be of interest in numerical approximations.

2. The penalized problem

In this section, we consider the penalized problem (P_ε) stated in the introduction and prove the existence of a minimizer and some regularity properties.

Theorem 2.1. *There exists a solution to the problem (P_ε). Moreover, any such solution u_ε has the following properties:*

(1) *u_ε is locally Lipschitz continuous in Ω.*
(2) *For every $D \subset\subset \Omega$, there exist constants $C, c > 0$ such that for every $x \in D \cap \{u_\varepsilon > 0\}$,*

$$c \operatorname{dist}(x, \partial\{u_\varepsilon > 0\}) \leq u_\varepsilon(x) \leq C \operatorname{dist}(x, \partial\{u_\varepsilon > 0\}).$$

(3) *For every $D \subset\subset \Omega$, there exists a constant $c > 0$ such that for $x \in \partial\{u > 0\}$ and $B_r(x) \subset D$,*

$$c \leq \frac{|B_r(x) \cap \{u_\varepsilon > 0\}|}{|B_r(x)|} \leq 1 - c.$$

The constants may depend on ε.

The proof will be divided into a series of steps for the reader's convenience.

Proof of existence. Let $(u_n) \subset \mathcal{K}$ be a minimizing sequence for $\mathcal{J}_\varepsilon$. Then $\mathcal{J}_\varepsilon(u_n)$ is bounded and so $\|u_n\|_{H^1(\Omega)} \le C$. Therefore there exists a subsequence (that we still call u_n) and a function $u_\varepsilon \in H^1(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_\varepsilon \quad \text{weakly in } H^1(\Omega),\\ u_n &\to u_\varepsilon \quad \text{strongly in } L^q(\partial\Omega),\\ u_n &\to u_\varepsilon \quad \text{a.e. } \Omega. \end{aligned}$$

Thus,

$$\begin{aligned} &\|u_\varepsilon\|_{L^q(\Gamma_N)} = 1,\\ &u_\varepsilon = \varphi_0 \quad \text{on} \quad \Gamma_D,\\ &|\{u_\varepsilon > 0\}| \le \liminf_{n\to\infty} |\{u_n > 0\}| \quad \text{and}\\ &\|u_\varepsilon\|_{H^1(\Omega)} \le \liminf_{n\to\infty} \|u_n\|_{H^1(\Omega)}. \end{aligned}$$

Hence $u_\varepsilon \in \mathcal{K}$ and

$$\mathcal{J}_\varepsilon(u_\varepsilon) \le \liminf_{n\to\infty} \mathcal{J}_\varepsilon(u_n) = \inf_{v\in\mathcal{K}} \mathcal{J}_\varepsilon(v),$$

therefore u_ε is a minimizer of $\mathcal{J}_\varepsilon$ in $\mathcal{K}$. □

Remark 2.1. Any minimizer u_ε of $\mathcal{J}_\varepsilon$ satisfies the inequality

$$\Delta u - u \ge 0 \quad \text{in } \Omega. \tag{2.1}$$

In fact, this can be seen by performing one side perturbations. Namely, we let $v = u_\varepsilon - t\varphi$ with $t > 0$ and $\varphi \in C_0^\infty(\Omega)$, $\varphi \ge 0$ to get

$$\int_\Omega \nabla u_\varepsilon \nabla\varphi + u_\varepsilon\varphi \le 0.$$

In the remaining of the section we will remove the subscript ε from the solution of (P_ε).

For the proof of properties (1)–(3), we apply the ideas developed in [4]. To this end, we need a series of lemmas.

Lemma 2.1. *Let $u \in \mathcal{K}$ be a solution to (P_ε). There exists a constant $C = C(N, \Omega, \varepsilon)$ such that for every ball $B_r \subset\subset \Omega$*

$$\frac{1}{r}⨍_{\partial B_r} u \ge C \quad \textit{implies} \quad u > 0 \quad \textit{in} \quad B_r.$$

Proof. The idea is similar to that of Lemma 3.2 in [4]. Let v be the solution to

$$\begin{cases} v = u & \text{in } \overline{\Omega \setminus B_r},\\ \Delta v = v & \text{in } B_r. \end{cases} \tag{2.2}$$

Then $v \in \mathcal{K}$, $v > 0$ in B_r. We claim that

$$\|u - v\|^2_{H^1(\Omega)} = \|u\|^2_{H^1(\Omega)} - \|v\|^2_{H^1(\Omega)}. \tag{2.3}$$

In fact,

$$\int_{B_r} \nabla v \nabla(v - u) + v(v - u)\, dx = 0$$

since $v-u\in H^1_0(B_r)$. This implies

$$\int_{B_r} \nabla u\nabla v + uv\,dx = \int_{B_r} |\nabla v|^2 + v^2\,dx. \tag{2.4}$$

This equality implies the claim since $u=v$ in $\Omega\setminus B_r$.

By (2.1), $u\le v$ in B_r. Now, by (2.3), since u is a minimizer and $u=v$ in $\Omega\setminus B_r$, we have

$$\begin{aligned}\int_{\Omega} |\nabla(u-v)|^2 + (u-v)^2\,dx &\le -F_\varepsilon(|\{u>0\}|) + F_\varepsilon(|\{v>0\}|)\\ &\le C_\varepsilon\,|\{u=0\}\cap B_r|.\end{aligned} \tag{2.5}$$

Now, as in [4], the idea is to control $|\{u=0\}\cap B_r|$ from above by the left hand side of (2.5). By replacing $u(x)$ by $u(x_0+rx)/r$ we can assume that $B_r=B_1(0)$. For $|z|\le\frac12$ we consider the change of variables from B_1 into itself such that z becomes the new origin. We call $u_z(x)=u\big((1-|x|)z+x\big)$, $v_z(x)=v\big((1-|x|)z+x\big)$ and define

$$r_\xi = \inf\Big\{r \,/\, \frac18 \le r\le 1 \quad\text{and}\quad u_z(r\xi)=0\Big\},$$

if this set is nonempty. Observe that this change of variables leaves the boundary fixed.

Now, for almost every $\xi\in\partial B_1$ we have

$$v_z(r_\xi\xi) = \int_{r_\xi}^1 \frac{d}{dr}(u_z-v_z)(r\xi)\,dr \le \sqrt{1-r_\xi}\left(\int_{r_\xi}^1 |\nabla(u_z-v_z)(r\xi)|^2\,dr\right)^{1/2}. \tag{2.6}$$

Let us see that

$$v_z(r_\xi\xi)\ge C(N,\Omega)(1-r_\xi)\fint_{\partial B_1} u. \tag{2.7}$$

In fact $v_z(r_\xi\xi)=v\big((1-r_\xi)z+r_\xi\xi\big)$ and if $|(1-r_\xi)z+r_\xi\xi|\le\frac34$, by Harnack inequality applied to a solution to $\Delta v - r^2 v=0$ in B_1 with $r\le1$,

$$v_z(r_\xi\xi)\ge C_N v(0).$$

Clearly (2.7) follows from

$$v(0)\ge\alpha(N)\fint_{\partial B_1} v = \alpha(N)\fint_{\partial B_1} u. \tag{2.8}$$

But (2.8) is a consequence of the mean value property of solutions to the Schrödinger equation $\Delta v - r^2 v = 0$, namely

$$v(0) = \frac{1}{J(r)}\fint_{\partial B_1(0)} v$$

where $J(r)=\Gamma(N/2)\Big(\frac r2\Big)^{1-\frac N2} I_{\frac{N-2}{2}}(r)$ and $I_{\frac{N-2}{2}}$ is the Bessel function. In particular

$$J(0)=1.$$

See Theorem 9.9 in [18] for this result.

Now, if $|(1-r_\xi)z + r_\xi\xi| \geq \frac{3}{4}$ we prove by a comparison argument that inequality (2.7) also holds. In fact, first observe that we can assume that $\fint_{\partial B_1} v = \fint_{\partial B_1} u = 1$. So that, by (2.8), $v \geq C_N\alpha$ in $B_{3/4}$. Let $w(x) = e^{-\lambda|x|^2} - e^{-\lambda}$. There exists $\lambda = \lambda(N,\alpha)$ such that

$$\begin{cases} \Delta w \geq w & \text{in } B_1 \setminus B_{3/4}, \\ w \leq C_N\alpha & \text{in } \partial B_{3/4}, \\ w = 0 & \text{in } \partial B_1, \end{cases}$$

so that, since $\Delta v \leq v$, there holds that $v \geq w \geq C(1-|x|)$ in $B_1 \setminus B_{3/4}$. Therefore,

$$v_z(r_\xi\xi) \geq C\Big(1 - |(1-r_\xi)z + r_\xi\xi|\Big) \fint_{\partial B_1} u \geq C(1-r_\xi) \fint_{\partial B_1} u$$

since $|z| \leq \frac{1}{2}$. So that (2.7) holds for every $r_\xi \geq \frac{1}{8}$.

By (2.6) and (2.7) we have

$$c\sqrt{1-r_\xi} \fint_{\partial B_1} u \leq \left(\int_{r_\xi}^1 |\nabla(u_z - v_z)|^2(r\xi)\, dr \right)^{1/2}.$$

Hence

$$\begin{aligned} c^2 \int_{\partial B_1} (1-r_\xi)\, dS_\xi \left(\fint_{\partial B_1} u \right)^2 &\leq \int_{\partial B_1} \int_{r_\xi}^1 |\nabla(u_z - v_z)|^2(r\xi)\, dr\, dS_\xi \\ &\leq C \int_{B_1} |\nabla(u_z - v_z)|^2\, dx. \end{aligned}$$

Since

$$\int_{\partial B_1} (1-r_\xi)\, dS_\xi \geq \int_{B_1 \setminus B_{1/4}(z)} \chi_{\{u=0\}}\, dx,$$

we have

$$\begin{aligned} c^2 |\{x \in B_1 \setminus B_{1/4}(z) \,/\, u(x) = 0\}| \left(\fint_{\partial B_1} u \right)^2 &\leq C \int_{B_1} |\nabla(u_z - v_z)|^2\, dx \\ &\leq K \int_{B_1} |\nabla(u - v)|^2\, dx. \end{aligned}$$

Finally, we integrate over $z \in B_{1/2}(0)$ and use (2.5) to obtain

$$\begin{aligned} |B_1 \cap \{u=0\}| \left(\fint_{\partial B_1} u \right)^2 &\leq K \int_{B_1} |\nabla(u - v)|^2\, dx \\ &\leq K C_\varepsilon |B_1 \cap \{u=0\}|. \end{aligned} \tag{2.9}$$

Therefore we either have $u > 0$ almost everywhere in B_1 or else $\fint_{\partial B_1} u \leq \sqrt{KC_\varepsilon}$.

Hence we deduce that if

$$\fint_{\partial B_1} u \geq \sqrt{KC_\varepsilon} = C(N, \Omega, \varepsilon)$$

then $|B_1 \cap \{u=0\}| = 0$. So that by (2.5) $u = v > 0$ in B_1. □

Now we can prove the Lipschitz continuity of the minimizer u.

Proof of Theorem 2.1 (1). The proof follows as in [4] Lemma 3.3. In fact, let $D \subset\subset D' \subset\subset \Omega$ and $x \in D$. Let $r > 0$ be the largest number such that $B_r(x) \subset \{u > 0\} \cap D'$. As in [4] we prove by using Lemma 2.1 that $\{u > 0\}$ is open and

$$\frac{1}{r} \fint_{\partial B_r(x)} u \le C$$

with C independent of either u or x. Since $u > 0$ in $B_r(x)$, it is a solution to

$$\Delta u = u \quad \text{in } B_r(x).$$

In fact, let v be the solution to $\Delta v = v$ in $B_r(x)$, $v = u$ on $\Omega \setminus B_r(x)$. Then,

$$0 \le \|u - v\|^2_{H^1(\Omega)} = \|u\|^2_{H^1(\Omega)} - \|v\|^2_{H^1(\Omega)} = \mathcal{J}_\varepsilon(u) - \mathcal{J}_\varepsilon(v) \le 0.$$

So that, $u = v$ in $B_r(x)$.

Hence, there is a universal constant such that

$$|\nabla u(x)| \le C\Big\{ r\|u\|_{L^\infty(B_r(x))} + \frac{1}{r} \fint_{\partial B_r(x)} u \Big\}.$$

Now, since u is subharmonic in Ω and $D' \subset\subset \Omega$, there holds that u is bounded in D' by a constant that depends on the H^1 norm of u in Ω which is bounded by a constant that depends only on Ω and ε. Therefore,

$$|\nabla u(x)| \le C$$

with C depending only on N, Ω, ε, D and D'. □

In order to prove the nondegeneracy of u we need the following Lemma (see [4], Lemma 3.4).

Lemma 2.2. *Let $u \in \mathcal{K}$ be a solution to (P_ε). For $0 < \kappa < 1$ there exists a constant $c = c(\kappa, N, \Omega, \varepsilon)$ such that for every ball $B_r(x_0) \subset\subset \Omega$,*

$$\frac{1}{r} \fint_{\partial B_r} u \le c \quad \textit{implies that} \quad u = 0 \quad \textit{in} \quad B_{\kappa r}.$$

Proof. As in [4] Lemma 3.4, we consider the function

$$\phi_s^N(x) = \begin{cases} \dfrac{s}{N-2}\left(\left(\dfrac{s}{|x|}\right)^{N-2} - 1\right) & \text{for} \quad N \ge 3, \\ s \log \dfrac{s}{|x|} & \text{for} \quad N = 2, \\ s - |x| & \text{for} \quad N = 1. \end{cases} \tag{2.10}$$

For simplicity let us take $\bar u(x) = \frac{1}{r} u(x_0 + rx)$,

$$\bar F_\varepsilon(s) = \begin{cases} \frac{1}{\varepsilon}\big(s - \frac{\alpha}{r^N}\big) & \text{if } s > \frac{\alpha}{r^N}, \\ \varepsilon\big(s - \frac{\alpha}{r^N}\big) & \text{if } s \le \frac{\alpha}{r^N}, \end{cases}$$

and

$$\bar{\mathcal{J}}_\varepsilon(w) = \int_{\Omega^r} |\nabla w|^2 + r^2 w^2 + \bar F_\varepsilon(|\{w > 0\}|)$$

where $\Omega^r = \frac{1}{r}(\Omega - x_0)$. So that, $\mathcal{J}_\varepsilon(u) = r^N \bar{\mathcal{J}}_\varepsilon(\bar u)$.

Now, let $v(x) = \dfrac{\gamma\sqrt{\kappa}}{-\phi_\kappa^N(\sqrt{\kappa})}\max(-\phi_\kappa^N(x), 0)$ where, since $\bar u$ is subharmonic,

$$\gamma := \frac{1}{\sqrt{\kappa}} \sup_{B_{\sqrt{\kappa}}} \bar u \le C_1(N,\kappa) \fint_{\partial B_1} \bar u = C_1(N,\kappa) \frac{1}{r} \fint_{\partial B_r(x_0)} u.$$

Hence, $v \ge \bar u$ on $\partial B_{\sqrt{\kappa}}$, and therefore if

$$w = \begin{cases} \min(\bar u, v) & \text{in } \ B_{\sqrt{\kappa}}, \\ \bar u & \text{in } \ \Omega^r \setminus B_{\sqrt{\kappa}}, \end{cases}$$

there holds that,

$$\begin{aligned}
&\int_{B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\, dx + |B_\kappa \cap \{\bar u > 0\}| \\
&\quad = \bar{\mathcal{J}}_\varepsilon(\bar u) - \int_{\Omega^r \setminus B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\, dx + |B_\kappa \cap \{\bar u > 0\}| - \bar F_\varepsilon(|\{\bar u > 0\}|) \\
&\quad \le \bar{\mathcal{J}}_\varepsilon(w) - \int_{\Omega^r \setminus B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\, dx + |B_\kappa \cap \{\bar u > 0\}| - \bar F_\varepsilon(|\{\bar u > 0\}|) \\
&\quad = \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^2 + r^2 w^2\, dx - \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\, dx + |B_\kappa \cap \{\bar u > 0\}| \\
&\qquad\qquad + \bar F_\varepsilon(|\{w > 0\}|) - \bar F_\varepsilon(|\{\bar u > 0\}|) \\
&\quad \le \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^2 + r^2 w^2\, dx - \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\, dx + (1-\varepsilon)\, |B_\kappa \cap \{\bar u > 0\}|.
\end{aligned}$$

since $w = 0$ in B_κ, $w = \bar u$ in $\Omega^r \setminus B_{\sqrt{\kappa}}$. We have also used that $\bar F_\varepsilon(A) - \bar F_\varepsilon(B) \ge \varepsilon(A - B)$ if $A \ge B$ and $\{w > 0\} \subset \{\bar u > 0\}$. This inclusion following from the fact that $w \le \bar u$. Thus,

$$\begin{aligned}
&\int_{B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\, dx + \varepsilon |B_\kappa \cap \{\bar u > 0\}| \\
&\quad \le \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^2 + r^2 w^2\, dx - \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2 \\
&\quad = \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla \bar u - \nabla(\bar u - v)^+|^2 - |\nabla \bar u|^2\, dx + r^2 \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \big(\bar u - (\bar u - v)^+\big)^2 - \bar u^2\, dx \\
&\quad = -\int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \nabla(\bar u - v)^+ \nabla(\bar u + v)\, dx - r^2 \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (\bar u - v)^+ (\bar u + v)\, dx \\
&\quad = -\int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \nabla(\bar u - v)^+ \nabla \bar u\, dx - r^2 \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (\bar u - v)^+ \bar u\, dx \\
&\qquad\quad - \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \nabla(\bar u - v)^+ \nabla v\, dx - r^2 \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (\bar u - v)^+ v\, dx \\
&\quad \le -2 \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \nabla(\bar u - v)^+ \nabla v\, dx - 2r^2 \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (\bar u - v)^+ v\, dx \\
&\quad \le 2 \int_{\partial B_\kappa} \bar u \nabla v\, \eta\, dS \le C_2(N,\kappa)\, \gamma \int_{\partial B_\kappa} \bar u.
\end{aligned}$$

Therefore,

$$\int_{B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\,dx + \varepsilon |B_\kappa \cap \{\bar u > 0\}| \le C_2(N,\kappa)\,\gamma \int_{\partial B_\kappa} \bar u. \tag{2.11}$$

Here we have used that $\min(\bar u, v) = \bar u - (\bar u - v)^+$, $\Delta v = 0$ in $B_{\sqrt{\kappa}} \setminus B_\kappa$, $v = 0$ on ∂B_κ, and $(\bar u - v)^+ = 0$ on $\partial B_{\sqrt{\kappa}}$.

Recall that γ is controlled by $\frac{1}{r} \fint_{\partial B_r(x_0)} u$, so that γ will be small if $\frac{1}{r} \fint_{\partial B_r(x_0)} u$ is small.

On the other hand, by standard estimates,

$$\begin{aligned}
\int_{\partial B_\kappa} \bar u &\le C_3(N,\kappa) \int_{B_\kappa} |\nabla \bar u| + \bar u\,dx \\
&\le C_3(N,\kappa) \Big\{ \frac12 \int_{B_\kappa} |\nabla \bar u|^2\,dx + \frac12 |B_\kappa \cap \{\bar u > 0\}| + \gamma |B_\kappa \cap \{\bar u > 0\}| \Big\} \\
&\le\; C_3(N,\kappa) \Big\{ \int_{B_\kappa} |\nabla \bar u|^2 + r^2 \bar u^2\,dx + |B_\kappa \cap \{\bar u > 0\}| \Big\},
\end{aligned}$$

if $\gamma \le 1/2$.

So that, by (2.11), if γ is small enough ($\gamma \le 1/2$ and $C_2(N,\kappa) C_3(N,\kappa)\gamma < 1$), we deduce that $|B_\kappa \cap \{\bar u > 0\}| = 0$. This is, $u = 0$ in $B_{r\kappa}(x_0)$ and the lemma is proved. □

We can now prove the nondegeneracy of u.

Proof of Theorem 2.1, (2). Let $x \in \{u > 0\}$ and $r = \operatorname{dist}(x, \{u = 0\})$. As we proved in (2.8), since $\Delta u = u$ in $B_r(x)$, there holds that

$$u(x) \ge \alpha(N) \fint_{\partial B_r(x)} u.$$

Since $u(x) > 0$,

$$\frac{1}{r} \fint_{\partial B_r(x)} u \ge c$$

where c is the constant in Lemma 2.2 for $\kappa = 1/2$. Thus,

$$u(x) \ge c\alpha\, r.$$

The upper bound clearly follows from the Lipschitz continuity of u. Hence (2) is proved. □

Proof of Theorem 2.1, (3). In order to prove the uniform positive density of $\{u > 0\}$ and $\{u = 0\}$ at every free boundary point we proceed as in [4], Lemma 3.7. The only difference being that the function v that we have to take is the one in (2.2).

This ends the proof of Theorem 2.1. □

Corollary 2.1. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Let $D \subset\subset \Omega$. There exist constants $c, C > 0$ depending only on N, Ω, D and ε such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*

$$c \le \frac{1}{r} \fint_{\partial B_r(x)} u \le C. \tag{2.12}$$

Proof. It follows easily from Lemmas 2.1 and 2.2. □

Lemma 2.3. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then u satisfies for every $\varphi \in C_0^\infty(\Omega)$ such that $supp\,\varphi \subset \{u > 0\}$,*

$$\int_\Omega \nabla u \nabla \varphi + u\varphi \, dx = 0. \tag{2.13}$$

Moreover, the application

$$\lambda(\varphi) := -\int_\Omega \nabla u \nabla \varphi + u\varphi \, dx$$

from $C_0^\infty(\Omega)$ into $\mathbb{R}$ defines a nonnegative Radon measure with support on $\Omega \cap \partial\{u > 0\}$.

Proof. The proof follows exactly as in [4], Lemma 4.2. □

Theorem 2.2. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Let $D \subset\subset \Omega$. Then, there exist constants $C, c > 0$ such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*

$$c\, r^{N-1} \le \int_{B_r(x)} d\lambda \le C\, r^{N-1}.$$

Proof. For n large enough, let $u_n = u * \rho_n$ where ρ_n are the standard mollifiers. Then,

$$\begin{aligned}\int_{B_r(x)} \lambda * \rho_n \, dx = \int_{B_r(x)} \Delta u_n - u_n \, dx = \int_{\partial B_r(x)} \nabla u_n \cdot \nu \, dS - \int_{B_r(x)} u_n \\ \le \omega_{N-1} \sup_{\partial B_r(x)} |\nabla u_n|\, r^{N-1} \le C\, r^{N-1}\end{aligned}$$

since $|\nabla u_n| \le |\nabla u| \le C$ for a certain constant C depending on D. By taking limit for $n \to \infty$ we get

$$\int_{B_r(x)} d\lambda \le C\, r^{N-1}.$$

The other inequality follows as in the proof of Theorem 4.3 in [4] by taking as $G_y(z)$ the (positive) Green function of $-\Delta + Id$ with homogeneous Dirichlet boundary conditions in the ball $B_r(x)$. Then, for $0 < \kappa < 1/2$ and $y \in B_{\kappa r}(x)$ one uses the inequality

$$v(y) \ge C v(x) \ge C\alpha ⨍_{\partial B_r(x)} u$$

for v the solution to $\Delta v - v = 0$ in $B_r(x)$, $v = u$ on $\partial B_r(x)$, that follows from Harnack inequality and (2.8). □

Theorem 2.3 (Representation Theorem)**.** *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then,*

1. *$\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ for every $D \subset\subset \Omega$.*
2. *There exists a Borel function q_u such that*
$$\Delta u - u = q_u\, \mathcal{H}^{N-1} \lfloor \partial\{u > 0\}.$$
3. *For $D \subset\subset \Omega$ there are constant $0 < c \le C < \infty$ depending on N, Ω, D and the constants in (2.12) such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*
$$c \le q_u(x) \le C, \quad c\, r^{N-1} \le \mathcal{H}^{N-1}(B_r(x) \cap \partial\{u > 0\}) \le C\, r^{N-1}.$$

Proof. The proof follows exactly as that of Theorem 4.5 in [4]. □

Remark 2.2. Let $u \in \mathcal{K}$ be a solution to (P_ε) and $D \subset\subset \Omega$. Then $D \cap \partial\{u > 0\}$ has finite perimeter. Thus, the reduce boundary $\partial_{\text{red}}\{u > 0\}$ is defined as well as the measure theoretic normal $\nu(x)$ for $x \in \partial_{\text{red}}\{u > 0\}$. See [8].

If the free boundary $\partial\{u > 0\}$ is a regular surface then $q_u = -\partial_\nu u$. In Theorem 2.4 it is shown that this is true for almost all points in the reduce boundary.

Proposition 2.1. *Let $u \in \mathcal{K}$ be a solution to (P_ε) and let $B_{\rho_k}(x_k) \subset \Omega$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Let*

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

We call u_k a blow-up sequence with respect to $B_{\rho_k}(x_k)$. Since u is locally Lipschitz continuous, there exists a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ satisfying (2.12) with the same constants, when $x_k \in \partial\{u > 0\}$, and such that, for a subsequence,

$$\begin{aligned}
&u_k \to u_0 \quad \textit{in} \quad C^\alpha_{\text{loc}}(\mathbb{R}^N) \quad \textit{for every} \quad 0 < \alpha < 1,\\
&\nabla u_k \to \nabla u_0 \quad \textit{weakly star in} \quad L^\infty_{\text{loc}}(\mathbb{R}^N),\\
&\partial\{u_k > 0\} \to \partial\{u_0 > 0\} \quad \textit{locally in Hausdorff distance,}\\
&\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}} \quad \textit{in} \quad L^1_{\text{loc}}(\mathbb{R}^N),\\
&\Delta u_0 = 0 \quad \textit{in} \quad \{u_0 > 0\}.
\end{aligned}$$

Moreover if $x_k \in \partial\{u > 0\}$, then $0 \in \partial\{u_0 > 0\}$

Proof. It follows as in [4], Section 4.7 observing that $\Delta u_k - \rho_k^2 u_k = 0$ in $\{u_k > 0\}$. □

Theorem 2.4 (Identification of q_u)**.** *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then, for almost every $x_0 \in \partial_{\text{red}}\{u > 0\}$,*

$$u(x_0 + x) = q_u(x_0)\langle x, \nu(x_0)\rangle^- + o(|x|) \quad \textit{for} \quad x \to 0$$

with $\nu(x_0)$ the outward unit normal de $\partial\{u > 0\}$ in the measure theoretic sense.

Proof. It follows exactly as Theorem 4.8 and Remark 4.9 in [4]. □

Remark 2.3. Observe that by Theorem 2.1, (3)

$$\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0.$$

See [8].

Now we get a more precise identification of q_u.

Theorem 2.5. *Let $u \in \mathcal{K}$ be a solution to (P_ε) and q_u the function in Theorem 2.4. Then there exists a constant λ_u such that*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| = \lambda_u, \qquad \textit{for every } x_0 \in \Omega \cap \partial\{u > 0\} \tag{2.14}$$

$$q_u(x_0) = \lambda_u, \qquad \mathcal{H}^{N-1} - a.e\ x_0 \in \Omega \cap \partial\{u > 0\}. \tag{2.15}$$

Moreover, if B is a ball contained in $\{u = 0\}$ touching the boundary $\partial\{u > 0\}$ at x_0. Then

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \frac{u(x)}{\text{dist}(x, B)} = \lambda_u. \tag{2.16}$$

Proof. We follow the ideas of [2] Theorem 3 and [16] Theorem 5.1 and Lemma 5.2.

Let $x_0, x_1 \in \partial\{u > 0\}$ and $\rho_k \to 0^+$. For $i = 0, 1$ let $x_{i,k} \to x_i$ with $u(x_{i,k}) = 0$ such that $B_{\rho_k}(x_{i,k}) \subset \Omega$ and such that the blow-up sequence

$$u_{i,k}(x) = \frac{1}{\rho_k} u(x_{i,k} + \rho_k x)$$

has a limit $u_i(x) = \lambda_i \langle x, \nu_i\rangle^-$, with $0 < \lambda_i < \infty$ and ν_i a unit vector. We will prove that $\lambda_0 = \lambda_1$. From this, the Theorem will follow as in [16].

Assume that $\lambda_1 < \lambda_0$. Then, we will perturb the minimizer u near x_0 and x_1 and get an admissible function with less energy, which is a contradiction. We perform a perturbation that increases the measure of the positivity set in a neighborhood of $x_{0,k}$ and decreases its measure in a neighborhood of $x_{1,k}$. We perform this perturbation in such a way that we change the measure of the positivity set in an amount of essentially order $o(\rho_k^N)$.

To this end, we take a nonnegative C_0^∞ symmetric function Φ supported in the unit interval, and for $t > 0$ small, we define

$$\tau_k(x) = \begin{cases} x + t\rho_k \Phi\Big(\dfrac{|x - x_{0,k}|}{\rho_k}\Big)\nu_0 & \text{for } x \in B_{\rho_k}(x_{0,k}), \\[2ex] x - t\rho_k \Phi\Big(\dfrac{|x - x_{1,k}|}{\rho_k}\Big)\nu_1 & \text{for } x \in B_{\rho_k}(x_{1,k}), \\[2ex] x & \text{elsewhere,} \end{cases}$$

which is a diffeomorphism if t is small enough. Now, let

$$v_k(x) = u(\tau_k^{-1}(x)),$$

that are admissible functions. Moreover, since $\|D\tau_k^{-1}\| \le C$ independent of k for t small enough, there holds that

$$\|\nabla v_k\|_{L^\infty} \le C$$

independent of k.

Also, we have

$$F_\varepsilon(|\{v_k > 0\}|) - F_\varepsilon(|\{u > 0\}|) = o(t)\,\rho_k^N + o(\rho_k^N). \tag{2.17}$$

In fact, $v_k = u$ in $\Omega \setminus (B_{\rho_k}(x_{0,k}) \cup B_{\rho_k}(x_{1,k}))$ and

$$\begin{aligned} |\{v_k > 0\} \cap B_{\rho_k}(x_{i,k})| - |\{u > 0\} \cap B_{\rho_k}(x_{i,k})| = \\ = (-1)^i \rho_k^N \Big(t \int_{B_1 \cap \{y_1 = 0\}} \Phi(|y|)\, d\mathcal{H}_y^{N-1} + o_i(t)\Big) + o(\rho_k^N), \end{aligned}$$

since $\Phi(|y|)$ is radially symmetric and $\chi_{\{u_{i,k} > 0\}} \to \chi_{\{\langle x, \nu_i\rangle < 0\}}$ in $L^1_{\text{loc}}(\mathbb{R}^N)$.

Similar computations involving also the development of ∇v_k in terms of ∇u and $D\tau_k$ give

$$\begin{aligned} \int_\Omega |\nabla v_k|^2\, dx - \int_\Omega |\nabla u|^2\, dx = &\rho_k^N \Big((\lambda_1^2 - \lambda_0^2)\, t \int_{B_1(0) \cap \{y_1 = 0\}} \Phi(|y|)\, d\mathcal{H}_y^{N-1} + o(t)\Big) \\ &+ o(\rho_k^N). \end{aligned} \tag{2.18}$$

See, [2] or [16] for detailed computations.

It remains to estimate the difference of the L^2 norms. Since $u(x_{i,k}) = 0$ there holds that

$$u(x) \le C\rho_k^N \quad \text{in } B_{\rho_k}(x_{i,k}).$$

On the other hand,

$$0 = u(x_{i,k}) = v_k(\tau_k(x_{i,k})) = v_k(x_{i,k} + (-1)^i\, t\rho_k\Phi(0)\nu_i).$$

Thus,

$$v_k(z) \le C\big|z - x_{i,k} - (-1)^i\, t\rho_k\Phi(0)\nu_i\big| \le K\rho_k \quad \text{if } z \in B_{\rho_k}(x_{i,k}).$$

Therefore,

$$\int_\Omega v_k^2\, dx - \int_\Omega u^2\, dx = o(\rho_k^N). \tag{2.19}$$

Thus, we get from (2.17), (2.18) and (2.19), for t small enough and k large enough, that

$$\mathcal{J}_\varepsilon(v_k) < \mathcal{J}_\varepsilon(u),$$

a contradiction. □

Summing up, we have the following theorem,

Theorem 2.6. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then u is a weak solution to the following free boundary problem*

$$\begin{aligned} -\Delta u + u &= 0 && \text{in } \{u > 0\} \cap \Omega,\\ \frac{\partial u}{\partial \nu} &= \lambda_u && \text{on } \partial\{u > 0\} \cap \Omega, \end{aligned}$$

where λ_u is the constant in Theorem 2.5. More precisely, $\mathcal{H}^{N-1}-$a.e. point $x_0 \in \partial\{u > 0\}$ belongs to $\partial_{red}\{u > 0\}$ and

$$u(x_0 + x) = \lambda_u\langle x, \nu(x_0)\rangle^- + o(|x|) \quad \text{for} \quad x \to 0.$$

Finally, we get an estimate of the gradient of u that will be needed in order to get the regularity of the free boundary.

Theorem 2.7. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Given $D \subset\subset \Omega$, there exist constants $C = C(N, \varepsilon, D)$, $r_0 = r_0(N, D) > 0$ and $\gamma = \gamma(N, \varepsilon, D) > 0$ such that, if $x_0 \in D \cap \partial\{u > 0\}$ and $r < r_0$, then*

$$\sup_{B_r(x_0)} |\nabla u| \le \lambda_u(1 + Cr^\gamma).$$

Proof. The proof follows the lines of the proof of Theorem 4.1 in [5].

Let $U_k = \big(|\nabla u| - \lambda_u - \frac{1}{k}\big)^+$ and $U_0 = \big(|\nabla u| - \lambda_u\big)^+$. By (2.14) we know that U_k vanishes in a neighborhood of the free boundary. Also, the support of U_k is contained in $\{u > 0\}$. Therefore U_k satisfies

$$\Delta U_k \ge U_k \quad \text{in } \Omega \cap \{u > 0\}$$

and vanishes in a neighborhood of the free boundary. We extend U_k by zero into $\{u = 0\}$ and set

$$h_k(r) = \sup_{B_r(x_0)} U_k, \qquad h_0(r) = \sup_{B_r(x_0)} U_0,$$

for any $r < r_0 = \operatorname{dist}(D, \partial\Omega)$ and $x_0 \in D \cap \partial\{u > 0\}$.

Then, $h_k(r) - U_k$ is a supersolution of $\Delta v = v$ in the ball $B_r(x_0)$ and

$$\begin{aligned} h_k(r) - U_k \ &\geq 0 && \text{in } B_r(x_0) \\ &= h_k(r) && \text{in } B_r(x_0) \cap \{u = 0\}. \end{aligned}$$

Applying the weak Harnack inequality (see [12] p. 246) with $1 \leq p < N/(N-2)$, we get

$$\inf_{B_{r/2}(x_0)} \big(h_k(r) - U_k\big) \geq c r^{-N/p} \|h_k(r) - U_k\|_{L^p(B_r(x_0))} \geq c h_k(r),$$

since, by Theorem 2.1 (3), $|B_r(x_0) \cap \{u = 0\}| \geq c r^N$. Taking now $k \to \infty$ we obtain

$$\inf_{B_{r/2}(x_0)} \big(h_0(r) - U_0\big) \geq c h_0(r),$$

for some $0 < c < 1$, which is the same as

$$\sup_{B_{r/2}(x_0)} U_0 \leq (1 - c) h_0(r).$$

Therefore

$$h_0\big(\frac{r}{2}\big) \leq (1 - c) h_0(r),$$

from which it follows that $h_0(r) \leq C r^\gamma$ for some $C > 0$, $0 < \gamma < 1$ and now the conclusion of the Theorem follows. ☐

3. Regularity of the free boundary.

At this point we have that our minimizer u_ε meets the conditions of the regularity theory developed in [4]. The only difference being the equation satisfied by u_ε in $\{u_\varepsilon > 0\}$.

We will recall some definitions and we will point out the only significant difference with [4]. The rest of the proof of the regularity then follows as sections 7 and 8 of [4] with only minor modifications.

Throughout this section we will remove the subscript ε.

Definition 3.1 (Flat free boundary points)**.** *Let $0 < \sigma_+, \sigma_- \leq 1$ and $\tau > 0$. We say that u is of class*

$$F(\sigma_+, \sigma_-; \tau) \quad \textit{in} \quad B_\rho = B_\rho(0)$$

if

(1) $0 \in \partial\{u > 0\}$ *and*

$$\begin{aligned} &u = 0 && \textit{for} \quad x_N \geq \sigma_+ \rho, \\ &u(x) \geq -\lambda(x_N + \sigma_- \rho) && \textit{for} \quad x_N \leq -\sigma_- \rho. \end{aligned}$$

(2) $|\nabla u| \leq \lambda(1 + \tau)$ *in* B_ρ.

If the origin is replaced by x_0 and the direction e_N by the unit vector ν we say that u is of class $F(\sigma_+, \sigma_-; \tau)$ in $B_\rho(x_0)$ in direction ν.

Observe that the results in Section 2 imply that the minimizer u of $\mathcal{J}_\varepsilon$ is in the class $F(\sigma,1;\sigma)$ in $B_\rho(x_0)$ in direction $\nu_u(x_0)$ for every $x_0 \in \partial_{red}\{u>0\}$ with $\sigma = \sigma(\rho) \to 0$ as $\rho \to 0$.

The following lemma (Lemma 7.2 in [4]) is the only one that requires a non obvious modification.

Lemma 3.1. *There is a constant $C = C(N)$ such that $u \in F(\sigma,1;\sigma)$ in $B_\rho(x_0)$ in direction ν implies $u \in F(2\sigma, C\sigma;\sigma)$ in $B_{\rho/2}(x_0)$ in direction ν.*

Proof. Clearly, by a change of variables, we may assume that $x_0 = 0$ and $\nu = e_N$. Let $\bar u(x) = u(\rho x)/\lambda\rho$, then $|\nabla \bar u| \le 1+\sigma$ and $\bar u \in F(\sigma,1;\sigma)$ in B_1. That is $\bar u = 0$ if $x_N > \sigma$. Define

$$\eta(x') = \begin{cases} \exp\left(-\dfrac{9|x'|^2}{1-9|x'|^2}\right) & \text{for } |x'| < \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

and choose $s \ge 0$ maximal with the property that $\bar u = 0$ in $x_N > \sigma - s\eta(x')$.

Now, the proof follows as in Lemma 7.2 of [4] with the only difference that the comparison function v must be the solution to $\Delta v = \rho^2 \bar u$ in $D = B_1 \cap \{x_N < \sigma - s\eta(x')\}$ instead of a harmonic function. The estimate

$$\partial_{-\nu} v \le 1 + C\sigma$$

follows from

$$|\nabla(v + x_N)| \le C\Big[\sup_D (v + x_N) + \rho^2\Big] \le C\sigma$$

in $D \cap B_{1/2}$ if $\rho^2 \le C\sigma$, since

$$\Delta(v + x_N) = \rho^2 \bar u \qquad \text{in} \quad D,$$

$v + x_N \le C\sigma$ in D and $|\bar u(x)| \le 2$. ☐

Once this lemma is established the following regularity result follows.

Theorem 3.1. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then $\partial_{red}\{u>0\}$ is a $C^{1,\beta}$ surface locally in Ω and the remainder of the free boundary has $\mathcal{H}^{N-1}$–measure zero. Moreover, if $N = 2$ then the whole free boundary is a $C^{1,\beta}$ surface.*

4. Behavior of the minimizer for small ε.

To complete the analysis of the problem, we will now show that if ε is small enough, then

$$|\{u_\varepsilon > 0\}| = \alpha.$$

To this end, we need to prove that the constant $\lambda_\varepsilon := \lambda_{u_\varepsilon}$ is bounded from above and below by positive constants independent of ε. We perform this task in a series of lemmas.

Lemma 4.1. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε). Then, there exist constants $C, c > 0$ independent of ε such that*

$$c \le |\{u_\varepsilon > 0\}| \le \alpha + C\varepsilon. \tag{4.1}$$

Proof. As $\mathcal{J}_\varepsilon(u_\varepsilon)$ is bounded from above uniformly in ε we obtain

$$F_\varepsilon(|\{u_\varepsilon > 0\}|) \le C.$$

Hence

$$|\{u_\varepsilon > 0\}| \le \alpha + C\varepsilon.$$

For the lower bound, we proceed as follows; by the Sobolev trace embedding, for some $1 < p < 2$, such that $p(N-1)/(N-p) > q$,

$$1 \le \|u_\varepsilon\|_{L^q(\partial\Omega)} \le C\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C\|u_\varepsilon\|_{H^1(\Omega)}|\{u_\varepsilon > 0\}|^\theta,$$

for some exponent θ that depends only on p. Since $\|u_\varepsilon\|_{H^1(\Omega)}$ is uniformly bounded, the lower bound follows. ☐

Lemma 4.2. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε). Then, there exists a constant $C > 0$ independent of ε such that*

$$\lambda_\varepsilon := \lambda_{u_\varepsilon} \le C.$$

Proof. Let $D \subset\subset \Omega$ smooth, such that $\omega = |D| > \alpha$ and $|\Omega \setminus D| < c$ where c is the constant in Lemma 4.1. Then,

$$|D \cap \{u_\varepsilon > 0\}| \le \alpha + C\varepsilon < \omega$$

for ε small enough. On the other hand

$$|D \cap \{u_\varepsilon > 0\}| \ge |\{u_\varepsilon > 0\}| - |\Omega \setminus D| \ge c - |\Omega \setminus D| > 0.$$

Therefore by the relative isoperimetric inequality we have

$$\mathcal{H}^{N-1}(D \cap \partial\{u_\varepsilon > 0\}) \ge c_0 \min\{|D \cap \{u_\varepsilon > 0\}|, |D \cap \{u_\varepsilon = 0\}|\}^{\frac{N-1}{N}} \ge c_1 > 0.$$

Now, take $\varphi \in C_0^\infty(\Omega)$ as a test function in Lemma 2.3 such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in D and $\|\nabla\varphi\|_\infty \le C = C(\mathrm{dist}(D, \partial\Omega))$ to get, since $\|u_\varepsilon\|_{H^1(\Omega)}$ is bounded independently of ε,

$$C \ge \int_\Omega \nabla u^\varepsilon \nabla\varphi\, dx + \int_\Omega u_\varepsilon \varphi\, dx = \lambda_\varepsilon(\varphi) \ge \lambda_\varepsilon \mathcal{H}^{N-1}(D \cap \partial_{\mathrm{red}}\{u_\varepsilon > 0\}).$$

This completes the proof of the lemma. ☐

The uniform lower bound follows similarly to Lemma 6 in [2]. We only make a sketch of the proof for the reader's convenience. It is at this point where we need the hypothesis that $\Gamma_D \neq \emptyset$.

Lemma 4.3. *Let $\Gamma_D \neq \emptyset$ be the closure of a relatively open subset of $\partial\Omega$. Let $\varphi_0 \in H^1(\Omega)$ with $\varphi_0 \ge c_0 > 0$ in Γ_D. Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε). Then*

(1) *u_ε is positive in a neighborhood of Γ_D (depending on ε).*
(2) *There exists a constant $c > 0$ independent of ε such that*

$$c < \lambda_\varepsilon := \lambda_{u_\varepsilon}.$$

Proof. Let us first prove (1). In fact, arguing as in (2.9), given $y_0 \in \Gamma_D$ there exists a constant $K > 0$ independent of ε such that

$$|\Omega_r \cap \{u = 0\}| \left(\frac{1}{r} \fint_{\partial\Omega_r} u\right)^2 \le K \int_{\Omega_r} |\nabla(u - v)|^2\, dx,$$

where $\Omega_r = \Omega \cap B_r(y_0)$ and v is the solution of

$$\begin{cases} \Delta v = v & \text{in } \Omega_r \\ v = u & \text{on } \partial\Omega_r. \end{cases}$$

Therefore,

$$\left(\frac{c_0}{r}\right)^2 |\Omega_r \cap \{u=0\}| \leq K(\|u\|^2_{H^1(\Omega_r)} - \|v\|^2_{H^1(\Omega_r)}) \leq \frac{C}{\varepsilon}|\Omega_r \cap \{u=0\}|.$$

So that, $u > 0$ in Ω_r for small r depending on ε.

In order to see (2) we proceed as in [2] Lemma 6. Let $y_0 \in \Gamma_D$ and let D_t with $0 \leq t \leq 1$ be a family of open sets with smooth boundary and uniformly (in ε and t) bounded curvatures such that D_0 is an exterior tangent ball at y_0, D_1 contains a free boundary point, $D_t \cap \partial\Omega \subset \Gamma_D$ and $D_0 \subset\subset D_t$ for $t > 0$.

Let $t \in (0,1)$ be the first time such that D_t touches the free boundary and let $x_0 \in \partial D_t \cap \partial\{u_\varepsilon > 0\} \cap \Omega$. Now, take w the solution to $\Delta w = w$ in $D_t \setminus \overline{D}_0$ with $w = c_0$ on ∂D_0 and $w = 0$ on ∂D_t. Thus $w \leq u_\varepsilon$ in $D_t \cap \Omega$ and $\partial_{-\nu} w(x_0) \geq c\, c_0$ with c independent of ε, therefore, for r small enough

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u_\varepsilon \geq \frac{1}{r}\fint_{\partial B_r(x_0)} w \geq \bar{c}\, c_0$$

with $\bar{c}$ is independent of ε.

If v_0 is the solution to

$$\begin{cases} \Delta v = v & \text{in } B_r(x_0) \\ v = u & \text{on } \partial B_r(x_0), \end{cases}$$

then, by (2.9), we have

$$\begin{aligned} c|B_r(x_0) \cap \{u_\varepsilon = 0\}| &\leq |B_r(x_0) \cap \{u_\varepsilon = 0\}| \left(\frac{1}{r}\fint_{\partial B_r(x_0)} u_\varepsilon\right)^2 \\ &\leq K \int_{B_r(x_0)} |\nabla(u_\varepsilon - v_0)|^2\, dx \leq K(\|u_\varepsilon\|^2_{H^1(B_r(x_0))} - \|v_0\|^2_{H^1(B_r(x_0))}). \end{aligned}$$

Let now $\delta_r = |B_r(x_0) \cap \{u_\varepsilon = 0\}|$ and let $x_1 \in \partial\{u_\varepsilon > 0\}$ be such that the free boundary is smooth in a neighborhood of x_1. We perturb $\{u_\varepsilon > 0\}$ in a neighborhood of x_1 so that the measure of the perturbed set is increased an amount δ_r (cf. with Theorem 2.5).

Let Φ be a smooth nonnegative function supported in $B_\kappa(x_1)$ with $\kappa > 0$ small. For $x \in B_\kappa(x_1)$ we write $x = \sigma + s\nu(\sigma)$ with $\sigma \in \partial\{u_\varepsilon > 0\}$ and $s \in \mathbb{R}$ where $\nu(\sigma)$ is the outer unit normal to the free boundary at σ. We define the change of variables $y = x - \Phi(\sigma)\tau\nu(\sigma)$ with $\tau > 0$ small and the deformed set $\mathcal{D}_{\delta_r}$ such that $\mathcal{D}_{\delta_r} \cap B_\kappa(x_1) = \{y \,/\, x \in \{u_\varepsilon > 0\} \cap B_\kappa(x_1)\}$. Observe that if r is small we can perform this perturbation in such a way that it decreases the measure of $\{u_\varepsilon > 0\}$ in exactly δ_r. Also, observe that $\delta_r \to 0$ as $r \to 0$.

Now let v_r be the solution of

$$\begin{cases} \Delta v = v & \text{in } \mathcal{D}_{\delta_r}, \\ v = 0 & \text{on } \partial\mathcal{D}_{\delta_r} \cap B_\kappa(x_1), \\ v = u_\varepsilon & \text{on } \partial B_\kappa(x_1) \cap \overline{\mathcal{D}}_r, \end{cases} \tag{4.2}$$

then v_r verifies

$$\frac{\partial v_r}{\partial \nu} = -\lambda_\varepsilon + o(\delta_r).$$

On the other hand,

$$u_\varepsilon = \lambda_\varepsilon \delta_r + o_\varepsilon(\delta_r), \qquad \text{on } \partial\{v_r > 0\} \cap B_\kappa(x_1).$$

Thus

$$\begin{aligned}\int_{B_\kappa(x_1)} |\nabla v_r|^2 + v_r^2\,dx - \int_{B_\kappa(x_1)} |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2\,dx &= \int_{B_\kappa(x_1)} |\nabla(u_\varepsilon - v_r)|^2 + (u_\varepsilon - v_r)^2\,dx \\ &= -\int_{\partial\{v_r>0\}\cap B_\kappa(x_1)} \frac{\partial v_r}{\partial \nu} u_\varepsilon\,dS \\ &= \lambda_\varepsilon^2 \delta_r + o_\varepsilon(\delta_r).\end{aligned}$$

Now we extend v_r by zero to $B_\kappa(x_1) \setminus \mathcal{D}_{\delta_r}$ and define

$$w_r = \begin{cases} v_r & \text{in } B_\kappa(x_1), \\ v_0 & \text{in } B_r(x_0), \\ u & \text{elsewhere.} \end{cases}$$

Then $|\{w_r > 0\}| = |\{u_\varepsilon > 0\}|$ and $w_r = u_\varepsilon$ on $\partial\Omega$, thus

$$\begin{aligned} 0 \le \mathcal{J}_\varepsilon(w_r) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega |\nabla w_r|^2 + w_r^2\,dx - \int_\Omega |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2\,dx \\ &= \int_{B_r(x_0)} |\nabla v_0|^2 + v_0^2\,dx - \int_{B_\kappa(x_0)} |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2\,dx \\ &\quad + \int_{B_\kappa(x_1)} |\nabla v_r|^2 + v_r^2\,dx - \int_{B_\kappa(x_1)} |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2\,dx \\ &\le -c\delta_r + \lambda_\varepsilon^2 \delta_r + o_\varepsilon(\delta_r), \end{aligned}$$

for every $r > 0$ small. Therefore, $\lambda_\varepsilon^2 \ge c/2$. $\square$

Now we are in a position to prove the main result of this section, namely that for ε small the measure of the positivity set is exactly α.

Theorem 4.1. *Let $\Gamma_D \neq \emptyset$ be the closure of a relatively open subset of $\partial\Omega$. Let $\varphi_0 \in H^1(\Omega)$ with $\varphi_0 \ge c_0 > 0$ in Γ_D. Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε). Then, for ε small*

$$|\{u_\varepsilon > 0\}| = \alpha. \tag{4.3}$$

Proof. Arguing by contradiction, assume first that $|\{u_\varepsilon > 0\}| > \alpha$. Let $x_1 \in \partial\{u_\varepsilon > 0\} \cap \Omega$ be a regular point. We will proceed as in the proof of the previous lemma. Given $\delta > 0$, we perturb the domain $\{u_\varepsilon > 0\}$ in a neighborhood of x_1, $B_\kappa(x_1)$, decreasing its measure by δ. We choose δ small so that the measure of the perturbed set is still larger than α. Then we let v be the solution to (4.2) extended by zero to the rest of $B_\kappa(x_1)$ and equal to u in the rest of Ω. We have

$$\begin{aligned} 0 \le \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega |\nabla v|^2 + v^2 - \int_\Omega |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2 + F_\varepsilon(|\{v > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}|) \\ &\le \lambda_\varepsilon^2 \delta + o_\varepsilon(\delta) - \frac{1}{\varepsilon}\delta \le (C^2 - \frac{1}{\varepsilon})\delta + o_\varepsilon(\delta) < 0, \end{aligned}$$

if $\varepsilon < \varepsilon_0$ and then $\delta < \delta_0(\varepsilon)$. A contradiction.

Now assume that $|\{u_\varepsilon > 0\}| < \alpha$. We proceed as in the previous case but this time we perturb in a neighborhood of x_1 the set $\{u_\varepsilon > 0\}$ increasing the measure by δ. Then we construct the function v as before, and if δ is small enough $|\{v > 0\}| < \alpha$. Then

$$\begin{aligned} 0 \le \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega |\nabla v|^2 + v^2 - \int_\Omega |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2 + F_\varepsilon(|\{v > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}|) \\ &\le -\lambda_\varepsilon^2 \delta + o_\varepsilon(\delta) + \varepsilon\delta \le (-c^2 + \varepsilon)\delta + o_\varepsilon(\delta) < 0, \end{aligned}$$

if $\varepsilon < \varepsilon_1$ and then $\delta < \delta_0(\varepsilon)$. Again a contradiction that ends the proof. □

As a consequence of the previous theorem, we get

Corollary 4.1. *Let $\Gamma_D \neq \emptyset$ be the closure of a relatively open subset of $\partial\Omega$. Let $\varphi_0 \in H^1(\Omega)$ with $\varphi_0 \ge c_0 > 0$ in Γ_D. Then, there exists a minimizer u of $\mathcal{J}(v)$ in the set*

$$\mathcal{K}_\alpha = \{v \in H^1(\Omega) \,/\, \|v\|_{L^q(\Gamma_N)} = 1,\ v = \varphi_0 \text{ on } \Gamma_D,\ |\{v > 0\}| = \alpha\}.$$

This minimizer can be chosen in such a way that it is locally Lipschitz continuous in Ω and the free boundary $\partial\{u > 0\} \cap \Omega$ is locally a $C^{1,\beta}$ surface up to a set of $\mathcal{H}^{N-1}$ measure zero. In the case $N = 2$ the free boundary is locally a $C^{1,\beta}$ surface.

Proof. From our previous results we have (4.3) for every ε small enough. Therefore we can take $u = u_\varepsilon$ and the desired regularity of u and its free boundary follows from the results of Sections 2 and 3. □

5. Main results.

In this last section we go back to our original minimization problem related to the best Sobolev trace constant. Here we prove that any extremal is a locally Lispchitz continuous function and the boundary of the hole $\partial\{u > 0\} \cap \Omega$ is locally $C^{1,\beta}$ up to a set of $\mathcal{H}^{N-1}$ measure zero.

We begin with the following.

Theorem 5.1. *Let ϕ_0 be a minimizer for (P_α). Assume that there exists a positive constant c such that $\phi_0 > c$ in a ball $B_0' \subset \Omega$ (resp. on $B_0' \cap \partial\Omega$ where B_0' is a ball centered at $\partial\Omega$). Then ϕ_0 is a minimizer of $\mathcal{J}_\varepsilon$ in*

$$\mathcal{K}_2 = \{v \in H^1(\Omega) \,/\, \|v\|_{L^q(\partial\Omega)} = 1,\ v = \phi_0 \text{ in } B_0\}$$

(resp. ϕ_0/k is a minimizer of $\mathcal{J}_\varepsilon$ in $\mathcal{K} = \{v \in H^1(\Omega) \,/\, \|v\|_{L^q(\Gamma_N)} = 1,\ v = \phi_0/k \text{ on } \Gamma_D\}$ with $\Gamma_D = \partial\Omega \cap B_0$, $\Gamma_N = \partial\Omega \setminus \Gamma_D$). Here, B_0 is a ball compactly contained in B_0' and $k = \|\phi_0\|_{L^q(\Gamma_N)}$.

In particular, ϕ_0 is locally Lipschitz continuous in Ω and the free boundary $\partial\{\phi_0 > 0\} \cap \Omega$ is locally a $C^{1,\beta}$ surface up to a set of zero $\mathcal{H}^{N-1}$ measure. In the case $N = 2$ the free boundary is locally a $C^{1,\beta}$ surface.

Proof. We will make the proof for the first case, the second one follows in the same way.

Let ε be small enough so that any minimizer u_ε of $\mathcal{J}_\varepsilon$ in $\mathcal{K}_2$ verifies that $|\{u_\varepsilon > 0\}| = \alpha$. Then, it follows that ϕ_0 is one of such minimizers and so the conclusions of the theorem follow. In fact, as ϕ_0 minimizes (P_α) we have

$$\mathcal{J}_\varepsilon(\phi_0) = \int_\Omega |\nabla \phi_0|^2 + |\phi_0|^2\,dx \le \int_\Omega |\nabla v|^2 + |v|^2\,dx \tag{5.1}$$

for any $v \in H^1(\Omega)$ such that $\|v\|_{L^q(\partial\Omega)} = 1$ and $|\{v > 0\}| = \alpha$. In particular (5.1) holds for $v = u_\varepsilon$. Thus

$$\mathcal{J}_\varepsilon(\phi_0) \le \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v\in\mathcal{K}_2} \mathcal{J}_\varepsilon(v).$$

This ends the proof. □

In particular, by the symmetry results for minimizers of (P_α) in balls of [11] we have the following corollary.

Corollary 5.1. *Let $\Omega = B(x_0, r)$ be a ball and let ϕ_0 be a minimizer of (P_α). Then ϕ_0 is locally Lipschitz continuous in $B(x_0, r)$ and the free boundary $\partial\{\phi_0 > 0\} \cap B(x_0, r)$ is locally a $C^{1,\beta}$ surface up to a set of zero $\mathcal{H}^{N-1}$ measure. In the case $N = 2$ the free boundary is locally a $C^{1,\beta}$ surface.*

Proof. In [11] it was proved that any minimizer ϕ_0 of (P_α) in the case that Ω is a ball $B_r(x_0)$ satisfies that, for any $c_0 > 0$, $\{\phi_0 \ge c_0\} \cap \partial B_r(x_0)$ is a spherical cap. Since $\|\phi_0\|_{L^q(\partial\Omega)} = 1$, there exists $c_0 > 0$ such that $\{\phi_0 \ge c_0\} \cap \partial B_r(x_0) \ne \emptyset$. Hence the conditions of Theorem 5.1 are satisfied. □

In the general case, for the problem (P_α) we can prove that the set of α's for which there exists minimizers with smooth free boundary is dense in $(0, |\Omega|)$. More precisely,

Theorem 5.2. *For any $0 < \alpha < |\Omega|$ there exists $\alpha_\varepsilon \to \alpha$ as $\varepsilon \to 0$ such that there exists a solution ϕ_ε of (P_{α_ε}) which is locally Lipschitz continuous in Ω and has a locally $C^{1,\beta}$ free boundary up to a set of zero $\mathcal{H}^{N-1}$−measure. In the case $N = 2$ the free boundary is a locally $C^{1,\beta}$ surface.*

Proof. Let u_ε be a minimizer of $\mathcal{J}_\varepsilon$. We already know that $\alpha_\varepsilon := |\{u_\varepsilon > 0\}| \le \alpha + C\varepsilon$ (see (4.1)). Let us see that $\alpha_\varepsilon \to \alpha$ as $\varepsilon \to 0$. If not, there exists a sequence $\varepsilon_j \to 0$ such that $\alpha_{\varepsilon_j} = |\{u_{\varepsilon_j} > 0\}| \le \theta < \alpha$. Let ϕ_0 be a minimizer of (P_α). By the strict monotonicity of $S(\alpha)$ (see [11], Remark 2.2) we have

$$\begin{aligned}\mathcal{J}(\phi_0) = S(\alpha) < S(\theta) \le \mathcal{J}(u_{\varepsilon_j}) &= \mathcal{J}_{\varepsilon_j}(u_{\varepsilon_j}) - F_{\varepsilon_j}(\alpha_{\varepsilon_j})\\ &\le \mathcal{J}_{\varepsilon_j}(\phi_0) - F_{\varepsilon_j}(\alpha_{\varepsilon_j}) = \mathcal{J}(\phi_0) - F_{\varepsilon_j}(\alpha_{\varepsilon_j}) \le \mathcal{J}(\phi_0) + C\varepsilon_j\end{aligned}$$

a contradiction.

Now, taking $\phi_\varepsilon = u_\varepsilon$ we see that ϕ_ε is a minimizer of (P_{α_ε}). In fact, let v be an admissible function for (P_{α_ε}) then

$$\mathcal{J}(v) + F_\varepsilon(\alpha_\varepsilon) = \mathcal{J}_\varepsilon(v) \ge \mathcal{J}_\varepsilon(\phi_\varepsilon) = \mathcal{J}(\phi_\varepsilon) + F_\varepsilon(\alpha_\varepsilon)$$

therefore

$$\mathcal{J}(v) \ge \mathcal{J}(\phi_\varepsilon).$$

The theorem is proved. □

Finally, we have the following result.

Theorem 5.3. *Let u_ε be a minimizer of $\mathcal{J}_\varepsilon$ in $\mathcal{K}_1$. Then there exists $\phi_0 \in H^1(\Omega)$ a solution to (P_α) such that, up to a subsequence, $u_\varepsilon \to \phi_0$ in $H^1(\Omega)$.*

Proof. In the proof of Theorem 5.2 we showed that $|\{u_\varepsilon > 0\}| \to \alpha$ as $\varepsilon \to 0$.

It is easy to see that $\mathcal{J}_\varepsilon(u_\varepsilon)$ is bounded uniformly in ε and so u_ε is uniformly bounded in $H^1(\Omega)$. Therefore, passing to a subsequence if necessary, there exists $u_0 \in H^1(\Omega)$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega),\\ u_\varepsilon &\to u_0 \quad \text{strongly in } L^q(\partial\Omega),\\ u_\varepsilon &\to u_0 \quad \text{a.e. } \Omega. \end{aligned}$$

Thus,

$$\begin{aligned} &\|u_0\|_{L^q(\partial\Omega)} = 1,\\ &|\{u_0 > 0\}| \le \alpha = \lim_{\varepsilon\to 0} |\{u_\varepsilon > 0\}| \quad \text{and}\\ &\|u_0\|_{H^1(\Omega)} \le \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{H^1(\Omega)}. \end{aligned}$$

Let us call $\phi_0 = u_0$ and let us see that ϕ_0 is a solution to (P_α). In fact, let $v \in H^1(\Omega)$ be such that $|\{v > 0\}| = \alpha$ and $\|v\|_{L^q(\partial\Omega)} = 1$. Then

$$\mathcal{J}(v) = \mathcal{J}_\varepsilon(v) \ge \mathcal{J}_\varepsilon(u_\varepsilon).$$

Now, since $\liminf_{\varepsilon\to 0} F_\varepsilon(|\{u_\varepsilon > 0\}|) \ge 0$ there holds that

$$\mathcal{J}(v) \ge \liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} \mathcal{J}(u_\varepsilon) \ge \mathcal{J}(\phi_0). \tag{5.2}$$

It remains to see that $|\{\phi_0 > 0\}| = \alpha$. Assume not, then $\alpha_1 := |\{\phi_0 > 0\}| < \alpha$. So, by the strict monotonicity of $S(\cdot)$ there holds that $S(\alpha) < S(\alpha_1)$ but

$$S(\alpha) = \inf_v \mathcal{J}(v) \ge \mathcal{J}(\phi_0) \ge S(\alpha_1),$$

a contradiction.

Now, taking $v = \phi_0$ in (5.2),

$$\mathcal{J}(\phi_0) \le \liminf_{\varepsilon\to 0} \mathcal{J}(u_\varepsilon) \le \liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon) \le \mathcal{J}(\phi_0).$$

Hence, $\|\phi_0\|_{H^1(\Omega)} = \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{H^1(\Omega)}$ and so, by taking a further subsequence if necessary, the convergence is actually strong. □

Remark 5.1. We believe that, as in the previous cases, the minimizers u_ε of $\mathcal{J}_\varepsilon$ in $\mathcal{K}_1$ will already be solutions to (P_α) for ε small. Nevertheless, despite the fact that the result of Theorem 5.3 does not give regularity of the minimizer ϕ_0 we believe that it could be of interest in numerical approximations of the solution to (P_α).

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