EIGENVALUES OF THE P-LAPLACIAN IN FRACTAL STRINGS WITH INDEFINITE WEIGHTS

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Abstract. In this paper we study the spectral counting function of the weighted p-laplacian in fractal strings, where the weight is allowed to change sign. We obtain error estimates related to the interior Minkowski dimension of the boundary. We also find the asymptotic behavior of eigenvalues.

1. Introduction

In this paper we study the following eigenvalue problem:

\begin{equation}
-(\psi_p(u'))' = \lambda r(x)\psi_p(u) \quad \text{in } \Omega
\end{equation}

with zero Dirichlet boundary conditions, in a bounded open set \( \Omega \subset \mathbb{R} \). Here, the weight \( r \) is a given bounded function which may change sign, \( \lambda \) is a real parameter and

\[ \psi_p(s) = |s|^{p-2}s \]

for \( s \neq 0 \), and 0 if \( s = 0 \).

In [4, 10] it was proved that there exists a countable sequence of nonnegative eigenvalues \( \{\lambda_k\}_{k \in \mathbb{N}} \), tending to \( +\infty \) when \( r \) is a continuous function. For indefinite weights \( r \in L^1 \), the existence of a sequence of eigenvalues was proved in [3]. When \( N = 1 \), in [1], it was proved that the variational eigenvalues represents a complete list of eigenvalues.

We define the spectral counting function \( N(\lambda, \Omega) \) as the number of eigenvalues of problem (1.1) less than a given \( \lambda \):

\[ N(\lambda, \Omega) = \# \{ k : \lambda_k \leq \lambda \} . \]

We will write \( N_D(\lambda, \Omega) \) (resp., \( N_N(\lambda, \Omega) \)) whenever we need to stress the dependence on the Dirichlet (resp., Neumann) boundary conditions. Also, we will stress the dependence of problem (1.1) in the weight function, writing \( N(\lambda, \Omega, r) \).

In [6], we obtained the following asymptotic development when \( r > 0 \)

\[ N(\lambda, \Omega) \sim \frac{\lambda^{1/p}}{2\pi_p} \int_{\Omega} r^{1/p} \, dx \]

as \( \lambda \to \infty \), using variational arguments and a suitable extension of the method of ‘Dirichlet-Neumann bracketing’ in [2]. Here,

\[ \pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}} . \]

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For the remainder estimate \( R(\lambda, \Omega) = N(\lambda, \Omega) - \frac{\lambda^{1/p}}{2\pi^p} \int_{\Omega} r^{1/p} \, dx \) we showed that
\[
R(\lambda, \Omega) = O(\lambda^{\mu/p})
\]
where \( \mu \in (0, 1] \) depends on the regularity of the weight \( r \) and the boundary \( \partial \Omega \).

However, the parameter \( \mu \) does not reflect any geometric information about \( \partial \Omega \).

The goal of this paper is the study of the remainder term and the extension of the previous results to indefinite weights.

We improve the previous estimate in terms of the interior Minkowski dimension \( d \) of \( \partial \Omega \), i.e.
\[
R(\lambda, \Omega) = O(\lambda^{d/p})
\]
For indefinite weights, there exists a sequence of positive eigenvalues and a sequence of negative eigenvalues as well. Our main result is
\[
N^\pm(\lambda, \Omega) = \frac{\lambda^{1/p}}{2\pi^p} \int_{\Omega} \left( r^\pm \right)^{1/p} \, dx + O(\lambda^{d/p})
\]
where \( N^+(\lambda) \) denotes the number of positive eigenvalues of problem (1.1) less than a given \( \lambda \), and \( r^+(x) = \max\{r(x), 0\} \), and \( N^-(\lambda) \) denotes the number of negatives eigenvalues greater than \( -\lambda \). When \( p = 2 \), this asymptotic expansion was obtained in [7].

The paper is organized as follows: In section 2, we introduce the necessary notation and definitions. In section 3 we state and prove the main theorem. In section 4 we analyze the eigenvalue problem with indefinite weights.

2. Notation, hypotheses and preliminary results

2.1. Notation and hypotheses. Let \( A_\varepsilon \) denote the tubular neighborhood of radius \( \varepsilon \) of a set \( A \subset \mathbb{R}^n \), i. e.,
\[
A_\varepsilon = \{ x \in \mathbb{R} : \text{dist}(x, A) \leq \varepsilon \}
\]
We define the interior Minkowski dimension of \( \partial \Omega \) as
\[
d = \dim(\partial \Omega) = \inf\left\{ \delta \geq 0 : \limsup_{\varepsilon \to 0^+} \varepsilon^{-(n-\delta)} |(\partial \Omega)_\varepsilon \cap \Omega|_n = 0 \right\}
\]
We define the interior Minkowski content of \( \partial \Omega \) as the limit (whenever it exist):
\[(2.1) \quad M_{\text{int}}(\partial \Omega, d) = \lim_{\varepsilon \to 0^+} \varepsilon^{-(n-d)} |(\partial \Omega)_\varepsilon \cap \Omega|_n.
\]
Respectively, \( M^*_{\text{int}}(\partial \Omega, d)(M^*_{\text{int}}(\partial \Omega, d)) \) denotes the \( d \)--dimensional upper (lower) interior Minkowski content, replacing the limit in (2.1) by an upper (resp., lower) limit.

For the history about the right fractal dimension involved in this problem when \( p = 2 \), see [8].

Let \( \Omega \) be an open set in \( \mathbb{R} \). Then, \( \Omega = \bigcup_{n=1}^{\infty} I_n \), where \( I_n \) is an interval of length \( l_n \). We can assume that
\[
l_1 \geq l_2 \geq \cdots \geq l_n \geq \cdots > 0
\]
In [5, 9] was proved that \( \partial \Omega \) is \( d \)--Minkowski measurable if and only if \( l_n \sim Cn^{-1/d} \). Moreover, the Minkowski content of \( \partial \Omega \) is \( \frac{\chi^d}{1-d} C_d \).

Our assumption on the domain \( \Omega \) is,
\[
(\text{H1}) \quad \Omega \text{ is an open bounded set in } \mathbb{R} \text{ such that } M^*_{\text{int}}(\partial \Omega, d) < \infty.
\]
Observe that we do not make any assumption of self similarity about $\partial \Omega$.

Given any $\eta_0 > 0$ and $q \in \mathbb{N}$, we consider a tessellation of $\mathbb{R}$ by a countable family of open intervals $\{I_{\zeta} \}_{\zeta \in \mathbb{Z}}$ of length $\eta_0 = 2^{-q} \eta_0$. We define

$$I_0(\Omega) = \{ \zeta_0 \in \mathbb{Z} : I_{\zeta_0} \subset \Omega \},$$

$$\Omega_0 = \Omega \setminus (\cup_{\zeta_0 \in \mathbb{Z}} I_{\zeta_0}),$$

$$I_q(\Omega) = \{ \zeta_q \in \mathbb{Z} : I_{\zeta_q} \subset \Omega_{q-1} \},$$

and

$$(2.2) \quad \Omega_q = \Omega \setminus (\overline{\Omega}_{q-1} \cup \cup_{\zeta_q \in \mathbb{Z}} I_{\zeta_q}).$$

Let $r \in L^\infty(\Omega)$ be a positive function. Given $\gamma > 0$, we say that the function $r$ satisfies the “$\gamma$-condition” if there exist positive constants $c_1$ and $\eta_1$ such that for all $\zeta_q \in I_q(\Omega)$ and all $\eta \leq \eta_1$,

$$(H2) \quad \int_{I_{\zeta_q}} |r - r_{\zeta_q}|^{1/p} dx \leq c_1 \eta_1^\gamma,$$

where $r_{\zeta_q} = \left( |I_{\zeta_q}|^{1/p} \int_{I_{\zeta_q}} r^{1/p} dx \right)^p$ is the mean value of $r^{1/p}$ in $I_{\zeta_q}$.

**Remark 2.1.** The coefficient $\gamma$ enable us to measure the smoothness of $r$, the larger $\gamma$, the smoother $r$. When $r$ is Holder continuous of order $\theta > 0$ and is bounded away from zero on $\Omega$, it satisfies the $\gamma$-condition for $0 < \gamma \leq 1 + \theta/p$. If $r$ is only continuous and positive on $\Omega$, then it satisfies the $\gamma$-condition for $0 < \gamma \leq 1$.

2.2. Preliminary results. In this subsection we introduce the main tools to deal with our problem, the genus and the Dirichlet-Neumann bracketing. We remark that the results in this subsection hold in any dimension.

Most of the results in this subsection are contained in [6]. However, we include the proofs in order to make the paper self contained.

Let $X$ be a Banach space. We consider the class:

$$\Sigma = \{ A \subset X : A \text{ is compact} , A = -A \}.$$

Let us recall the definition of the Krasnoselskii genus $\gamma : \Sigma \to \mathbb{N} \cup \{ \infty \}$ as

$$\gamma(A) = \min \{ k \in \mathbb{N} : \text{there exist } f \in C(A, \mathbb{R}^k \setminus \{ 0 \}) \text{ s.t. } f(x) = -f(-x) \}.$$

By the Ljusternik-Schnirelmann theory, we have a sequence of nonlinear eigenvalues of problem (1.1) with Dirichlet (resp. Neumann) boundary condition, given by

$$(2.3) \quad \lambda_k = \inf_{F \in C_k} \sup_{u \in F} \int_\Omega |u'|^p dx$$

where

$$C_k = \{ C \subset M : C \text{ is compact} , C = -C, \gamma(C) \geq k \},$$

$$M = \{ u \in W^{1,p}_0(\Omega) \text{ (resp., } W^{1,p}(\Omega) \text{ ) : } \int_\Omega r(x)|u|^p dx = 1 \}.$$
where

\[ R(u) = \frac{\int_\Omega |u|^p \, dx}{\int_\Omega r(x)|u|^p \, dx} \]

When \( r \in L^\infty \), we need to use a comparison result that is essentially contained in [1] but we include the arguments for the sake of completeness.

**Theorem 2.2.** Let \( r_1 \) and \( r_2 \) be two positive functions in \( L^\infty(\Omega) \), with \( r_1(x) \leq r_2(x) \). Then,

\[ \lambda_k^1 \geq \lambda_k^2. \]

**Proof.** It follows from (2.4), using that

\[ \sup_{u \in F} R_2(u) \leq \sup_{u \in F} R_1(u), \]

hence,

\[ \lambda_k^2 = \inf_{F \in C_k} \sup_{u \in F} R_2(u) \leq \inf_{F \in C_k} \sup_{u \in F} R_1(u) = \lambda_k^1, \]

as we wanted to show. \( \square \)

In a similar way, we prove the Dirichlet–Neumann bracketing,

**Theorem 2.3.** Let \( \Omega_1, \Omega_2 \in \mathbb{R}^N \) be disjoint open sets such that \( (\Omega_1 \cup \Omega_2)^c = \Omega \) and \( |\Omega \setminus \Omega_1 \cup \Omega_2| = 0 \), then

\[ N_D(\lambda, \Omega_1 \cup \Omega_2) \leq N_D(\lambda, \Omega) \leq N_N(\lambda, \Omega) \leq N_N(\lambda, \Omega_1 \cup \Omega_2) \]

**Proof.** It is an easy consequence of the following inclusions

\[ W^{1,p}_0(\Omega_1 \cup \Omega_2) = W^{1,p}_0(\Omega_1) + W^{1,p}_0(\Omega_2) \subset W^{1,p}_0(\Omega) \]

and

\[ W^{1,p}(\Omega) \subset W^{1,p}(\Omega_1) \oplus W^{1,p}(\Omega_2) = W^{1,p}(\Omega_1 \cup \Omega_2), \]

and the variational formulation (2.3). Here, using that

\[ M(X) = \{ u \in X : \int_\Omega r(x)|u|^p \, dx = 1 \} \subset M(Y) = \{ u \in Y : \int_\Omega r(x)|u|^p \, dx = 1 \}, \]

and \( C_k(X) \subset C_k(Y) \), we obtain the desired inequality, where \( X = W^{1,p}_0(\Omega_1 \cup \Omega_2) \) (or \( X = W^{1,p}(\Omega) \)) and \( Y = W^{1,p}_0(\Omega) \) (or \( Y = W^{1,p}(\Omega_1 \cup \Omega_2) \)). \( \square \)

The Dirichlet–Neumann bracketing is a powerful tool combined with the following result:

**Proposition 2.4.** Let \( \{ \Omega_j \}_{j \in \mathbb{N}} \) be a pairwise disjoint family of bounded open sets in \( \mathbb{R}^N \). Then,

\[ N(\lambda, \bigcup_{j \in \mathbb{N}} \Omega_j) = \sum_{j \in \mathbb{N}} N(\lambda, \Omega_j). \]

**Proof.** Let \( \lambda \) be an eigenvalue of problem (1.1) in \( \Omega \), and let \( u \) be the associated eigenfunction. For all \( v \in W^{1,p}_0(\Omega) \) we have

\[ \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{p-2} uv \, dx = 0. \]

Choosing \( v \) with compact support in \( \Omega_j \), we conclude that \( u|_{\Omega_j} \) is an eigenfunction of problem (1.1) in \( \Omega_j \) with eigenvalue \( \lambda \).

In the other hand, an eigenfunction \( u \) of \( \Omega_j \) extended by zero outside is an eigenfunction of \( \Omega \). \( \square \)
3. Main result

**Theorem 3.1.** Let $\Omega \in \mathbb{R}$ be an open, bounded set, and $d \in (0,1)$ such that $M^*_\text{int}(\partial \Omega, d) < +\infty$. Let $r \in L^\infty$ be a positive function satisfying (H2) with $d < \gamma$. Then,

$$N(\lambda, \Omega) = \frac{\lambda^{1/p}}{2\pi p} \int_{\Omega} r^{1/p} \, dx + O(\lambda^{d/p}).$$

**Remark 3.2.** Observe that for $\gamma > d$ the remainder term does not depend on $\gamma$. So Theorem 3.1 improves the results of [7].

**Proof.** For a fixed $\lambda > 1$, let us choose $a > 0$ and $\eta_0$ such that $\eta_0 = \lambda - a$. Since $M^*_\text{int}(\partial \Omega, d) < +\infty$, there exists a positive constant $C$ such that

$$(3.1) \quad \# I_q(\Omega) \leq C \eta_0^{-d}.$$ 

Let us define the Weyl term:

$$\varphi(\lambda, \Omega, r) = \frac{\lambda^{1/p}}{2\pi p} \int_{\Omega} r^{1/p} \, dx.$$ 

As $r \in L^\infty(\Omega)$ we have that $r(x) \leq M$ for almost all $x \in \Omega$. Thus, $\lambda$ being fixed, there exist $k \in \mathbb{N}$ such that

$$N_D(\lambda, I_{\zeta_q}, r) = 0,$$

for all $q > k$. We define $K = \max\{q \in \mathbb{N} : N_D(\lambda, I_{\zeta_q}, r) \neq 0\}$ (let us observe that $K$ depends on $\lambda$).

The proof falls naturally into two steps, i.e., to find a lower and an upper bound for $R(\lambda, \Omega)$.

**Step 1:** From Theorem 2.3 we obtain

$$(3.2) \quad \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_D(\lambda, I_{\zeta_q}, r) - \varphi(\lambda, \Omega, r) \leq N_D(\lambda, \Omega, r) - \varphi(\lambda, \Omega, r).$$

We can rewrite (3.2) as:

$$\sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_D(\lambda, I_{\zeta_q}, r) - \varphi(\lambda, \Omega, r) = A_1 + A_2 + A_3 + A_4,$$

with

$$A_1 = \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left( N_D(\lambda, I_{\zeta_q}, r) - N_D(\lambda, I_{\zeta_q}, r_{\zeta_q}) \right),$$

$$A_2 = \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left( N_D(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) \right),$$

$$A_3 = \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left( \varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r) \right),$$

$$A_4 = -\varphi(\lambda, \Omega_K, r),$$

where $\Omega_K$ is given by (2.2).
Using the monotonicity of the eigenvalues with respect to the weight (see Theorem 2.2), and $r \leq r_{\zeta_q} + |r - r_{\zeta_q}|$, a simple computation shows that
\[ N(\lambda, I_{\zeta_q}, r) \leq N(\lambda, I_{\zeta_q}, r_{\zeta_q}) + N(\lambda, I_{\zeta_q}, |r - r_{\zeta_q}|), \]
which gives:
\[ |N_D(\lambda, I_{\zeta_q}, r) - N_D(\lambda, I_{\zeta_q}, r_{\zeta_q})| \leq N(\lambda, I_{\zeta_q}, |r - r_{\zeta_q}|) \leq c_1 c_2 \lambda^{1/p}. \]

Hence, by (3.1),
\[ |A_1| \leq c_1 \sum_{q=0}^{K} #(I_q) \eta_{q}\lambda^{1/p} \leq c_1 \eta_0^{\gamma-d} \lambda^{1/p} \sum_{q=0}^{K} 2^{-q(\gamma-d)} \leq c_1 \lambda^{1/p} - a(\gamma-d). \]

If $\gamma > d$ we take $a > 1/p(\gamma - d)$ and we obtain $|A_1| = O(1)$.

We now consider $A_2$. But
\[ |N(\lambda, (0, T), M) - \frac{(M\lambda)^{1/p}}{2\pi d T}| = \left| \frac{(M\lambda)^{1/p}}{2\pi d T} - \frac{(M\lambda)^{1/p}}{2\pi d T} \right| \leq 1, \]
which is non positive. Therefore,
\[ |A_2| \leq \sum_{q=0}^{K} #(I_q) \leq C \lambda^{d/p}. \]

Here, we are using that there exists a positive constant $C$ such that
\[ \frac{C}{2} \lambda^{1/p} \leq 2^K \leq C \lambda^{1/p}. \]

Clearly, by the definition of $r_{\zeta_q}$ in $(H_2)$, $A_3 = 0$.

In order to bound $A_4$, let us note that $\Omega_K \subset \{ x \in \Omega : d(x, \partial \Omega) \leq \eta_K \}$. So, the definition of Minkowski measure gives
\[ |A_4| = \varphi(\lambda, \Omega_K, r) = c \int_{\Omega_K} (r\lambda)^{1/p} dx \leq c \lambda^{1/p} \eta_K^{1-d} \leq C \lambda^{d/p}. \]

**Step 2**: In a similar way, we can find an upper bound for $R(\lambda, \Omega, r)$. As in the previous step, we introduce
\[ J_q(\Omega) = \{ \zeta_q \in \mathbb{Z} : I_{\zeta_q} \cap \partial \Omega \neq \emptyset \}, \]
\[ \Omega \subset \bigcup_{q=0}^{K} \bigcup_{\zeta_q \in I_q} I_{\zeta_q}, \]
and again,
\[ \#J(\Omega) \leq C \eta_K^{d}. \]

From Theorem 2.3 we have
\[ N_D(\lambda, \Omega, r) \leq \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_N(\lambda, I_{\zeta_q}, r) + \sum_{\zeta_k \in J_K} N_N(\lambda, I_{\zeta_k}, r). \]

Subtracting the Weyl term from the expression above we have
\[ \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_N(\lambda, I_{\zeta_q}, r) + \sum_{\zeta_k \in J_K} N_N(\lambda, I_{\zeta_k}, r) - \varphi(\lambda, \Omega, r) \leq B_1 + B_2 + B_3 + B_4 + B_5, \]
with
\[ B_1 = \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} (N_N(\lambda, I_{\zeta_q}, r) - N_N(\lambda, I_{\zeta_q}, r_{\zeta_q})), \]
\[ B_2 = \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} (N_N(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r_{\zeta_q})), \]
\[ B_3 = \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} (\varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r)), \]
\[ B_4 = - \varphi(\lambda, \Omega_K, r), \]
\[ B_5 = \sum_{\zeta \in J_K} N_N(\lambda, I_{\zeta}, r). \]

The terms \( B_1, B_2, B_3 \) and \( B_4 \) can be handled in much the same way, the only difference being in the analysis of \( B_5 \). However, as \( r \in L^\infty \),
\[ B_5 \leq \sum_{\zeta \in J_K} N_N(\lambda, I_{\zeta}, 1) \leq #(J_K)C\lambda^{1/p}r_K \leq C\lambda^{d/p}. \]

This completes the proof. \( \square \)

4. Indefinite weights.

Let us begin recalling the existence of a sequence of variational eigenvalues with an indefinite weight:

**Theorem 4.1.** Let \( r \in L^\infty \), with \( r^+ \neq 0 \). Then every eigenvalue of problem (1.1) is given by (2.3). If we consider \( \Sigma^+ = \{\lambda_k^+\}_{k \in \mathbb{N}} \) the set of positive eigenvalues and \( \Sigma^- = \{\lambda_k^-\}_{k \in \mathbb{N}} \) the set of negative eigenvalues, we have that \( \Sigma^- \neq \emptyset \) if \( r^- \neq 0 \) and \( \lambda_k^+ \to +\infty \) and \( \lambda_k^- \to -\infty \) as \( k \to +\infty \).

The proof can be found in [1].

To obtain the asymptotic behaviour of \( N(\lambda) \) we need to impose some conditions in \( r^+ = \max\{r, 0\} \), and \( r^- = r - r^+ \), let us suppose that \( r^+ \) (resp. \( r^- \)) satisfy (H2) for certain \( \gamma^+ \) (resp. \( \gamma^- \)). Let \( \Omega^+_K \) be the interior of \( \Omega^+_1 = \{x \in \Omega : r(x) > 0\} \) and let \( d^+ \) be the interior Minkowski dimension of \( \partial \Omega^+_K \), analogously, let \( d^- \) be the dimension of \( \partial \Omega^-_K \).

Clearly, it suffices to obtain the asymptotic expansion for the number of positive eigenvalues. The negatives ones may be studied in much the same way. Let us note, however, that it is possible to have \( d^- \neq d^+ \), or \( \gamma^- \neq \gamma^+ \).

**Theorem 4.2.** Let \( \Omega \in \mathbb{R} \) be an open, bounded set, and \( d^+ \in (0,1) \) such that \( M_{int}(\partial \Omega, d^+) < +\infty \). Let \( r \in L^\infty(\Omega) \) be a function with \( r^+ \) satisfying (H2) for certain \( \gamma^+ > d^+ \). Then,
\[ N^+(\lambda, \Omega, r) = \frac{\lambda^{1/p}}{2\pi p} \int_{\Omega^+} (r^+)^{1/p} dx + O(\lambda^{d^+/p}). \]

**Proof.** We only need to find lower and upper bounds for \( \lambda_k^+ \) having the same asymptotic. We achieve this with the help of monotonicity that allows us to reduce the problem to the case of positive weights.
Let $\rho$ be fixed. Now, applying Theorem 2.2, we have
\begin{equation}
\lambda_n(r^+ + \rho, \Omega) \leq \lambda_n^+(r, \Omega) \leq \lambda_n(r^+, \Omega).
\end{equation}
The first inequality, together with Theorem 3.1, implies that
\[ N^+(\lambda, \Omega, r) \leq \frac{\lambda^{1/p}}{2\pi} \int_{\Omega} (r^+ + \rho)^{1/p} + O(\lambda^{d^*/p}). \]
Now the proof follows by choosing $\rho = \lambda^{d^*/2} - 1$.

For the lower bound we use the second inequality in (4.1) which gives
\[ N^+(\lambda, \Omega, r) \geq \frac{\lambda^{1/p}}{2\pi} \int_{\Omega} (r^+)^{1/p} + O(\lambda^{d^*/p}). \]
The proof is now complete.

Remark 4.3. From Theorem 4.2 it is easy to see that
\[
\lim_{n \to \infty} n^{-p} \lambda_n^+(r^+ + \rho, \Omega) = \left( \frac{\pi_p}{\int_{\Omega} (r^+ + \rho)^{1/p}} \right)^p
\]
\[
\lim_{n \to \infty} n^{-p} \lambda_n^+(r, \Omega^+) = \left( \frac{\pi_p}{\int_{\Omega^+} r^{1/p}} \right)^p
\]
Combined with the previous inequalities for the eigenvalues, we have
\[
\left( \frac{\pi_p}{\int_{\Omega} (r^+ + \rho)^{1/p}} \right)^p \leq \liminf_{n \to \infty} n^{-p} \lambda_n^+(r, \Omega) \leq \limsup_{n \to \infty} n^{-p} \lambda_n^+(r, \Omega) \leq \left( \frac{\pi_p}{\int_{\Omega^+} r^{1/p}} \right)^p
\]
Clearly, when $\rho \to 0$, the first integral converges to $\int_{\Omega} (r^+)^{1/p}$, and we obtain
the asymptotic formula for the positives eigenvalues
\[ \lambda_n^+(r, \Omega) \sim \left( \frac{\pi_p}{\int_{\Omega} (r^+)^{1/p}} \right)^p. \]

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