EIGENVALUES OF THE P-LAPLACIAN IN FRACTAL STRINGS WITH INDEFINITE WEIGHTS

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Abstract. In this paper we study the spectral counting function of the weighted p-laplacian in fractal strings, where the weight is allowed to change sign. We obtain error estimates related to the interior Minkowski dimension of the boundary. We also find the asymptotic behavior of eigenvalues.

1. Introduction

In this paper we study the following eigenvalue problem:

$$-(\psi_p(u'))' \quad = \quad \lambda r(x)\psi_p(u) \quad \text{in } \Omega \tag{1.1}$$

with zero Dirichlet boundary conditions, in a bounded open set $\Omega \subset \mathbb{R}$. Here, the weight r is a given bounded function which may change sign, λ is a real parameter and

$$\psi_p(s) = |s|^{p-2}s$$

for $s \neq 0$, and 0 if $s = 0$.

In [4, 10] it was proved that there exists a countable sequence of nonnegative eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$, tending to $+\infty$ when r is a continuous function. For indefinite weights $r \in L^1$, the existence of a sequence of eigenvalues was proved in [3]. When $N = 1$, in [1], it was proved that the variational eigenvalues represents a complete list of eigenvalues.

We define the spectral counting function $N(\lambda, \Omega)$ as the number of eigenvalues of problem (1.1) less than a given λ :

$$N(\lambda, \Omega) = \#\{k \ : \ \lambda_k \leq \lambda\}.$$

We will write $N_D(\lambda, \Omega)$ (resp., $N_N(\lambda, \Omega)$) whenever we need to stress the dependence on the Dirichlet (resp., Neumann) boundary conditions. Also, we will stress the dependence of problem (1.1) in the weight function, writing $N(\lambda, \Omega, r)$.

In [6], we obtained the following asymptotic development when $r > 0$

$$N(\lambda, \Omega) \sim \frac{\lambda^{1/p}}{2\pi_p} \int_\Omega r^{1/p}\, dx$$

as $\lambda \to \infty$, using variational arguments and a suitable extension of the method of 'Dirichlet-Neumann bracketing' in [2]. Here,

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}.$$

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For the remainder estimate $R(\lambda,\Omega) = N(\lambda,\Omega) - \frac{\lambda^{1/p}}{2\pi_p}\int_\Omega r^{1/p}\,dx$ we showed that

$$R(\lambda,\Omega) = O(\lambda^{\mu/p})$$

where $\mu \in (0,1]$ depends on the regularity of the weight r and the boundary $\partial\Omega$.

However, the parameter μ does not reflect any geometric information about $\partial\Omega$.

The goal of this paper is the study of the remainder term and the extension of the previous results to indefinite weights.

We improve the previous estimate in terms of the interior Minkowski dimension d of $\partial\Omega$, i.e.

$$R(\lambda,\Omega) = O(\lambda^{d/p})$$

For indefinite weights, there exists a sequence of positive eigenvalues and a sequence of negative eigenvalues as well. Our main result is

$$N^{\pm}(\lambda,\Omega) = \frac{\lambda^{1/p}}{2\pi_p}\int_\Omega (r^{\pm})^{1/p}\,dx + O(\lambda^{d/p}),$$

where $N^+(\lambda)$ denotes the number of positive eigenvalues of problem (1.1) less than a given λ, and $r^+(x) = \max\{r(x),0\}$, and $N^-(\lambda)$ denotes the number of negatives eigenvalues greater than $-\lambda$. When $p=2$, this asymptotic expansion was obtained in [7].

The paper is organized as follows: In section 2, we introduce the necessary notation and definitions. In section 3 we state and prove the main theorem. In section 4 we analyze the eigenvalue problem with indefinite weights.

2. Notation, hypotheses and preliminary results

2.1. **Notation and hypotheses.** Let A_ε denote the tubular neighborhood of radius ε of a set $A \subset \mathbb{R}^n$, i. e.,

$$A_\varepsilon = \{x \in \mathbb{R} : \mathrm{dist}(x,A) \le \varepsilon\}$$

We define the interior Minkowski dimension of $\partial\Omega$ as

$$d = \dim(\partial\Omega) = \inf\{\delta \ge 0 \ : \ \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-\delta)}|(\partial\Omega)_\varepsilon \cap \Omega|_n = 0\}$$

We define the interior Minkowski content of $\partial\Omega$ as the limit (whenever it exist):

$$M_{int}(\partial\Omega,d) = \lim_{\varepsilon\to 0^+} \varepsilon^{-(n-d)}|(\partial\Omega)_\varepsilon \cap \Omega|_n. \tag{2.1}$$

Respectively, $M^*_{int}(\partial\Omega,d)(M_{*int}(\partial\Omega,d))$ denotes the $d-$dimensional upper (lower) interior Minkowski content, replacing the limit in (2.1) by an upper (resp., lower) limit.

For the history about the right fractal dimension involved in this problem when $p=2$, see [8].

Let Ω be an open set in $\mathbb{R}$. Then, $\Omega = \cup_{n=1}^\infty I_n$, where I_n is an interval of length l_n. We can assume that

$$l_1 \ge l_2 \ge \cdots \ge l_n \ge \cdots > 0$$

In [5, 9] was proved that $\partial\Omega$ is d-Minkowski measurable if and only if $l_n \sim Cn^{-1/d}$. Moreover, the Minkowski content of $\partial\Omega$ is $\frac{2^{1-d}}{1-d}C^d$.

Our assumption on the domain Ω is,

(H1) Ω is an open bounded set in $\mathbb{R}$ such that $M^*_{int}(\partial\Omega,d) < \infty$.

Observe that we do not make any assumption of self similarity about $\partial\Omega$.

Given any $\eta_0 > 0$ and $q \in \mathbb{N}$, we consider a tessellation of $\mathbb{R}$ by a countable family of open intervals $\{I_{\zeta_q}\}_{\zeta_q\in\mathbb{Z}}$ of length $\eta_q = 2^{-q}\eta_0$. We define

$$I_0(\Omega) = \{\zeta_0 \in \mathbb{Z} \ : \ I_{\zeta_0} \subset \Omega\},$$
$$\Omega_0 = \Omega \setminus (\cup_{\zeta_0\in\mathbb{Z}}\overline{I_{\zeta_0}}),$$
$$I_q(\Omega) = \{\zeta_q \in \mathbb{Z} \ : \ I_{\zeta_q} \subset \Omega_{q-1}\},$$

and

$$\Omega_q = \Omega \setminus (\overline{\Omega}_{q-1} \bigcup \cup_{\zeta_q\in\mathbb{Z}}\overline{I_{\zeta_q}}). \tag{2.2}$$

Let $r \in L^\infty(\Omega)$ be a positive function.

Given $\gamma > 0$, we say that the function r satisfies the "γ-condition" if there exist positive constants c_1 and η_1 such that for all $\zeta_q \in I_q(\Omega)$ and all $\eta \le \eta_1$,

$$\int_{I_{\zeta_q}} |r - r_{\zeta_q}|^{1/p}\, dx \le c_1\eta_q^\gamma, \tag{H2}$$

where $r_{\zeta_q} = \left(|I_{\zeta_q}|^{-1}\int_{I_{\zeta_q}} r^{1/p}\, dx\right)^p$ is the mean value of $r^{1/p}$ in I_{ζ_q}.

Remark 2.1. The coefficient γ enable us to measure the smoothness of r, the larger γ, the smoother r. When r is Holder continuous of order $\theta > 0$ and is bounded away from zero on Ω, it satisfies the γ-condition for $0 < \gamma \le 1 + \theta/p$. If r is only continuous and positive on $\overline{\Omega}$, then it satisfies the γ-condition for $0 < \gamma \le 1$.

2.2. **Preliminary results.** In this subsection we introduce the main tools to deal with our problem, the genus and the Dirichlet-Neumann bracketing. We remark that the results in this subsection hold in any dimension.

Most of the results in this subsection are contained in [6]. However, we include the proofs in order to make the paper self contained.

Let X be a Banach space. We consider the class:

$$\Sigma = \{A \subset X \ : \ A \ \text{ is compact }, \ A = -A\}.$$

Let us recall the definition of the Krasnoselskii genus $\gamma : \Sigma \to \mathbb{N}\cup\{\infty\}$ as

$$\gamma(A) = \min\{k \in \mathbb{N} \ \text{ there exist } \ f \in C(A, \mathbb{R}^k \setminus \{0\}), \ f(x) = -f(-x)\}.$$

By the Ljusternik-Schnirelmann theory, we have a sequence of nonlinear eigenvalues of problem (1.1) with Dirichlet (resp. Neumann) boundary condition, given by

$$\lambda_k = \inf_{F\in C_k}\sup_{u\in F}\int_\Omega |u'|^p\, dx \tag{2.3}$$

where

$$C_k = \{C \subset M \ : \ C \ \text{ is compact }, \ C = -C, \ \gamma(C) \ge k\},$$
$$M = \{u \in W_0^{1,p}(\Omega) \ (\text{ resp., } W^{1,p}(\Omega) \ ) \ : \ \int_\Omega r(x)|u|^p\, dx = 1\}.$$

Due to the homogeneity of equation (1.1), we have an equivalent formula for the eigenvalues,

$$\lambda_k = \inf_{F\in C_k}\sup_{u\in F} R(u) \tag{2.4}$$

where

$$R(u) = \frac{\int_\Omega |u'|^p\,dx}{\int_\Omega r(x)|u|^p\,dx}$$

When $r \in L^\infty$, we need to use a comparison result that is essentially contained in [1] but we include the arguments for the sake of completeness.

Theorem 2.2. *Let r_1 and r_2 be two positive functions in $L^\infty(\Omega)$, with $r_1(x) \le r_2(x)$. Then,*

$$\lambda_k^1 \ge \lambda_k^2.$$

Proof. It follows from (2.4), using that $R_2(u) \le R_1(u)$ for all $u \in W^{1,p}(\Omega)$. For each $F \in C_k$, we have $\sup_{u\in F} R_2(u) \le \sup_{u\in F} R_1(u)$, hence,

$$\lambda_k^2 = \inf_{F\in C_k} \sup_{u\in F} R_2(u) \le \inf_{F\in C_k} \sup_{u\in F} R_1(u) = \lambda_k^1,$$

as we wanted to show. □

In a similar way, we prove the Dirichlet–Neumann bracketing,

Theorem 2.3. *Let $\Omega_1, \Omega_2 \in \mathbb{R}^N$ be disjoint open sets such that $(\overline{\Omega_1 \cup \Omega_2})^\circ = \Omega$ and $|\Omega \setminus \Omega_1 \cup \Omega_2|_n = 0$, then*

$$N_D(\lambda, \Omega_1 \cup \Omega_2) \le N_D(\lambda, \Omega) \le N_N(\lambda, \Omega) \le N_N(\lambda, \Omega_1 \cup \Omega_2)$$

Proof. It is an easy consequence of the following inclusions

$$W_0^{1,p}(\Omega_1 \cup \Omega_2) = W_0^{1,p}(\Omega_1) \oplus W_0^{1,p}(\Omega_2) \subset W_0^{1,p}(\Omega)$$

and

$$W^{1,p}(\Omega) \subset W^{1,p}(\Omega_1) \oplus W^{1,p}(\Omega_2) = W^{1,p}(\Omega_1 \cup \Omega_2),$$

and the variational formulation (2.3). Here, using that

$$M(X) = \{u \in X \ : \ \int_\Omega r(x)|u|^p\,dx = 1\} \subset M(Y) = \{u \in Y \ : \ \int_\Omega r(x)|u|^p\,dx = 1\},$$

and $C_k(X) \subset C_k(Y)$, we obtain the desired inequality, where $X = W_0^{1,p}(\Omega_1 \cup \Omega_2)$ (or $X = W^{1,p}(\Omega)$) and $Y = W_0^{1,p}(\Omega)$ (or $Y = W^{1,p}(\Omega_1 \cup \Omega_2)$). □

The Dirichlet–Neumann bracketing is a powerful tool combined with the following result:

Proposition 2.4. *Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a pairwise disjoint family of bounded open sets in $\mathbb{R}^N$. Then,*

$$N(\lambda, \bigcup_{j\in\mathbb{N}} \Omega_j) = \sum_{j\in\mathbb{N}} N(\lambda, \Omega_j).$$

Proof. Let λ be an eigenvalue of problem (1.1) in Ω, and let u be the associated eigenfunction. For all $v \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx - \lambda \int_\Omega |u|^{p-2}uv\,dx = 0.$$

Choosing v with compact support in Ω_j, we conclude that $u|_{\Omega_j}$ is an eigenfunction of problem (1.1) in Ω_j with eigenvalue λ.

In the other hand, an eigenfunction u of Ω_j extended by zero outside is an eigenfunction of Ω. □

3. Main result

In this section we prove our main result,

Theorem 3.1. *Let $\Omega \in \mathbb{R}$ be an open, bounded set, and $d \in (0,1)$ such that $M^*_{Int}(\partial\Omega, d) < +\infty$. Let $r \in L^\infty$ be a positive function satisfying* (H2) *with $d < \gamma$. Then,*

$$N(\lambda, \Omega) = \frac{\lambda^{1/p}}{2\pi_p} \int_\Omega r^{1/p}\, dx + O(\lambda^{d/p}).$$

Remark 3.2. Observe that for $\gamma > d$ the remainder term does not depends on γ. So Theorem 3.1 improves the results of [7].

Proof. For a fixed $\lambda > 1$, let us choose $a > 0$ and η_0 such that $\eta_0 = \lambda^{-a}$. Since $M^*_{Int}(\partial\Omega, d) < +\infty$, there exist a positive constant C such that

$$\#I_q(\Omega) \le C\eta_q^{-d}. \tag{3.1}$$

Let us define the Weyl term:

$$\varphi(\lambda, \Omega, r) = \frac{\lambda^{1/p}}{2\pi_p} \int_\Omega r^{1/p}\, dx.$$

As $r \in L^\infty(\Omega)$ we have that $r(x) \le M$ for almost all $x \in \Omega$. Thus, λ being fixed, there exist $k \in \mathbb{N}$ such that

$$N_D(\lambda, I_{\zeta_q}, r) = 0,$$

for all $q > k$. We define $K = \max\{q \in \mathbb{N} \ : \ N_D(\lambda, I_{\zeta_q}, r) \neq 0\}$ (let us observe that K depends on λ).

The proof falls naturally into two steps, i.e., to find a lower and an upper bound for $R(\lambda, \Omega)$.

Step 1: From Theorem 2.3 we obtain

$$\sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_D(\lambda, I_{\zeta_q}, r) - \varphi(\lambda, \Omega, r) \le N_D(\lambda, \Omega, r) - \varphi(\lambda, \Omega, r). \tag{3.2}$$

We can rewrite (3.2) as:

$$\sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_D(\lambda, I_{\zeta_q}, r) - \varphi(\lambda, \Omega, r) = A_1 + A_2 + A_3 + A_4,$$

with

$$\begin{aligned}
A_1 &= \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left(N_D(\lambda, I_{\zeta_q}, r) - N_D(\lambda, I_{\zeta_q}, r_{\zeta_q}) \right), \\
A_2 &= \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left(N_D(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) \right), \\
A_3 &= \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left(\varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r) \right), \\
A_4 &= -\,\varphi(\lambda, \Omega_K, r),
\end{aligned}$$

where Ω_K is given by (2.2).

Using the monotonicity of the eigenvalues with respect to the weight (see Theorem 2.2), and $r \le r_{\zeta_q} + |r - r_{\zeta_q}|$, a simple computation shows that

$$N(\lambda, I_{\zeta_q}, r) \le N(\lambda, I_{\zeta_q}, r_{\zeta_q}) + N(\lambda, I_{\zeta_q}, |r - r_{\zeta_q}|),$$

which gives:

$$|N_D(\lambda, I_{\zeta_q}, r) - N_D(\lambda, I_{\zeta_q}, r_{\zeta_q})| \le N(\lambda, I_{\zeta_q}, |r - r_{\zeta_q}|) \le c_1 \eta_q^\gamma \lambda^{1/p}.$$

Hence, by (3.1),

$$|A_1| \le c_1 \sum_{q=0}^{K} \#(I_q) \eta_q^\gamma \lambda^{1/p} \le c_1 \eta_0^{\gamma-d} \lambda^{1/p} \sum_{q=0}^{K} 2^{-q(\gamma-d)} \le c\lambda^{(1/p)-a(\gamma-d)}.$$

If $\gamma > d$ we take $a > 1/p(\gamma - d)$ and we obtain $|A_1| = O(1)$.

We now consider A_2. But

$$\left| N(\lambda, (0,T), M) - \frac{(M\lambda)^{1/p}}{2\pi_p T} \right| = \left| \left[\frac{(M\lambda)^{1/p}}{2\pi_p T} \right] - \frac{(M\lambda)^{1/p}}{2\pi_p T} \right| \le 1,$$

which is non positive. Therefore,

$$|A_2| \le \sum_{q=0}^{K} \#(I_q) \le C\lambda^{d/p}.$$

Here, we are using that there exists a positive constant C such that

$$\frac{C}{2}\lambda^{1/p} \le 2^K \le C\lambda^{1/p}.$$

Clearly, by the definition of r_{ζ_q} in (H_2), $A_3 = 0$.

In order to bound A_4, let us note that $\Omega_K \subset \{x \in \Omega \ : \ d(x, \partial\Omega) \le \eta_K\}$. So, the definition of Minkowski measure gives

$$|A_4| = \varphi(\lambda, \Omega_K, r) = c \int_{\Omega_K} (r\lambda)^{1/p}\, dx \le c\lambda^{1/p} \eta_K^{1-d} \le c\lambda^{d/p}$$

Step 2: In a similar way, we can find an upper bound for $R(\lambda, \Omega, r)$. As in the previous step, we introduce

$$J_q(\Omega) = \{\zeta_q \in \mathbb{Z} \ : \ I_{\zeta_q} \cap \partial\Omega \neq \emptyset\},$$

$$\Omega \subset \bigcup_{q=0}^{K} \bigcup_{\zeta_q \in I_q} I_{\zeta_q} \cup \bigcup_{\zeta_k \in J_K} I_{\zeta_k},$$

and again,

$$\#J_K(\Omega) \le C\eta_K^{-d}.$$

From Theorem 2.3 we have

$$N_D(\lambda, \Omega, r) \le \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_N(\lambda, I_{\zeta_q}, r) + \sum_{\zeta_k \in J_K} N_N(\lambda, I_{\zeta_k}, r).$$

Subtracting the Weyl term from the expression above we have

$$\sum_{q=0}^{K} \sum_{\zeta_q \in I_q} N_N(\lambda, I_{\zeta_q}, r) + \sum_{\zeta_k \in J_K} N_N(\lambda, I_{\zeta_k}, r) - \varphi(\lambda, \Omega, r) \le B_1 + B_2 + B_3 + B_4 + B_5,$$

with

$$\begin{aligned}
B_1 &= \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left(N_N(\lambda, I_{\zeta_q}, r) - N_N(\lambda, I_{\zeta_q}, r_{\zeta_q}) \right), \\
B_2 &= \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left(N_N(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) \right), \\
B_3 &= \sum_{q=0}^{K} \sum_{\zeta_q \in I_q} \left(\varphi(\lambda, I_{\zeta_q}, r_{\zeta_q}) - \varphi(\lambda, I_{\zeta_q}, r) \right), \\
B_4 &= -\,\varphi(\lambda, \Omega_K, r), \\
B_5 &= \sum_{\zeta_k \in J_K} N_N(\lambda, I_{\zeta_k}, r).
\end{aligned}$$

The terms B_1, B_2, B_3 and B_4 can be handled in much the same way, the only difference being in the analysis of B_5. However, as $r \in L^\infty$,

$$B_5 \le \sum_{\zeta_k \in J_K} N_N(\lambda, I_{\zeta_k}, 1) \le \#(J_K) C \lambda^{1/p} \eta_K \le C \lambda^{d/p}.$$

This completes the proof. □

4. Indefinite weights.

Let us begin recalling the existence of a sequence of variational eigenvalues with an indefinite weight:

Theorem 4.1. *Let $r \in L^\infty$, with $r^+ \not\equiv 0$. Then every eigenvalue of problem* (1.1) *is given by* (2.3)*. If we consider $\Sigma^+ = \{\lambda_k^+\}_{k\in\mathbb{N}}$ the set of positive eigenvalues and $\Sigma^- = \{\lambda_k^-\}_{k\in\mathbb{N}}$ the set of negative eigenvalues, we have that $\Sigma^- \ne \emptyset$ if $r^- \not\equiv 0$ and $\lambda_k^+ \to +\infty$ and $\lambda_k^- \to -\infty$ as $k \to +\infty$.*

The proof can be found in [1].

To obtain the asymptotic behaviour of $N(\lambda)$ we need to impose some conditions in $r^+ = \max\{r, 0\}$, and $r^- = r - r^+$, let us suppose that r^+ (resp. r^-) satisfy (H2) for certain γ^+ (resp. γ^-). Let Ω_+° be the interior of $\Omega_+ = \{x \in \Omega : r(x) > 0\}$ and let d^+ be the interior Minkowski dimension of $\partial\Omega_+^\circ$, analogously, let d^- be the dimension of $\partial\Omega_-^\circ$.

Clearly, it suffices to obtain the asymptotic expansion for the number of positives eigenvalues. The negatives ones may be studied in much the same way. Let us note, however, that it is possible to have $d^- \ne d^+$, or $\gamma^- \ne \gamma^+$.

Theorem 4.2. *Let $\Omega \in \mathbb{R}$ be an open, bounded set, and $d^+ \in (0,1)$ such that $M^*_{Int}(\partial\Omega_+, d^+) < +\infty$. Let $r \in L^\infty(\Omega)$ be a function with r^+ satisfying* (H2) *for certain $\gamma^+ > d^+$. Then,*

$$N^+(\lambda, \Omega, r) = \frac{\lambda^{1/p}}{2\pi_p} \int_{\Omega_+} (r^+)^{1/p}\, dx + O(\lambda^{d^+/p}).$$

Proof. We only need to find lower and upper bounds for λ_n^+ having the same asymptotic. We achieve this with the help of monotonicity that allows us to reduce the problem to the case of positive weights

Let ρ be fixed. Now, applying Theorem 2.2, we have

$$\lambda_n(r^+ + \rho, \Omega) \le \lambda_n(r^+, \Omega) \le \lambda_n^+(r, \Omega_+^\circ). \tag{4.1}$$

The first inequality, together with Theorem 3.1, implies that

$$N^+(\lambda, \Omega, r) \le N(\lambda, \Omega, r^+ + \rho) = \frac{\lambda^{1/p}}{2\pi_p} \int_\Omega (r^+ + \rho)^{1/p} + O(\lambda^{d^+/p}).$$

Now the proof follows by choosing $\rho = \lambda^{d^+-1}$.

For the lower bound we use the second inequality in (4.1) which gives

$$\begin{aligned} N^+(\lambda, \Omega, r) \ge N(\lambda, \Omega_+^0, r) &= \frac{\lambda^{1/p}}{2\pi_p} \int_{\Omega_+^0} r^{1/p} + O(\lambda^{d^+/p}) \\ &= \frac{\lambda^{1/p}}{2\pi_p} \int_\Omega (r^+)^{1/p} + O(\lambda^{d^+/p}). \end{aligned}$$

The proof is now complete. □

Remark 4.3. From Theorem 4.2 it is easy to see that

$$\lim_{n\to\infty} n^{-p} \lambda_n^+(r^+ + \rho, \Omega) = \left(\frac{\pi_p}{\int_\Omega (r^+ + \rho)^{1/p}} \right)^p$$

$$\lim_{n\to\infty} n^{-p} \lambda_n^+(r, \Omega^{+\circ}) = \left(\frac{\pi_p}{\int_{\Omega^{+\circ}} r^{1/p}} \right)^p$$

Combined with the previous inequalities for the eigenvalues, we have

$$\left(\frac{\pi_p}{\int_\Omega (r^+ + \rho)^{1/p}} \right)^p \le \liminf_{n\to\infty} n^{-p} \lambda_n^+(r, \Omega) \le \limsup_{n\to\infty} n^{-p} \lambda_n^+(r, \Omega) \le \left(\frac{\pi_p}{\int_{\Omega^{+\circ}} r^{1/p}} \right)^p$$

Clearly, when $\rho \to 0$, the first integral converges to $\int_\Omega (r^+)^{1/p}$, and we obtain the asymptotic formula for the positives eigenvalues

$$\lambda_n^+(r, \Omega) \sim \left(\frac{\pi_p}{\int_\Omega (r^+)^{1/p}} \right)^p.$$

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