THE BEST SOBOLEV TRACE CONSTANT IN PERIODIC MEDIA FOR CRITICAL AND SUBCRITICAL EXPONENTS

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Abstract. In this paper we study homogenization problems for the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ in a bounded smooth domain. When $q = 2$ this leads to a Steklov-like eigenvalue problem. We deal with the best constant of the Sobolev trace embedding in rapidly oscillating periodic media, we consider H^1 and L^q spaces with weights that are periodic in space. We find that extremals for these embeddings converge to a solution of an homogenized limit problem and the best trace constant converges to a homogenized best trace constant. Our results are in fact more general, we can also consider general operators of the form $a^\varepsilon(x, \nabla u)$ with nonlinear Neumann boundary conditions. In particular, we can deal with the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$.

1. Introduction.

Sobolev inequalities have been studied by many authors and is by now a classical subject. It at least goes back to [3], for more references see [10]. Relevant for the study of boundary value problems for differential operators is the Sobolev trace inequality that has been intensively studied, see for example, [11, 12, 14, 15, 16]. Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$, we deal with the best constant of the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$. When $q = 2$ this leads to an eigenvalue problem of the Steklov type.

Our main goal here is to consider the Sobolev trace inequality for H^1 and L^q spaces with weights that oscillate periodically. We find that extremals for these embeddings converge as the oscillations go to infinity to a solution of an homogenized limit problem and the best trace constant converges to an homogenized best trace constant.

Let us consider the following coefficients

$$\begin{cases} a_{ij} \in L^\infty_\#(\mathbb{T}), \text{ where } \mathbb{T} = [0,1]^N, \text{ i.e., each } a_{ij} \text{ is a } \mathbb{T}\text{-periodic} \\ \text{bounded measurable function defined on } \mathbb{R}^N, \\ \exists \alpha, \beta > 0 \text{ such that } \alpha|\eta|^2 \le a_{ij}(x)\eta_i\eta_j \le \beta|\eta|^2 \ \forall \eta \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{T}, \\ a_{ij} = a_{ji} \quad \forall i,j = 1, \dots, N, \end{cases} \tag{1.1}$$

$$\begin{cases} a_0 \in L^\infty_\#(\mathbb{T}), \text{ i.e., } a_0 \text{ is } \mathbb{T}\text{-periodic, and} \\ \exists a_-, a_+ \in \mathbb{R}_+, \text{ such that } 0 < a_- \le a_0(x) \le a_+, \text{ a.e. } x \in \mathbb{T} \end{cases} \tag{1.2}$$

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and

$$(1.3)\qquad \begin{cases} b \in L^\infty_\#(\mathbb{T}), \text{ i.e., } b \text{ is } \mathbb{T}\text{-periodic, and} \\ \exists b_-, b_+ \in \mathbb{R}_+, \text{ such that } 0 < b_- \le b(x) \le b_+, \text{ a.e. } x \in \mathbb{T}. \end{cases}$$

Associated to these coefficients and a parameter $\varepsilon > 0$, we consider for every critical or subcritical exponent, $1 \le q \le 2_* := 2(N-1)/(N-2)$, the Sobolev trace inequality,

$$S(\varepsilon) \int_{\partial\Omega} b^\varepsilon |v|^q dS \le \int_\Omega \left(a^\varepsilon_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} + a^\varepsilon_0 v^2 \right) dx,$$

valid for all $v \in H^1(\Omega)$. Here $a^\varepsilon_{ij}(x) := a_{ij}(x/\varepsilon)$, $a^\varepsilon_0(x) := a_0(x/\varepsilon)$ and $b^\varepsilon(x) := b(x/\varepsilon)$.

The best Sobolev trace constant is the largest $S(\varepsilon)$ such that the above inequality holds, that is,

$$(1.4)\qquad S(\varepsilon) := \inf_{v \in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega \left(a^\varepsilon_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} + a^\varepsilon_0 v^2 \right) dx}{\left(\displaystyle\int_{\partial\Omega} b^\varepsilon |v|^q \, dS \right)^{2/q}}.$$

For subcritical exponents, $1 \le q < 2_*$, the embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact, so we have existence of extremals, i.e. functions where the infimum is attained. These extremals are strictly positive in Ω (see [14]) and smooth up to the boundary (see [6]). When one normalize the extremals with

$$(1.5)\qquad \int_{\partial\Omega} b^\varepsilon |u_\varepsilon|^q dS = 1,$$

it follows that they are weak solutions of the following problem

$$(1.6)\qquad \begin{cases} \dfrac{\partial}{\partial x_i} \left(a^\varepsilon_{ij} \dfrac{\partial u_\varepsilon}{\partial x_j} \right) = a^\varepsilon_0 u_\varepsilon & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu^\varepsilon} := a^\varepsilon_{ij} \dfrac{\partial u_\varepsilon}{\partial x_j} \nu_i = S(\varepsilon) b^\varepsilon |u_\varepsilon|^{q-2} u_\varepsilon & \text{on } \partial\Omega, \end{cases}$$

where ν is the unit outward normal vector. Of special importance is the case $q = 2$. In this case, (1.6) is an eigenvalue problem of Steklov type, see [20]. In the rest of this article we will assume that the extremals are normalized according to (1.5).

Our first result is the following:

Theorem 1. *Let $1 \le q < 2_*$. Assume that Ω is a generic domain, that is, assume that the boundary of Ω, $\partial\Omega$, does not contain flat pieces or that it contains finitely many flat pieces with conormal not proportional to any $m \in \mathbb{Z}^N$. Then, the function $S(\varepsilon)$ converges as $\varepsilon \to 0$ to S^* the best Sobolev trace constant of the homogenized problem that is defined by*

$$(1.7)\qquad S^* = \inf_{v \in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega \left(a^*_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} + a^*_0 v^2 \right) dx}{\left(\displaystyle\int_{\partial\Omega} b^* |v|^q \, dS \right)^{2/q}},$$

where the homogenized coefficients are defined by: a_0^* and b^* are the mean value of a_0 and b respectively, i.e.,

$$a_0^* := \int_{\mathbb{T}} a_0(y)\,dy, \qquad b^* := \int_{\mathbb{T}} b(y)\,dy. \tag{1.8}$$

The coefficients a_{ij}^* are given by

$$a_{ij}^* := \int_{\mathbb{T}} \left(a_{ij} - \frac{\partial a_{i\ell}}{\partial y_\ell} \chi_j \right) dy, \tag{1.9}$$

where, for any $k = 1, \dots, d$, χ_k is the unique solution of the cell problem

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(a_{ij} \dfrac{\partial \chi_k}{\partial y_j} \right) = \dfrac{\partial a_{k\ell}}{\partial y_\ell} & in \quad \mathbb{T}, \\ \chi_k \in H^1_\#(\mathbb{T}), \qquad m(\chi_k) = 0. \end{cases} \tag{1.10}$$

Moreover, as $\varepsilon \to 0$ the sequence of extremals $\{u_\varepsilon\}$ of (1.4) converges (along subsequences) weakly in $H^1(\Omega)$ to a limit u^* that is an extremal of the homogenized problem (1.7) and so, it verifies

$$\begin{cases} \dfrac{\partial}{\partial x_i}\left(a_{ij}^* \dfrac{\partial u^*}{\partial x_j} \right) = a_0^* u^* & in\ \Omega, \\ \dfrac{\partial u^*}{\partial \nu^*} := a_{ij}^* \dfrac{\partial v^*}{\partial x_j}\nu_i = S^* b^* |u^*|^{q-2} u^* & on\ \partial\Omega. \end{cases} \tag{1.11}$$

Remark 1.1. *The homogenized coefficients are related to the original coefficients by the usual homogenization rules (see [5]). Concerning boundary terms, in [17], it is proved that for generic domains there exists a limit. However for non-generic domains there exist different limits for different sequences of $\varepsilon \to 0$. In Theorem 1 we consider the generic case, that is, we impose that the boundary of Ω does not contain flat pieces or that it contains finitely many flat pieces with conormal not proportional to any $m \in \mathbb{Z}^N$.*

Remark 1.2. *This result can be generalized using H-convergence. If we have a sequence of coefficients (a_{ij}^ε) that converges to (a_{ij}^*) in the sense of H-convergence (see [18]) then the corresponding extremals u_ε converge weakly in $H^1(\Omega)$ to an extremal of the limit problem. To see this fact we only have to observe that, using H-convergence, we can pass to the limit in the weak form of the equation (1.6).*

Also, this result can also be seen from the Γ-convergence of functionals point of view. The functionals describe the stored energy of the portion of the ε-periodic composite material occupying a region Ω of $\mathbb{R}^N$. The Γ-convergence provide the behavior of the extremals and the shape of the limit of the functionals (see [9] for an extensive study of this method).

Our second result deals with the critical exponent, $q = 2_*$. In this case, under a geometric assumption on the domain, we have a similar result.

Theorem 2. *Assume that Ω is a generic domain (see Theorem 1) and that*

$$\frac{\alpha(N-2)|B(0,1)|^{1/(N-1)}}{2(b_+)^{2/2_*}} > \frac{|\Omega| a_+}{|\partial\Omega| b_-}, \tag{1.12}$$

where the constants α, a_+ and $b_\pm$ are given in (1.1)–(1.3).

Then, the function $S(\varepsilon)$ converges as $\varepsilon \to 0$ to S^* the best Sobolev trace constant of the homogenized problem that is defined by (1.7) Moreover, as $\varepsilon \to 0$ the sequence *of extremals* $\{u_\varepsilon\}$ *of* (1.4) *converges (along subsequences) weakly in* $H^1(\Omega)$ *to a limit* u^* *that is an extremal of the homogenized problem* (1.7) *(and so, a solution of* (1.11)*).*

Remark 1.3. *In the proof of Theorem 2, what is actually used is that there exists* $\delta > 0$ *(independent of* ε*) such that* $S(\varepsilon)$ *satisfies*

$$\frac{\alpha(N-2)|B(0,1)|^{1/(N-1)}}{2(b_+)^{2/2_*}} - \delta > S(\varepsilon). \tag{1.13}$$

This condition is implied by (1.12) *taking* $u \equiv 1$ *as a test function in* (1.4)*.*

Arguing as in [15]*, one can check that the hypothesis* (1.13) *implies the existence of an extremal* u_ε *for* (1.4)*.*

Our results are in fact more general. For the sake of clarity we choose to present first the linear case with periodic coefficients in full detail. However, using ideas from [4], we can deal with more general (nonlinear) operators.

Let $a^\varepsilon(x,\xi)$ and $b^\varepsilon(x,u)$, with $x \in \Omega$, $\xi \in \mathbb{R}^N$ and $u \in \mathbb{R}$ be general nonlinear functions verifying convenient hypotheses (see Section 5). We consider

$$\lambda_1 = \inf_{v \in W^{1,p}(\Omega)\setminus W^{1,p}_0(\Omega)} \frac{\displaystyle\int_\Omega a(x,\nabla v)\cdot\nabla v + b(x,v)v\,dx}{\displaystyle\int_{\partial\Omega} |v|^q\,dS}. \tag{1.14}$$

Theorem 3. *Assume that* a^ε *and* b^ε *satisfy the hypotheses* (A1)–(A4), (B1)–(B3) *in Section 5 and that there exist two limit functions* $a_{\rm hom} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ *and* $b_{\rm hom} : \Omega \times \mathbb{R} \to \mathbb{R}$*, that satisfy the same hypotheses.*

Also assume that the operators $\mathcal{A}^\varepsilon$ *G-converge to the operator* $\mathcal{A}_{\rm hom}$ *associated to these functions. Let* λ_1^ε *and* $\lambda_1^{\rm hom}$ *be as in* (1.14) *with* a, b *replaced by* $a^\varepsilon, b^\varepsilon$ *and* $a_{\rm hom}, b_{\rm hom}$ *respectively.*

(1) *If* $1 \le q < p_* := p(N-1)/(N-p)$ *then,* $\lambda_1^\varepsilon \to \lambda_1^{\rm hom}$ *as* $\varepsilon \to 0$*.*
Moreover, the extremals $\{u_\varepsilon\}$ *converge (along subsequences) weakly in* $W^{1,p}(\Omega)$ *to a limit* u^* *that is an extremal of the homogenized problem.*

(2) *For the critical case,* $q = p_*$*, assume that* Ω *verifies*

$$\frac{|\Omega|}{|\partial\Omega|^{p^*/p}} < \frac{c}{K(N,p)}, \tag{1.15}$$

where $K(N,p)$ *is the best Sobolev trace constant in a half-space*

$$K(N,p) = \inf_{\nabla v \in L^p(\mathbb{R}^N_+),\, w \in L^{p^*}(\partial\mathbb{R}^N_+)} \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla v|^p dx}{\left(\displaystyle\int_{\partial\mathbb{R}^N_+} |v|^{p^*}\,dS\right)^{p/p^*}},$$

and c *depend on the family of coefficients. Then, the conclusions of the previous item hold true.*

To end this introduction, let us mention that homogenization results for the Sobolev trace constant in domains with holes for critical and subcritical exponents have been recently considered in [13] in the spirit of [8].

The rest of the paper is organized as follows, in Section 2 we recall some preliminary results that are needed in the proof of the main theorems, in Section 3 we deal with the subcritical case (Theorem 1), in Section 4 with the critical case (Theorem 2) and, finally, in Section 5 we prove the extension for the nonlinear case (Theorem 3).

2. Preliminaries

In this subsection we present some results and techniques in homogenization of periodic media. We briefly recall the notion of two-scale convergence (see [2], [19]).

Proposition 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ and w_ε be a bounded sequence in $L^2(\Omega)$. There exist a subsequence, still denoted by ε, and a limit $w(x,y) \in L^2(\Omega; L^2_\#(\mathbb{T}))$ such that w_ε two-scale converges to w in the sense that*

$$\lim_{\varepsilon\to 0} \int_\Omega w_\varepsilon(x)\phi(x, x/\varepsilon)\, dx = \int_\Omega \int_{\mathbb{T}} w(x,y)\phi(x,y)\, dxdy$$

for every function $\phi(x,y) \in L^2(\Omega; C_\#(\mathbb{T}))$. The two-scale convergence is denoted by $w_\varepsilon \rightharpoonup w$ in 2s. Furthermore, if $\{w_\varepsilon\}$ is a bounded sequence that converges weakly to a limit w in $H^1(\Omega)$. Then, w_ε two-scale converges to w, and there exists a function $w_1(x,y) \in L^2(\Omega; H^1_\#(\mathbb{T}))$ such that, up to a subsequence, we have the following two-scale convergence

$$\nabla w_\varepsilon(x) \rightharpoonup \nabla_x w(x) + \nabla_y w_1(x,y) \qquad \text{in 2-scale.}$$

This two-scale convergence result is a powerful tool to deal with our problem, the study of the limit as $\varepsilon \to 0$ in (1.4).

Another important tool is the weak star convergence in $L^\infty(\Omega)$. In general, if $g_\varepsilon, g \in L^\infty(\Omega)$, we say that g_ε converges to g weak star in $L^\infty(\Omega)$, denoted by $g_\varepsilon \overset{*}{\rightharpoonup} g$ in $L^\infty(\Omega)$, if

$$\int_\Omega g_\varepsilon \phi\, dx \to \int_\Omega g\phi\, dx, \qquad \forall \phi \in L^1(\Omega).$$

We note immediately, see [8], that an $\varepsilon\mathbb{T}$-periodic function converges weak$-*$ in L^∞ to its mean value. Thus

$$a_0^\varepsilon \overset{*}{\rightharpoonup} a_0^* \qquad \text{in } L^\infty(\Omega).$$

Moreover, if Ω is a generic domain, i.e. $\partial\Omega$ does not contain flat pieces or that it contains finitely many flat pieces with conormal not proportional to any $m \in \mathbb{Z}^N$, we have that

$$b^\varepsilon \overset{*}{\rightharpoonup} b^* \qquad \text{in } L^\infty(\partial\Omega), \tag{2.1}$$

where b^* is given by (1.8), see Remark 1.1.

3. Subcritical case

In this section we assume that q is subcritical, that is $1 \le q < 2_*$, so the immersion $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact.

Proof of Theorem 1. First, let us prove that the best constants $S(\varepsilon)$ and the extremals u_ε are bounded in H^1 independently of ε. Indeed, by the definition of $S(\varepsilon)$ in (1.4) and our assumptions on the coefficients (1.1), (1.2), there exist two constants $0 < c < C$ such that

$$c\,\lambda_0 \,\le\, S(\varepsilon) \,\le\, C\,\lambda_0, \tag{3.1}$$

with λ_0 defined by

$$\lambda_0 = \inf_{v\in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 + v^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{2/q}}. \tag{3.2}$$

Now, we show that the extremals u_ε, the weak solutions of (1.6), are bounded in H^1-norm independently of ε. To prove this fact recall that we have normalized the extremals by (1.5). By our assumptions on the coefficients (1.1), (1.2), we have

$$S(\varepsilon) = \int_\Omega \left(a^\varepsilon_{ij} \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_i} + a^\varepsilon_0 |u_\varepsilon|^2 \right) dx \geq \int_\Omega \left(\alpha|\nabla u_\varepsilon|^2 + a_-|u_\varepsilon|^2\right) dx.$$

By (3.1), we obtain that u_ε is bounded in H^1 independently of ε. Hence there exists a subsequence (that we still call u_ε) and a function $u_0 \in H^1(\Omega)$ such that $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$ and $u_\varepsilon \to u_0$ strongly in $L^q(\partial\Omega)$ for $1 \leq q < 2*$. By the above mentioned convergence and (2.1) we have that

$$\int_{\partial\Omega} b^*|u_0|^q\,dS = 1.$$

Moreover, using Proposition 2.1, we obtain that $u_\varepsilon \rightharpoonup u_0$ in 2-scale and there exists u_1 such that

$$\nabla u_\varepsilon \rightharpoonup \nabla_x u_0(x) + \nabla_y u_1(x,y), \quad \text{in 2-scale.}$$

We use $\phi(x) + \varepsilon\phi_1(x, x/\varepsilon)$ with $\phi \in H^1(\Omega)$ and $\phi_1 \in H^1(\Omega; C_\#(\mathbb{T}))$ as a test function in the weak form of (1.6). As $S(\varepsilon)$ is bounded, we can assume that $S(\varepsilon) \to S_0$, for an appropriate subsequence. Then we pass to the limit the weak formulation and, by the two-scale convergence, we get

$$\begin{aligned}\int_\Omega\int_{\mathbb{T}} a_{ij}(y)\left(\frac{\partial u_0}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x,y)\right)\left(\frac{\partial \phi}{\partial x_i}(x) + \frac{\partial \phi_1}{\partial y_i}(x,y)\right) dy\,dx \\ + \int_\Omega\int_{\mathbb{T}} a_0(y)u_0(x)\phi(x)dx\,dy = S_0\int_{\partial\Omega}\int_{\mathbb{T}} b(y)|u_0|^{q-2}u_0\phi(x)\,dS\,dy.\end{aligned}$$

Integrating by parts we obtain that (u_0, u_1) is the weak solution of the system

$$\frac{\partial}{\partial x_i}\left(\int_{\mathbb{T}} a_{ij}(y)\left(\frac{\partial u_0}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x,y)\right)dy\right) = a^*_0\,u_0(x) \qquad \text{in } \Omega, \tag{3.3}$$

$$\nu_i\frac{\partial}{\partial x_i}\left(\int_{\mathbb{T}} a_{ij}(y)\left(\frac{\partial u^*}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x,y)\right)\,dy\right) = S_0\,b^*\,|u_0|^{q-2}u_0(x) \quad \text{on } \partial\Omega, \tag{3.4}$$

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y)\left(\frac{\partial u_0}{\partial x_i}(x) + \frac{\partial u_1}{\partial y_i}(x,y)\right)\right) = 0 \qquad \text{in } \Omega\times\mathbb{T}, \tag{3.5}$$

with a^*_0 and b^* defined in (1.8). Considering

$$u_1(x,y) = \sum_{i=1}^N \frac{\partial u_0}{\partial x_i}(x)\chi_i(y),$$

we note that u_1 satisfies (3.5) for any u_0 since χ_1 is solution of (1.10). Moreover, with this function u_1 in (3.3) and (3.4), we obtain that u_0 is a solution of

$$\begin{cases} \dfrac{\partial}{\partial x_i}\left(a^*_{ij}\dfrac{\partial u_0}{\partial x_j}\right) = a^*_0 u_0 & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu^*} = S_0 b^* |u_0|^{q-2} u_0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where the coefficients a^*_{ij} are given by (1.9) and the derivative normal $\partial/\partial\nu^*$ is defined in (1.11). Now, since S_0 satisfies (3.6), we get $S_0 \geq S^*$ with S^* defined in (1.7). To conclude the proof of Theorem 1 we need to show that $S_0 = S^*$. In fact, let u^* be an extremal of (1.7) and consider

$$v_\varepsilon = u^* + \varepsilon \chi^\varepsilon_k \frac{\partial u^*}{\partial x_k}$$

as a test function in (1.4), where $\chi^\varepsilon_k(x) = \chi_k(x/\varepsilon)$. From the maximum principle and Hopf's Lemma we get that u^* is strictly positive in $\overline{\Omega}$. Therefore the regularity results of [6] are applicable and we obtain that $u^* \in C^\infty(\overline{\Omega})$. Thus, since the functions $\chi_k \in W^{1,\infty}$ (this is a consequence of the hypotheses on the coefficients), we have immediately $v_\varepsilon \rightharpoonup u^*$ weakly in $H^1(\Omega)$ and $v_\varepsilon \to u^*$ strongly in $L^q(\partial\Omega)$ for $1 \leq q < 2^*$. Now, we obtain

$$\begin{aligned} \int_\Omega \left(a^\varepsilon_{ij}\frac{\partial v_\varepsilon}{\partial x_j}\frac{\partial v_\varepsilon}{\partial x_i} + a^\varepsilon_0 v^2_\varepsilon\right) dx &= \int_\Omega \left(a^\varepsilon_{ij} + a^\varepsilon_{ik}\frac{\partial \chi^\varepsilon_j}{\partial y_k}\right)\frac{\partial u^*}{\partial x_j}\frac{\partial u^*}{\partial x_i}\, dx \\ &\quad + \int_\Omega \left(a^\varepsilon_{ij}\frac{\partial \chi^\varepsilon_k}{\partial y_j}\frac{\partial \chi^\varepsilon_\ell}{\partial y_i} + a^\varepsilon_{ik}\frac{\partial \chi^\varepsilon_\ell}{\partial y_i}\right)\frac{\partial u^*}{\partial x_k}\frac{\partial u^*}{\partial x_\ell}\, dx \\ &\quad + \int_\Omega a^\varepsilon_0 (u^*)^2 dx + O(\varepsilon). \end{aligned}$$

Passing to the limit, using that χ_k is a solution of (1.10) and by the weak$-*$ convergence in L^∞, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \left(a^\varepsilon_{ij}\frac{\partial v_\varepsilon}{\partial x_j}\frac{\partial v_\varepsilon}{\partial x_i} + a^\varepsilon_0 v^2_\varepsilon\right) dx = \int_\Omega \left(a^*_{ij}\frac{\partial u^*}{\partial x_j}\frac{\partial u^*}{\partial x_i} + a^*_0 (u^*)^2\right) dx,$$

where a^*_0 and a^*_{ij} are defined by (1.8) and (1.9), respectively. Moreover, again by the weak$-*$ convergence in L^∞, we have

$$\int_{\partial\Omega} b^\varepsilon |v_\varepsilon|^q \to \int_{\partial\Omega} b^* |u^*|^q.$$

Therefore, passing to the limit in (1.4) with test function v_ε, we prove $S_0 \leq S^*$ and we conclude the proof of Theorem 1. □

Remark 3.1. *Results on correctors of the extremals are easily obtained with the two scale convergence method. Considering the solutions of the cell problem* (1.10)*, the corrector term is defined by*

$$u^\varepsilon_1(x) = \chi_k(x/\varepsilon)\frac{\partial u^*}{\partial x_k}(x),$$

where u^* is an extremal of the homogenized problem (1.11)*. Hence, by Proposition 2.1 and following the same lines as* [2]*, $(u^\varepsilon - u^* - \varepsilon u^\varepsilon_1)$ converges strongly to zero in $H^1(\Omega)$.*

4. Critical case

In this section we deal with the critical exponent $q = 2_* = 2(N-1)/(N-2)$.

Proof of Theorem 2. Recall that, as observed in Remark 1.3, hypothesis (1.12) implies the existence of an extremal u_ε for (1.4).

As before, by the definition of $S(\varepsilon)$ in (1.4) and our assumptions on the coefficients (1.1), (1.2), we have (3.1). Hence, the extremals u_ε are bounded in $H^1(\Omega)$ and we have, for a subsequence,

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 \qquad \text{weakly in } H^1(\Omega), \\ u_\varepsilon &\to u_0 \qquad \text{strongly in } L^q(\partial\Omega), \quad \text{with } 1 \le q < 2_*. \end{aligned}$$

Arguing exactly as in the previous section we obtain that $u_\varepsilon \rightharpoonup u_0$ in $2s$ and moreover, that u_0 is a weak solution to

$$\begin{cases} \dfrac{\partial}{\partial x_i}\left(a^*_{ij}\dfrac{\partial u_0}{\partial x_j}\right) = a^*_0 u_0 & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu^*} = Sb^*|u_0|^{q-2}u_0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where S is the limit of a subsequence of $S(\varepsilon)$, the coefficients a^*_{ij} are given by (1.9) and the derivative normal $\partial/\partial\nu^*$ is defined in (1.11).

Let us prove that $u_0 \neq 0$. To this end we use the following Theorem due to [16].

Theorem 4. *There exists a constant $B > 0$ such that,*

$$\left(\int_{\partial\Omega} v^{2_*}\, dS\right)^{2/2_*} \le A\int_\Omega |\nabla v|^2\, dx + B\int_\Omega v^2\, dx$$

for every $v \in H^1(\Omega)$, where

$$A = \frac{2}{(N-2)|B(0,1)|^{1/(N-1)}}.$$

Remark 4.1. *The constant A in Theorem 4 is sharp.*

Now, as $u_\varepsilon \ge 0$, it follows that $u_0 \ge 0$ and, by classical regularity theory, u_0 is smooth up to the boundary. By the strong maximum principle and Hopf's lemma, it follows that either $u_0 > 0$ or $u_0 \equiv 0$. In order to prove of the result, we have to exclude this last possibility. To this end, we use the argument given in [15] to show that $\|u_0\|_{L^2(\Omega)} \neq 0$. In fact, by Theorem 4, we have that there exists a constant B such that

$$\left(\int_{\partial\Omega} v^{2_*}\, d\sigma\right)^{2/2_*} \le A\int_\Omega |\nabla v|^2\, dx + B\int_\Omega |v|^2\, dx$$

for every $v \in H^1(\Omega)$. Recall that u_ε are normalized such that satisfies (1.5), so, by (1.3),

$$1 = \left(\int_{\partial\Omega} b^\varepsilon u_\varepsilon^{2_*}\, d\sigma\right)^{2/2_*} \le (b_+)^{2/2_*}\left(A\int_\Omega |\nabla u_\varepsilon|^2\, dx + B\int_\Omega u_\varepsilon^2\, dx\right).$$

Hence, for some suitable $\tilde{B}$ we get,

$$\frac{1}{(b_+)^{2/2_*}} \le \frac{A}{\alpha}\left(\int_\Omega a^\varepsilon_{ij}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_i}\, dx + \int_\Omega a^\varepsilon_0|u_\varepsilon|^2\, dx\right) + \tilde{B}\left(\int_\Omega u_\varepsilon^2\, dx\right).$$

Therefore,

$$\frac{1}{(b_+)^{2/2_*}} \le \frac{A}{\alpha} S(\varepsilon) + \tilde{B} \int_\Omega |u_\varepsilon|^2 \, dx \tag{4.2}$$

Passing to the limit $\varepsilon \to 0$ in (4.2) we arrive to

$$\frac{1}{(b_+)^{2/2_*}} \le \frac{A}{\alpha} S + \tilde{B} \int_\Omega |u_0|^2 \, dx$$

therefore, as we have assumed (1.13), that implies

$$\frac{\alpha}{A(b_+)^{2/2_*}} > S,$$

we conclude $u_0 \ne 0$.

Now, multiplying (4.1) by u_0 and integrating by parts, we obtain

$$\int_\Omega a_{ij}^* \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_i} + a_0^* u_0^2 \, dx = S \int_{\partial\Omega} u_0^{2_*} \, dS.$$

As $u_0 \ne 0$ it follows that $S \ne 0$ and $\|u_0\|_{L^{2_*}(\partial\Omega)} \ne 0$. Therefore, we conclude that

$$S_0 \le \frac{\displaystyle\int_\Omega a_{ij}^* \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_i} + a_0^* u_0^2 \, dx}{\left(\displaystyle\int_{\partial\Omega} u_0^{2_*} \, dS \right)^{2/2_*}} = S \left(\int_{\partial\Omega} u_0^{2_*} \, dS \right)^{1/(N-1)} \le S.$$

Now, arguing exactly as in the end of Section 3, we conclude the desired result. □

5. The nonlinear case

Finally, in this section we consider the extension of our previous results to a more general class of nonlinear operators, including the $p-$Laplacian with oscillating coefficients. The main ideas for these extensions are similar to the ones used before combined with those of [4].

We consider nonlinear monotone operators $\mathcal{A} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ of the form

$$\mathcal{A}u = -\operatorname{div}(a(x, \nabla u)) + b(x, u),$$

whose coefficients $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ belong to the class of functions satisfying the following hypotheses:

- (A1) $a(\cdot, \cdot)$ is of Carathéodory type.
- (A2) Monotonicity: $0 \le (a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2) \quad \forall \xi_1, \xi_2$, a.e. x.
- (A3) Uniform ellipticity: $\alpha |\xi|^p \le a(x, \xi) \cdot \xi \quad \forall \xi$, a.e. x.
- (A4) Growth: $|a(x, \xi)| \le \beta |\xi|^{p-1} \quad \forall \xi$, a.e. x.

and the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses

- (B1) $b(\cdot, \cdot)$ is of Carathéodory type.
- (B2) Uniform $\alpha |u|^p \le b(x, u) u \quad \forall u$, a.e. x.
- (B3) Growth: $|b(x, u)| \le \beta |u|^{p-1} \quad \forall u$, a.e. x.

For a and b satisfying the above hypotheses, we consider the eigenvalue problem

$$\begin{aligned} \operatorname{div}(a(x, \nabla u)) &= b(x, u) && \text{in } \Omega, \\ a(x, \nabla u) \cdot \nu &= \lambda |u|^{q-2} u && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

If there exist λ and u solutions of (5.1), taking u as a test function in the eigenvalue problem, we note that

$$\lambda = \frac{\displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla u + b(x,u)u dx}{\displaystyle\int_{\partial\Omega} |u|^q dS}. \tag{5.2}$$

Moreover, the infimum in (1.14) is attained and is called the first eigenvalue λ_1 for the problem (5.1). This fact is indeed by the lower semi-continuity property of the functinal associated to $\mathcal{A}$ for the minimizing sequence.

Let $\varepsilon > 0$ be a small parameter which represents the scale of heterogeneity. We consider a family of functions a^ε, b^ε satisfying the previous hypotheses, for example, $a^\varepsilon(x,\xi) = a(x/\varepsilon,\xi)$ and $b^\varepsilon(x,u) = b(x/\varepsilon,u)$ which are, in addition, periodic in x. Thus, we deal with the minimization problem

$$\lambda_1^\varepsilon = \inf_{v\in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega a^\varepsilon(x,\nabla v)\cdot\nabla v + b^\varepsilon(x,v)v\,dx}{\displaystyle\int_{\partial\Omega} |v|^q\,dS}. \tag{5.3}$$

First, assume that q is subcritical. Then, since the embedding $W^{1,p}(\Omega)\hookrightarrow L^q(\partial\Omega)$ is compact there exist extremals for (5.3). We normalize the extremals with the condition

$$\int_{\partial\Omega} |u_\varepsilon|^q dS = 1. \tag{5.4}$$

The normalized extremals are weak solutions of the problem

$$\begin{aligned} \operatorname{div}(a^\varepsilon(x,\nabla u_\varepsilon)) &= b^\varepsilon(x,u_\varepsilon)u_\varepsilon && \text{in } \Omega,\\ a(x,\nabla u_\varepsilon)\cdot\nu &= \lambda_1^\varepsilon |u_\varepsilon|^{q-2}u_\varepsilon && \text{on } \partial\Omega. \end{aligned} \tag{5.5}$$

Since in the statement of Theorem 3 we have assumed the G-convergence of the operators the conclusions concerning the convergence of the first eigenvalue and its associated extremals follows.

Note that this assumption is not restrictive, since if a^ε and b^ε are measurable coefficients which satisfy (A1)-(A3) and (B1)–(B3), then the operators $\mathcal{A}^\varepsilon$ G-converge (up to a subsequence) to a maximal monotone operator $\mathcal{A}_{\text{hom}}$ whose coefficients, a_{hom} and b_{hom}, are measurable and satisfies (A1)-(A3) and (B1)–(B3). We refer to Theorem 4.1 of [7] for this well-known compactness result for the G-convergence on the class of multivalued functions of the type a.

For the critical case $p* = p(N-1)/(N-2)$ we can argue exactly as before in section 4, noting that condition (1.15) on the domain and the coefficients involved implies that there are minimizers of (5.3) since some compactness is recovered.

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References

[1] Adimurthi, S.L. Yadava. *Positive solution for Neumann problem with critical non linearity on boundary*. Comm. Partial Differential Equations, 16 (11) (1991), 1733–1760.

[2] G. Allaire. *Homogenization and two-scale convergence*. SIAM J. Math. Anal., 23, (1992), 1482–1518.

[3] T. Aubin. *Équations différentielles non linéaires et le problème de Yamabe concernant la courbure scalaire*. J. Math. Pures et Appl., 55 (1976), 269–296.

[4] L. Baffico, C. Conca, M. Rajesh. *Homogenization of a class of nonlinear eigenvalue problems*. Proc. of the Royal Society of Edinburgh, 136A, 722, 2006

[5] A. Bensoussan, J. L. Lions and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam, 1978.

[6] P. Cherrier, *Problèmes de Neumann non linéaires sur les variétés Riemanniennes*. J. Funct. Anal. Vol. 57 (1984), 154-206.

[7] V. Chiado Piat, G. Dal Maso and A. Defranceschi. *G-convergence of monotone operators*. Annls Inst. H. Poincare 7 (1990), 123-160.

[8] D. Cioranescu and F. Murat. *A Strange Term Coming from Nowhere*, in Topics in the Mathematical Modelling of Composite Materials, A. Cherkaev et. al. eds., Progress in Nonlinear Differential Equations and Their Applications, 31, Birkhäuser, Boston, (1997), 45–93.

[9] G. Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.

[10] O. Druet and E. Hebey. *The AB program in geometric analysis: sharp Sobolev inequalities and related problems*. Mem. Amer. Math. Soc. 160 (761) (2002).

[11] J. F. Escobar. *Sharp constant in a Sobolev trace inequality*. Indiana Math. J., 37 (3) (1988), 687–698.

[12] J. Fernández Bonder, E. Lami Dozo and J.D. Rossi. *Symmetry properties for the extremals of the Sobolev trace embedding*. Ann. Inst. H. Poincaré. Anal. Non Linéaire, 21 (2004), no. 6, 795–805.

[13] J. Fernández Bonder, R. Orive and J.D. Rossi. *The best Sobolev trace constant in domains with holes for critical or subcritical exponents*. Submitted.

[14] J. Fernández Bonder and J.D. Rossi. *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains*. Comm. Pure Appl. Anal., 1 (3) (2002), 359–378.

[15] J. Fernández Bonder and J.D. Rossi. *On the existence of extremals for the Sobolev trace embedding theorem with critical exponent*. Bull. London Math. Soc., 37 (2005), no. 1, 119–125.

[16] Y. Li and M. Zhu. *Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries*. Comm. Pure Appl. Math., 50 (1997), 449–487.

[17] J.-L. Lions. *Some methods in the mathematical analysis of systems and their control*. Kexue Chubanshe (Science Press), Beijing; Gordon & Breach Science Publishers, New York, 1981.

[18] F. Murat and L. Tartar, *H-convergence*, in Topics in the Mathematical Modelling of Composite Materials, A. Cherkaev et. al. eds., Progress in Nonlinear Differential Equations and Their Applications, 31, Birkhäuser, Boston, (1997), 21–43.

[19] G. Nguetseng. *A general convergence result for a functional related to the theory of homogenization*. SIAM J. Math. Anal., 20, (1989), 608–623.

[20] M. W. Steklov, *Sur les problèmes fondamentaux en physique mathématique*, Ann. Sci. Ecole Norm. Sup., Vol. 19 (1902), 455–490.

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