ON THE BEST SOBOLEV TRACE CONSTANT AND EXTREMALS IN DOMAINS WITH HOLES

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Abstract. We study the dependence on the subset $A \subset \Omega$ of the Sobolev trace constant for functions defined in a bounded domain Ω that vanish in the subset A. First we find that there exists an optimal subset that makes the trace constant smaller among all the subsets with prescribed and positive Lebesgue measure. In the case that Ω is a ball we prove that there exists an optimal hole that is spherically symmetric. In the case $p = 2$ we prove that every optimal hole is spherically symmetric. Then, we study the behavior of the best constant when the hole is allowed to have zero Lebesgue measure. We show that this constant depends continuously on the subset and we discuss when it is equal to the Sobolev trace constant without the vanishing restriction.

1. Introduction.

In this paper we are interested in the best Sobolev trace constant from $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$ for functions that vanish on a subset A of Ω.

The properties of the best Sobolev trace constant have been widely studied from different points of view. Of special interest has been its dependence on the set Ω. Parallel to the study of the constant is the study of extremals for the trace inequality and their behavior. See, for instance, [5, 7, 9].

Our interest is on the behavior of this constant and extremals for the trace inequality when we restrict the test functions to those that vanish in the subset A. It is our main concern to understand the behavior of this constant and extremals with respect to A.

In order to start our discussion let us define the constant S_A that is the object of our investigation. For $A \subset \Omega$ we let

$$S_A = \inf \left\{ \frac{\int_\Omega (|\nabla u|^p + |u|^p)\, dx}{\left(\int_{\partial\Omega} |u|^q \, dS \right)^{p/q}} ,\, u \in W^{1,p}(\Omega) \text{ s.t. } u \not\equiv 0 \text{ on } \partial\Omega \text{ and } u = 0 \text{ a.e. } A \right\}. \tag{1.1}$$

In this work we restrict ourselves to the subcritical case. This is, we consider exponents $1 \le q < p_* = p(N-1)/(N-p)$ for $p < N$, $1 \le q < \infty$ for $p \ge N$, so that the immersion

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$W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact. Therefore, it is easy to see that there exist extremals for S_A. When A is closed an extremal for S_A is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \setminus A, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega \setminus A, \\ u = 0 & \text{in } A, \end{cases} \tag{1.2}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$laplacian, $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative and λ depends on the normalization of u. For instance, if $\|u\|_{L^q(\partial\Omega)} = 1$, then $\lambda = S_A$. For results related to (1.2) see [8, 15, 17].

One of the problems we are interested in is the optimization of S_A among subsets A of Ω of a given positive measure $\alpha < |\Omega|$. We prove that there exist extremals for this optimization problem. A natural question then is what can be said about the extremals u and the "optimal holes" $\{u = 0\}$ (regularity, location, symmetry, etc...).

In this paper we prove that, when Ω is a ball, there exist an extremal and an optimal hole that are spherically symmetric. In the case $p = 2$ we prove that all the optimal holes and extremals are spherically symmetric (see Section 2 for the definition of spherical symmetry). For general domains Ω, $1 < p < \infty$, we prove that when $q \geq p$ the complement of the optimal hole is (measure-theoretic) connected. In a companion paper, [10], we prove regularity of the optimal holes and extremals in the case $p = 2$.

Problems of optimal design related to eigenvalue problems like (1.2) appear in applications. For instance, in problems of minimization of the energy stored in the design under a prescribed loading. Solutions of these problems are unstable to perturbations of the loading. The stable optimal design problem is formulated as minimization of the stored energy of the project under the most unfavorable loading. This most dangerous loading is one that maximizes the stored energy over the class of admissible functions. The problem is reduced to minimization of Steklov eigenvalues. This is, (1.2) when $p = q = 2$. See [2].

We want to stress that the results in this paper are new, even in the linear case, $p = q = 2$.

Optimization problems for eigenvalues of elliptic operators have been widely studied in the past, and are still an area of intensive research. For a comprehensive description of the current developments in the field and very interesting open problems, we refer to [11]. In [3] the author studies an optimization problem for the second Neumann eigenvalue of the laplacian with $A \subset \partial\Omega$. He proves the existence of an optimal window A_0 and shows that – when Ω is a ball – A_0 is spherically symmetric in the sense of [16]. Our approach to the optimization problem follows closely the one in [3]. Optimal design problems have been widely studied not only for eigenvalue problems. See for instance [1, 12, 14].

When trying to give sense to a best Sobolev trace constant for functions that vanish in a set of zero Lebesgue measure a different approach has to be made. We consider the space

$$W_A^{1,p}(\Omega) = \overline{C_0^\infty(\overline{\Omega} \setminus A)}, \tag{1.3}$$

where the closure is taken in $W^{1,p}-$norm. That is, $W^{1,p}_A(\Omega)$ stands for the set of functions of the Sobolev space $W^{1,p}(\Omega)$ that can be approximated by smooth functions that vanish in a neighborhood of A.

In this context the best Sobolev trace constant is defined as

$$\mathbf{S}_A = \inf_{u\in W^{1,p}_A(\Omega)\setminus W^{1,p}_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u|^q\,dS\right)^{p/q}}. \tag{1.4}$$

In this case we prove that this problem only makes sense if A is a set with positive $p-$capacity (see (1.6)). More precisely, we prove that $\mathbf{S}_A = \mathbf{S}_\emptyset$ if and only if the $p-$capacity of A is zero. Note that $\mathbf{S}_\emptyset$ is the usual Sobolev trace constant from $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$.

Observe that the constants S_A and $\mathbf{S}_A$ need not be the same. For a discussion on this relation, see the end of Section 3.

Finally, we address the problem of the continuity of $\mathbf{S}_A$ with respect to A. Here the natural topology for the sets A is the Hausdorff distance and in fact, we prove that $\mathbf{S}_A$ is continuous in this topology. Also, we prove the continuity of the extremals of $\mathbf{S}_A$ in $W^{1,p}$ norm with respect to the Hausdorff distance of the sets A.

In order to finish this introduction, let us comment briefly on related work. First we comment on works related to the dependence of the Sobolev trace constant with respect to variations of the domain. In [5] the authors analyze the behavior of extremals and best Sobolev trace constants in expanding domains for $p = 2$ and $q > 2$. They prove that the extremals develop a peak near the point where the mean curvature of the boundary attains a maximum. In [9] the authors analyze the dependance of the best Sobolev trace constant and extremals in expanding and contracting domains for $p > 1$ and $1 \le q < p_*$. Also, in [7] the behavior of the Sobolev trace constant and extremals in thin domains is analyzed.

Finally, see [6] and [13] where symmetry and symmetry breaking properties of extremals of the Sobolev trace constant in balls are analyzed.

1.1. **Statements of the results.** Now we state the main results of the paper.

Our first result is the sequential lower semicontinuity of S_A.

Theorem 1.1. *Let $A_n \subset \Omega$ be sets of positive measure such that*

$$\chi_{A_n} \overset{*}{\rightharpoonup} \chi_{A_0} \quad \textit{in } L^\infty(\Omega),$$

where χ_A is the characteristic function of the set A. Then

$$S_{A_0} \le \liminf_{n\to\infty} S_{A_n},$$

where S_A is given by (1.1).

We remark that the continuity is not true in general. See Remark 2.1.

This semicontinuity result suggest that a minimizer for S_A among sets A of fixed positive Lebesgue measure exists. However, there is a major difficulty here because of the fact that sets of prescribed positive Lebesgue measure are not compact with respect to the topology of Theorem 1.1. We overcome this difficulty in the next theorem.

Our result concerning the existence of an optimal design for the constant S_A is as follows.

Theorem 1.2. *Given $0 < \alpha < |\Omega|$, let us define*

$$S(\alpha) := \inf_{A\subset\Omega,\ |A|=\alpha} S_A. \tag{1.5}$$

Then, there exists a set $A_0 \subset \Omega$ such that $|A_0| = \alpha$ and $S_{A_0} = S(\alpha)$.

On the other hand, there is no upper bound for S_A.

Theorem 1.3. *Let $0 < \alpha < |\Omega|$. Then,*

$$\sup_{A\subset\Omega,\ |A|=\alpha} S_A = \infty.$$

Next we study symmetry properties of optimal sets A_0 in the special case where Ω is a ball. To this end, we need the definition of *spherical symmetrization* (see [16]). Given a measurable set $A \subset \mathbb{R}^N$, the spherical symmetrization A^* of A is defined as follows: for each r, take $A \cap \partial B(0, r)$ and replace it by the spherical cap of the same area and center re_N. The union of these caps is A^*.

We have the following result,

Theorem 1.4. *Let $\Omega = B(0, 1)$ and $0 < \alpha < |B(0, 1)|$. Then, there exists an optimal hole of measure α which is spherically symmetric, that is $A^* = A$. Moreover, when $p = 2$, every optimal hole is spherically symmetric and $\{u > 0\}$ is a connected set for every minimizer u.*

For general domains, we can prove that the complement of the optimal hole is (measure-theoretic) connected, if $q \geq p$.

Theorem 1.5. *Let A_0 be an optimal hole for $S(\alpha)$, and u be the corresponding extremal. Then, if $q \geq p$, $\Omega \setminus A_0 = \{u > 0\}$ is measure-theoretic connected. That is, if $\{u > 0\} \subset U_1 \cup U_2$, where U_i, $i = 1, 2$, are nonempty, disjoint open sets, then $|\{u > 0\} \cap U_i| = 0$ for some $i = 1, 2$.*

Now we state the results that allow us to consider the case $|A| = 0$. For simplicity we will consider closed sets A.

First, we study when $\mathbf{S}_A$ is equal to the usual Sobolev trace constant, that is, when $\mathbf{S}_A = \mathbf{S}_\emptyset$. For this purpose we recall de definition of $p-$capacity (see [4]). For $A \subset \Omega$ closed, define

$$\mathrm{Cap}_p(A) = \inf\{\int_{\mathbb{R}^n} |\nabla\phi|^p\,dx \,/\, \phi \in W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N) \text{ and } A \subset \{\phi \geq 1\}^\circ\}. \tag{1.6}$$

We have

Proposition 1.1. $\mathbf{S}_A = \mathbf{S}_\emptyset$ *if and only if* $\mathrm{Cap}_p(A) = 0$.

Next, we look at the dependence of $\mathbf{S}_A$ on perturbations of A. We find that $\mathbf{S}_A$ is continuous with respect to A in the topology given by the Haussdorff distance.

Theorem 1.6. *Let $A, A_n \subset \Omega$ be closed sets such that $d(A_n, A) \to 0$ as $n \to \infty$ where $d(A_n, A)$ is the Hausdorff distance between A_n and A. Then*

$$|\mathbf{S}_{A_n} - \mathbf{S}_A| \to 0 \qquad \textit{when} \quad n \to \infty$$

and if we denote by u_n an extremal for $\mathbf{S}_{A_n}$ normalized such that $\|u_n\|_{L^q(\partial\Omega)} = 1$, there exists a subsequence u_{n_k} such that

$$\lim_{k\to\infty} u_{n_k} = u, \qquad \textit{strongly in } W^{1,p}(\Omega), \tag{1.7}$$

and u is an extremal for $\mathbf{S}_A$.

The rest of the paper is organized as follows, in Section 2 we prove the existence of an optimal set A and its symmetry properties when Ω is a ball. In Section 3 we prove our results involving the $p-$capacity of a subset and the continuous dependence of the Sobolev constant on the subset A.

Throughout the paper, by C we mean a constant that may vary from line to line but remains independent of the relevant quantities.

2. The optimization problem

In this section, we prove the sequential lower semicontinuity of the best Sobolev trace constant S_A with respect to A (Theorem 1.1). Then following ideas from [3], we prove that if we consider holes with positive and fixed measure a minimizing hole does exist (Theorem 1.2). Moreover, we prove that a maximizing hole does not exist (Theorem 1.3). Finally, we study the symmetry properties of minimizing holes when Ω is a ball (Theorem 1.4) and the connectivity of $\{u > 0\}$ in the case $q \geq p$ (Theorem 1.5).

2.1. **Semicontinuity result.**

Proof of Theorem 1.1. Let $A_n, A \subset \Omega$ such that $\chi_{A_n} \overset{*}{\rightharpoonup} \chi_A$ in $L^\infty(\Omega)$.

Let $u_n \in W^{1,p}(\Omega)$ be an extremal for S_{A_n} normalized such that $\|u_n\|_{L^q(\partial\Omega)} = 1$ and $u_n \geq 0$. Let $a = \liminf S_{A_n}$. Without loss of generality, we may assume that $S_{A_n} \to a$.

Then, there exists a constant C such that

$$\|u_n\|_{W^{1,p}(\Omega)} \leq C.$$

Thus, there exists a function $u \in W^{1,p}(\Omega)$ such that, for a subsequence that we still call u_n,

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } W^{1,p}(\Omega), \\ u_n &\to u && \text{strongly in } L^p(\Omega), \\ u_n &\to u && \text{strongly in } L^q(\partial\Omega). \end{aligned} \tag{2.1}$$

In particular, $\|u\|_{L^q(\partial\Omega)} = 1$, $u \ge 0$ and

$$\|u\|_{W^{1,p}(\Omega)} \le \liminf \|u_n\|_{W^{1,p}(\Omega)}.$$

We claim that u is an admissible function in the characterization of S_A. To this end, observe that by (2.1) and by our hypotheses on A_n and A,

$$\int_A u\,dx = \lim \int_{A_n} u_n\,dx = 0.$$

As $u \ge 0$, the claim follows.

Now,

$$S_A \le \|u\|^p_{W^{1,p}(\Omega)} \le \liminf \|u_n\|^p_{W^{1,p}(\Omega)} = \liminf S_{A_n}.$$

Also, this last inequality implies that

$$\|u_n\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)},$$

so $u_n \to u$ in $W^{1,p}(\Omega)$. The proof is now complete. □

Remark 2.1. The continuity of S_A with respect to A in the sense of Theorem 1.1 does not hold in general.

For instance, take $A_n = \overline{B_r(x^0)} \cup \overline{B_{1/n}(x^1)}$ and $A = \overline{B_r(x^0)}$. It is easy to see that $\chi_{A_n} \overset{*}{\rightharpoonup} \chi_A$. Let u_n be extremals for S_{A_n}. As in the proof of Theorem 1.1, we may assume that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. If $p > N$, as $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, then $u_n \to u$ uniformly. Since an extremal u_n for S_{A_n} vanishes in x^1 there holds that $u(x^1) = 0$. On the other hand, if $S_A = \lim S_{A_n}$ then u is an extremal for S_A. Thus u is a weak solution of (1.2) which implies, by the maximum principle (see [19]), that $u > 0$ in $\Omega \setminus A$. A contradiction.

For general p, we can take $A_n = \overline{B_r(x^0)} \cup \big([0, 1/n] \times \Sigma\big)$ where Σ is a closed portion of the hyperplane $\{x_1 = 0\}$ and $A = \overline{B_r(x^0)}$. Arguing in a similar way as before, using that $W^{1,p}(\Omega) \hookrightarrow L^p(\Sigma)$, we get that $S_A < \lim S_{A_n}$.

2.2. **Existence of an optimal hole.**

Proof of Theorem 1.2. In order to prove Theorem 1.2, we define the functional

$$J(u) = \int_\Omega |\nabla u|^p + |u|^p\,dx.$$

Our problem is to find extremals for

$$S(\alpha) = \inf\{S_A \,/\, A \subset \Omega\,,\, |A| = \alpha\}.$$

It is easy to see that

$$S(\alpha) = \inf\{S_A \,/\, A \subset \Omega\,,\; |A| \geq \alpha\}.$$

In fact, it is clear that

$$\inf\{S_A \,/\, A \subset \Omega\,,\; |A| = \alpha\} \geq \inf\{S_A \,/\, A \subset \Omega\,,\; |A| \geq \alpha\}.$$

On the other hand, if v is a test function for a set of measure greater than or equal to α it is also a test function for a set of measure α. Thus, the two infima coincide. Now,

$$\begin{aligned} S(\alpha) &= \inf\{S_A \,/\, A \subset \Omega\,,\; |A| \geq \alpha\} \\ &= \inf\big\{J(v) \,/\, v \in W^{1,p}(\Omega),\; v \geq 0,\; \|v\|_{L^q(\partial\Omega)} = 1,\; |\{v = 0\}| \geq \alpha\big\}. \end{aligned}$$

Note that we can always restrict ourselves to nonnegative test functions since a minimizer of S_A does not change sign.

So, let $\{u_n\}$ be a minimizing sequence of $J(v)$ in

$$\big\{v \in W^{1,p}(\Omega),\; v \geq 0,\; \|v\|_{L^q(\partial\Omega)} = 1,\; |\{v = 0\}| \geq \alpha\big\}.$$

Observe that as u_n is a minimizing sequence, we get

$$\|u_n\|_{W^{1,p}(\Omega)} \leq C.$$

Hence, by taking a subsequence if necessary, we may assume that there exists $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u, \text{ weakly in } W^{1,p}(\Omega), \\ &u_n \to u, \text{ strongly in } L^q(\partial\Omega), \\ &u_n \to u, \text{ strongly in } L^p(\Omega). \end{aligned} \tag{2.2}$$

So that, $u \in W^{1,p}(\Omega)$, $u \geq 0$ and $\|u\|_{L^q(\partial\Omega)} = 1$.

Now, let $A_n = \{u_n = 0\}$. Then, again by taking a subsequence if necessary we have that there exists a function $0 \leq \phi \leq 1$ such that

$$\chi_{A_n} \rightharpoonup \phi, \text{ weakly in } L^{p'}(\Omega). \tag{2.3}$$

So that, in particular, for $A = \{\phi > 0\}$,

$$|A| \geq \int_\Omega \phi = \lim \int_\Omega \chi_{A_n} = |A_n| \geq \alpha.$$

Since $u \geq 0$, $\phi \geq 0$ and

$$\int_\Omega u\,\phi = \lim \int_\Omega u_n\,\chi_{A_n} = 0$$

there holds that $u = 0$ almost everywhere in A. Thus, u is an admissible function and

$$J(u) = \|u\|^p_{W^{1,p}(\Omega)} \leq \lim \|u_n\|^p_{W^{1,p}(\Omega)} = S(\alpha).$$

Thus, u is an extremal for $S(\alpha)$.

It only remains to see that $|\{u = 0\}| = \alpha$. In fact, suppose by contradiction that u vanishes in a set A with $|A| > \alpha$. By taking a subset we may assume that A is closed. Let us take a small ball B so that $|A \setminus B| > \alpha$ with B centered at a point in $\partial A \cap \partial\Omega_1$ where

Ω_1 is the connected component of $\Omega \setminus A$ such that $\partial\Omega \subset \partial\Omega_1$. We can pick the ball B in such a way that $|A \cap B| > 0$. In particular, $|\{u = 0\} \cap B| > 0$. (See the figure below).

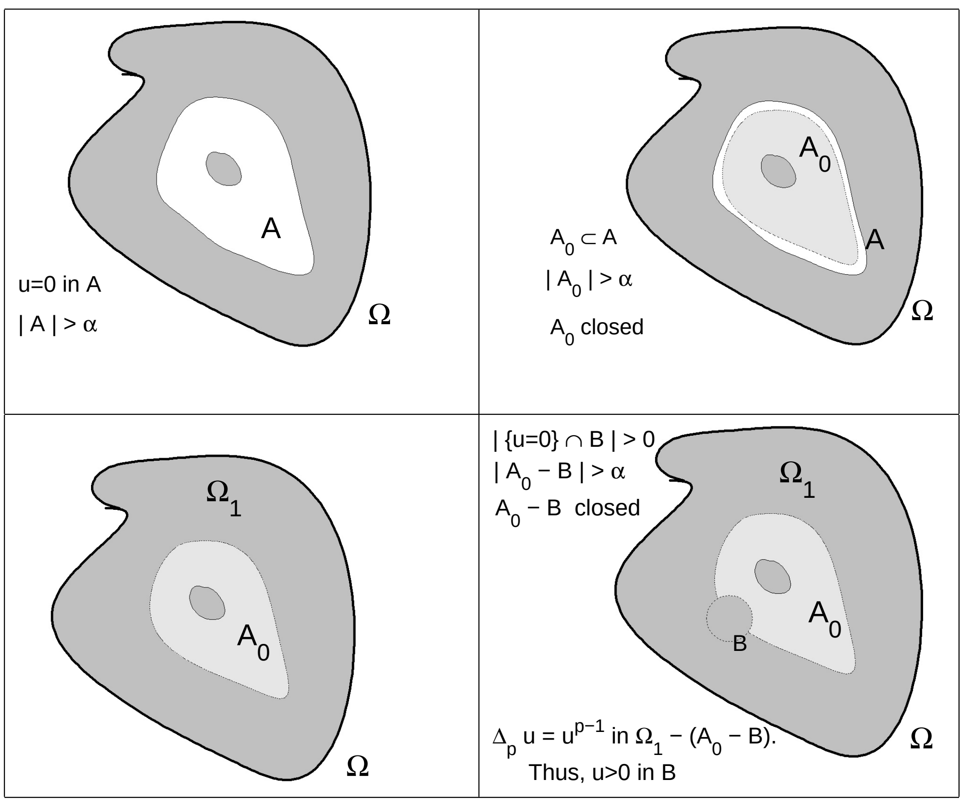
Since u is an extremal for $S(\alpha)$ and $|A \setminus B| > \alpha$, it is an extremal for $S_{A\setminus B}$. Thus, there holds that

$$\Delta_p u = u^{p-1} \quad \text{in} \quad \Omega \setminus (A \setminus B) = (\Omega \setminus A) \cup B.$$

Now, as $u \geq 0$ there holds that either $u \equiv 0$ or $u > 0$ in each connected component of $(\Omega \setminus A) \cup B$. Since $u \neq 0$ on $\partial\Omega$ there holds, in particular, that $u > 0$ in B. This is a contradiction to the choice of the ball B. Therefore,

$$|\{u = 0\}| = \alpha.$$

The theorem is proved. $\square$



Construction in the proof of Theorem 1.2

Remark 2.2. As a consequence of the proof of Theorem 1.2 we deduce that $S(\alpha)$ is a strictly increasing function of α. In fact, it is immediate to see that $S(\alpha)$ is nondecreasing since test functions for $\alpha_1 > \alpha$ are also test functions for α. On the other hand, if $S(\alpha_1) = S(\alpha)$ and u is an extremal for $S(\alpha_1)$ then $|\{u = 0\}| = \alpha_1$. But u is an admissible function for $S(\alpha)$ so that it is an extremal for $S(\alpha)$ with $|\{u = 0\}| > \alpha$. This is a contradiction with our results. Thus, S is strictly increasing.

Now we prove that a maximal hole does not exist.

Proof of Theorem 1.3. Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon)$ be such that

$$A_\varepsilon = \{x \in \Omega \,/\, \varepsilon \le \operatorname{dist}(x, \partial\Omega) \le \delta\}$$

has measure α. We will see that $S_{A_\varepsilon} \to \infty$ as $\varepsilon \to 0$. In fact, let $u_\varepsilon \in W^{1,p}(\Omega)$ be an extremal for S_{A_ε} normalized so that $\|u_\varepsilon\|_{L^q(\partial\Omega)} = 1$.

For each $\kappa > 0$, let

$$\Omega_\kappa = \{x \in \Omega \,/\, \operatorname{dist}(x, \partial\Omega) > \kappa\}.$$

Observe that u_ε is a solution to

$$\begin{cases} \Delta_p v = |v|^{p-2} v & \text{in } \Omega_\delta, \\ v = 0 & \text{on } \partial\Omega_\delta. \end{cases}$$

Thus, $u_\varepsilon = 0$ in Ω_δ. On the other hand, by construction $u_\varepsilon = 0$ in $\Omega_\varepsilon \setminus \Omega_\delta$. Thus, $u_\varepsilon = 0$ in Ω_ε and so $u_\varepsilon \to 0$ a.e. Ω.

If S_{A_ε} were bounded then, up to a subsequence, there would exist a function $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \\ u_\varepsilon &\to u \quad \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega, \\ u_\varepsilon &\to u \quad \text{strongly in } L^q(\partial\Omega). \end{aligned}$$

Since $u_\varepsilon \to 0$ a.e. Ω and $\|u_\varepsilon\|_{L^q(\partial\Omega)} = 1$ we arrive at a contradiction. $\square$

2.3. **Properties of optimal holes.** In this subsection, we consider the case $\Omega = B(0,1)$ and investigate if the optimal hole constructed in the previous section inherits some symmetry from the domain.

To this end, we recall the definition of *spherical symmetrization* that we have given in the introduction. Given a measurable set $A \subset \mathbb{R}^N$, the spherical symmetrization A^* of A is constructed as follows: for each r, take $A \cap \partial B(0,r)$ and replace it by the spherical cap of the same area and center re_N. This can be done for almost every r. The union of these caps is A^*. Now, the spherical symmetrization u^* of a measurable function $u \ge 0$ is constructed by symmetrizing the super-level sets so that, for every t, $\{u^* \ge t\} = \{u \ge t\}^*$. See [16] for more details.

The following theorem is proved in [16].

Theorem 2.1. *Let $u \in W^{1,p}(B(0,1))$ and let u^* be its spherical symmetrization. Then $u^* \in W^{1,p}(B(0,1))$ and*

$$\int_{B(0,1)} |\nabla u^*|^p \, dx \le \int_{B(0,1)} |\nabla u|^p \, dx, \tag{2.4}$$

$$\int_{B(0,1)} |u^*|^p \, dx = \int_{B(0,1)} |u|^p \, dx, \tag{2.5}$$

$$\int_{\partial B(0,1)} |u^*|^q \, dS = \int_{\partial B(0,1)} |u|^q \, dS. \tag{2.6}$$

With these preliminaries, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let u be an extremal for J and u^* the spherical symmetrization of u. Now, by Theorem 2.1, u^* is an admissible function for the minimization problem and $J(u^*) \le J(u)$. So the first part of the theorem follows.

Assume now that $p = 2$. In this case, it is proved in [3] that if equality holds in (2.4) then for each $0 < r \le 1$ there exists a rotation R_r such that

$$u \mid_{\partial B(0,r)} = (u^* \circ R_r) \mid_{\partial B(0,r)} . \tag{2.7}$$

We can assume that the axis of symmetry e_N was taken so that $R_1 = Id$.

Observe that by the results of [10], u and u^* as well as any other minimizer v are locally Lipschtiz continuous in $B(0,1)$ and they are solutions to

$$\Delta v = v \tag{2.8}$$

in $B(0,1) \cap \{v > 0\}$.

Let us first see that $\{v > 0\}$ is connected.

Since $v = R\, v^*$ on $\partial B(0,1)$ for a certain rotation R, there holds that $\{v > 0\} \cap \partial B(0,1)$ is a connected set. Thus, there can only be one connected component of $\{v > 0\}$ that touches the boundary of the ball. In any other component $\mathcal{O}$, (2.8) holds and $v = 0$ on the boundary of $\mathcal{O}$. Therefore, $v = 0$ in $\mathcal{O}$ which is a contradiction to the fact that $\mathcal{O} \subset \{v > 0\}$.

Now let us see that $u = u^*$. In fact, Let $w = u - u^*$. Then w satisfies

$$\begin{cases} \Delta w = w & \text{in } \{u > 0\} \cap \{u^* > 0\}, \\ w = 0 & \text{on } \{u > 0\} \cap \partial B(0,1), \end{cases}$$

On the other hand, $\partial u/\partial \nu = S(\alpha) u^{q-1} = S(\alpha)(u^*)^{q-1} = \partial u^*/\partial \nu$ on $\{u > 0\} \cap \partial B(0,1)$. Hence, $\partial w/\partial \nu = 0$ on $\{u > 0\} \cap \partial B(0,1)$. Thus, by Holmgren's uniqueness theorem there holds that $u = u^*$ in a neighborhood of $\{u > 0\} \cap \partial B(0,1)$ inside $B(0,1)$. Now, analytic continuation gives that $u = u^*$ in $\{u > 0\} \cap \{u^* > 0\}$. This implies that necessarily $\{u > 0\} = \{u^* > 0\}$. In fact, if this is not the case we have that at least one of the following holds:

(1) $\partial\{u>0\}\cap\{u^*>0\}\cap B(0,1)\neq\emptyset$.

or

(2) $\partial\{u^*>0\}\cap\{u>0\}\cap B(0,1)\neq\emptyset$.

In the first case, since $u=u^*$ in $\{u>0\}\cap\{u^*>0\}$ and $u=0$ on $\partial\{u>0\}\cap B(0,1)$, there holds that $u^*=0$ in $\partial\{u>0\}\cap\{u^*>0\}\cap B(0,1)\neq\emptyset$ which is a contradiction. Analogously, we arrive at a contradiction in the second case.

Thus, $\{u>0\}=\{u^*>0\}$ and since both functions are solutions of the same boundary value problem in this set, they have to coincide. This is, $u=u^*$.

The theorem is proved. □

We end this section with the proof of Theorem 1.5.

Proof of Theorem 1.5. The proof is done by contradiction. Assume that $\{u>0\}\cap\Omega\subset U_1\cup U_2$, where $U_i,\ i=1,2$ are nonempty, disjoint, open sets with $|U_i\cap\{u>0\}|>0$.

First, we claim that

$$\int_{\partial\Omega\cap\overline{U_i}}|u|^q\,dS<1,\quad i=1,2.$$

In fact, if $\int_{\partial\Omega\cap\overline{U_1}}|u|^q\,dS=1$, then it follows that $u|_{\partial U_2}\equiv 0$.

But then, if we take $\phi:=u\chi_{U_1}$, we get that $\phi\in W^{1,p}(\Omega)$, $|\{\phi=0\}|>|\{u=0\}|$ and $\|\phi\|_{L^q(\partial\Omega)}=1$, so

$$\begin{aligned}J(\phi)\geq J(u)&=\int_{U_1}|\nabla u|^p+|u|^p\,dx+\int_{U_2}|\nabla u|^p+|u|^p\,dx\\&=J(\phi)+\int_{U_2}|\nabla u|^p+|u|^p\,dx\end{aligned}$$

Therefore, $u|_{U_2}\equiv 0$ which is a contradiction and so the claim follows.

Now, taking

$$\phi_i:=\frac{u\chi_{U_i}}{\left(\int_{\partial\Omega\cap\overline{U_i}}|u|^q\,dS\right)^{p/q}},$$

we have that each ϕ_i is admissible in the characterization of $S(\alpha)$ and so

$$J(u)=J(u\chi_{U_1})+J(u\chi_{U_2})<J(\phi_i)$$

The inequality is strict because of the strict monotonicity of $S(\alpha)$.

Now, for any $0<\lambda<1$,

$$\begin{aligned}J(u\chi_{U_1})+J(u\chi_{U_2})&<\lambda J(\phi_1)+(1-\lambda)J(\phi_2)\\&=\lambda\frac{\int_{U_1}|\nabla u|^p+|u|^p\,dx}{\left(\int_{\partial\Omega\cap\overline{U_1}}|u|^q\,dS\right)^{p/q}}+(1-\lambda)\frac{\int_{U_2}|\nabla u|^p+|u|^p\,dx}{\left(\int_{\partial\Omega\cap\overline{U_2}}|u|^q\,dS\right)^{p/q}}.\end{aligned}$$

So if we take $\lambda = \left(\int_{\partial\Omega\cap\overline{U_1}} |u|^q\, dS\right)^{p/q}$, we get, as $q \ge p$, $(1-\lambda) \le \left(\int_{\partial\Omega\cap\overline{U_2}} |u|^q\, dS\right)^{p/q}$, therefore

$$\begin{aligned} J(u) = J(u\chi_{U_1}) + J(u\chi_{U_2}) &< \lambda \frac{\int_{U_1} |\nabla u|^p + |u|^p\, dx}{\left(\int_{\partial\Omega\cap\overline{U_1}} |u|^q\, dS\right)^{p/q}} + (1-\lambda)\frac{\int_{U_2} |\nabla u|^p + |u|^p\, dx}{\left(\int_{\partial\Omega\cap\overline{U_2}} |u|^q\, dS\right)^{p/q}} \\ &\le \int_{U_1} |\nabla u|^p + |u|^p\, dx + \int_{U_2} |\nabla u|^p + |u|^p\, dx = J(u), \end{aligned}$$

a contradiction.

This finishes the proof. □

3. Capacitary setting

In this section we consider general sets that may have zero Lebesgue measure. First, we analyze when the Sobolev trace constant sees the set A. Then, we prove the continuity of $\mathbf{S}_A$ with respect to A in Hausdorff distance. We end this section with a discussion about the relationship between S_A and $\mathbf{S}_A$.

We need a result that may be found in [4]. We prove it here for the sake of completeness.

Lemma 3.1. *Let $A \subset \Omega$. Then, $W^{1,p}_A(\Omega) = W^{1,p}(\Omega)$ if and only if $\mathrm{Cap}_p(A) = 0$.*

Proof. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and take A a subset of Ω such that $\mathrm{Cap}_p(A) = 0$.

From (1.6), it follows that $\mathrm{Cap}_p(A) = 0$ if and only if, for every $\varepsilon > 0$ there exists $\phi_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ such that $\|\nabla\phi_\varepsilon\|^p_{L^p(\mathbb{R}^N)} < \varepsilon$ and $\phi_\varepsilon \equiv 1$ in a neighborhood of A.

Take now $u \in C^\infty(\overline{\Omega})$ and define $u_\varepsilon = (1-\phi_\varepsilon)u \in W^{1,p}_A(\Omega)$. Then we have

$$\|u - u_\varepsilon\|_{W^{1,p}(\Omega)} = \|\phi_\varepsilon u\|_{W^{1,p}(\Omega)}.$$

We will show that $\|\phi_\varepsilon u\|_{W^{1,p}(\Omega)} \to 0$ as $\varepsilon \to 0$. From this fact, the result will follow since $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Now,

$$\begin{aligned} \|\phi_\varepsilon u\|^p_{W^{1,p}(\Omega)} &\le C\left(\int_\Omega |\nabla\phi_\varepsilon|^p |u|^p\, dx + \int_\Omega |\phi_\varepsilon|^p |\nabla u|^p\, dx + \int_\Omega |\phi_\varepsilon|^p |u|^p\, dx\right) \\ &= C(I + II + III). \end{aligned}$$

We will bound each term separately.

$$I \le \|u\|_\infty \int_\Omega |\nabla\phi_\varepsilon|^p\, dx < \|u\|_\infty \varepsilon. \tag{3.1}$$

The terms II and III are bounded in the same way. We perform the computations for II. By Hölder's inequality we have

$$II \le \left(\int_\Omega |\phi_\varepsilon|^{p\alpha}\, dx \right)^{1/\alpha} \left(\int_\Omega |\nabla u|^{p\alpha'}\, dx \right)^{1/\alpha'}.$$

So, if we take $\alpha = p^*/p$ where $p^* = Np/(N-p)$ is the critical Sobolev exponent, it follows that

$$II \le \left(\int_{\mathbb{R}^N} |\phi_\varepsilon|^{p^*}\, dx \right)^{p/p^*} \left(\int_\Omega |\nabla u|^{N}\, dx \right)^{p/N} \le \|\nabla u\|_\infty^p |\Omega|^{p/N} \left(\int_{\mathbb{R}^N} |\phi_\varepsilon|^{p^*}\, dx \right)^{p/p^*}.$$

Then, by the Gagliardo–Nirenberg–Sobolev inequality,

$$II \le C\|\nabla u\|_\infty^p |\Omega|^{p/N} \int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^p\, dx < C\|\nabla u\|_\infty^p |\Omega|^{p/N}\varepsilon. \tag{3.2}$$

Analogously, for III we have

$$III \le C\|u\|_\infty^p |\Omega|^{p/N}\varepsilon. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we have

$$\|u - u_\varepsilon\|_{W^{1,p}(\Omega)}^p = \|\phi_\varepsilon u\|_{W^{1,p}(\Omega)}^p \le C\varepsilon. \tag{3.4}$$

Assume now that $C_0^\infty(\overline{\Omega} \setminus A)$ is dense in $W^{1,p}(\Omega)$. We will show that $\mathrm{Cap}_p(A) = 0$.

By hypotheses, for every $\varepsilon > 0$, there exists $u_\varepsilon \in C^\infty(\overline{\Omega})$ such that

$$A \cap \mathrm{supp}(u_\varepsilon) = \emptyset \quad \text{and} \quad \|1 - u_\varepsilon\|_{W^{1,p}(\Omega)} < \varepsilon.$$

Take $\phi_\varepsilon = 1 - u_\varepsilon$, then $\phi_\varepsilon \in W^{1,p}(\Omega)$. As Ω is smooth, it has the extension property. Then, there exists $\bar\phi_\varepsilon \in W^{1,p}(\mathbb{R}^N)$ such that

$$\|\bar\phi_\varepsilon\|_{W^{1,p}(\mathbb{R}^N)} \le E\|\phi_\varepsilon\|_{W^{1,p}(\Omega)}.$$

But

$$\int_{\mathbb{R}^N} |\nabla \bar\phi_\varepsilon|^p\, dx \le \|\bar\phi_\varepsilon\|_{W^{1,p}(\mathbb{R}^N)}^p \le E^p \|\phi_\varepsilon\|_{W^{1,p}(\Omega)}^p < E^p\varepsilon^p,$$

as we wanted to show. □

With this lemma, we can prove Proposition 1.1.

Proof of Proposition 1.1. We only need to see that if $\mathbf{S}_A = \mathbf{S}_\emptyset$ then $\mathrm{Cap}_p(A) = 0$. Let u be an extremal for $\mathbf{S}_A$. As $\mathbf{S}_A = \mathbf{S}_\emptyset$, u is also an extremal for $\mathbf{S}_\emptyset$. Therefore, u is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

By known regularity theory (see [18]) and the maximum principle (see [19]) it follows that $u \in C^{1,\alpha}(\overline{\Omega})$ and $u > 0$ in $\overline{\Omega}$.

As $u \in W_A^{1,p}(\Omega)$, there exists a sequence $u_n \in C_0^\infty(\overline{\Omega} \setminus A)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$.

Let $\phi_n = 1 - \dfrac{u_n}{u}$. Observe that $\phi_n \equiv 1$ in a neighborhood of A. Moreover,

$$\begin{aligned}\|\nabla \phi_n\|_{L^p(\Omega)} &= \Big\| \frac{1}{u}\nabla(u_n - u) + \frac{1}{u^2}\nabla u(u-u_n)\Big\|_{L^p(\Omega)}\\ &\le C\|\nabla(u_n-u)\|_{L^p(\Omega)} + C\|\nabla u\|_{L^\infty(\Omega)}\|u-u_n\|_{L^p(\Omega)},\end{aligned}$$

where we used the fact that $u \ge \alpha > 0$ in Ω. Then, for every $\varepsilon > 0$ there exists n_0 such that

$$\|\nabla\phi_n\|_{L^p(\Omega)} < \varepsilon \qquad \text{if } n \ge n_0.$$

Now the result follows by extending ϕ_n to $\mathbb{R}^N$ and regularizing. $\square$

Remark 3.1. If the set A is a regular surface of dimension k, then there exists a trace operator

$$T : W^{1,p}(\Omega) \to L^p(A)$$

if $k > N - p$. This relates to Lemma 3.1 by the fact that sets of Hausdorff dimension less than or equal to $N - p$ have zero $p-$capacity (see [4]).

Now we prove the continuity of $\mathbf{S}_A$ with respect to A in Hausdorff distance. Recall that the Hausdorff distance is definded by,

$$d(A_1, A_2) = \inf\{r > 0\,,\ A_1 \subset B_r(A_2) \text{ and } A_2 \subset B_r(A_1)\}.$$

Here $B_r(A) = \bigcup_{x\in A} B_r(x)$ is the usual fattening of A.

Proof of Theorem 1.6. Let $A_\varepsilon = B_\varepsilon(A)$. Assume that $d(A_n, A) \to 0$ as $n \to \infty$, then given $\varepsilon > 0$ there exists n_0 such that $A, A_n \subset A_\varepsilon$ if $n \ge n_0$ and it follows that

$$W^{1,p}_{A_\varepsilon}(\Omega) \subset W^{1,p}_A(\Omega) \cap W^{1,p}_{A_n}(\Omega). \tag{3.5}$$

First, observe that by (1.4), we have

$$S = \mathbf{S}_\emptyset \le \mathbf{S}_A,\, \mathbf{S}_{A_n} \le \mathbf{S}_{A_\varepsilon}, \tag{3.6}$$

if $n \ge n_0$.

Now, let $u \in W^{1,p}_A(\Omega)$ be an extremal for $\mathbf{S}_A$ normalized such that $\|u\|_{L^q(\partial\Omega)} = 1$. As $u \in W^{1,p}_A(\Omega)$, given $\delta > 0$ there exists $u_\delta \in C_0^\infty(\overline{\Omega}\setminus A)$ such that

$$\|u - u_\delta\|_{W^{1,p}(\Omega)} \le \delta$$

and, moreover, we can assume that

$$\operatorname{supp}(u_\delta) \subset \overline{\Omega}\setminus A_\varepsilon \tag{3.7}$$

if ε is small enough.

Now, by (3.5) and (3.7), $u_\delta \in W^{1,p}_{A_n}(\Omega)$ for $n \ge n_0$, so

$$\begin{aligned}\mathbf{S}_{A_n} &\le \frac{\|u_\delta\|^p_{W^{1,p}(\Omega)}}{\|u_\delta\|^p_{L^q(\partial\Omega)}} \le \frac{\big(\|u\|_{W^{1,p}(\Omega)} + \|u_\delta - u\|_{W^{1,p}(\Omega)}\big)^p}{\big(\|u\|_{L^q(\partial\Omega)} - \|u_\delta - u\|_{L^q(\partial\Omega)}\big)^p}\\ &\le \frac{(\mathbf{S}_A^{1/p} + \delta)^p}{(1 - S^{-1/p}\delta)^p} \le \mathbf{S}_A + C_1\delta,\end{aligned}$$

where C_1 is a constant that depends on S and $\mathbf{S}_A$.

By symmetry, we get

$$\mathbf{S}_A \le \mathbf{S}_{A_n} + C_2\delta,$$

where C_2 depends on S and $\mathbf{S}_{A_n}$, but by (3.6) it can be taken depending on S and $\mathbf{S}_{A_\varepsilon}$ only.

So

$$|\mathbf{S}_{A_n} - \mathbf{S}_A| \le C\delta \qquad \text{if} \qquad n \ge n_0,$$

as we wanted to show.

It remains to see the convergence of the extremals. Let u_n be an extremal for $\mathbf{S}_{A_n}$ normalized such that $\|u_n\|_{L^q(\partial\Omega)} = 1$. Then, as

$$\|u_n\|_{W^{1,p}(\Omega)} = \mathbf{S}_{A_n}^{1/p} \to \mathbf{S}_A^{1/p},$$

it follows that u_n is bounded in $W^{1,p}(\Omega)$. So there exists a sequence (that we still denote u_n) and a function $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),\\ u_n &\to u \quad \text{strongly in } L^q(\partial\Omega). \end{aligned} \tag{3.8}$$

By the definition of the spaces $W^{1,p}_{A_n}(\Omega)$, $W^{1,p}_A(\Omega)$ and by (3.5), it is straightforward to see that $u \in W^{1,p}_A(\Omega)$.

Now, by (3.8), $\|u\|_{L^q(\partial\Omega)} = 1$ and, also by (3.8),

$$\mathbf{S}_A \le \|u\|^p_{W^{1,p}(\Omega)} \le \lim_{n\to\infty} \|u_n\|^p_{W^{1,p}(\Omega)} = \lim_{n\to\infty} \mathbf{S}_{A_n} = \mathbf{S}_A.$$

Therefore, u is an extremal for $\mathbf{S}_A$ and $u_n \to u$ strongly in $W^{1,p}(\Omega)$. The proof is now complete. □

3.1. **Relationship between S_A and $\mathbf{S}_A$.** It is easy to see that $S_A \le \mathbf{S}_A$. The other inequality is not true in general. Moreover, the constant S_A is not modified if we change the set A by a set of zero Lebesgue measure while $\mathbf{S}_A$ will be modified unless the variation on A is of zero $p-$capacity.

For example, if A is a hypersurface contained in Ω, then $S_A = S_\emptyset$. On the other hand $\mathbf{S}_A > \mathbf{S}_\emptyset = S_\emptyset = S_A$ since an extremal for $\mathbf{S}_A$ has zero trace on A.

However, if A is the closure of an open set with regular boundary, both constants agree.

This leads to the question of the existence of a representative of the set A for both constants to agree. We believe that this is an interesting problem.

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