ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF THE ONE DIMENSIONAL WEIGHTED $p-$LAPLACE OPERATOR

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Abstract. In this paper we study the spectral counting function for the weighted p-laplacian in one dimension. First, we prove that all the eigenvalues can be obtained by a mini-max characterization and then we show the existence of a Weyl-type leading term. Finally we find estimates for the remainder term.

1. Introduction

In this paper we study the following eigenvalue problem:

$$-(\psi_p(u'))' = \lambda r(x)\psi_p(u), \tag{1.1}$$

in a bounded open set $\Omega \subset \mathbb{R}$, with Dirichlet or Neumann boundary conditions. Here, the weight r is a real-valued, bounded, positive continuous function, λ is a real parameter, $1 < p < +\infty$ and

$$\psi_p(s) = |s|^{p-2}s,$$

for $s \neq 0$ and 0 if $s = 0$.

From [7] Theorem 1.1 pag. 233, we know that the spectrum consists on a countable sequence of nonnegative eigenvalues $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ (repeated according multiplicity) tending to $+\infty$. See also [15] were a similar result is obtained for the radial p-laplacian and for the one-dimensional p-laplacian with mixed boundary conditions. With the same ideas as in [3], Theorem 4.1 it is easy to prove that the sequence $\{\lambda_k\}_k$ coincides with the eigenvalues obtained by the Ljusternik-Schnirelmann theory. We recall that the variational characterization of the eigenvalues is as follows:

$$\lambda_k^\Omega = \inf_{F\in C_k^\Omega} \sup_{u\in F} \int_\Omega |u'|^p, \tag{1.2}$$

where

$$C_k^\Omega = \left\{C \subset M^\Omega \ : \ C \ \text{ compact }, \ C = -C, \ \gamma(C) \geq k\right\},$$

$$M^\Omega = \left\{u \in W_0^{1,p}(\Omega) \ (\text{ resp., } W^{1,p}(\Omega)\ ) \ : \ \int_\Omega r(x)|u|^p = 1\right\}$$

and $\gamma(C)$ is the Krasnoselskii genus (see [16] for the definition and properties of γ).

So our first result is,

Theorem 1.1. *Every eigenvalue of problem* (1.1) *is given by* (1.2).

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We define the spectral counting function $N(\lambda, \Omega)$ as the number of eigenvalues of problem (1.1) less than a given λ :

$$N(\lambda, \Omega) = \#\{k \ : \ \lambda_k \le \lambda\}.$$

We will write $N_D(\lambda, \Omega)$ (resp., $N_N(\lambda, \Omega)$) whenever we need to stress the dependence on the Dirichlet (resp., Neumann) boundary conditions.

The problem of estimating the spectral counting function has a long history, special in the linear case (p=2). See for instance [5, 9, 10, 12] and the references therein.

However, up to our knowledge, for $p \neq 2$ there is a lack of information about the behavior of $N(\lambda, \Omega)$. The only known result is due to [8]. In that paper, the authors show that the eigenvalues of the p-laplacian in $\mathbb{R}^n$ (with $r = 1$) obtained by the mini-max theory satisfy

$$c_1(\Omega) k^{p/n} \le \lambda_k \le c_2(\Omega) k^{p/n}. \tag{1.3}$$

It is easy to see that this eigenvalue inequality is equivalent to

$$C_1(\Omega)\lambda^{n/p} \le N(\lambda, \Omega) \le C_2(\Omega)\lambda^{n/p},$$

for certain positive constants when $\lambda \to \infty$, see Lemma 3.2 below.

Our next result is concerned with the asymptotic behavior of the eigenvalues of (1.1) and begins our analysis of the function $N(\lambda, \Omega)$.

We obtain the following asymptotic expansion:

$$N(\lambda, \Omega) \sim \frac{\lambda^{1/p}}{\pi_p} \int_\Omega r^{1/p}, \tag{1.4}$$

as $\lambda \to \infty$, where π_p is defined as

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}. \tag{1.5}$$

The proof is based on variational arguments, including a suitable extension of the 'Dirichlet-Neumann bracketing' method, see [2]. We prove,

Theorem 1.2. *Let $r(x)$ be a real-valued, positive and bounded continuous function in Ω. Then,*

$$N(\lambda, \Omega) = \frac{\lambda^{1/p}}{\pi_p} \int_\Omega r^{1/p} + o(\lambda^{1/p}). \tag{1.6}$$

Observe that by Theorem 1.2, the asymptotic behavior of the eigenvalues (1.3) is improved. In fact, what (1.6) implies is

$$\lambda_k \sim c k^p.$$

Once we found the first order asymptotics of $N(\lambda, \Omega)$, it is natural to try to improve these estimates and look for a second order term.

Following the ideas of [5], we analyze the remainder term $R(\lambda, \Omega) = N(\lambda, \Omega) - \frac{1}{\pi_p}\int_\Omega (\lambda r)^{1/p}$. We show that

$$R(\lambda, \Omega) = O(\lambda^{\delta/p}), \tag{1.7}$$

where $\delta \in (0, 1]$ depends on the regularity of the boundary $\partial\Omega$ and on the smoothness of the weight r measured in a subtle way. To this end, let us introduce the following definitions:

Given any $\eta > 0$ sufficiently small, we consider a tessellation of $\mathbb{R}$ by a countable family of disjoint open intervals $\{I_\zeta\}_{\zeta\in\mathbb{Z}}$, of length η.

Definition 1.3. Let Ω be a bounded open set in $\mathbb{R}$. Given $\beta > 0$, we say that the boundary $\partial\Omega$ satisfies the "β-condition" if there exist positive constants c_0 and $\eta_0 < 1$ such that for all $\eta \le \eta_0$,

$$\frac{\#(J \setminus I)}{\#I} \le c_0\eta^\beta, \tag{1.8}$$

where

$$I = I(\Omega) = \{\zeta \in \mathbb{Z} \ : \ I_\zeta \subset \Omega\}, \tag{1.9}$$

$$J = J(\Omega) = \{\zeta \in \mathbb{Z} \ : \ I_\zeta \cap \overline{\Omega} \ne \emptyset\}. \tag{1.10}$$

It is easy to see that if the set is Jordan contented (i.e., it is well approximated from within and without by a finite union of intervals), then it verifies the "β-condition" for $\beta = 1$. The coefficient β allows us to measure the smoothness of $\partial\Omega$.

Definition 1.4. Given $\gamma > 0$, we say that the function r satisfies the "γ-condition" if there exist positive constants c_1 and $\eta_1 < 1$ such that for all $\zeta \in I(\Omega)$ and all $\eta \le \eta_1$,

$$\int_{I_\zeta} |r - r_\zeta|^{1/p} \le c_1\eta^\gamma, \tag{1.11}$$

where $r_\zeta = \Big(|I_\zeta|^{-1}\int_{I_\zeta} r^{1/p}\Big)^p$ is the mean value of $r^{1/p}$ in I_ζ.

Remark 1.5. 1. The coefficient γ enable us to measure the smoothness of r, the larger γ, the smoother r.

2. When r is Hölder continuous of order $\theta > 0$ and is bounded away from zero on Ω, then it satisfies the γ-condition for $0 < \gamma \le 1 + \theta/p$.

If r is only continuous and positive on $\overline{\Omega}$, then it satisfies the γ-condition for $0 < \gamma \le 1$

Now we are ready to state the theorem,

Theorem 1.6. *Let Ω be a bounded open set in $\mathbb{R}$ with boundary $\partial\Omega$ satisfying the "β-condition" for some $\beta > 0$, and let r be a bounded, positive and continuous function satisfying the "γ-condition" for some $\gamma > 1$. Set $\nu = \min(\beta, \gamma - 1)$. Then, for all $\delta \in [1/(\nu + 1), 1]$, we have*

$$N(\lambda, \Omega) - \frac{1}{\pi_p}\int_\Omega (\lambda r)^{1/p} = O(\lambda^{\delta/p}) \tag{1.12}$$

Finally, we end this article with some examples where we compute the remainder term explicitly.

The paper is organized as follows. In §2, we introduce the genus in a version due to Krasnoselskii and prove the variational characterization of all the eigenvalues together with some auxiliary lemmas. In §3, we prove the asymptotic expansion (1.4). We analyze the remainder estimate in §4. Finally, in §5, we compute explicitly a non-trivial second term for $r = 1$ and analyze the asymptotic behavior of the eigenvalues.

2. Variational characterization of the eigenvalues

In this section we first show that every eigenvalue of (1.1) is given by a variational characterization and then we prove the Dirichlet-Neumann bracketing method that will be the main tool in the remaining of the paper.

So let us begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof follows the lines of Theorem 4.1 of [3].

By [7] the spectrum is countable and we can assume that it is given by $\mu_1 < \mu_2 \leq \cdots$. Given (u_k, μ_k) an eigenpair of (1.1), we claim that u_k has k nodal domains. It is clear that the number of nodal domains is less or equal than k (see for instance [1]). Now the claim follows by induction, since the first eigenfunction has exactly one nodal domain, and by [15] Theorem 4.1 (b) if u_k has k nodal domains, u_{k+1} has at least $k+1$ nodal domains.

Now, if (u_k, μ_k) is an eigenpair of (1.1), we can consider $w_i(x) = u_k(x)$ if x belong to the i^{th} nodal domain, and $w_i(x) = 0$ elsewhere. Let S_t be the sphere in $W^{1,p}(\Omega)$ of radius t. Then, the set $C_k = \text{span}\{w_1, \dots, w_k\} \cap S_t$ has genus k and is an admissible set in the characterization (1.2) of the k^{th} variational eigenvalue λ_k, from where it follows that $\lambda_k \leq \mu_k$ and then $\lambda_k = \mu_k$. The proof is now complete. □

The remaining of the section is devoted to the proof of the so called Dirichlet-Neumann bracketing method. We want to remark that these results hold for arbitrary dimensions $n \geq 1$ if one consider only the variational eigenvalues.

Theorem 2.1. *Let $U_1, U_2 \in \mathbb{R}^n$ be disjoint open sets such that $(\overline{U_1 \cup U_2})^{int} = U$ and $|U \setminus U_1 \cup U_2|_n = 0$, then*

$$N_D(\lambda, U_1 \cup U_2) \leq N_D(\lambda, U) \leq N_N(\lambda, U) \leq N_N(\lambda, U_1 \cup U_2).$$

Here $|A|_n$ stands for the n-dimensional Lebesgue measure of the set A.

Proof. It is an easy consequence of the following inclusions

$$W_0^{1,p}(U_1 \cup U_2) = W_0^{1,p}(U_1) \oplus W_0^{1,p}(U_2) \subset W_0^{1,p}(U) \tag{2.1}$$

and

$$W^{1,p}(U) \subset W^{1,p}(U_1) \oplus W^{1,p}(U_2) = W^{1,p}(U_1 \cup U_2), \tag{2.2}$$

and the variational formulation (1.2). In fact, using that

$$M^U(X) = \left\{ u \in X \ : \ \int_U r(x)|u|^p = 1 \right\} \subset M^U(Y) = \left\{ u \in Y \ : \ \int_U r(x)|u|^p = 1 \right\},$$

and that $C_k^U(X) \subset C_k^U(Y)$ where $X = W_0^{1,p}(U_1 \cup U_2)$ or $W^{1,p}(U)$ and $Y = W_0^{1,p}(U)$ or $W^{1,p}(U_1 \cup U_2)$ respectively, we obtain the desired inequality. □

The Dirichlet-Neumann bracketing method is a powerful tool when combined with the following result:

Proposition 2.2. *Let $\Omega = \cup_j \Omega_j$, where $\{\Omega_j\}_j$ is a pairwise disjoint family of bounded open sets in $\mathbb{R}^n$. Then,*

$$N(\lambda, \Omega) = \sum_j N(\lambda, \Omega_j). \tag{2.3}$$

Proof. Let λ be an eigenvalue of problem (1.1) in Ω, and let u be the associated eigenfunction. For all $v \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int_\Omega |u|^{p-2}uv = 0. \tag{2.4}$$

Choosing v with compact support in Ω_j, we conclude that $u|_{\Omega_j}$ is an eigenfunction of problem (1.1) in Ω_j with eigenvalue λ.

For the other inclusion, it is sufficient to extend an eigenfunction u in Ω_j by zero outside, which gives an eigenfunction in Ω. □

3. The function $N(\lambda)$

In this section we prove the asymptotic expansion given by Theorem 1.2.

First let us recall the following lemma.

Lemma 3.1. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the eigenvalues of (1.1) in $(0,T)$, with Dirichlet boundary condition and $r = 1$. Then,*

$$\lambda_k = \frac{\pi_p^p}{T^p} k^p. \tag{3.1}$$

Let $\{\mu_k\}_{k\in\mathbb{N}}$ be the eigenvalues of (1.1) in $(0,T)$, with Neumann boundary condition and $r = 1$. Then,

$$\mu_k = \frac{\pi_p^p}{T^p}(k-1)^p. \tag{3.2}$$

Proof. This result was proved in [14]. □

With the aid of Lemma 3.1 we can prove the following.

Lemma 3.2. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the eigenvalues of (1.1) in $(0,T)$ and suppose that $m \le r(x) \le M$. Then,*

$$\frac{1}{M}\frac{\pi_p^p}{T^p}k^p \le \lambda_k \le \frac{1}{m}\frac{\pi_p^p}{T^p}k^p, \tag{3.3}$$

and

$$\frac{Tm^{1/p}}{\pi_p}\lambda^{1/p} - 1 \le N(\lambda,(0,T)) \le \frac{TM^{1/p}}{\pi_p}\lambda^{1/p}. \tag{3.4}$$

Proof. Equation (3.3) is an easy consequence of the Sturmian Comparison principle in [15] (pag. 182 Theorem 4.1 (b) and the subsequent Corollary) and the explicit formula for the eigenvalues with constant weight. Now,

$$\#\left\{k : \frac{\pi_p^p k^p}{T^p M} \le \lambda\right\} \le \#\{k : \lambda_k \le \lambda\} \le \#\left\{k : \frac{\pi_p^p k^p}{T^p m} \le \lambda\right\}. \tag{3.5}$$

The left hand side is greater than

$$\frac{Tm^{1/p}}{\pi_p}\lambda^{1/p} - 1,$$

which gives the lower bound. In the same way, we obtain

$$N(\lambda,(0,T)) \le \left[\frac{Tm^{1/p}}{\pi_p}\lambda^{1/p}\right] \le \frac{Tm^{1/p}}{\pi_p}\lambda^{1/p}.$$

This finishes the proof. □

Now we prove a proposition that is the key ingredient in the proof of Theorem 1.2.

Proposition 3.3. *Let $r(x)$ be a real-valued, positive continuous function in $[0,T]$. Then,*

$$N(\lambda,(0,T)) = \frac{\lambda^{1/p}}{\pi_p}\int_0^T r^{1/p} + o(\lambda^{1/p}). \tag{3.6}$$

Proof. Let $[0,T] = \overline{\cup_{1\le j\le J} I_j},\ I_j\cap I_k = \emptyset$ with $|I_j| = T/J = \eta$. We define

$$m_j = \inf_{x\in I_j} r(x), \qquad M_j = \sup_{x\in I_j} r(x).$$

We can choose $\eta>0$ such that

$$\sum_{j=1}^J \eta m_j^{1/p} = \int_0^T r^{1/p} - \varepsilon_1, \qquad \sum_{j=1}^J \eta M_j^{1/p} = \int_0^T r^{1/p} + \varepsilon_2,$$

with $\varepsilon_1,\varepsilon_2>0$ arbitrarily small.

From Theorem 2.1 and Proposition 2.2, we obtain

$$\sum_{j=1}^J N_D(\lambda, I_j) \le N(\lambda,(0,T)) \le \sum_{j=1}^J N_N(\lambda, I_j).$$

Hence, using that

$$N_D(\lambda,I_j) \ge m_j^{1/p}\frac{\lambda^{1/p}}{\pi_p} - 1 \quad \text{and} \quad N_N(\lambda,I_j) \le M_j^{1/p}\frac{\lambda^{1/p}}{\pi_p},$$

we have

$$\frac{\lambda^{1/p}}{\pi_p}\left(\int_0^T r^{1/p} - \varepsilon_1\right) - J \le N(\lambda,(0,T)) \le \frac{\lambda^{1/p}}{\pi_p}\left(\int_0^T r^{1/p} + \varepsilon_2\right).$$

Letting $\lambda\to\infty$, we have

$$\frac{N(\lambda,(0,T))}{\frac{\lambda^{1/p}}{\pi_p}\int_0^T r^{1/p}} \to 1$$

and the proof is complete. □

Finally, we arrive at the proof of Theorem 1.2.

Proof of Theorem 1.2. It is an easy consequence of Proposition 2.2 and Proposition 3.3. Let $\Omega = \cup_{j=1}^\infty I_j$, then

$$N(\lambda,\Omega) = \sum_{j=1}^\infty N(\lambda, I_j) \sim \sum_{j=1}^\infty \frac{\lambda^{1/p}}{\pi_p}\int_{I_j} r^{1/p} = \frac{\lambda^{1/p}}{\pi_p}\int_\Omega r^{1/p}. \tag{3.7}$$

This completes the proof. □

4. Remainder estimates

As we mentioned in the introduction, we now look for an improvement in the asymptotic expansion of $N(\lambda,\Omega)$. This is the content of Theorem 1.6.

Proof of Theorem 1.6. For the convenience of the reader, the proof is divided into several steps.

Moreover, we will stress the dependence of the spectral counting function with respect to the weight function by writing $N(\lambda,\Omega,r)$.

Step 1. Let $\eta>0$ be fixed. We define

$$\varphi(\lambda)=\pi_p^{-1}\int_\Omega(\lambda r)^{1/p},\qquad \varphi(\lambda,\zeta)=\eta\pi_p^{-1}(\lambda r_\zeta)^{1/p}, \tag{4.1}$$

where $r_\zeta=\left(|I_\zeta|^{-1}\int_{I_\zeta}r^{1/p}\right)^p$.

From Theorem 2.1 we obtain

$$\sum_{\zeta\in I}N_D(\lambda,I_\zeta,r)-\varphi(\lambda)\le N_D(\lambda,\Omega,r)-\varphi(\lambda) \tag{4.2}$$

and

$$N_D(\lambda,\Omega,r)-\varphi(\lambda)\le\sum_{\zeta\in I}N_N(\lambda,I_\zeta,r)+\sum_{\zeta\in J\setminus I}N_N(\lambda,I_\zeta\cap\Omega,r)-\varphi(\lambda). \tag{4.3}$$

We are reduced to find a bound for the left (resp., right) term of (4.2) (resp., (4.3)).

Step 2. We can rewrite (4.2) as:

$$\begin{aligned}\sum_{\zeta\in I}N_D(\lambda,I_\zeta,r)-\varphi(\lambda)\le&\sum_{\zeta\in I}N_D(\lambda,I_\zeta,r_\zeta)-\varphi(\lambda,\zeta)+\sum_{\zeta\in I}\varphi(\lambda,\zeta)-\varphi(\lambda)\\&+\sum_{\zeta\in I}N_D(\lambda,I_\zeta,r)-\sum_{\zeta\in I}N_D(\lambda,I_\zeta,r_\zeta).\end{aligned} \tag{4.4}$$

Let us note that both $\sum_{\zeta\in I}N_D(\lambda,I_\zeta,r_\zeta)-\varphi(\lambda,\zeta)$ and $\sum_{\zeta\in I}\varphi(\lambda,\zeta)-\varphi(\lambda)$ are negative. Now, by Lemma 3.2:

$$\sum_{\zeta\in I}|N_D(\lambda,I_\zeta,r_\zeta)-\varphi(\lambda,\zeta)|\le\#(I)M\le\eta^{-1}|\Omega|. \tag{4.5}$$

We can bound

$$\sum_{\zeta\in I}\varphi(\lambda,\zeta)-\varphi(\lambda)=\pi_p^{-1}\lambda^{1/p}\left(\sum_{\zeta\in I}\int_{I_\zeta}(r^{1/p}-r_\zeta^{1/p})+\sum_{\zeta\in J\setminus I}\int_{I_\zeta\cap\Omega}r^{1/p}\right)$$

as

$$C\lambda^{1/p}\#(J\setminus I)\eta M\le C\lambda^{1/p}\eta^\beta. \tag{4.6}$$

Here we have used that $r\le M$, and that $\partial\Omega$ satisfies the β-condition.

Finally, the third term in (4.4) can be handled using the monotonicity of the eigenvalues with respect to the weight (see [15]). Using that $r\le r_\zeta+|r-r_\zeta|$, a simple computation shows that

$$N(\lambda,I_\zeta,r)\le N(\lambda,I_\zeta,r_\zeta)+N(\lambda,I_\zeta,|r-r_\zeta|),$$

which gives

$$\sum_{\zeta\in I} N_D(\lambda, I_\zeta, r) - N_D(\lambda, I_\zeta, r_\zeta) \le \sum_{\zeta\in I} N(\lambda, I_\zeta, |r-r_\zeta|) \le C\lambda^{1/p}\#(I)\eta^\gamma$$

and using the same arguments as above and the fact that r satisfies the γ-condition, we obtain

$$\sum_{\zeta\in I} N_D(\lambda, I_\zeta, r) - N_D(\lambda, I_\zeta, r_\zeta) \le C\lambda^{1/p}\eta^{\gamma-1}. \tag{4.7}$$

Collecting (4.5), (4.6) and (4.7) we have the lower bound

$$C\lambda^{1/p}(\eta^\beta + \eta^{\gamma-1}) + C\eta^{-1}. \tag{4.8}$$

Step 3. In a similar way, we can find an upper bound for (4.3),

$$\left(\sum_{\zeta\in I} N_N(\lambda, I_\zeta, r) - \varphi(\lambda)\right) + \sum_{\zeta\in J\setminus I} N_N(\lambda, I_\zeta\cap\Omega, r). \tag{4.9}$$

We only need to estimate the last term, but

$$N_N(\lambda, I_\zeta\cap\Omega, r) \le C\lambda^{1/p}\int_{I_\zeta\cap\Omega} r^{1/p} \le C(M\eta\lambda)^{1/p}$$

and again, using the β-condition, we have

$$\sum_{\zeta\in J\setminus I} N_N(\lambda, I_\zeta\cap\Omega, r) \le C\lambda^{1/p}\eta^\beta. \tag{4.10}$$

Hence, we obtain an upper bound for (4.3):

$$C\lambda^{1/p}(\eta^\beta + \eta^{\gamma-1}) + C\eta^{-1}. \tag{4.11}$$

Step 4. From (4.8) and (4.11) we obtain

$$|N(\lambda,\Omega) - \frac{1}{\pi_p}\int_\Omega (\lambda r)^{1/p}| \le C\lambda^{1/p}(\eta^\beta + \eta^{\gamma-1}) + C\eta^{-1}. \tag{4.12}$$

We now choose $\eta = \lambda^{-a}$, with $0 < a \le \delta$. It is clear that the last term in (4.12) is bounded by $C\lambda^\delta$. Also, it is easy to see that, if $a \ge \frac{1}{\beta}(\frac{1}{p} - \delta)$, then $\lambda^{1/p}\eta^\beta \le \lambda^\delta$. Likewise, choosing $a \ge \frac{1}{\gamma-1}(\frac{1}{p} - \delta)$, we have $\lambda^{1/p}\eta^{\gamma-1} \le \lambda^\delta$. When $\beta = 0$, or $\gamma = 1$, we must choose $a = 1/p$.

This completes the proof. □

5. Concluding remarks

We end this paper showing a family of examples with a power-like second term, and an example with an irregular second term. Finally, we discuss the asymptotic behavior of the eigenvalues.

In the examples below, the parameter d provides some geometrical information about $\partial\Omega$. In both cases, d is the interior Minkowski (or box) dimension of the boundary, we refer the reader to [4] and references therein for the definition and properties of the Minkowski dimension.

Examples of explicit second term. Let $\Omega = \cup_j I_j$, where $|I_j| = j^{-1/d}$, and $0 < d < 1$. We have the following asymptotic expansion for the spectral counting function when $r = 1$:

$$N(\lambda, \Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + C(d)\lambda^{d/p} + O(\lambda^{d/p(2+d)}). \tag{5.1}$$

The proof can be obtained with number-theoretic methods. We have:

$$N(\lambda, \Omega) = \sum_{j=1}^{\infty} \Big[\frac{j^{-1/d}}{\pi_p}\lambda^{1/p}\Big] = \#\{(m,n) \in \mathbb{N}^2 \ : \ m.n^{1/d} \le \pi_p^{-1}\lambda^{1/p}\}.$$

In fact, for each j we can draw the vertical segment of length $j^{-1/d}\lambda^{1/p}/\pi_p$ in the plane, and the series in the left is the number of lattice points below the function $y(x) = \frac{\lambda^{1/p}}{\pi_p}x^{-1/d}$. See [13] for a detailed proof.

When $p = 2$ and $|I_j| \sim j^{-1/d}$, it is shown in [11] that

$$N(\lambda, \Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + C(d)\lambda^{d/p} + o(\lambda^{d/p}),$$

without the lattice point theory, the same result is valid for $p \neq 2$. However, let us note that the error in equation (5.1) is better, which enables us to obtain more precise estimates whenever we know more about the asymptotic behavior of $|I_j|$. On the other hand, the result in [11] holds for more general domains that the ones considered here.

Example of irregular second term. Let Ω be the complement of the ternary Cantor set, and $r = 1$. We have:

$$N(\lambda, \Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} - f(\ln(\lambda))\lambda^{\ln(2)/p\ln(3)} + O(1). \tag{5.2}$$

Here $f(x)$ is a bounded, periodic function. Our proof follows closely [6], where the usual Laplace operator on a self-similar set in $\mathbb{R}^n$ was studied for every $n \ge 2$.

Let us define $\rho(x) = x - [x]$, it is evident that $|\rho(x)| \le \min(x, 1)$. Hence,

$$N(\lambda, \Omega) - \frac{|\Omega|}{\pi_p}\lambda^{1/p} = -\sum_{j=0}^{\infty} 2^j \rho\left(\frac{\lambda^{1/p}}{3^{j+1}\pi_p}\right) \le C\lambda^{1/p}. \tag{5.3}$$

It remains to prove the periodicity of f. We write the error term as

$$\sum_{j=-\infty}^{\infty} 2^j \rho\left(\frac{\lambda^{1/p}}{3^{j+1}\pi_p}\right) - \sum_{j=-\infty}^{-1} 2^j \rho\left(\frac{\lambda^{1/p}}{3^{j+1}\pi_p}\right). \tag{5.4}$$

Using that $|\rho(x)| \le 1$, the second series converges and it is bounded by a constant.

Let us introduce the new variable

$$y = \frac{\ln(\lambda^{1/p}) - \ln(\pi_p)}{\ln(3)}, \tag{5.5}$$

which gives $3^y = \lambda^{1/p}/\pi_p$ and $2^y = (\lambda^{1/p}/\pi_p)^d$, where

$$d = \frac{\ln(2)}{\ln(3)}. \tag{5.6}$$

Replacing in (5.4), we obtain

$$\frac{1}{2}\sum_{j=-\infty}^{\infty} 2^j \rho\left(\frac{\lambda^{1/p}}{3^j\pi_p}\right) = \frac{1}{2}\left(\frac{\lambda^{1/p}}{3^j\pi_p}\right)^d \sum_{j=-\infty}^{\infty} 2^{j-y}\rho(3^{y-j}). \tag{5.7}$$

Thus, as $j-(y-1)=(j+1)-y$, we deduce that $f(x)$ is periodic with period equal to one.

Asymptotics of eigenvalues. From Theorem (1.6) it is easy to prove the following asymptotic formula for the eigenvalues:

$$\lambda_k \sim ck^p.$$

It follows immediately since $k \sim N(\lambda_k)$, which gives:

$$\lambda_k \sim \left(\frac{\pi_p}{\int_\Omega r^{1/p}}\right)^p k^p.$$

Using the Dirichlet-Neumann bracketing method, it is possible to improve the constants in equation (1.3). In [8] the authors only consider two cubes $Q_1 \subset \Omega \subset Q_2$, and they obtain a lower and an upper bound for the eigenvalues in cubes which depends on the measure of the cubes Q_1, Q_2 instead of the measure of Ω.

A similar argument as in [8], changing the functions $\{\sin(kx)\}_k$ for $\{\sin_p(kx)\}_k$, gives the upper bound:

$$\lambda^k \leq \left(\frac{\pi_p}{|\Omega|}\right)^{p/n} k^{p/n}.$$

However, it seems difficult to improve the lower bound obtained with the aid of the Bernstein's Lemma.

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