

# AN OPTIMIZATION PROBLEM RELATED TO THE BEST SOBOLEV TRACE CONSTANT IN THIN DOMAINS

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ABSTRACT. Let  $\Omega \subset \mathbb{R}^N$  be a bounded, smooth domain. We deal with the best constant of the Sobolev trace embedding  $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$  for functions that vanish in a subset  $A \subset \Omega$ , which we call the hole, i.e., we deal with the minimization problem  $S_A = \inf \|u\|_{W^{1,p}(\Omega)}^p / \|u\|_{L^q(\partial\Omega)}^p$  for functions that verify  $u|_A = 0$ . It is known that there exists an optimal hole that minimizes the best constant  $S_A$  among subsets of  $\Omega$  of prescribed volume.

In this paper we look for optimal holes and extremals in thin domains. We find a limit problem (when the thickness of the domain goes to zero), that is a standard Neumann eigenvalue problem with weights and prove that when the domain is contracted to a segment it is better to concentrate the hole in one side of the domain.

## 1. INTRODUCTION

Sobolev inequalities have been studied by many authors and it is by now a classical subject due to their applications in the study of boundary value problems for differential operators. This subject at least goes back to [1], for more references see [5]. In particular, the Sobolev trace inequality has been intensively studied in recent years, see [2, 6, 7, 8, 12, 16, 17, 18], etc.

Let  $\Omega$  be a smooth and bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . For any subcritical exponent  $q$ , that is  $1 \leq q < p_* := \frac{p(N-1)}{N-p}$  if  $1 < p < N$  and  $1 \leq q < \infty$  if  $p \geq N$ , we have the compact embedding  $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$  and hence the following inequality holds:

$$S \|u\|_{L^q(\partial\Omega)}^p \leq \|u\|_{W^{1,p}(\Omega)}^p$$

for all  $u \in W^{1,p}(\Omega)$ . This is known as the Sobolev trace embedding theorem. The best constant for this embedding is the largest  $S$  such that the above inequality holds, that is

$$S = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla u|^p + |u|^p dx}{\left(\int_{\partial\Omega} |u|^q dS\right)^{p/q}}.$$

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Here we are interested in the best Sobolev trace constant for functions that vanish in a subset  $A$  of  $\Omega$ , that we will call the *hole*. That is,

$$S_A = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \left\{ \frac{\int_{\Omega} |\nabla u|^p + |u|^p dx}{\left( \int_{\partial\Omega} |u|^q dS \right)^{p/q}} : u|_A = 0 \right\}.$$

Since the embedding is compact it is easy to prove that there exist extremals for  $S_A$ , see [12]. When  $A$  is closed an extremal  $u$  for  $S_A$  is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega \setminus A, \\ |u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega \setminus A, \\ u = 0 & \text{in } A, \end{cases}$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian and  $\frac{\partial}{\partial \nu}$  is the outer normal derivative. If the extremal is normalized as  $\|u\|_{L^q(\partial\Omega)} = 1$  the Lagrange multiplier  $\lambda$  verifies  $\lambda = S_A$ .

Our main concern in this paper is to look for the behavior of this constant and extremals with respect to  $A$ . Namely we are interested in the optimization of  $S_A$  among subsets  $A$  of  $\Omega$  of a given positive measure. In [13] the existence of an optimal hole  $A^*$  is shown,

**Theorem** ([13], Theorem 1.2) *Given  $0 < \alpha < 1$ . let us define*

$$(1.1) \quad S(\alpha) := \inf_{A \subset \Omega, |A| = \alpha|\Omega|} S_A.$$

*Then, there exists a set  $A^* \subset \Omega$  such that  $|A^*| = \alpha|\Omega|$  and  $S_{A^*} = S(\alpha)$ . Moreover, every corresponding extremal  $u^*$  verifies that  $|\{u^* = 0\}| = \alpha|\Omega|$ .*

A natural question now is what can be said about extremals  $u^*$  and optimal holes  $A^* = \{u^* = 0\}$ . One method to get more information about the best Sobolev trace constant and its corresponding extremals is to consider the limiting problem in thin domains. This can be seen as a *dimension reduction* technique. To this end, let  $N = n + k$  and define the family

$$\Omega_\mu = \left\{ (x, \mu y) : (x, y) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^k \right\}.$$

For small values of  $\mu$ ,  $\Omega_\mu$  is a narrow domain in  $y$  direction.

We will call  $S_\mu(\alpha)$  the constant defined in (1.1) with  $\Omega$  replaced by  $\Omega_\mu$ . Observe that  $S_\mu(\alpha)$  has the following variational characterization

$$S_\mu(\alpha) = \inf_{u \in W^{1,p}(\Omega_\mu) \setminus W_0^{1,p}(\Omega_\mu)} \left\{ \frac{\int_{\Omega_\mu} |\nabla u|^p + |u|^p dx}{\left( \int_{\partial\Omega_\mu} |u|^q dS \right)^{p/q}} : |\{u = 0\}| \geq \alpha|\Omega_\mu| \right\}.$$

As a first approach assume that  $\Omega$  is a product,  $\Omega = \Omega_1 \times \Omega_2$  where  $\Omega_1 \subset \mathbb{R}^n$  and  $\Omega_2 \subset \mathbb{R}^k$ . Then  $\Omega_\mu = \Omega_1 \times \mu\Omega_2$ . As  $\mu$  gets smaller, the domain  $\Omega$  is approaching  $\Omega_1$ , so it is natural to expect that the problem

(1.1) is converging to an optimal design problem in  $\Omega_1$ . It turns out that this is the case. Moreover, the limit problem is

$$(1.2) \quad \bar{S}(\alpha) := \inf_{v \in W^{1,p}(\Omega_1) \setminus \{0\}} \left\{ \frac{\int_{\Omega_1} |\nabla v|^p + |v|^p dx}{\left( \int_{\Omega_1} |v|^q dx \right)^{p/q}} : |\{v = 0\}| \geq \alpha |\Omega_1| \right\}.$$

Observe that the limit problem is no longer a trace problem but an immersion one.

Our first result is as follows,

**Theorem 1.1.** *Let  $\Omega = \Omega_1 \times \Omega_2$ . Given  $0 < \alpha < 1$ , it holds*

$$(1.3) \quad \lim_{\mu \rightarrow 0} \frac{S_\mu(\alpha)}{\mu^{(kq-kp+p)/q}} = \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}} \bar{S}(\alpha).$$

Moreover, if we scale the extremals  $u_\mu$  of  $S_\mu(\alpha)$  to the original domain  $\Omega$  as  $v_\mu(x, y) = u_\mu(x, \mu y)$ ,  $x \in \Omega_1$ ,  $y \in \Omega_2$ , and normalize with

$$\mu \int_{\partial\Omega_1 \times \Omega_2} |v_\mu|^q dS_x dy + \int_{\Omega_1 \times \partial\Omega_2} |v_\mu|^q dx dS_y = 1,$$

then

$$v_\mu \rightarrow v = v(x) \quad \text{strongly in } W^{1,p}(\Omega) \text{ as } \mu \rightarrow \infty,$$

where  $v \in W^{1,p}(\Omega_1)$  is an extremal for (1.2). Finally, if  $A_\mu$  is an optimal hole for  $S_\mu(\alpha)$  and we scale it back to  $\Omega$  as  $\bar{A}_\mu = \{(x, y) : (x, \mu y) \in A_\mu\}$  then, up to subsequences,

$$|\bar{A}_\mu \Delta \bar{A}| \rightarrow 0, \quad \text{as } \mu \rightarrow 0,$$

or, equivalently,

$$\chi_{\bar{A}_\mu} \rightarrow \chi_{\bar{A}} \quad \text{in } L^1(\Omega),$$

where  $\bar{A} = A^* \times \Omega_2$  and  $A^* \subset \Omega_1$  is optimal for (1.2).

**Remark 1.1.** Observe that  $\bar{S}(\alpha)$  is the best constant for the Sobolev embedding  $W^{1,p}(\Omega_1) \hookrightarrow L^q(\Omega_1)$  for functions that vanish in a set of measure at least  $\alpha |\Omega_1|$ .

To see how the best hole looks like for thin domains, we further simplify the problem and consider the simplest case in which we contract the domain to an interval. That is, we consider  $\Omega_1 = (a, b)$ . In this case we can compute the optimal limit constant  $\bar{S}(\alpha)$  in (1.2) and also every optimal hole  $A^*$  with measure  $|A^*| = \alpha(b - a)$ . We have the following result,

**Theorem 1.2.** *The optimal limit constant  $\bar{S}(\alpha)$  is attained only for a hole  $A^* = (a, a + \alpha(b - a))$  or  $A^* = (b - \alpha(b - a), b)$ , that is the best hole is an interval concentrated on one side of the interval  $(a, b)$ . Moreover, the optimal limit constant is given by*

$$\bar{S}(\alpha) = \frac{(2\pi)^p (p-1)}{(2\alpha(b-a)p \sin(\frac{\pi}{p}))^p} + 1.$$

The result for the one-dimensional limiting case in Theorem 1.2 gives us also an idea of how the optimal hole for very thin domains that are almost an interval looks like. It is better to concentrate the hole on one side of the domain.

**Corollary 1.1.** *For  $\mu$  small enough the best hole  $A_\mu$  for the domain  $\Omega_\mu = (a, b) \times \mu\Omega_2$  with measure  $|A_\mu| = \alpha|\Omega_\mu|$  looks like  $A_\mu \simeq (a, a + \alpha(b - a)) \times \mu\Omega_2$  or like  $A_\mu \simeq (b - \alpha(b - a), b) \times \mu\Omega_2$ .*

If  $\Omega_1$  has more than one dimension we are not able to prove a result like Theorem 1.2. For  $n \geq 2$ , extremals for the limiting problem can look different than in the one dimensional case. Even for the unit cube in  $\mathbb{R}^3$  contracted to the two dimensional square in  $\mathbb{R}^2$  the optimal hole is not analogue to its projections onto  $(0, 1)$ . In order to see how an optimal hole looks like in higher dimensions, we consider the unit square with three different holes of measure  $1/2$  in Section 5.

We can also generalize Theorem 1.1 and prove a result for general geometries. In this case we arrive to a problem like (1.2) but with weights that are the volume of the sections and the surface of the boundary of the sections. We have,

**Theorem 1.3.** *Let  $\Omega$  be a bounded and smooth domain in  $\mathbb{R}^N$ . Let  $\Omega_x$  be the  $x$ -section of  $\Omega$  and  $P(\Omega)$  be the projection onto the  $x$  variables, i.e.*

$$\Omega_x := \{y \in \mathbb{R}^k : (x, y) \in \Omega\} \quad \text{and} \quad P(\Omega) := \{x \in \mathbb{R}^n : \Omega_x \neq \emptyset\}.$$

*Then, if we call  $\rho(x) = |\Omega_x|$  and  $\beta(x) = |\partial\Omega_x|$  we have that*

$$\lim_{\mu \rightarrow 0} \frac{S_\mu(\alpha)}{\mu^{(kq - kp + p)/q}} = \bar{S}(\alpha, \rho, \beta),$$

where

$$\bar{S}(\alpha, \rho, \beta) := \inf_{v \in W^{1,p}(P(\Omega), \rho)} \left\{ \frac{\int_{P(\Omega)} (|\nabla v|^p + |v|^p) \rho(x) dx}{\left( \int_{P(\Omega)} |v|^q \beta(x) dx \right)^{p/q}} \right\}$$

with  $|\{v > 0\}| \leq \alpha|P(\Omega)|$ .

Here  $W^{1,p}(P(\Omega), \rho)$  is the weighted Sobolev space,

$$W^{1,p}(P(\Omega), \rho) = \left\{ v : P(\Omega) \rightarrow \mathbb{R} : \int_{P(\Omega)} (|\nabla v|^p + |v|^p) \rho(x) dx < +\infty \right\}.$$

**Organization of the paper.** To simplify and clarify the exposition, we prove in Sections 2 and 3 our main results in the product case. In Section 2 we look at contractions in the product case and prove Theorem 1.1. In Section 3 we deal with the limit problem in one space dimension, i.e. Theorem 1.2. In section 4 we indicate how to modify our arguments to deal with general geometries. Finally, in Section 5, we discuss the difficulties in order

to extend Theorem 1.2 to more than one dimension and illustrate our results with some examples.

## 2. DIMENSION REDUCTION. PROOF OF THEOREM 1.1

In this section we analyze the limit, as  $\mu \rightarrow 0$  of  $S_\mu(\alpha)$ . We consider the case where  $\Omega = \Omega_1 \times \Omega_2$  and leave the extension to more general geometries to the final section.

The following notation will be used

$$Q_\mu(u) = \frac{\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dx \, dy}{\left( \int_{\partial\Omega_\mu} |u|^q \, dS \right)^{p/q}},$$

with  $u \in W^{1,p}(\Omega_\mu) \setminus W_0^{1,p}(\Omega_\mu)$ .

We are interested in the limiting problem for  $S_\mu(\alpha)$  when  $\mu \rightarrow 0$ . For this, let us call  $u_\mu$  an extremal for  $S_\mu(\alpha)$ . The optimal hole is  $A_\mu := \{u_\mu = 0\}$  and  $|A_\mu| = \alpha|\Omega_\mu|$ . We define the rescaled extremals as  $v_\mu(x, y) = u_\mu(x, \mu y)$ . We have that  $v_\mu \in W^{1,p}(\Omega)$ ,  $v_\mu = 0$  a.e. in  $\bar{A}_\mu := \{(x, y) \mid (x, \mu y) \in A_\mu\}$  and, by a simple change of variables,

$$\frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} = \frac{\int_{\Omega} |(\nabla_x v_\mu, \mu^{-1} \nabla_y v_\mu)|^p + |v_\mu|^p \, dx \, dy}{\left( \mu \int_{\partial\Omega_1 \times \Omega_2} |v_\mu|^q \, dS_x \, dy + \int_{\Omega_1 \times \partial\Omega_2} |v_\mu|^q \, dx \, dS_y \right)^{p/q}},$$

where  $\nabla_x u = (u_{x_1}, \dots, u_{x_n})$  and  $\nabla_y u = (u_{y_1}, \dots, u_{y_k})$ . Observe that the rescaled holes verify that  $|\bar{A}_\mu| = \alpha|\Omega|$ . We can assume that the extremals  $u_\mu$  have been chosen so that the rescaled extremals  $v_\mu$  are normalized with

$$(2.1) \quad \mu \int_{\partial\Omega_1 \times \Omega_2} |v_\mu|^q \, dS_x \, dy + \int_{\Omega_1 \times \partial\Omega_2} |v_\mu|^q \, dx \, dS_y = 1.$$

We will also need the following lemma, that has been proven in [10]. We only make a sketch of the proof for the reader's convenience.

**Lemma 2.1** ([10], Lemma 3.1). *Let  $f_n, f : \Omega \rightarrow \mathbb{R}_{\geq 0}$  be measurable functions such that  $f_n(x) \rightarrow f(x)$  a.e. in  $\Omega$ . Suppose that  $|\{f_n = 0\}| \rightarrow |\{f = 0\}|$ . Then*

$$|\{f_n = 0\} \Delta \{f = 0\}| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

*Proof.* By Egoroff's Theorem, given  $\varepsilon > 0$ , there exists a closed set  $C_\varepsilon \subset \Omega$  such that

$$|C_\varepsilon| < \varepsilon \quad \text{and} \quad f_n \rightarrow f \quad \text{uniformly in } \Omega \setminus C_\varepsilon.$$

We call  $E_\varepsilon := \Omega \setminus C_\varepsilon$ . By the uniform convergence, for any  $\delta > 0$ ,

$$(2.2) \quad \{f_n = 0\} \cap E_\varepsilon \subset \{f \leq \delta\} \cap E_\varepsilon,$$

if  $n$  is large enough.

Now,  $\{f = 0\} \setminus \{f_n = 0\} \subset \left( (\{f \leq \delta\} \setminus \{f_n = 0\}) \cap E_\varepsilon \right) \cup C_\varepsilon$ . Therefore, by (2.2), we obtain

$$|\{f = 0\} \setminus \{f_n = 0\}| \leq |\{f \leq \delta\}| - |\{f_n = 0\}| + 2\varepsilon.$$

Taking limit first as  $n \rightarrow \infty$  and then as  $\delta \rightarrow 0$  we obtain, as  $\varepsilon > 0$  is arbitrary,

$$\lim_{n \rightarrow \infty} |\{f = 0\} \setminus \{f_n = 0\}| = 0.$$

The fact that  $\lim_{n \rightarrow \infty} |\{f_n = 0\} \setminus \{f = 0\}| = 0$  follows in the same way.  $\square$

Now, we proceed with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** First, let us prove that  $Q_\mu(u_\mu)/\mu^{(kq-kp+p)/q}$  is bounded independently of  $\mu$ .

To this end we choose a test function depending only on the  $x$  variable, that is  $w(x, y) = f(x) \in W^{1,p}(\Omega)$  with  $|\{x \in \Omega_1 : f(x) = 0\}| = \alpha|\Omega_1|$ . Then,  $|\{w = 0\}| = \alpha|\Omega|$  and

$$\begin{aligned} \frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} &\leq \frac{|\Omega_2| \int_{\Omega_1} |\nabla_x f|^p + |f|^p dx}{\underbrace{\left( \mu|\Omega_2| \int_{\partial\Omega_1} |f|^q dS_x + |\partial\Omega_2| \int_{\Omega_1} |f|^q dx \right)^{p/q}}_{\geq 0}} \\ &\leq \frac{|\Omega_2| \int_{\Omega_1} |\nabla_x f|^p + |f|^p dx}{|\partial\Omega_2|^{p/q} \left( \int_{\Omega_1} |f|^q dx \right)^{p/q}} \leq C \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}}, \end{aligned}$$

with a constant  $C$  independent of  $\mu$ . Observe that  $C$  can be taken to be  $\bar{S}(\alpha)$  minimizing among all possible choices for  $f$ . So we have obtained

$$(2.3) \quad \frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} \leq \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}} \bar{S}(\alpha).$$

Next, we show convergence of  $v_\mu$ . For an extremal  $v_\mu$  normalized by (2.1) it follows that

$$\int_{\Omega} |(\nabla_x v_\mu, \mu^{-1} \nabla_y v_\mu)|^p + |v_\mu|^p dx dy \leq C,$$

and therefore  $v_\mu$  is bounded in  $W^{1,p}(\Omega)$  independently of  $\mu$ . Extracting a subsequence  $\mu_j \rightarrow 0$  we get

$$\begin{aligned} v_{\mu_j} &\rightharpoonup v_0 \quad \text{weakly in } W^{1,p}(\Omega), \\ v_{\mu_j} &\rightarrow v_0 \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

Additionally we get

$$\int_{\Omega} |\mu^{-1} \nabla_y v_\mu|^p dx dy \leq C,$$

where  $C$  is a constant independent of  $\mu$ . Hence

$$\int_{\Omega} |\nabla_y v_\mu|^p dx dy \leq \mu^p C \rightarrow 0 \quad \text{as } \mu \rightarrow 0,$$

and it follows that the limit  $v_0$  does not depend on  $y$ , that is  $v_0 = v_0(x)$ . As  $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ , we further have that  $v_{\mu_j} \rightarrow v_0$  strongly in  $L^q(\partial\Omega)$ .

Considering now the normalized boundary terms (2.1) and taking the limit, we obtain that  $v_0$  verifies

$$\int_{\Omega_1} |v_0|^q dx = \frac{1}{|\partial\Omega_2|}.$$

Finally we want to see what limiting problem  $v_0$  satisfies. We begin by considering what happens to the rescaled holes  $\bar{A}_\mu$  when  $\mu \rightarrow 0$ . As  $\bar{A}_\mu$  are bounded, its characteristic functions  $\chi_{\bar{A}_\mu}$  are bounded in  $L^p(\Omega)$ . Therefore there exists a subsequence,  $\mu_j \rightarrow 0$ , and a function  $\phi \in L^\infty(\Omega)$ ,  $0 \leq \phi \leq 1$  such that

$$\chi_{\bar{A}_{\mu_j}} \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega).$$

So that, in particular, for  $\bar{A} = \{\phi > 0\}$

$$|\bar{A}| \geq \int_{\Omega} \phi = \lim \int_{\Omega} \chi_{\bar{A}_{\mu_j}} = \lim |\bar{A}_{\mu_j}| = \alpha|\Omega|.$$

Since we can always restrict ourselves to nonnegative test functions by changing  $v_0, \phi$  by its absolute value,  $v_0, \phi \geq 0$  and

$$\int_{\Omega} v_0 \phi = \lim \int v_{\mu_j} \chi_{\bar{A}_{\mu_j}} = 0$$

it follows that  $v_0$  vanishes almost everywhere in  $\bar{A}$  and  $|\bar{A}| \geq \alpha|\Omega|$ . Therefore  $|\{v_0 > 0\}| \leq (1 - \alpha)|\Omega|$ .

Now we consider the quotient  $Q_\mu(u_\mu)/\mu^{(kq-kp+p)/q}$  with the rescaled extremals  $v_\mu$  normalized by (2.1). Hence

$$\begin{aligned} \frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} &= \int_{\Omega} |(\nabla_x v_\mu, \mu^{-1} \nabla_y v_\mu)|^p + |v_\mu|^p dx dy \\ &\geq \int_{\Omega} |\nabla_x v_\mu|^p + \underbrace{\mu^{-p} |\nabla_y v_\mu|^p}_{\geq 0} + |v_\mu|^p dx dy \\ &\geq \int_{\Omega} |\nabla_x v_\mu|^p + |v_\mu|^p dx dy. \end{aligned}$$

Taking the limit as  $\mu \rightarrow 0$  we get

$$\begin{aligned}
\liminf_{\mu \rightarrow 0} \frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} &\geq |\Omega_2| \int_{\Omega_1} |\nabla_x v_0|^p + |v_0|^p dx \\
&= \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}} \frac{\int_{\Omega_1} |\nabla_x v_0|^p + |v_0|^p dx}{\left(\int_{\Omega_1} |v_0|^q dx\right)^{p/q}} \\
&\geq \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}} \inf_{\substack{v \in W^{1,p}(\Omega_1) \\ |\{v>0\}| \leq (1-\alpha)|\Omega_1|}} \frac{\int_{\Omega_1} |\nabla_x v|^p + |v|^p dx}{\left(\int_{\Omega_1} |v|^q dx\right)^{p/q}} \\
&= \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}} \bar{S}(\alpha).
\end{aligned}$$

So, combining this with (2.3), we arrive at

$$(2.4) \quad \lim_{\mu \rightarrow 0} \frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} = \frac{|\Omega_2|}{|\partial\Omega_2|^{p/q}} \bar{S}(\alpha)$$

with  $v_0$  is an extremal for the limiting problem. Moreover, since  $v_\mu \rightharpoonup v_0$  weakly in  $W^{1,p}(\Omega)$  and, by (2.4),  $\|v_\mu\|_{W^{1,p}(\Omega)} \rightarrow \|v_0\|_{W^{1,p}(\Omega)}$ , it follows that  $v_\mu \rightarrow v_0$  strongly in  $W^{1,p}(\Omega)$ .

The fact that  $|\bar{A}_\mu \triangle \bar{A}| \rightarrow 0$  as  $\mu \rightarrow 0$  is a consequence of Lemma 2.1. This finishes the proof.  $\square$

### 3. THE LIMIT PROBLEM IN ONE DIMENSION

In this section we investigate the limit problem (1.2) in the one dimensional case. This case is obviously simpler than the higher dimensional one, mainly because two facts: first, the geometry is easier and second, the Sobolev spaces  $W^{1,p}((a, b))$  are contained in the space of continuous functions.

**Proof of Theorem 1.2.** In this case,  $\Omega_1 = (a, b)$ . Now, if  $u \in W^{1,p}((a, b))$  is an extremal for (1.2), then the set  $\{u > 0\}$  is open and  $u$  verifies

$$(3.1) \quad \begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u & \text{in } (a, b) \cap \{u > 0\}, \\ u' = 0 & \text{on } \partial(a, b) \cap \{u > 0\}, \end{cases}$$

where

$$\lambda = \frac{|\partial\Omega_2|}{|\Omega_2|} \bar{S}(\alpha) - 1.$$

So, if we denote by  $A_0 = \{u = 0\}$ , this is a standard eigenvalue problem for the  $p$ -Laplacian with mixed boundary conditions on  $(a, b) \setminus A_0$ .



Now, the problem of computing  $\bar{S}(\alpha)$  can be formulated in terms of optimizing the first eigenvalue of (3.1), i.e.

$$(3.2) \quad \inf_{\substack{A_0 \subset (a,b) \\ |A_0| \geq \alpha(b-a)}} \lambda(A_0).$$

In the following we solve (3.2).

To this end we introduce some general results for the  $p$ -Laplacian eigenvalue problem. The first eigenvalue for the  $p$ -Laplacian on an interval  $I$  with Dirichlet boundary conditions can be explicitly computed, see [3], and is given by

$$(3.3) \quad \lambda_1 = \frac{(2\pi)^p(p-1)}{(\ell(I)p \sin(\frac{\pi}{p}))^p},$$

where  $\ell(I)$  stands for the length of the interval. This formula shows that  $\lambda_1$  is decreasing as the length of the interval  $I$  increases.

Now, as the set  $\{u > 0\}$  is open we can write it as an union of disjoint open intervals

$$\{u > 0\} = \bigcup_{k=1}^{\infty} I_k.$$

Considering problem (3.1) on each interval  $I_k = (a_k, b_k)$  gives us eigenvalues  $\lambda(I_k)$  greater than or equal to the eigenvalues  $\lambda$  of the original problem (3.1). For the location of an interval  $(a_k, b_k)$  there exist two possibilities

- (1)  $(a_k, b_k) \subset (a, b)$  lies within the interval  $(a, b)$ , that is  $a < a_k < b_k < b$  and  $b_k - a_k \leq \alpha(b - a)$ ,
- (2)  $(a_k, b_k) \subset (a, b)$  touches the boundary of  $(a, b)$ , that is  $a_k = a$  and  $b_k \leq a + (1 - \alpha)(b - a)$  or the symmetric case  $a_k \geq b - (1 - \alpha)(b - a)$  and  $b_k = b$ .

We will denote by  $\ell(I_k) = b_k - a_k$  the length of the interval  $I_k$ .

We begin with case (1) and the corresponding eigenvalue problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u & \text{in } (a_k, b_k), \\ u(a_k) = u(b_k) = 0. \end{cases}$$

The first eigenvalue for this problem is given by

$$\lambda_1(I_k) = \frac{(2\pi)^p(p-1)}{(\ell(I_k)p \sin(\frac{\pi}{p}))^p}.$$

If we consider on the other hand case (2), the corresponding eigenvalue problem is

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u & \text{in } (a, b_k), \\ u'(a) = 0, \\ u(b_k) = 0. \end{cases}$$

By reflecting the interval  $(a, b_k)$  via the Neumann boundary we get an equivalent problem as follows

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda |u|^{p-2}u & \text{in } (a - (b_k - a), b_k), \\ u(a - (b_k - a)) = u(b_k) = 0. \end{cases}$$

The first eigenvalue of this problem is

$$\lambda_1(I_k) = \frac{(2\pi)^p(p-1)}{(2\ell(I_k)p \sin(\frac{\pi}{p}))^p}.$$

So  $\lambda(A_0) = \inf_k \lambda_1(I_k)$  is realized as the first eigenvalue of the interval of largest length. That is, if we call

$$\ell_0 = \max\{\ell(I_k) : \bar{I}_k \cap \{a, b\} = \emptyset\}$$

and

$$\ell_1 = \max\{\ell(I_k) : \bar{I}_k \cap \{a, b\} \neq \emptyset\},$$

then we have obtained that

$$\lambda(A_0) = \frac{(2\pi)^p(p-1)}{(\max\{\ell_0, 2\ell_1\}p \sin(\frac{\pi}{p}))^p}.$$

Now, it is obvious that this quantity is minimized among closed sets  $A_0$  of measure  $|A_0| \geq \alpha(b-a)$  when its complement is an interval of length  $\alpha(b-a)$  concentrating on the boundary of  $(a, b)$ , that is,

$$(a, b) \setminus A_0 = \begin{cases} (a, a + \alpha(b-a)) & \text{or} \\ (b - \alpha(b-a), b) \end{cases}$$

and in this case,

$$\lambda(A_0) = \frac{(2\pi)^p(p-1)}{(2\alpha(b-a)p \sin(\frac{\pi}{p}))^p}.$$

This ends the proof.  $\square$

**Proof of Corollary 1.1.** The corollary is a consequence of the convergence of the optimal holes  $A_\mu$  as  $\mu \rightarrow 0$  proved in Theorem 1.2.  $\square$

**Remark 3.1.** In the special case of contracting the unit square  $(0, 1)^2$  in  $\mathbb{R}^2$  to the interval  $(0, 1)$  we have  $\Omega_\mu = (0, 1) \times (0, \mu)$ . In the case  $p = 2$  and  $\alpha = 1/2$  the optimal eigenvalue for the limit problem is given by  $\lambda^* = \pi^2$  with corresponding eigenfunction

$$u^*(x) = \begin{cases} \cos(\sqrt{\lambda^*}x) & \text{in } [0, \frac{1}{2}), \\ 0 & \text{in } [\frac{1}{2}, 1]. \end{cases}$$

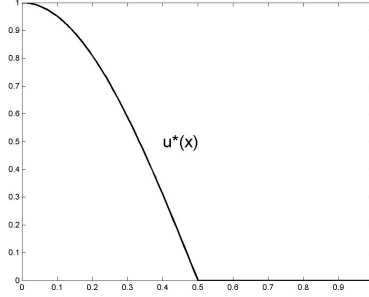


FIGURE 1. The extremal for  $\bar{S}_A$  for the unit interval with an optimal hole of measure  $\frac{1}{2}$ .

#### 4. GENERAL GEOMETRIES

In this section, we show how to modify our previous arguments in order to generalize the results when  $\Omega$  is a general bounded domain in  $\mathbb{R}^N$  and not necessarily a product. As we mentioned in the introduction, what we get as the limit of the best Sobolev trace constant is the best constant of a weighted Sobolev type inequality. To prove our result we use the same ideas as in [11], but we include the main arguments here for the sake of completeness.

Let  $\Omega \subset \mathbb{R}^N = \{(x, y) \mid x \in \mathbb{R}^n, y \in \mathbb{R}^k\}$  be a general bounded smooth domain and we consider  $\Omega_\mu = \{(x, \mu y) \mid (x, y) \in \Omega\}$ .

We want, as in the product case, to write the integrals involved in the quotient  $Q_\mu(u)$  as integrals over the projection of  $\Omega$  over  $y$ . To do this, we define

$$\Omega_x = \{y \in \mathbb{R}^k : (x, y) \in \Omega\}$$

and

$$P(\Omega) = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^k \text{ with } (x, y) \in \Omega\}.$$

For a given function  $u_\mu \in W^{1,p}(\Omega_\mu)$ , we call  $v_\mu(x, y) = u_\mu(x, \mu y)$ . Then  $v_\mu \in W^{1,p}(\Omega)$  and, by Fubini's theorem,

$$\begin{aligned} \int_{\Omega_\mu} |\nabla u_\mu|^p + |u_\mu|^p dx dy &= \mu^k \int_{\Omega} |(\nabla_x v_\mu, \mu^{-1} \nabla_y v_\mu)|^p + |v_\mu|^p dx dy \\ &= \mu^k \int_{P(\Omega)} \left( \int_{\Omega_x} |(\nabla_x v_\mu, \mu^{-1} \nabla_y v_\mu)|^p + |v_\mu|^p dy \right) dx. \end{aligned}$$

Observe that if  $v_\mu = v_\mu(x)$ , we obtain

$$\int_{\Omega_\mu} |\nabla u_\mu|^p + |u_\mu|^p dx dy = \mu^k \int_{P(\Omega)} (|\nabla_x v_\mu|^p + |v_\mu|^p) |\Omega_x| dx.$$

To deal with the boundary, by our assumptions on the domain,  $\partial\Omega$  can be locally described as the graph of a smooth function. So we have that

$$\partial\Omega = \bigcup_{i=1}^l S_i \cup \bigcup_{j=1}^r T_j \quad (\text{disjoint union}),$$

where, after relabelling the variables if necessary,

$$S_i = \{(x, y) : y_1 = h_i(x, y')\}, \quad \text{where } h_i : D_i \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}$$

and the terms labelled  $T_j$  collect the “vertical” parts of the boundary

$$T_j = \{(x, y) : x_1 = g_j(x', y)\}, \quad \text{where } g_j : E_j \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}.$$

As  $T_j$  is “vertical”, we can assume that the parametrization has been taken such that, in the case  $x_1 = g_j(x', y)$ , the function  $g_j$  verifies  $\nabla_y g_j \equiv 0$  in  $E_j$ .

Observe that

$$P(\Omega) = \bigcup_{i=1}^l P(D_i) \quad (\text{not necessarily disjoint}).$$

Hence,  $\partial\Omega_\mu$  is described as

$$\partial\Omega_\mu = \bigcup_{i=1}^l S_{i,\mu} \cup \bigcup_{j=1}^r T_{j,\mu} \quad (\text{disjoint union}),$$

where  $S_{i,\mu} = \{(x, y) : y_1 = \mu h_i(x, \mu^{-1}y')\}$ , with  $h_i : D_i \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  and  $T_{j,\mu} = \{(x, y) : x_1 = g_j(x', \mu^{-1}y)\}$ , with  $g_j : E_j \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ . We have

$$\int_{\partial\Omega_\mu} |u_\mu|^q dS = \sum_{i=1}^l \int_{S_{i,\mu}} |u_\mu|^q dS + \sum_{j=1}^r \int_{T_{j,\mu}} |u|^q dS.$$

In the first case,

$$\begin{aligned} \int_{S_{i,\mu}} |u_\mu|^q dS &= \mu^{k-1} \int_{D_i} |v_\mu|^q \sqrt{1 + \mu^2 |\nabla_x h_i|^2 + |\nabla_{y'} h_i|^2} dx dy' \\ &= \mu^{k-1} \int_{D_i} |v_\mu|^q \omega_{i,\mu} dx dy'. \end{aligned}$$

It is easy to see that  $\omega_{i,\mu} \rightarrow \omega_i$  uniformly in  $D_i$ , where

$$\omega_i = \sqrt{1 + |\nabla_{y'} h_i|^2}.$$

In the second case, using that  $\nabla_y g_j \equiv 0$  in  $E_j$ , we get

$$\begin{aligned} \int_{T_{j,\mu}} |u_\mu|^q dS &= \mu^k \int_{E_j} |v_\mu|^q \sqrt{1 + |\nabla_{x'} g_j|^2 + \mu^{-2} |\nabla_y g_j|^2} dx' dy \\ &= \mu^k \int_{E_j} |v_\mu|^q \sqrt{1 + |\nabla_{x'} g_j|^2} dx' dy \\ &= \mu^k \int_{E_j} |v_\mu|^q \gamma_j dx' dy. \end{aligned}$$

Collecting all these facts, we have that

$$\begin{aligned} \frac{Q_\mu(u_\mu)}{\mu^{(kq-kp+p)/q}} &= \frac{1}{\mu^{(kq-kp+p)/q}} \frac{\int_{\Omega_\mu} |\nabla u_\mu|^p + |u_\mu|^p dx dy}{\left( \int_{\partial\Omega_\mu} |u_\mu|^q dS \right)^{p/q}} \\ &= \frac{\int_{\Omega} |(\nabla_x v_\mu, \mu^{-1} \nabla_y v_\mu)|^p + |v_\mu|^p dx dy}{\left( \sum_{i=1}^l \int_{D_i} |v_\mu|^q \omega_{i,\mu} dx dy' + \mu \sum_{j=1}^r \int_{E_j} |v_\mu|^q \gamma_j dx' dy \right)^{p/q}}. \end{aligned}$$

Once these observations have been made, using the obtained expression for  $Q_\mu(u)$ , all the arguments given in the previous sections follow without major changes.

After performing the computations, we get that the weights that appear in the statement of Theorem 1.3 are given by

$$\rho(x) = |\Omega_x| \quad \text{and} \quad \beta(x) = \sum_{i=1}^l \int_{(D_i)_x} \omega_i dy'.$$

Note that, from the explicit expression of  $\beta(x)$  we get

$$\beta(x) = \sum_{i=1}^l \int_{(D_i)_x} \sqrt{1 + |\nabla_{y'} h_i|^2} dy' = |\partial\Omega_x|.$$

Finally, observe that by our assumptions on  $\partial\Omega$ , the functions  $\omega_i \in L^\infty(D_i)$ , so  $\beta \in L^\infty(P(\Omega))$ .

## 5. THE LIMIT PROBLEM IN HIGHER DIMENSIONS. EXAMPLES

The difference in more than one dimension is the difficulty of computing the optimal hole for the limit problem. Theorem 1.2 is not true for  $n \geq 2$  and extremals for the limiting problem can look different than in the case of an interval. Even for the unit square contracted to two dimensions the optimal hole is not analogue to its projections on  $(0, 1)$ . To see this we consider the unit square with three different holes of measure  $1/2$ , compare Figure 2.

**1st hole** We begin with the unit square  $\Omega = (0, 1)^2$  with the hole  $A_1$  plotted in the first image of Figure 2. The corresponding eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & (0, \frac{1}{2}) \times (0, 1), \\ u_x = 0 & x = 0, \\ u_y = 0 & y = 0 \wedge y = 1, \\ u = 0 & (\frac{1}{2}, 1) \times (0, 1). \end{cases}$$

can be solved explicitly and gives the first eigenvalue  $\lambda = \pi^2$ . Taking  $\lambda = 2S_{A_1} - 1$  we get

$$(5.1) \quad S_{A_1} \approx 5.4348.$$

**2nd hole** Now we consider the same problem with a hole  $A_2$  like in the second picture of Figure 2. This gives an eigenvalue problem for the laplacian on a triangle with mixed boundary conditions. Since the solution of this problem is much more complicated than in the first example we only consider the value of the quotient in  $S_{A_2}$  for a test function. For this let  $u(x, y) = y - x$  and

$$Q(u) = \frac{\int_{(0,1)^2} |u|^2 + |\nabla u|^2 \, dx \, dy}{\int_{(0,1)^2} |u|^2 \, dx \, dy}$$

the quotient of  $S_{A_2}$  for this particular choice of  $u$ . This quotient  $Q(u)$  can be easily computed and gives an upper bound for  $S_{A_2}$

$$(5.2) \quad S_{A_2} \leq Q(u) = 4.$$

The comparison between the value for  $S_{A_1}$  in (5.1) with the upper bound for  $S_{A_2}$  in (5.2) shows that  $A_1$  is not the optimal hole for the limiting problem of the unit square.

**3rd hole** Our last example is the unit square  $\Omega = (0, 1)^2$  with a hole  $A_3 = (0, 1) \setminus B(0, \sqrt{2/\pi})$  and the cone centered in one corner of the square as a test function. The cone  $c(r, \theta)$  written in polar coordinates is given by

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases}$$

with  $r \in (0, \sqrt{2/\pi})$ ,  $\theta \in (0, \frac{\pi}{2})$ . Again the quotient  $Q$  in  $S_{A_3}$  can be computed explicitly and gives an upper bound on  $S_{A_3}$

$$S_{A_3} \leq Q(c(r, \theta)) \approx 7.1730.$$

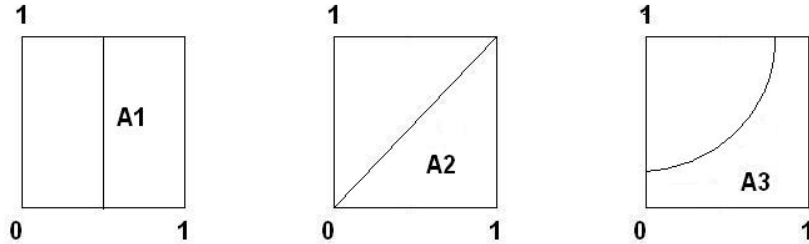


FIGURE 2. Three examples of unit squares with holes (f.l.t.r.)  $A_1$ ,  $A_2$  and  $A_3$  of measure  $1/2$  respectively.

**General geometries** Now we turn our attention to general geometries. To illustrate the influence of the weight in the limiting problem for general domains we consider two examples of domains with nonconstant weights, compare with Figure 3.

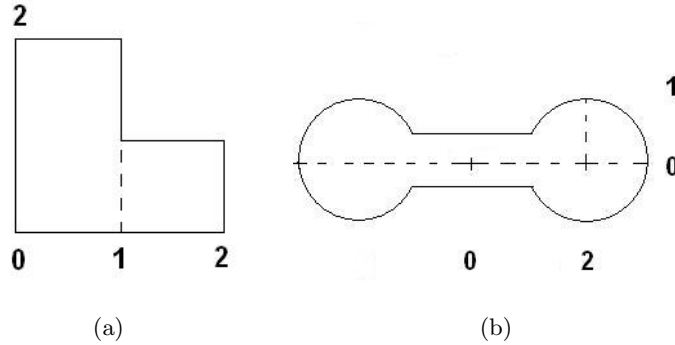


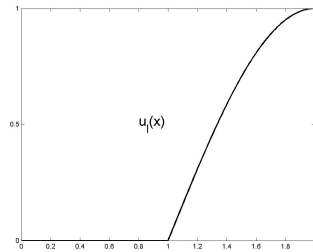
FIGURE 3. Two examples of domains in  $\mathbb{R}^2$  with nonconstant weight.

**Example 1** The corresponding weight for the L-shaped domain in Figure 3(a) is

$$\rho(x) = \begin{cases} 2 & x \in (0, 1) \\ 1 & x \in (1, 2), \end{cases}$$

and the weight for the boundary in the denominator of the limiting quotient  $\bar{Q}$  is constant  $\beta(x) = 2$ . For a fixed hole  $A$  in the interval  $(0, 2)$  and corresponding function  $u$ ,  $\bar{Q}$  can be computed explicitly. Concentrating the hole of measure 1 (that is  $|\Omega_0|/2 = |(0, 2)|/2$ ) on the left side of the interval with

$$u_l(x) = \begin{cases} 0 & x \in (0, 1], \\ -\cos\left(\frac{\pi}{2}x\right) & x \in (1, 2), \end{cases}$$



we get  $\bar{Q}(u_l) = (\frac{\pi^2}{4} + 1)/2$ . Comparing this with the same hole placed on the right side of the interval with  $\bar{Q}(u_r) = \frac{\pi^2}{4} + 1$  we end up with the following inequality

$$Q(u_l) < Q(u_r).$$

This supports the conjecture that in the case of a constant boundary weight  $\beta$  the hole should be placed where  $\rho$  attains its maximum values. Note that a nonsymmetric domain  $\Omega$  like in the case of the L-shaped domain also results

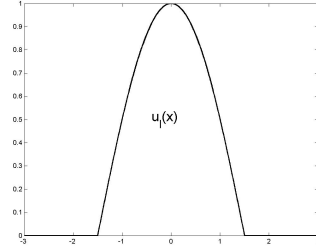
in a nonsymmetric limiting problem that depends on the geometry of the domain.

**Example 2** Now, we consider the example of a domain like the one in Figure 3(b) consisting of two balls of radius 1 and a rectangle of size  $2 \times 1/2$  we want to discuss three cases of locating a hole of measure  $|\Omega_0|/2 = 3$  in the interval  $(-3, 3)$  (that is the limit domain in this case). The weight in the domain  $(-3, 3)$  this time is given by

$$\rho(x) = \begin{cases} 1 & \text{within the support of the rectangle} \\ 2\sqrt{1 - (x \pm 2)^2} & \text{within the support of the two balls.} \end{cases}$$

The boundary weight  $\beta$  is again constant and equal to 2. The following results were computed numerically with *Maple*. We begin with considering two examples again pointing out the role of the weight  $\rho$ . In the first case we concentrate our hole in the middle of the interval on the support of the rectangle,

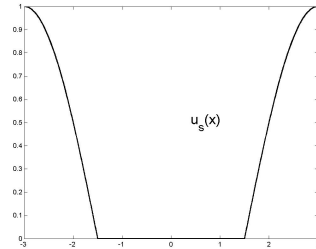
$$u_c(x) = \begin{cases} 0 & x \in (-3, -1.5], \\ \cos(\frac{\pi}{3}x) & x \in (-1.5, 1.5), \\ 0 & x \in [1.5, 3). \end{cases}$$



This gives  $\bar{Q}(u_c) = 1.04831$ .

Now consider the quotient for the hole splitted in two parts each located on one side of the interval

$$u_s(x) = \begin{cases} \cos(\frac{\pi}{3}(x+3)) & x \in (-3, -1.5), \\ 0 & x \in [-1.5, 1.5], \\ \cos(\frac{\pi}{3}(x-3)) & x \in (1.5, 3). \end{cases}$$



This gives  $\bar{Q}(u_s) = 1.73887$ . The role of the weight  $\rho$  appears again, namely that it is better to place the hole where  $\rho$  is large.

To conclude with this example, we now concentrate the hole on one side of the interval. Because our problem is symmetric it does not matter on which side we place it. The quotient gives  $\bar{Q}(u_l) \approx 0.9094$  which is, as follows from our theoretical results, the best that can be obtained.



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