INTERIOR AND UP TO THE BOUNDARY REGULARITY FOR THE FRACTIONAL g-LAPLACIAN: THE CONVEX CASE

JULIÁN FERNÁNDEZ BONDER, ARIEL SALORT* AND HERNÁN VIVAS

Abstract. We establish interior and up to the boundary Hölder regularity estimates for weak solutions of the Dirichlet problem for the fractional $g-$Laplacian with bounded right hand side and g convex. These are the first regularity results available in the literature for integro-differential equations in the context of fractional Orlicz-Sobolev spaces.

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1. Introduction and main results

The aim of this work is to prove interior and up to the boundary Hölder regularity for solutions of the Dirichlet problem for the fractional $g-$Laplacian:

$$\begin{cases} (-\Delta_g)^s u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω a bounded open subset of $\mathbb{R}^n$ with $C^{1,1}$ boundary, $f \in L^\infty(\Omega)$ and $(-\Delta_g)^s$ is the fractional $g-$Laplacian defined in [18]:

$$(-\Delta_g)^s u(x) := \text{p.v.} \int_{\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{dy}{|x-y|^{n+s}} \tag{1.2}$$

with $g = G'$ the derivative of a Young function G (see Section 2 for definitions) and p.v. stands for the principal value.

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Fractional (or nonlocal) operators arise naturally in the study of Lévy processes with jumps, where the infinitesimal generator of a stable pure jump process is given, through the Lévy-Khintchine formula, by a nonlocal operator. These processes have attracted much interest both in the PDE and Probability communities, as they appear to capture accurately the features of a wide range of phenomena in Physics, Finance, Image processing, or Ecology; see [1, 12, 29, 33] and references therein.

Although the development of some aspects of the theory of nonlocal operators from the point of view of Analysis can be traced back several decades, either from a Harmonic Analysis or a potential theoretic approach, it was the seminal program developed by Caffarelli and Silvestre in [7, 8, 9] that opened up the way for a (so to speak) modern regularity theory for elliptic nonlocal equations. Indeed, they developed a regularity theory for fully nonlinear nonlocal uniformly elliptic equations in a parallel fashion to that of the nowadays classic second order theory, cf. [6]. Since then, the subject has been and continues to be a very active research area, and is hardly possible to give a complete account of the developments; the interested reader is referred to [4, 20, 30] further discussions and references.

The interior regularity results in Caffarelli and Silvestre's works relied heavily on a notion of ellipticity that constrains the behavior of the kernels of the operators to have certain smoothness and/or decay properties. These works were extended by Kriventsov [26] and Serra [34] to the class of *rough* kernels, which can be highly oscillatory and irregular and further developments where made by Chang Lara and Dávila [13], where the symmetry assumption on the kernels is dropped. Also, non-translation invariant operators were considered by Jin and Xiong in [37]. A parallel line of work was developed by Ros-Oton and Serra, whom in [31] established sharp interior regularity estimates for general stable operators, i.e. operators in which the spectral measure is not necessarily absolutely continuous with respect to the Lebesgue measure but satisfies a natural ellipticity assumption. It is also worth highlighting the work of Yu [38], where a first step is taken towards the development of a Calderón-Zygmund type theory for nonlinear, nonlocal operators.

Once the interior regularity is understood, the next step is to study the the boundary regularity of solutions, which presents many differences and challenges with respect to the case of local second order equations. Some of the most important advances in that direction are Ros-Oton and Serra's cited paper [31] and its fully nonlinear counterpart [32]. In these works they study Dirichlet problems of the form

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{1.3}$$

with L a stable operator. Their main results are that if Ω is $C^{1,1}$ and $f \in L^\infty(\Omega)$, then $u/d^s \in C^{s-\varepsilon}(\overline{\Omega})$ for all $\varepsilon > 0$ for the linear case and if Ω is $C^{2,\gamma}$, $a \in C^{1,\gamma}(S^{n-1})$, then if $f \in C^\gamma(\overline{\Omega})$ then $u/d^s \in C^{\gamma+s}(\overline{\Omega})$ for $\gamma \in (0,s)$ whenever $\gamma+s$ is not an integer for the nonlinear case. Here $d = d(x)$ is the distance to $\partial\Omega$ and a is the spectral measure of the operator.

On the other hand, using a rather different approach Grubb proved in [21, 22], for a problem like (1.3), that if Ω is C^∞ and $a \in C^\infty(S^{n-1})$, then

$$f \in C^\gamma(\overline{\Omega}) \quad \Longrightarrow \quad u/d^s \in C^{\gamma+s}(\overline{\Omega}) \qquad \text{for all } \gamma > 0,$$

whenever $\gamma+s \notin \mathbb{Z}$ for elliptic pseudodifferential operators satisfying the s-transmission property. In particular, $u/d^s \in C^\infty(\overline{\Omega})$ whenever $f \in C^\infty(\overline{\Omega})$. Furthermore, when $s+\gamma$ is an integer, more information is given in [22] in terms of Hölder-Zygmund spaces C^k_*.

An common feature of all of the above cited papers is that they deal with the *uniformly elliptic* case, in which, roughly speaking, the operator under study lies between two multiples of the *fractional Laplacian*:

$$(-\Delta)^s u(x) \cong \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy$$

arguably the most canonical example of nonlocal operators. The archetypal example of a *degenerate elliptic* operator is given by the *fractional $p-$Laplacian*:

$$(-\Delta)^s_p u(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} dy, \quad p > 2.$$

This operator arises, for instance, when studying minimizers of the Gagliardo seminorm

$$[u]_{s,p} := \iint \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dxdy \tag{1.4}$$

which are distinctive of the well-known fractional Sobolev spaces, see for instance [17].

Such scenario has been investigated thoroughly in the last years. Just to cite some relevant examples, we point out the work by Di Castro, Kuusi and Palatucci, that proved basic Hölder regularity following De Giorgi's approach in [15] and a Harnack inequality in [16]. Higher Hölder estimates were then obtained by Brasco, Lindgren and Schikorra and higher integrability type results by Brasco and Lindgren [2]. The eigenvalue problem was also successfully addressed by Lindgren and Lindqvist [28]. All of these papers concern the interior regularity of solutions; regarding global regularity, the most important work available seems to be the paper by Iannizzotto, Mosconi and Squassina on global Hölder regularity for the fractional $p-$Laplacian [24], which has been a mayor source of inspiration for our manuscript.

A rather natural question of interest is to replace the power growth in (1.4) by another type of behavior; this immediately gives way to consider the fractional order Orlicz-Sobolev spaces, introduced by the first and the second author in [18]. These spaces have received a great amount of attention in the last years, either in the study of their structural properties [10, 11, 5] or the PDE questions that arise such as eigenvalue problems [35, 36] and Pólya-Szegö type results [14].

However, to the best of the authors' knowledge *there are no results at all available in the literature dealing with regularity issues in this context.* The main goal of this paper is, therefore, to fill that gap. The main result we prove is global (i.e. interior and up to the boundary) Hölder regularity for solutions of (1.1). This is achieved under a rather natural structural assumption on the smoothness of the boundary of Ω and a structural "ellipticity" assumption on g given in terms of two constants λ and Λ (see equation (2.1) below). The result is stated as follows:

Theorem 1.1. *There exist $\alpha \in (0, s]$ depending only on n, s, Ω, Λ and λ such that for any weak solution $u \in W_0^{s,G}(\Omega)$ of (1.1), $u \in C^\alpha(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C \tag{1.5}$$

where C is a constant depending on $s, n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}$ and Ω.

Remark 1.2. In order to have the interior regularity, there is no need to require the so-called *complementary condition* $u = 0$ in $\mathbb{R}^n \setminus \Omega$. As is customary in the nonlocal literature, it suffices for u to verify $(-\Delta_g)^s u = f$ in Ω and some integrability conditions at infinity. See Section 3 (and Theorem 3.4) for a precise statement.

The strategy of the proof of Theorem 1.1 is divided in two steps:

(1) first tackle the interior regularity of solutions. This is done by first proving an appropriate weak Harnack inequality and then an oscillation decay argument;
(2) the boundary regularity, in turn, is achieved by constructing barrier functions that behave like the distance function close to $\partial\Omega$; such a construction is possible due to the fact that the one dimensional profile $(x_n)_+^s$ satisfies $(-\Delta_g)^s(x_n)_+^s = 0$ so that straightening the boundary gives that $(-\Delta_g)^s d^s$ is bounded.

The rest of the paper is organized as follows: in Section 2 we give the preliminary notions and definitions that will be used throughout the paper. In Section 3 we prove a weak Harnack inequality, from which an oscillation decay results and the corresponding interior regularity estimates follow. In Section 4 we investigate the behavior near the boundary of solutions to (1.1) and in particular we prove that they are controlled by the distance function in the whole Ω; putting together the results from Sections 3 and 4, the global Hölder regularity follows and Theorem 1.1 is proven in Section 5. Finally, in the Appendix we gather, for the reader's convenience, the proof of some technical results that are used in the main body of the paper.

2. Preliminaries

2.1. **Young functions.** An application $G\colon [0,\infty) \longrightarrow [0,\infty)$ is said to be a *Young function* if it admits the integral representation $G(t) = \int_0^t g(\tau)\, d\tau$, where the right continuous function g defined on $[0,\infty)$ has the following properties:

$$g(0) = 0, \quad g(t) > 0 \text{ for } t > 0, \tag{g_1}$$

$$g \text{ is nondecreasing on } (0,\infty) \tag{g_2}$$

$$\lim_{t\to\infty} g(t) = \infty. \tag{g_3}$$

From these properties it is easy to see that a Young function G is continuous, nonnegative, strictly increasing and convex on $[0,\infty)$. Further, we recall that we may extend g to the whole $\mathbb{R}$ in an odd fashion: for $t<0$ $g(t) = -g(-t)$.

We will consider the class of Young functions such that $g = G'$ is an absolutely continuous function that satisfies the condition

$$1 < \lambda \leq \frac{tg'(t)}{g(t)} \leq \Lambda < \infty. \tag{2.1}$$

This condition was first considered in the seminal work of G. Lieberman [27] and is the analogous to the ellipticity condition in the linear theory as it will be apparent later on.

This condition implies that $g(t)$ is bounded between powers. In fact, (2.1) implies that

$$c_1 t^\lambda \leq g(t) \leq c_2 t^\Lambda, \tag{2.2}$$

for $t \geq 0$ and some constants $c_1, c_2 > 0$.

By a simple integration, it is easy to check that condition (2.1) implies that G verifies

$$2 < p^- \leq \frac{tg(t)}{G(t)} \leq p^+ < \infty, \tag{2.3}$$

where $p^- = \lambda + 1$ and $p^+ = \Lambda + 1$. Hence, we readily obtain

$$c_1 \frac{t^{p^-}}{p^-} \leq G(t) \leq c_2 \frac{t^{p^+}}{p^+} \qquad \text{for } t \geq 0. \tag{2.4}$$

In [25, Theorem 4.1] it is shown that the upper bound in (2.1) (or in (2.3)) is equivalent to the so-called Δ_2 *condition (or doubling condition)*, namely

$$g(2t) \leq 2^\Lambda g(t), \qquad G(2t) \leq 2^{p^+} G(t) \qquad t \geq 0. \tag{Δ_2}$$

Further, since (2.1) implies $G''(t) > 0$ we have that G is a convex function. For for our purposes we will make the further assumption on g:

$$g \text{ is a convex function on } (0,\infty). \tag{g_4}$$

We point out that this is analogous of dealing with the degenerate case $p \geq 2$ for the fractional $p-$Laplacian.

Throughout the paper, a constant will be called *universal* if it depends only on n, s, λ and Λ. Also, from now on, and until the end of the paper assumption (g_4) will be enforced.

The convexity of G and its derivative in particular gives the following:

Lemma 2.1. *When $\alpha \in [0,1]$ and $t \geq 0$*

$$G(\alpha t) \leq \alpha G(t), \qquad g(\alpha t) \leq \alpha g(t)$$

and for $\alpha \geq 1$ and $t \geq 0$

$$G(\alpha t) \geq \alpha G(t) \qquad g(\alpha t) \geq \alpha g(t).$$

More generally, for any, $\alpha, t \geq 0$

$$\min\{\alpha^\lambda, \alpha^\Lambda\} g(t) \leq g(\alpha t) \leq \max\{\alpha^\lambda, \alpha^\Lambda\} g(t) \tag{2.5}$$

and

$$G(t)\min\{\alpha^{p^-}, \alpha^{p^+}\} \leq G(\alpha t) \leq G(t)\max\{\alpha^{p^-}, \alpha^{p^+}\}. \tag{2.6}$$

The proof of this lemma is elementary and the reader can see a simple proof in [19, Lemma 2.1].

2.2. **Fractional Orlicz-Sobolev spaces.** Recall that for a Young function G the Orlicz spaces are given by

$$L^G(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable}\colon \int_\Omega G(u(x))\,dx < \infty \right\}.$$

These are Banach spaces under the assumption that condition (Δ_2) holds.

Given a fractional parameter $s \in (0,1)$ we will work with the fractional Orlicz-Sobolev spaces

$$W^{s,G}(\Omega) := \left\{ u \in L^G(\Omega)\colon D_s u \in L^G(\Omega \times \Omega, d\mu) \right\}$$

where

$$d\mu = \frac{dxdy}{|x-y|^n}$$

and

$$D_s u(x,y) := \frac{u(x) - u(y)}{|x-y|^s} \tag{2.7}$$

denotes the $s-$Hölder quotient.

Over the space $W^{s,G}(\Omega)$ we define the Luxemburg type norm

$$\|u\|_{s,G,\Omega} := \|u\|_{G,\Omega} + [u]_{s,G,\Omega}, \tag{2.8}$$

where

$$\|u\|_{G,\Omega} := \inf\left\{ \lambda > 0\colon \int_\Omega G\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\}$$
$$[u]_{s,G,\Omega} := \inf\left\{ \lambda > 0\colon \iint_{\Omega\times\Omega} G\left(\frac{D_s u}{\lambda}\right) d\mu \leq 1 \right\}.$$

Given $\Omega \subset \mathbb{R}^n$ bounded we define

$$\widetilde{W}^{s,G}(\Omega) := \left\{ u \in L^G_{loc}(\mathbb{R}^n)\colon \exists U \supset\supset \Omega \text{ s.t. } \|u\|_{s,G,U} + \int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right) \frac{dx}{(1+|x|)^{n+s}} < \infty \right\}.$$

If Ω is unbounded

$$\widetilde{W}^{s,G}_{\rm loc}(\Omega) := \{u \in L^G_{loc}(\mathbb{R}^n)\colon u \in \widetilde{W}^{s,G}(\Omega') \text{ for any bounded } \Omega' \subset \Omega\}.$$

Observe that, when $u \in L^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} &\leq C(\|u\|_\infty)\int_0^\infty g\left(\frac{1}{(1+r)^s}\right)\frac{dr}{(1+r)^{1+s}}\\ &\leq C(\|u\|_\infty)G(1)\int_0^\infty (1+r)^{-s-1}\,dr < \infty,\end{aligned}$$

where we have used (Δ_2), (2.1) and (g_2). Observe that a similar estimate can be obtained for functions that are not bounded but grow in a controlled way. For instance, a similar bound is obtained if u verifies

$$|u(x)| \leq C(1+|x|)^{s+a}, \quad \text{for } a < \frac{s}{\Lambda}.$$

Further, we define the spaces

$$\begin{aligned} W^{s,G}_0(\Omega) &:= \left\{u \in W^{s,G}(\mathbb{R}^n)\colon u = 0 \text{ in } \Omega^c\right\},\\ W^{-s,\tilde{G}}(\Omega) &:= \left(W^{s,G}_0(\Omega)\right)^* \qquad \text{(the topological dual space)}\end{aligned}$$

where $\tilde{G}$ denotes the complementary Young function of G is defined as

$$\tilde{G}(t) := \sup\{tw - G(w)\colon w > 0\}. \tag{2.9}$$

For all measurable $u\colon \mathbb{R}^n \to \mathbb{R}$ the *nonlocal tail* centered at $x \in \mathbb{R}^n$ is defined as

$$\mathrm{Tail}_g(u;x,1) = g^{-1}\left(\int_{B_1^c(x)} g\left(\frac{u(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\right),$$

$$\mathrm{Tail}_g(u;x,R) = g^{-1}\left(R^s\int_{B_R^c(x)} g\left(R^s\frac{u(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\right).$$

When $x=0$ we just write $\mathrm{Tail}_g(u;x,R) = \mathrm{Tail}_g(u;R)$. Moreover, then $G(t)=t^p$ for some $p \geq 2$ we write

$$\mathrm{Tail}_p(u;x,R) = \left(R^{sp}\int_{B_R^c(x)} \frac{|u(y)|^{p-1}}{|x-y|^{n+sp}}\,dy\right)^{\frac{1}{p-1}}$$

and $\mathrm{Tail}_p(u;x,R) = \mathrm{Tail}_p(u;R)$ when $x=0$.

2.3. **Notions of solutions.** In this section we give the appropriate notions of solutions that will be used. We start with the definition of weak solution.

Definition 2.2. *Let* Ω *be a domain in* $\mathbb{R}^n$*,* $u \in \widetilde{W}^{s,G}(\Omega)$ *and* $f \in W^{-s,\tilde{G}}(\Omega)$*. We say that* u *is a weak subsolution of* $(-\Delta_g)^s u = f$ *in* Ω *if for any test function* $\varphi \in W_0^{s,G}(\Omega)$ *satisfying* $\varphi \geq 0$ *a.e. in* Ω *there holds*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu \leq \int_\Omega f\varphi\,dx. \tag{2.10}$$

We say that u *is a supersolution if* $-u$ *is a subsolution and that* u *is a solution if it is both a sub and a supersolution. In particular, if* u *is a solution*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu = \int_\Omega f\varphi\,dx \tag{2.11}$$

any test function $\varphi \in W_0^{s,G}(\Omega)$*.*

If Ω *is unbounded,* $u \in \widetilde{W}^{s,G}_{loc}(\Omega)$ *is a weak subsolution, supersolution or solution of* $(-\Delta_g)^s u = f$ *if* (2.10)-(2.11) *hold for any bounded open set* $\widetilde{\Omega} \subset \Omega$*.*

Observe that Lemma B.1 implies that the integrals in (2.10) and (2.11) are well defined.

Next, we define pointwise and strong solutions:

Definition 2.3. *Let* Ω *be a domain in* $\mathbb{R}^n$*,* $u \in \widetilde{W}^{s,G}_{loc}(\Omega)$ *and* f *a measurable function in* Ω*. We say that* u *is a pointwise subsolution of* $(-\Delta_g)^s u = f$ *in* Ω *if the limit*

$$\limsup_{\varepsilon\to 0^+} 2\int_{B^c_\varepsilon(x)} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}} \leq f(x) \tag{2.12}$$

for almost every Lebesgue point of u *in* Ω *(in particular, a.e. in* Ω*). We say that* u *is a pointwise supersolution if* $-u$ *is a subsolution and that* u *is a solution if it is both a sub and a supersolution.*

Moreover, if $f \in L^1_{loc}(\Omega)$ *we say that* u *is a strong subsolution of* $(-\Delta_g)^s u = f$ *if the limit* (2.12) *holds in* $L^1_{loc}(\Omega)$*. Analogous definitions hold for strong supersolution and solutions.*

3. Weak Harnack and Hölder regularity

In this section we prove a weak Harnack inequality and, as a consequence, get interior C^α estimates for solutions of (1.1) (see Corollary 3.6). We start with the following technical Lemma, that gives control on the fractional $g-$Laplacian of a bounded C^2 function:

Lemma 3.1. *Let* $\varphi \in C^2(\mathbb{R}^n)$ *such that* $\|\varphi\|_{C^2(\mathbb{R}^n)} < \infty$*. Then*

$$|(-\Delta_g)^s\varphi(x)| \leq K,$$

where K *is a (computable) positive constant depending on* $\|\varphi\|_{C^2(\mathbb{R}^n)}$*,* n*,* s*,* g *and* g'*.*

Proof. Let $\varphi \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be fixed. Denote

$$D_z\varphi(x) := \varphi(x) - \varphi(x+z), \qquad D_z^2\varphi(x) := \varphi(x+z) - 2\varphi(x) + \varphi(x-z).$$

Let us write

$$\begin{aligned}(-\Delta_g)^s\varphi(x) &= \left(\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x-y|<1} + \int_{|x-y|\geq 1}\right) g\left(\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}} \\ &:= \lim_{\varepsilon\to 0^+}(a_\varepsilon) + (b).\end{aligned}$$

(Notice that the smoothness and boundedness of φ already make $(-\Delta_g)^s\varphi(x)$ well defined.)

Taking into account that g is nondecreasing we readily get that

$$(b) \leq g(2\|\varphi\|_\infty)\int_{|x-y|\geq 1}|x-y|^{-n-s}\,dy = \frac{n\omega_n}{s}g(2\|\varphi\|_\infty).$$

Expression (a_ε) can be rewritten as

$$\begin{aligned}(a_\varepsilon) &= \int_{\varepsilon<|z|<1} g\left(\frac{D_z\varphi(x)}{|z|^s}\right)\frac{dy}{|z|^{n+s}} \\ &= \int_{\varepsilon<|z|<1} g\left(\frac{D_{-z}\varphi(x)}{|z|^s}\right)\frac{dy}{|z|^{n+s}} \\ &= \frac{1}{2}\int_{\varepsilon<|z|<1}\left(g\left(\frac{D_z\varphi(x)}{|z|^s}\right) + g\left(\frac{D_{-z}\varphi(x)}{|z|^s}\right)\right)\frac{dy}{|z|^{n+s}} \\ &= -\frac{1}{2}\int_{\varepsilon<|z|<1}(\psi(1)-\psi(0))\frac{dz}{|z|^{n+s}} \\ &= -\frac{1}{2}\int_{\varepsilon<|z|<1}\int_0^1\psi'(t)\,dt\frac{dz}{|z|^{n+s}}\end{aligned} \tag{3.1}$$

where we have denoted

$$\psi(t) = g\left(\frac{(1-t)D_z\varphi(x) - tD_{-z}\varphi(x)}{|z|^s}\right).$$

Since the derivative of ψ can be computed to be

$$\psi'(t) = g'\left(\frac{(1-t)D_z\varphi(x) - tD_{-z}\varphi(x))}{|z|^s}\right)\cdot\frac{D_z^2\varphi(x)}{|z|^s},$$

and since we have $|D_z\varphi| \leq \|\nabla\varphi\|_\infty|z|$, $|D_z^2\varphi| \leq \|D^2\varphi\|_\infty|z|^2$ we get

$$|\psi'(t)| \leq g'(2\|\nabla\varphi\|_\infty|z|^{1-s})\|D^2\varphi\|_\infty|z|^{2-s},$$

where we have used that g' increasing due to (g_4). Then, expression (3.1) can be bounded as follows

$$\begin{aligned}(a_\varepsilon) &\leq \int_{\varepsilon<|z|<1} g'(2\|\nabla\varphi\|_\infty|z|^{1-s})\|D^2\varphi\|_\infty|z|^{2-n-2s}\,dz\\ &\leq g'(2\|\nabla\varphi\|_\infty)\|D^2\varphi\|_\infty n\omega_n\int_\varepsilon^1 r^{1-2s}\,ds\\ &= g'(2\|\nabla\varphi\|_\infty)\|D^2\varphi\|_\infty n\omega_n\frac{1-\varepsilon^{2(1-s)}}{2(1-s)}.\end{aligned}$$

Finally, combining the bounds for (a_ε) and (b) we get that

$$|(-\Delta_g)^s\varphi(x)| \leq \frac{n\omega_n}{s}g(2\|\varphi\|_\infty) + \frac{n\omega_n}{2(1-s)}g'(2\|\nabla\varphi\|_\infty)\|D^2\varphi\|_\infty$$

as desired. □

We are now in position to prove our weak Harnack inequality.

Theorem 3.2 (Weak Harnack inequality). *If $u \in \widetilde{W}^{s,G}(B_{R/3})$ satisfies weakly*

$$\begin{cases}(-\Delta_g)^s u \geq -K & \text{in } B_{R/3}\\ u \geq 0 & \text{in } \mathbb{R}^n\end{cases} \tag{3.2}$$

for some $K \geq 0$, then there exists universal $\sigma \in (0,1)$, and an explicit constant $C_0 > 0$ such that

$$\inf_{B_{R/4}} u \geq \sigma R^s g^{-1}\left(\fint_{B_R\setminus B_{R/2}} g(R^{-s}|u|)\,dx\right) - R^s g^{-1}\left(C_0K\right).$$

Proof. We prove first the result for $R = 1$ and then get (3.2) by scaling from Lemma C.1.

Let $\varphi \in C^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^n$, $\varphi = 1$ in $B_{1/4}$ and $\varphi = 0$ in $B^c_{1/3}$. It follows that $\|\varphi\|_\infty \leq 1$ and we can assume that $\|\nabla\varphi\|_\infty \leq 2$. Then, in light of Lemma 3.1

$$|(-\Delta_g)^s\varphi| \leq C \quad \text{in } B_{1/3} \tag{3.3}$$

for some C depending of $\|\varphi\|_{C^2}$, g' and s. We define

$$L = g^{-1}\left(\fint_{B_1\setminus B_{1/2}} g(|u|)\,dx\right)$$

and for any $\sigma > 0$ set $w = \sigma L\varphi + u\chi_{B_1\setminus B_{1/2}}$. Then, in $B_{1/3}$ we can compute, by means of Lemma C.5,

$$(-\Delta_g)^s w(x) = (a) + (b),$$

where

$$(a) = (-\Delta_g)^s(\sigma L\varphi)(x)$$

and

$$(b) = 2\int_{B_1\setminus B_{1/2}} \left(g\left(\frac{\sigma L\varphi(x)-u(y)}{|x-y|^s}\right) - g\left(\frac{\sigma L\varphi(x)}{|x-y|^s}\right)\right)\frac{dy}{|x-y|^{n+s}}.$$

Expression (a) can be bounded by using Lemma 3.1, (Δ_2) and (2.1) as

$$\begin{aligned}(a) &\leq \frac{n\omega_n}{s} g(2\|\sigma L\varphi\|_\infty) + g'(2\|\nabla\sigma L\varphi\|_\infty)\|D^2\sigma L\varphi\|_\infty \frac{n\omega_n}{1-s}\\ &\leq C_1(n,s,\Lambda)(g(\sigma L)+g'(\sigma L)\sigma L)\\ &\leq C_1 g(\sigma L).\end{aligned}$$

For expression (b), by applying Lemma A.1, we get

$$\begin{aligned}(b) &\leq -2^{2-\Lambda}\int_{B_1\setminus B_{1/2}} g\left(\frac{|u(x)|}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\\ &\leq -2^{1-\Lambda}\int_{B_1\setminus B_{1/2}} g\left(|u(x)|\right)\,dy\\ &\leq -2^{2-\Lambda} n\omega_n(1-\tfrac{1}{2^n})g(L)\\ &\leq -C_2(n,s,\Lambda)g(L).\end{aligned}$$

Assume that $\sigma<1$, then from the bounds for (a) and (b) we get

$$(-\Delta_g)^s w(x) \leq C_1\sigma g(L) - C_2 g(L).$$

Then, if we further assume

$$\sigma < \min\left\{1, \frac{C_2}{2C_1}\right\},$$

we get the upper estimate

$$(-\Delta_g)^s w(x) \leq -\frac{C_2}{2} g(L) \quad \text{ in } B_{1/3}. \tag{3.4}$$

We distinguish two cases:

- if $L\leq g^{-1}\left(\frac{2K}{C_2}\right)$, then

$$\inf_{B_{1/4}} u \geq 0 \geq \sigma L - g^{-1}\left(\frac{2K}{C_2}\right),$$

- if $L > g^{-1}\left(\frac{2K}{C_2}\right)$, then, from (3.4)

$$(-\Delta_g)^s w(x) \leq -K \leq (-\Delta_g)^s u \quad \text{ in } B_{1/3},$$

moreover, by construction $w=\chi_{B_1\setminus B_{1/3}} u \leq u$ in $B_{1/3}^c$. In light of the comparison principle stated in Proposition C.4, the last two relations imply that

$$\inf_{B_{1/4}} u \geq \sigma L \geq \sigma L - g^{-1}\left(\frac{2K}{C_2}\right)$$

and the proof concludes. □

Next we extend Theorem 3.2 to supersolutions which are only nonnegative in a ball.

Lemma 3.3. *There exists* $\sigma \in (0,1)$, $C > 0$, *and for all* $\varepsilon > 0$ *a constant* $C_\varepsilon > 0$ *such that, if* $u \in \widetilde{W}^{s,G}(B_{R/3})$ *satisfies*

$$\begin{cases} (-\Delta_g)^s u \geq -K & \text{in } B_{R/3} \\ u \geq 0 & \text{in } B_R \end{cases}$$

for some $K \geq 0$, *then*

$$\inf_{B_{R/4}} u \geq R^s \sigma g^{-1}\left(\fint_{B_R \setminus B_{R/2}} g(R^{-s}u)\,dx \right) - \tilde{C} R^s g^{-1}(K) - C_\varepsilon \operatorname{Tail}_g(u_-;R) - \varepsilon \sup_{B_R} u.$$

Proof. We prove first the result for $R = 1$ and then the result follows for any $R > 0$ by scaling from Lemma C.1.

We apply Lemma C.5 to $v = u_-$, $u + v = u_+$ and $\Omega = B_{1/3}$. Then, in the weak sense in $B_{1/3}$ we have that

$$\begin{aligned}
(-\Delta_g)^s u_+(x) &= (-\Delta_g)^s u(x) + 2\int_{B_{1/3}^c} \left[g\left(\frac{u(x) - u_+(y)}{|x-y|^s} \right) - g\left(\frac{u(x)-u(y)}{|x-y|^s} \right) \right] \frac{dy}{|x-y|^{n+s}} \\
&\geq -K + 2\int_{u<0} \left[g\left(\frac{u(x)}{|x-y|^s} \right) - g\left(\frac{u(x)-u(y)}{|x-y|^s} \right) \right] \frac{dy}{|x-y|^{n+s}} \\
&\geq -K + C\int_{u<0} \left[g\left(\frac{u(x)}{|y|^s} \right) - g\left(\frac{u(x)-u(y)}{|y|^s} \right) \right] \frac{dy}{|y|^{n+s}}
\end{aligned}$$

where in the last inequality we have used that $|x-y| > \frac{2}{3}|y|$.

By using Lemma A.3 we have that, for any $\theta > 0$ there exists $C_\theta > 0$ such that, given that g is odd

$$g\left(\frac{u(x)}{|y|^s} \right) - g\left(\frac{u(x)-u(y)}{|y|^s} \right) \geq c_\theta g\left(\frac{u(x)}{|y|^s} \right) + C_\theta g\left(\frac{u(y)}{|y|^s} \right)$$

with $c_\theta = 1 - (1-\theta)^\Lambda < 1$. Then

$$\begin{aligned}
(-\Delta_g)^s u_+(x) &\geq -K - \int_{u<0} g\left(\frac{u(x)}{|y|^s} \right) \frac{dy}{|y|^{n+s}} - C_\theta \int_{u<0} g\left(\frac{u(y)}{|y|^s} \right) \frac{dy}{|y|^{n+s}} \\
&\geq -K - \int_{u<0} g\left(\frac{\sup_{B_1} u}{|y|^s} \right) \frac{dy}{|y|^{n+s}} - C_\theta g(\operatorname{Tail}_g(u_-;1)) \\
&\geq -K - g\Big(\sup_{B_1} u \Big) \int_{u<0} \frac{dy}{|y|^{n+s(\Lambda+1)}} - C_\theta g(\operatorname{Tail}_g(u_-;1)) \\
&\geq -K - C(\Lambda) g\Big(\sup_{B_1} u \Big) - C_\theta g(\operatorname{Tail}_g(u_-;1)) \\
&:= -\tilde{K}.
\end{aligned}$$

Now, we can apply Theorem 3.2 to u_+,

$$\inf_{B_{1/4}} u \geq \sigma g^{-1}\left(\fint_{B_1\setminus B_{1/2}} g(u)\,dx\right) - g^{-1}\left(C_0\tilde{K}\right).$$

Observe that from Lemma A.4, for $\theta < \frac{\varepsilon^\lambda}{4CC_0} \leq 1$, we get

$$\begin{aligned} g^{-1}\left(C_0\tilde{K}\right) &\leq 2^{\frac{2}{\lambda}}\left[g^{-1}(C_0K) + g^{-1}\Big(C_0C\theta g\Big(\sup_{B_1} u\Big)\Big) + g^{-1}(C_0C_\theta g(\mathrm{Tail}_g(u_-;1)))\right] \\ &\leq 2^{\frac{2}{\lambda}}\Big[\max\{1, C_0^{\frac{1}{\lambda}}\}g^{-1}(K) \;+\; \max\{1,(C_0C\theta)^{\frac{1}{\lambda}}\}\sup_{B_1} u \\ &\qquad + \max\{1,(C_0C_\theta)^{\frac{1}{\lambda}}\}\,\mathrm{Tail}_g(u_-;1)\Big] \\ &\leq \tilde{C}g^{-1}(K) + C_\varepsilon\,\mathrm{Tail}_g(u_-;1) + \varepsilon\sup_{B_1} u. \end{aligned}$$

Therefore

$$\inf_{B_{1/4}} u \geq \sigma g^{-1}\left(\fint_{B_1\setminus B_{1/2}} g(u)\,dx\right) - \tilde{C}g^{-1}(K) - C_\varepsilon\,\mathrm{Tail}_g(u_-;1) - \varepsilon\sup_{B_1} u$$

as required. □

As a consequence of the weak Harnack inequality, we get the following oscillation decay result:

Theorem 3.4. *There exist $\alpha \in (0,1)$ and a universal constant $C > 0$ with the following property: if $R_0 \in (0,1)$ and $u \in \widetilde{W}^{s,G}(B_{R_0}) \cap L^\infty(B_{R_0})$ satisfies*

$$|(-\Delta_g)^s u| \leq K$$

weakly in B_{R_0} and that $\mathrm{Tail}_{p^+}(u;R_0) < \infty$, then for all $r \in (0,R_0)$ it holds that

$$\operatorname*{osc}_{B_r} u \leq C(R_0^{s-\alpha}g^{-1}(K) + R_0^{-s}Q(u;R_0))r^\alpha$$

where

$$Q(u;R_0) = (\|u\|_{L^\infty(B_0)} + \mathrm{Tail}_{p^+}(u;R_0) + \mathrm{Tail}_{p^-}(u;R_0))^\beta$$

for some $\beta = \beta(\Lambda,\lambda) > 0$.

Remark 3.5. The hypothesis $\mathrm{Tail}_{p^+}(u;R_0) < \infty$ seems to be a technical one. As a matter of fact, what we expect in Theorem 3.4 is that the terms $\mathrm{Tail}_{p^+}(u;R_0)$ and $\mathrm{Tail}_{p^-}(u;R_0)$ to be replaced by $\mathrm{Tail}_g(u;R_0)$.

Nevertheless, in most applications (c.f. Section 4), this requirement is enough,

Proof. Given $R_0 \in (0,1)$, for any $j \in \mathbb{N}_0$ set the quantities

$$R_j = \frac{R_0}{4^j}, \quad B_j = B_{R_j}, \quad \frac{1}{2}B_j = B_{R_j/2}.$$

Let us prove that there exists a universal constant $\alpha \in (0,1)$, a number $L > 0$, a nondecreasing sequence $\{m_j\}$ and a nonincreasing sequence $\{M_j\}$ such that, for all $j \geq 0$,

$$m_j \leq \inf_{B_j} u \leq \sup_{B_j} u \leq M_j, \qquad M_j - m_j = LR_j^\alpha.$$

We proceed by induction on j.

Step 1. When $j = 0$, set $M_0 = \sup_{B_0} u$ and $m_0 = M_0 - LR_0^\alpha$, where $L > 0$ satisfies that

$$L \geq \frac{2\|u\|_{L^\infty(B_0)}}{R_0^\alpha}, \tag{3.5}$$

which implies that $m_0 \leq \inf_{B_0} u \leq M_0$.

Step 2. Inductive step: assume that sequences $\{m_j\}$ and $\{M_j\}$ are constructed up to the index j. Then

$$M_j - m_j = \fint_{B_j \setminus \frac{1}{2}B_j} (M_j - u)\, dx + \fint_{B_j \setminus \frac{1}{2}B_j} (u - m_j)\, dx$$

by using the convexity of g,

$$g(R_j^{-s}(M_j - m_j)) \leq C \fint_{B_j \setminus \frac{1}{2}B_j} g(Rj^{-s}(M_j - u))\, dx + C \fint_{B_j \setminus \frac{1}{2}B_j} g(R_j^{-s}(u - m_j))\, dx$$

from where

$$\begin{aligned} M_j - m_j \leq & CR_j^s g^{-1}\left(\fint_{B_j \setminus \frac{1}{2}B_j} g(R_j^{-s}(M_j - u))\, dx \right) \\ & + CR_j^s g^{-1}\left(\fint_{B_j \setminus \frac{1}{2}B_j} g(R_j^{-s}(u - m_j))\, dx \right). \end{aligned}$$

Let $\sigma \in (0,1)$, $\tilde{C} > 0$ be as in Lemma 3.3. From the previous inequality and Lemma 3.3 we get

$$\begin{aligned} C\sigma(M_j - m_j) \leq & \, \sigma R_j^s g^{-1}\left(\fint_{B_j \setminus \frac{1}{2}B_j} g(R_j^{-s}(M_j - u))\, dx \right) \\ & + \sigma R_j^s g^{-1}\left(R^{-s} \fint_{B_j \setminus \frac{1}{2}B_j} g(R_j^{-s}(u - m_j))\, dx \right) \\ \leq & \inf_{B_{j+1}} (M_j - u) + \inf_{B_{j+1}} (u - m_j) + 2\tilde{C} R_j^s g^{-1}(K) \\ & + C_\varepsilon [\mathrm{Tail}_g((M_j - u)_-; R_j) + \mathrm{Tail}_g((u - m_j)_-; R_j)] \\ & + \varepsilon \left[\sup_{B_{R_j}} (M_j - u) + \sup_{B_{R_j}} (u - m_j) \right]. \end{aligned}$$

Setting $\varepsilon=\sigma/4$, $C=\max\{2\tilde{C},C_\varepsilon\}$ and rearranging terms we get

$$\begin{aligned}\operatorname*{osc}_{B_{j+1}} u &\leq \left(1-\frac{\sigma}{2}\right)(M_j-m_j)\\ &\quad + C[(R_0^s g^{-1}(K)+\mathrm{Tail}_g((M_j-u)_-;R_j)+\mathrm{Tail}_g((u-m_j)_-;R_j)).\end{aligned} \tag{3.6}$$

Step 3. Let us estimate the tails. Observe that we can write

$$\begin{aligned}R_j^{-s}g(\mathrm{Tail}_g((u-m_j)_-;R_j)) &= \sum_{k=0}^{j-1}\int_{B_k\setminus B_{k+1}} g\left(R_j^s\frac{(u(y)-m_j)_-}{|y|^s}\right)\frac{dy}{|y|^{n+s}}\\ &\quad+\int_{B_0^c} g\left(R_j^s\frac{(u(y)-m_j)_-}{|y|^s}\right)\frac{dy}{|y|^{n+s}}.\end{aligned} \tag{3.7}$$

Let us deal with the first term. By inductive hypothesis, for all $0\leq k\leq j-1$ we have in $B_k\setminus B_{k+1}$

$$(u-m_j)_-\leq m_j-m_k\leq(m_j-M_j)+(M_k-m_k)=L(R_k^\alpha-R_j^\alpha),$$

hence

$$\begin{aligned}(i) := \sum_{k=0}^{j-1}\int_{B_k\setminus B_{k+1}} g\left(R_j^s\frac{(u(y)-m_j)_-}{|y|^s}\right)\frac{dy}{|y|^{n+s}} &\leq \sum_{k=0}^{j-1}\int_{B_k\setminus B_{k+1}} g\left(LR_j^{\alpha+s}\frac{(4^{\alpha(j-k)}-1)}{|y|^s}\right)\frac{dy}{|y|^{n+s}}\\ &\leq \sum_{k=0}^{j-1} g\left(LR_j^{\alpha+s}R_k^{-s}(4^{\alpha(j-k)}-1)\right)\int_{B_k\setminus B_{k+1}}\frac{dy}{|y|^{n+s}}\\ &\leq C(s,n)g\left(LR_j^{\alpha+s}R_j^{-s}\right)R_0^{-s}\sum_{k=0}^{j-1}(4^{\alpha(j-k)}-1)^\Lambda 4^{sk}.\end{aligned}$$

Then, if we choose $\alpha<\frac{s}{\Lambda}$

$$\begin{aligned}(i) &\leq C(s,n)g\left(LR_j^{\alpha+s}R_0^{-s}R_k^{-s}\right)4^{sj}\sum_{k=0}^{j-1}(4^{\alpha(j-k)}-1)^\Lambda 4^{-s(j-k)}\\ &\leq C(s,n)g\left(LR_j^\alpha\right)R_0^{-s}R_j^{-s}\sum_{h=1}^{\infty}(4^{h\alpha}-1)^\Lambda 4^{-sh}\\ &\leq C(s,n)g\left(LR_j^\alpha\right)R_0^{-s}R_j^{-s}S(\alpha)\end{aligned}$$

where

$$S(\alpha):=\sum_{h=1}^{\infty}(4^{h\alpha}-1)^\Lambda 4^{-sh}\leq\sum_{h=1}^{\infty}4^{h[\alpha\Lambda-s]}=\frac{1}{1-4^{\alpha\Lambda-s}}<\infty.$$

Moreover, observe that $S(\alpha)\to 0$ as $\alpha\to 0^+$. Therefore

$$g^{-1}(R_j^s\cdot(i))\leq CR_0^{-\frac{s}{\Lambda}}S(\alpha)^{\frac{1}{\Lambda}}LR_j^\alpha. \tag{3.8}$$

Let us deal with the second integral. Since by inductive hypothesis

$$m_j \le \inf_{B_j} u \le \sup_{B_j} u \le \|u\|_{L^\infty(B_0)},$$

we have

$$\begin{aligned}(ii) &:= \int_{B_0^c} g\left(R_j^s \frac{(u(y)-m_j)_-}{|y|^s}\right)\frac{dy}{|y|^{n+s}} \le \int_{B_0^c} g\left(R_j^s \frac{|u(y)|+\|u\|_{L^\infty(B_0)}}{|y|^s}\right)\frac{dy}{|y|^{n+s}}\\ &\le C\int_{B_0^c} g\left(R_j^s\frac{|u(y)|}{|y|^s}\right)\frac{dy}{|y|^{n+s}} + C\int_{B_0^c} g\left(R_j^s\frac{\|u\|_{L^\infty(B_0)}}{R_0^s}\right)\frac{dy}{|y|^{n+s}}\\ &\le (a)+(b).\end{aligned}$$

Expression (a) can be bounded by using (Δ_2) as

$$\begin{aligned}(a) &\le g(R_j^s R_0^s R_0^{-s})\left(\int_{B_0^c\cap\{y\colon u(y)|y|^{-s}\ge 1\}} \frac{|u(y)|^\Lambda}{|y|^{n+sp^+}}\,dy + \int_{B_0^c\cap\{y\colon u(y)|y|^{-s}<1\}} \frac{|u(y)|^\lambda}{|y|^{n+sp^-}}\,dy\right)\\ &\le Cg(R_j^sR_0^{-s})R_0^{sp^+}\int_{B_0^c}\frac{|u(y)|^\Lambda}{|y|^{n+sp^+}}\,dy + Cg(R_j^sR_0^{-s})R_0^{s(p^+-p^-)}R_0^{sp^-}\int_{B_0^c}\frac{|u(y)|^\lambda}{|y|^{n+sp^-}}\,dy\\ &\le g(R_j^sR_0^{-s})R_0^{-s}\left(\mathrm{Tail}_{p^+}(u;R_0)^{p^+-1} + \mathrm{Tail}_{p^-}(u;R_0)^{p^--1}\right),\end{aligned}$$

since $R_0<1$. Moreover, it is easy to see that

$$(b) \le Cg(\|u\|_{L^\infty(B_0)}R_j^sR_0^{-s})R_0^{-s}$$

Then, form the last two expression we get

$$\begin{aligned}(ii) &\le C(n,s)g(R_j^sR_0^{-s})R_0^{-s}(\mathrm{Tail}_{p^+}(u;R_0)^\Lambda) + \mathrm{Tail}_{p^-}(u;R_0)^\lambda))\\ &\quad + C(n,s)g(R_j^sR_0^{-s})R_0^{-s}\max\{\|u\|^\Lambda_{L^\infty(B_0)}, \|u\|^\lambda_{L^\infty(B_0)}\}\\ &\le C(n,s)Q(u;R_0)g(R_j^sR_0^{-s})R_0^{-s},\end{aligned}$$

where we have denoted

$$Q(u;R_0) = (\|u\|_{L^\infty(B_0)} + \mathrm{Tail}_{p^+}(u;R_0) + \mathrm{Tail}_{p^-}(u;R_0))^\beta$$

for some $\beta=\beta(\lambda,\Lambda)>0$.

From the last inequality we get

$$g^{-1}(R_j^s\cdot(ii)) \le C(n,s,\lambda,\Lambda)Q(u;R_0)R_0^{-s}R_j^s. \tag{3.9}$$

Plugging (3.8) and (3.9) in (3.7) gives that

$$\begin{aligned}\mathrm{Tail}_g((u-m_j)_-;R_j) &\le C(g^{-1}(R_j^s\cdot(i)) + g^{-1}(R_j^s\cdot(ii)))\\ &\le C(n,s,\lambda,\Lambda)R_0^{-s}R_j^s[S(\alpha)^{\frac{1}{\Lambda}}L + Q(u;R_0)]\end{aligned}$$

since $\alpha<s<sp^+$. The power $\beta>0$ in the definition of $Q(u;R_0)$ may have changed form line to line but still positive.

A similar estimate holds for $\mathrm{Tail}_g((M_j-u)_-;R_j)$.

Step 5. By using the previous computations we bound expression (3.6):

$$\begin{aligned}\operatorname*{osc}_{B_{j+1}} u &\leq \left(1-\frac{\sigma}{2}\right) L R_j^\alpha + C[(R_0^s g^{-1}(K) + \mathrm{Tail}_g((M_j-u)_-;R_j) + \mathrm{Tail}_g((u-m_j)_-;R_j))\\ &\leq 4^\alpha L R_{j+1}^\alpha \left[\left(1-\frac{\sigma}{2}\right) + C R_0^{-s} S(\alpha)^{\frac{1}{\Lambda}}\right] + 4^\alpha C R_{j+1}^\alpha [R_0^{s-\alpha} g^{-1}(K) + R_0^{-s} Q(u,R_0)].\end{aligned}$$

We choose $\alpha \in (0,s)$ universally such that

$$4^\alpha \left[\left(1-\frac{\sigma}{2}\right) + C R_0^{-s} S(\alpha)^{\frac{1}{\Lambda}}\right] \leq 1 - \frac{\sigma}{4}.$$

Then

$$\operatorname*{osc}_{B_{j+1}} u \leq \left(\left(1-\frac{\sigma}{4}\right) L + C[R_0^{s-\alpha} g^{-1}(K) + R_0^{-s} Q(u;R_0)]\right) R_{j+1}^\alpha.$$

Now we choose

$$L = \frac{4}{\sigma} C (R_0^{s-\alpha} g^{-1}(K) + R_0^{-s} Q(u;R_0))$$

which implies (3.5) as $4^{\alpha+1} C/\sigma > 2$ and gives

$$\operatorname*{osc}_{B_{j+1}} u \leq L R_{j+1}^\alpha.$$

Step 6. We may pick m_{j+1}, M_{j+1} such that

$$m_j \leq m_{j+1} \leq \inf_{B_{j+1}} u \leq \sup_{B_{j+1}} u \leq M_{j+1} \leq M_j, \quad M_{j+1} - m_{j+1} = L R_{j+1}^\alpha,$$

which completes the induction and proves the claim.

Now fix $r \in (0, R_0)$ and find an integer $j \geq 0$ such that $R_{j+1} \leq r < R_j$, thus $R_j \leq 4r$. Hence, by the claim and the election of L we have that, for some $C = C(n,s,\lambda,\Lambda)$,

$$\begin{aligned}\operatorname*{osc}_{B_r} u \leq \operatorname*{osc}_{B_j} u \leq L R_j^\alpha &\leq C(R_0^{s-\alpha} g^{-1}(K) + R_0^{-s} Q(u;R_0)) r^\alpha\\ &\leq C(R_0^{s-\alpha} g^{-1}(K) + R_0^{-s} Q(u;R_0)) r^\alpha\end{aligned}$$

which concludes the argument. □

An immediate corollary of Theorem 3.4 is the interior Hölder continuity of solutions:

Corollary 3.6. *Assume that the hypothesis of Theorem 3.4 are in force. Then,*

$$[u]_{C^\alpha(B_{R_0/2})} \leq C(n,s,\lambda,\Lambda)(R_0^{s-\alpha} g^{-1}(K) + R_0^{-s} Q(u;R_0)). \tag{3.10}$$

4. Boundary behavior

In this section we discuss the boundary behavior of solutions. Throughout this section, $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ will be defined by

$$u_0(x) := x_+^s. \tag{4.1}$$

We start by showing that u_0 is $(s,g)-$harmonic in the positive half line:

Lemma 4.1. *$u_0 \in \widetilde{W}^{s,G}_{loc}(\mathbb{R})$ and $(-\Delta_g)^s u_0 = 0$ weakly and strongly in $\mathbb{R}_+$.*

Proof. First we show that $u_0 \in \widetilde{W}^{s,G}_{\mathrm{loc}}(\mathbb{R})$. This follows rather straightforwardly since

$$G\left(\frac{|u_0(x)-u_0(y)|}{|x-y|^s}\right)\frac{1}{|x-y|} \le C\frac{|u_0(x)-u_0(y)|^{p^+}}{|x-y|^{1+sp^+}}$$

and we can apply the homogeneous result [24, Lemma 3.1].

To see that u_0 is a solution, let $\rho \in (0,1)$ take any compact set $K \subset (\rho, \rho^{-1})$ and let $x \in K$. For any $\varepsilon > 0$ consider the integral

$$I_\varepsilon := \int_{|x-y|>\varepsilon} g\left(\frac{|u_0(x)-u_0(y)|}{|x-y|^s}\right)\frac{u_0(x)-u_0(y)}{|u_0(x)-u_0(y)|}\frac{dy}{|x-y|^{1+s}}.$$

We will show that

$$I_\varepsilon \longrightarrow 0 \text{ uniformly as } \varepsilon \to 0^+. \tag{4.2}$$

For this, and notice that for $\varepsilon < x$

$$0 < x-\varepsilon < x+\varepsilon < \frac{x^2}{x-\varepsilon}$$

and split

$$I_\varepsilon = \int_{-\infty}^0 + \int_{x+\varepsilon}^{\frac{x^2}{x-\varepsilon}} + \int_0^{x-\varepsilon} + \int_{\frac{x^2}{x-\varepsilon}}^\infty = I_1 + I_2 + I_3 + I_4.$$

We will estimate each term separately. For this, we will use the following simple but useful identities:

$$\left(G\left(\frac{1}{|1-t|^s}\right)\right)' = g\left(\frac{1}{|1-t|^s}\right)\frac{s}{|1-t|^{1+s}} \quad t \neq 1 \tag{4.3}$$

$$\left(G\left(\frac{1-t^s}{|1-t|^s}\right)\right)' = sg\left(\frac{1-t^s}{|1-t|^s}\right)\left(\frac{1-t^{s-1}}{(1-t)^{1+s}}\right) \quad t < 1. \tag{4.4}$$

Notice that u_0 vanishes identically in $(-\infty,0)$ so, using the change of variables $t=\frac{y}{x}$ and (4.3), we have

$$\begin{aligned}I_1&=\int_{-\infty}^{0} g\left(\frac{x^s}{|x-y|^s}\right)\frac{dy}{|x-y|^{1+s}}=\int_{-\infty}^{0} g\left(\frac{1}{|1-\frac{y}{x}|^s}\right)\frac{x^{-1-s}dy}{|1-\frac{y}{x}|^{1+s}}\\&=x^{-s}\int_{-\infty}^{0} g\left(\frac{1}{|1-t|^s}\right)\frac{dt}{|1-t|^{1+s}}=\frac{x^{-s}}{s}\int_{-\infty}^{0}\left(G\left(\frac{1}{|1-t|^s}\right)\right)' dt\\&=\frac{x^{-s}}{s}G(1).\end{aligned}$$

To bound I_2 notice that using (2.3) and (2.4) we have

$$\begin{aligned}g\left(\frac{x^s-y^s}{|x-y|^s}\right)\frac{1}{|x-y|^{1+s}}&\le\frac{1}{(x^s-y^s)|x-y|}G\left(\frac{x^s-y^s}{|x-y|^s}\right)\\&\le\frac{(x^s-y^s)^{p^+-1}}{|x-y|^{1+sp^+}}\end{aligned}$$

so, using the Hölder continuity of the power s,

$$\begin{aligned}|I_2|&\le\int_{x+\varepsilon}^{\frac{x^2}{x-\varepsilon}}\frac{|x^s-y^s|^{p^+-1}}{|x-y|^{1+sp^+}}\,dy\le C\int_{x+\varepsilon}^{\frac{x^2}{x-\varepsilon}}\frac{1}{|x-y|^{1+s}}\,dy\\&=C\frac{x^{-s}}{s}\frac{(x^s-(x-\varepsilon)^s)}{\varepsilon^s}.\end{aligned}$$

Next, I_3 is estimated using the same change of variables as that for I_1:

$$\begin{aligned}I_3&=\int_0^{x-\varepsilon} g\left(\frac{x^s-y^s}{|x-y|^s}\right)\frac{dy}{|x-y|^{1+s}}=x^{-(1+s)}\int_0^{x-\varepsilon} g\left(\frac{1-\left(\frac{y}{x}\right)^s}{|1-\left(\frac{y}{x}\right)|^s}\right)\frac{dy}{|1-\left(\frac{y}{x}\right)|^{1+s}}\\&=x^{-s}\int_0^{1-\frac{\varepsilon}{x}} g\left(\frac{1-t^s}{|1-t|^s}\right)\frac{dt}{|1-t|^{1+s}}.\end{aligned}$$

Similarly, recalling that g is odd and making one further change of variables (for simplicity of notation we do not change the name of the variable),

$$\begin{aligned}I_4&=\int_{\frac{x^2}{x-\varepsilon}}^{\infty} g\left(\frac{x^s-y^s}{|x-y|^s}\right)\frac{dy}{|x-y|^{1+s}}=-x^{-(1+s)}\int_{\frac{x^2}{x-\varepsilon}}^{\infty} g\left(\frac{\left(\frac{y}{x}\right)^s-1}{|\frac{y}{x}-1|^s}\right)\frac{dy}{|\frac{y}{x}-1|^{1+s}}\\&=-x^{-s}\int_{(1-\frac{\varepsilon}{x})^{-1}}^{\infty} g\left(\frac{t^s-1}{|t-1|^s}\right)\frac{dt}{|t-1|^{1+s}}=-x^{-s}\int_0^{1-\frac{\varepsilon}{x}} g\left(\frac{t^{-s}-1}{|t^{-1}-1|^s}\right)\frac{dt}{t^2|t^{-1}-1|^{1+s}}\\&=-x^{-s}\int_0^{1-\frac{\varepsilon}{x}} g\left(\frac{t^{-s}-1}{|t^{-1}-1|^s}\right)\frac{t^{s-1}dt}{|1-t|^{1+s}}\end{aligned}$$

therefore, using (4.4)

$$\begin{aligned} I_3+I_4 &= x^{-s}\int_0^{1-\frac{\varepsilon}{x}} \left[g\left(\frac{1-t^s}{|1-t|^s}\right) - t^{s-1} g\left(\frac{t^{-s}-1}{|t^{-1}-1|^s}\right)\right] \frac{dt}{|1-t|^{1+s}} \\ &= x^{-s}\int_0^{1-\frac{\varepsilon}{x}} g\left(\frac{1-t^s}{|1-t|^s}\right) \frac{(1-t^{s-1})}{|1-t|^{1+s}}\,dt = \frac{x^{-s}}{s}\int_0^{1-\frac{\varepsilon}{x}} \left(G\left(\frac{1-t^s}{|1-t|^s}\right)\right)'\,dt \\ &= \frac{x^{-s}}{s}\left(G\left(\frac{x^s-(x-\varepsilon)^s}{\varepsilon^s}\right) - G(1)\right). \end{aligned}$$

Putting all the estimates together we get

$$|I_\varepsilon| \le \frac{x^{-s}}{s}\left(C\frac{(x^s-(x-\varepsilon)^s)}{\varepsilon^s} + G\left(\frac{x^s-(x-\varepsilon)^s}{\varepsilon^s}\right)\right) \tag{4.5}$$

and hence the desired convergence. In particular, u_0 is a strong (and thanks to Corollary B.3 a weak) solution. □

Next, we want to study the one-dimensional profile $u(x) = u_0(x_n)$ in the half space. Recall that $GL(n)$ stands for the general linear group that consists of all invertible $n \times n$ matrices.

Lemma 4.2. *Let $A \in GL(n)$ and define, for $\varepsilon > 0$ and $x \in \mathbb{R}^n_+ := \{x_n > 0\}$*

$$h_\varepsilon(x, A) := \int_{B_\varepsilon^c} g\left(\frac{u_0(x_n) - u_0(z+x_n)}{|Az|^s}\right) \frac{dz}{|Az|^{n+s}}$$

and $u(x) = u_0(x_n)$.

Then $h_\varepsilon \to 0^+$ uniformly in any compact $K \subset \mathbb{R}^n_+ \times GL(n)$. As a consequence, $u \in \widetilde{W}^{s,G}(\mathbb{R}^n)$ and $(-\Delta_g)^s u = 0$ weakly and strongly in $\mathbb{R}^n_+$.

Proof. Let $A \in GL(n)$ and $K = H \times H'$ a compact subset of $\mathbb{R}^n_+ \times GL(n)$. By the singular value decomposition we have that AS^{n-1} is an ellipse with diameter bounded by the spectral norm of A (here $S^{n-1} = \partial B_1$). The elliptical coordinates are given for any $y \in \mathbb{R}^n \setminus \{0\}$ by

$$y = \rho\omega, \quad \rho > 0, \quad \omega \in AS^{n-1}.$$

Then $dy = \rho^{n-1} d\rho d\omega$ with $d\omega$ the surface element of AS^{n-1}. Let us further call

$$e_A := A^{-1}e_n, \quad E_A := \{x \in \mathbb{R}^n : x \cdot e_A > 0\}.$$

Now, with the change of variables $Az = y$ we compute

$$\begin{aligned} h_\varepsilon(x,A) &= \int_{B_\varepsilon^c} g\left(\frac{u(x)-u(x+z)}{|Az|^s}\right)\,\frac{dz}{|Az|^{n+s}} \\ &= \int_{(AB_\varepsilon)^c} g\left(\frac{u(x)-u(x+A^{-1}y)}{|y|^s}\right)\,\frac{dy}{|\det A||y|^{n+s}} \\ &= \int_{AS^{n-1}} \frac{1}{|\det A||\omega|^{n+s}} \int_\varepsilon^\infty g\left(\frac{u_0(x_n)-u_0(x_n+\omega\cdot e_A\rho)}{|\omega\rho|^s}\right)\,\frac{d\rho}{\rho^{1+s}}d\omega \\ &= \int_{AS^{n-1}\cap E_A} \frac{1}{|\det A||\omega|^{n+s}} \int_{(-\varepsilon,\varepsilon)^c} g\left(\frac{u_0(x_n)-u_0(x_n+\omega\cdot e_A\rho)}{|\omega\rho|^s}\right)\,\frac{d\rho}{|\rho|^{1+s}}d\omega \\ &= \int_{AS^{n-1}\cap E_A} \frac{|\omega\cdot e_A|^{1+s}}{|\det A||\omega|^{n+s}} \int_{(-\varepsilon,\varepsilon)^c} g\left(\sigma(\omega)\frac{u_0(x_n)-u_0(x_n+\omega\cdot e_A\rho)}{|\omega\cdot e_A\rho|^s}\right)\,\frac{d\rho}{|\omega\cdot e_A\rho|^{1+s}}d\omega \end{aligned}$$

where

$$\sigma(\omega) := \left(\frac{\omega}{|\omega|}\cdot e_A\right)^s.$$

Notice that thanks to the bound (4.5) we have, for ε small enough depending on the norm of A,

$$\begin{aligned} \int_{(-\varepsilon,\varepsilon)^c} g\left(\frac{u_0(x_n)-u_0(x_n+\omega\cdot e_A\rho)}{|\omega\cdot e_A\rho|^s}\right)\,\frac{d(\omega\cdot e_A\rho)}{|\omega\cdot e_A\rho|^{1+s}} &\le \frac{x_n^{-s}}{s}\left(C\left(\frac{x_n^s-(x_n-\varepsilon)^s}{\varepsilon^s}\right)\right. \\ &\quad\left. + G\left(\frac{x_n^s-(x_n-\varepsilon)^s}{\varepsilon^s}\right)\right) \\ &=: Cx_n^{-s}\psi(x_n,\varepsilon). \end{aligned}$$

Therefore, using (2.5) we get

$$|h_\varepsilon(x,A)| \le Cx_n^{-s}\int_{AS^{n-1}\cap E_A} \frac{|\omega\cdot e_A|^{1+s}\max\{\sigma(\omega)^{sp^-},\sigma(\omega)^{sp^+}\}}{|\det A||\omega|^{n+s}}\psi(x_n,\omega\cdot e_A\varepsilon)d\omega.$$

Now for $\varepsilon > 0$

$$\frac{\partial}{\partial\varepsilon}\psi(x_n,\varepsilon) = \frac{s}{\varepsilon^{s+1}}\left(\varepsilon(x_n-\varepsilon)^{s-1} - x_n^s + (x_n-\varepsilon)^s\right) \ge 0.$$

Also, changing back the variables we have

$$\begin{aligned} &\int_{AS^{n-1}} \frac{|\omega\cdot e_A|^{1+s}\max\{\sigma(\omega)^{sp^-},\sigma(\omega)^{sp^+}\}}{|\det A||\omega|^{n+s}}d\omega \\ &\qquad = \int_{S^{n-1}} \frac{|\omega\cdot e_n|^{1+s}\max\left\{\left(\frac{\omega\cdot e_n}{|A\omega|}\right)^{sp^-},\left(\frac{\omega\cdot e_n}{|A\omega|}\right)^{sp^+}\right\}}{|A\omega|^{n+s}}d\omega \\ &\qquad \le \mathcal{H}^{n-1}(S^{n-1})\|A^{-1}\|^{n+s}\max\left\{\|A^{-1}\|^{sp^-},\|A^{-1}\|^{sp^+}\right\} \end{aligned}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)-$dimensional Hausdorff measure and we used that

$$|\omega\cdot e_A|\le|\omega||A^{-1}e_n|\le\|A\|\|A^{-1}\|.$$

Therefore, we get

$$|h_\varepsilon(x,A)|\le Cx_n^{-s}\psi(x_n,\|A\|\|A^{-1}\|\varepsilon)$$

and the result follows by taking $\varepsilon\to0$. From this result, the conclusions on $u(x)=u_0(x_n)$ follow straightforwardly. ☐

Once the $1-$d profile is known to be a solution, we wish to straighten the boundary of Ω. The following lemma asserts that when we do that the fractional $g-$Laplacian of the profile remains bounded:

Lemma 4.3. *Let Φ be a $C^{1,1}$ diffeomorphism in $\mathbb{R}^n$ such that $\Phi=Id$ in B_r^c for some $r>0$ and define*

$$v(x):=(\Phi^{-1}(x)\cdot e_n)_+^s. \tag{4.6}$$

Then $v\in\widetilde{W}^{s,G}_{loc}(\mathbb{R}^n)$ and $(-\Delta_g)^s v=f$ weakly in $\Phi(\mathbb{R}^n_+)$ with

$$\|f\|_\infty\le C\left(\|D\Phi\|_\infty,\|D\Phi^{-1}\|_\infty,r\right)\|D^2\Phi\|_\infty. \tag{4.7}$$

Proof. We want to show that

$$h_\varepsilon(x):=\int_{\{|\Phi^{-1}(x)-\Phi^{-1}(y)|>\varepsilon\}}g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}} \tag{4.8}$$

converges in $L^1(K)$ for any compact set $K\subset\mathbb{R}^n_+$ to some $f\in L^\infty(\mathbb{R}^n_+)$ satisfying (4.7). Once this is proven Lemma B.2 gives the result.

Let us make the change of variables $\Phi(\bar x)=x$, denote $J(\cdot)=|\det D\Phi(\cdot)|$ and write (4.8) as

$$\begin{aligned}
h_\varepsilon(x)&=\int_{B^c_\varepsilon(\bar x)}g\left(\frac{v(\Phi(\bar x))-v(\Phi(\bar y))}{|\Phi(\bar x)-\Phi(\bar y)|^s}\right)\frac{J(\bar y)d\bar y}{|\Phi(\bar x)-\Phi(\bar y)|^{n+s}}\\
&=\int_{B^c_\varepsilon(\bar x)}g\left(\frac{u_0(\bar x)-u_0(\bar y)}{|\Phi(\bar x)-\Phi(\bar y)|^s}\right)\frac{\zeta(\bar x,\bar y)d\bar y}{|D\Phi(\bar x)(\bar x-\bar y)|^{n+s}}\\
&\quad+\int_{B^c_\varepsilon(\bar x)}g\left(\frac{u_0(\bar x)-u_0(\bar y)}{|\Phi(\bar x)-\Phi(\bar y)|^s}\right)\frac{J(\bar x)d\bar y}{|D\Phi(\bar x)(\bar x-\bar y)|^{n+s}}\\
&=I_1+I_2
\end{aligned}$$

with

$$\zeta(\bar x,\bar y):=\frac{|D\Phi(\bar x)(\bar x-\bar y)|^{n+s}}{|\Phi(\bar x)-\Phi(\bar y)|^{n+s}}J(\bar y)-J(\bar x).$$

Now, we use ellipticity and the fact that

$$\frac{|D\Phi(\bar x)(\bar x-\bar y)|^{s\Lambda}}{|\Phi(\bar x)-\Phi(\bar y)|^{s\Lambda}}\le C(\|D\Phi\|_\infty,\|D\Phi^{-1}\|_\infty).$$

to get

$$\begin{aligned}|I_1| &\leq \int_{B^c_\varepsilon(\bar{x})} g\left(\frac{u_0(\bar{x})-u_0(\bar{y})}{|\Phi(\bar{x})-\Phi(\bar{y})|^s}\right)\frac{|\zeta(\bar{x},\bar{y})|d\bar{y}}{|D\Phi(\bar{x})(\bar{x}-\bar{y})|^{n+s}}\\ &\leq \int_{B^c_\varepsilon(\bar{x})} \frac{(u_0(\bar{x})-u_0(\bar{y}))^\Lambda}{|\Phi(\bar{x})-\Phi(\bar{y})|^{s\Lambda}}\frac{|\zeta(\bar{x},\bar{y})|d\bar{y}}{|D\Phi(\bar{x})(\bar{x}-\bar{y})|^{n+s}}\\ &\leq \int_{B^c_\varepsilon(\bar{x})} \frac{(u_0(\bar{x})-u_0(\bar{y}))^\Lambda}{|D\Phi(\bar{x})(\bar{x}-\bar{y})|^{n+sp^+}}|\zeta(\bar{x},\bar{y})|d\bar{y}.\end{aligned}$$

Therefore I can be bound exactly as the second term of (3.6) in [24].

I_2 on the other hand vanishes identically as $\varepsilon \to 0^+$ by means of Lemma 4.2; indeed, since $D\Phi(\mathbb{R}^n)$ is a compact subset of $GL(n)$ the integral vanishes uniformly in any compact set $\Phi^{-1}(K) \subset \mathbb{R}^n_+$, therefore in any compact set $K \subset \Phi(\mathbb{R}^n_+)$. $\square$

We will need the following Lemma regarding the geometric properties of a $C^{1,1}$ domain (cf. [23]). It essentially says that any point on $\partial\Omega$ has an interior and an exterior tangent ball and the distance function behaves like $|\cdot|$ close to such a point inside Ω. Recall that the distance function to $\partial\Omega$ is given by

$$d(x) := \operatorname{dist}(x, \Omega^c).$$

Lemma 4.4. *Let Ω be an bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. Then, there exists $\rho > 0$ such that for any $x_0 \in \partial\Omega$ there exist $x_1, x_2 \in \mathbb{R}^n$ in the normal line to $\partial\Omega$ at x_0 such that*

(1) $B_\rho(x_1) \subset \Omega$ and $B_\rho(x_2) \subset \Omega^c$;
(2) $\overline{B}_\rho(x_1) \cap \overline{B}_\rho(x_2) = \{x_0\}$;
(3) $d(x) = |x - x_0|$ for any $x = (1-t)x_0 + tx_1$, $t \in [0, 1]$.

Now we are in position to show that $(-\Delta_g)^s d^s$ is bounded in a neighborhood of $\partial\Omega$:

Proposition 4.5. *Let Ω be a bounded domain in of $\mathbb{R}^n$ with $C^{1,1}$ boundary. Then there exists $\rho > 0$ such that*

$$(-\Delta_g)^s d^s = f \qquad \textit{weakly in } \Omega_\rho$$

for some $f \in L^\infty(\Omega_\rho)$ with $\Omega_\rho := \{x \in \Omega : d(x) < \rho\}$.

Proof. By taking a finite covering of Ω_ρ by balls centered at points in $\partial\Omega$ and a partition of unity, it is enough to show that $(-\Delta_g)^s d^s = f$ holds weakly in $\Omega \cap B_{2\rho}$ with ρ small enough, depending only in the geometry of Ω. To that aim, we are going to flatten the boundary of Ω near the origin: let $\Phi(\bar{x}) = x$ be a $C^{1,1}$ diffeomorphism such that $\Phi = Id$ in $B^c_{4\rho}$ such that

$$\Omega \cap B_{2\rho} \subset\subset \Phi(B_{3\rho} \cap \mathbb{R}^n_+), \quad d(\Phi(\bar{x})) = (\bar{x}_n)_+ \text{ for } \bar{x} \in B_{3\rho}.$$

We will show that

$$h_\varepsilon(x) := \int_{\{|\Phi^{-1}(x)-\Phi^{-1}(y)|>\varepsilon\}} g\left(\frac{d^s(x)-d^s(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}} \longrightarrow f \text{ in } L^1_{\text{loc}}(\Omega \cap B_{2\rho})$$

for a function $f \in L^\infty$ and the result will follow from Lemma B.2.

Setting $\bar{x} = \Phi^{-1}(x)$ and changing variables we can compute (with J defined as in Lemma 4.3) we can write

$$
\begin{aligned}
h_\varepsilon(x) &= \int_{B^c_\varepsilon(x)} g\left(\frac{d^s(\Phi(\bar{x})) - d^s(\Phi(\bar{y}))}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right) \frac{J(\bar{y})d\bar{y}}{|\Phi(\bar{x}) - \Phi(\bar{y})|^{n+s}} \\
&= \int_{B^c_\varepsilon(x)\cap B_{3\rho}} g\left(\frac{d^s(\Phi(\bar{x})) - d^s(\Phi(\bar{y}))}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right) \frac{J(\bar{y})d\bar{y}}{|\Phi(\bar{x}) - \Phi(\bar{y})|^{n+s}} \\
&+ \int_{B^c_{3\rho}} g\left(\frac{d^s(\Phi(\bar{x})) - d^s(\Phi(\bar{y}))}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right) \frac{J(\bar{y})d\bar{y}}{|\Phi(\bar{x}) - \Phi(\bar{y})|^{n+s}} \\
&= \int_{B^c_\varepsilon(\bar{x})} g\left(\frac{u_0(\bar{x}_n) - u_0(\bar{y}_n)}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right) \frac{J(\bar{x})d\bar{y}}{|\Phi(\bar{x})(\bar{x} - \bar{y})|^{n+s}} \\
&+ \int_{B^c_{3\rho}} \left(g\left(\frac{d^s(\Phi(\bar{x})) - d^s(\Phi(\bar{y}))}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right) - g\left(\frac{u_0(\bar{x}_n) - u_0(\bar{y}_n)}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right)\right) \frac{J(\bar{y})d\bar{y}}{|\Phi(\bar{x}) - \Phi(\bar{y})|^{n+s}} \\
&= f_{1,\varepsilon}(\bar{x}) + f_2(\bar{x}).
\end{aligned}
$$

As in Lemma 4.3 (and using Lemma 4.2) we have that

$$\lim_{\varepsilon\to 0^+} f_{1,\varepsilon} = f_1 \text{ in } L^1_{\text{loc}}(\Omega \cap B_{2\rho})$$

with $f_1 \in L^\infty_{\text{loc}}(\mathbb{R}^n_+)$.

It remains to bound f_2. To do that, we note that

$$\operatorname{dist}(\Phi^{-1}(\Omega \cap B_{2\rho}), \Phi(B^c_{3\rho}) \geq \theta > 0$$

for some θ depending only on ρ and Φ. Now using that $d^s \circ \Phi$ is s-Hölder continuous (and so is u_0) and the properties of Φ we have

$$\left| g\left(\frac{d^s(\Phi(\bar{x})) - d^s(\Phi(\bar{y}))}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right) - g\left(\frac{u_0(\bar{x}_n) - u_0(\bar{y}_n)}{|\Phi(\bar{x}) - \Phi(\bar{y})|^s}\right)\right| \leq C$$

so that

$$|f_2(\bar{x})| \leq C \int_{B^c_{3\rho}} \frac{d\bar{y}}{|\Phi(\bar{x}) - \Phi(\bar{y})|^{n+s}} \leq C$$

and we get the result. □

Next, we want to construct the appropriate barriers to get bounds on our solutions in terms if the distance function d^s. We start by considering functions whose fractional $g-$Laplacian is constant in the unit ball.

Lemma 4.6. *The equation*

$$\begin{cases} (-\Delta_g)^s v = 1 & \text{in } B_1 \\ v = 0 & \text{in } B^c_1 \end{cases} \tag{4.9}$$

has a unique solution $v_0 \in W^{s,G}_0(\Omega)$. Moreover, $v_0 \in L^\infty(\mathbb{R}^n)$, is radially symmetric, nonincreasing and for any $r \in (0,1)$ it holds that $\inf_{B_r} v_0 > 0$.

Proof. First, weak solutions of (4.9) are constructed as minimizers in $W_0^{s,G}(B_1)$ of the functional

$$J(v) := \iint G\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n} - \int_{B_1} v\,dx$$

so existence and uniqueness follow from the direct method of the Calculus of Variations. Thanks to the rotational invariance of the equation given in Lemma C.3 we also have $v_0(x) = \psi(|x|)$ for some $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Further, by the Pólya-Szegö principle proved in [14] we have that ψ is nonincreasing.

Now let

$$r_0 := \inf\{r \in \mathbb{R}_+ : \psi(r) = 0\}$$

and let us show the last assertion by proving that $r_0 = 1$. It is clear that $r_0 \in (0,1]$, as ψ vanishes for $r > 1$. Let us then assume by contradiction that $r_0 \in (0,1)$. Then,

$$\begin{cases} (-\Delta_g)^s v_0 &= 1 \text{ in } B_{r_0} \\ v_0 &= 0 \text{ in } B_{r_0}^c. \end{cases}$$

Next denote $\tilde{v}_0(x) := v_0(r_0 x)$ and notice that Lemma C.1 gives that

$$\begin{cases} (-\Delta_{g_{r_0}})^s \tilde{v}_0 &= 1 \text{ in } B_1 \\ \tilde{v}_0 &= 0 \text{ in } B_1^c \end{cases}$$

which implies

$$\begin{cases} (-\Delta_g)^s \tilde{v}_0 &\leq r_0^s < 1 \text{ in } B_1 \\ \tilde{v}_0 &= 0 \text{ in } B_1^c \end{cases}$$

and the comparison principle implies that $v_0(x) \geq v_0(r_0 x)$, or $\psi(r) \geq \psi(r_0 r)$ for any $r \in (0, r_0)$. In particular,

$$0 \leq \psi(r_0^2) \leq \psi(r_0) = 0$$

so that $\psi(r_0^2) = 0$ which is a contradiction with the definition of r_0.

It remains to show that $v_0 \in L^\infty(\mathbb{R}^n)$. Let

$$w(x) := \min\{(2-x_n)_+^s, 5^s\} \in C^s(\mathbb{R}^n) \cap \widetilde{W}^{s,G}(B_1)$$

and notice that, for $x \in B_2$, $w(x) = u_0(2-x_n)$ with u_0 as defined in (4.1). Then, we can apply Lemma C.5 in $B_{3/2}$ with

$$u(x) = u_0(2-x_n),\ f \equiv 0 \text{ and } v(x) = (u_0(2-x_n) - 5^s)_+$$

to get, using Lemma 4.2,

$$(-\Delta_g)^s w(x) = 2\int_{\{y_n \leq -3\}} \left[g\left(\frac{(2-x_n)_+^s - 5^s}{|x-y|^s}\right) - g\left(\frac{(2-x_n)_+^s - (2-y_n)_+^s}{|x-y|^s}\right)\right] \frac{dy}{|x-y|^{n+s}}$$

weakly in B_1. The right hand side of this expression is a positive continuous function of x and hence bounded below in B_1 by some positive constant η, i.e.

$$(-\Delta_g)^s w \geq \eta > 0$$

weakly in B_1. Now, choose $c>0$ so that $\min\{c^\lambda, c^\Lambda\} = \eta^{-1}$ and use (2.5) to get

$$(-\Delta_g)^s(cw) \geq 1$$

which means that

$$(-\Delta_g)^s(cw) \geq (-\Delta_g)^s v_0.$$

This, together with the fact that

$$v_0 = 0 \leq cw \quad \text{in } B_1^c$$

gives, through Proposition C.4,

$$0 \leq v_0 \leq cw \quad \text{in } \mathbb{R}^n$$

and hence

$$0 \leq v_0 \leq \frac{5^s}{c} \quad \text{in } \mathbb{R}^n$$

as desired. □

As a consequence of the previous lemma we obtain that function with bounded fractional $g-$Laplacian are themselves bounded.

Proposition 4.7. *Let $u \in W_0^{s,G}(\Omega)$ be a weak solution of*

$$|(-\Delta_g)^s u| \leq K \quad \textit{in } \Omega$$

for some $K>0$.

Then

$$\|u\|_{L^\infty(B_1)} \leq C \tag{4.10}$$

where C is a positive constant depending only on $s, n, \lambda, \Lambda, K$ and diam(Ω).

Proof. Let $d >$diam(Ω) and take $x_0 \in \Omega$ such that $\Omega \subset\subset B_d(x_0)$. Consider v_0 as in the previous Lemma and notice that thanks the translation invariance, the scaling from Lemma C.1 and (2.5)

$$(-\Delta_g)^s v_0 \left(\frac{x-x_0}{d}\right) \geq \frac{1}{\max\{d^{s\lambda}, d^{s\Lambda}\}} \quad \text{weakly in } B_d(x_0).$$

As in the previous theorem, multiplying v_0 by a constant C (which will depend only on universal parameters and d)

$$(-\Delta_g)^s C v_0 \left(\frac{x-x_0}{d}\right) \geq K \quad \text{weakly in } \Omega$$

and since $u = 0 \leq Cv_0$ in Ω^c, the comparison principle gives $u \leq Cv_0$ in $\mathbb{R}^n$. We analogously bound $-u$ to get (4.10) and the proof concludes. □

In the next lemma we construct the barrier that we need to compare u with d^s:

Lemma 4.8. *There exist $w \in C^s(\mathbb{R}^n)$, $R > 0$, $\eta \in (0,1)$ and $c > 1$ such that*

$$(-\Delta_g)^s w \geq \eta \quad \text{weakly in } B_R(e_n) \setminus \overline{B_1}$$

and

$$c^{-1}(|x|-1)_+^s \leq w \leq c(|x|-1)_+^s \quad \text{in } \mathbb{R}^n.$$

Proof. Since the fractional $g-$Laplacian is translation invariant (and also rotation invariant, recall Lemma C.3) by using a similar scaling argument as the one of Lemma 4.6 it suffices to prove the result for any ball of radius $R > 2$ and any point $\bar{x}_R$ on its boundary. Let us set $\tilde{x}_R := (0, -(R^2-4)^{1/2})$ and $\bar{x}_R = \tilde{x}_R + Re_n$. In this way, $B_R(\bar{x}_R)$ intersects the hyperplane $\{x_n = 0\}$ at the $n-1$ dimensional ball $\{|x'| < 2\}$ where we denote as usual $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

For $R > 2$ there exists $\varphi \in C^{1,1}(\mathbb{R}^{n-1})$ with $\|\varphi\|_{C^{1,1}(\mathbb{R}^{n-1})} \leq C/R$ and

$$\varphi(x') = ((R^2 - |x'|)^{1/2} - (R^2-4)^{1/2})_+, \qquad \text{for all } |x'| \in [0,1] \cup [3,\infty)$$

and set

$$U_+ := \{x \in \mathbb{R}^{n-1} : \varphi(x') < x_n\}.$$

Further, by the same construction as in [24, Lemma 4.3] we have a diffeomorphism $\Phi \in C^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\Phi(0) = \bar{x}_R$, $\Phi = Id$ in B_4^c,

$$\|\Phi - Id\|_{C^{1,1}(\mathbb{R}^n,\mathbb{R}^n)} + \|\Phi^{-1} - Id\|_{C^{1,1}(\mathbb{R}^n,\mathbb{R}^n)} \leq \frac{C}{R}, \quad \Phi(\mathbb{R}^n_+) = U_+.$$

Next, let v be defined as in (4.6), so that Lemma 4.3 gives that

$$(-\Delta_g)^s v = f \text{ weakly in } U_+ \text{ and } \|f\|_\infty \leq \frac{C}{R}.$$

and note that, by the properties of Φ,

$$v(x) = u_0(x_n) \text{ in } B_4^c, \quad v \leq 4^s \text{ in } B_4.$$

Let us further truncate $\hat{v} := \min\{v, 5^s\}$ and observe that

$$v(x) - \hat{v}(x) = (x_n)_+^s - 5^s \text{ in } \{x_n \geq 5\}, \quad v - \hat{v} = 0 \text{ in } \{x_n < 5\}$$

and in particular $v - \hat{v}$ vanishes identically in B_4 so that, using Lemma C.5

$$(-\Delta_g)^s \hat{v} = (-\Delta_g)^s(v + (\hat{v} - v)) = f + h \text{ weakly in } B_4$$

with

$$\begin{aligned} h(x) &= 2\int_{B_4^c} \left[g\left(\frac{v(x) - \hat{v}(y)}{|x-y|^s} \right) - g\left(\frac{v(x) - v(y)}{|x-y|^s} \right) \right] \frac{1}{|x-y|^{n+s}}\, dy \\ &\geq 2\int_{\{y_n \geq 5\}} \left[g\left(\frac{(x_n)_+^s - 5^s}{|x-y|^s} \right) - g\left(\frac{(x_n)_+^s - (y_n)_+^s}{|x-y|^s} \right) \right] \frac{1}{|x-y|^{n+s}}\, dy \end{aligned}$$

for $x \in B_4$. Choosing an appropriate constant η as in the proof of Lemma 4.6 we have

$$(-\Delta_g)^s \hat{v} = f + g \geq -\frac{C}{R} + \eta \text{ weakly in } U_+ \cap B_4$$

and taking R large enough we get that

$$(-\Delta_g)^s \hat{v} \geq \frac{\eta}{2} \text{ weakly in } U_+ \cap B_2(\bar{x}_R). \tag{4.11}$$

We are ready to estimate $\hat{v}$. To do that, let us define

$$d_R(x) := (|x - \tilde{x}_R| - R)_+$$

and notice that we can immediately find $\tilde{c} > 1$ such that

$$\hat{v}(x) \leq \tilde{c} d_R^s(x) \quad \text{ for any } x \in \mathbb{R}^n. \tag{4.12}$$

In fact, no equation is used here since $\hat{v}$ vanishes in $U_+^c \supset B_R(\tilde{x}_R)$ and it is $s-$Hölder continuous in $\mathbb{R}^n$.

To get the lower bound, notice that if $x \in B_1(\bar{x}_R)$ then either

$$x \in B_1(\bar{x}_R) \cap U_+^c \subset B_R(\tilde{x}_R)$$

and $d_R^s(x) = \tilde{c}\hat{v}(x) = 0$ or

$$x \in B_1(\bar{x}_R) \cap U_+ \subset B_R^c(\tilde{x}_R).$$

In the latter case, we can let (X', X_n) be such that $x = \Phi(X)$, $Z = (X', 0)$ and $z = \Phi(Z)$. Then $|X'| < 1$ and $z \in \partial B_R(\tilde{x}_R)$ so that

$$d_R^s(x) \leq |x - z|^s \leq \tilde{c}|X - Z|^s = \tilde{c} X_n^s = \tilde{c}\hat{v}(x)$$

so, taking $\tilde{c} > 1$ bigger if needed

$$\hat{v} \geq \frac{1}{\tilde{c}} d_R^s \quad \text{ in } B_R(\tilde{x}_R). \tag{4.13}$$

We want to extend (4.13) but keeping (4.11) and (4.12). Take $\varepsilon \in (0, 1/\tilde{c})$ and set

$$v_\varepsilon := \max\{\hat{v}, \varepsilon d_R^s\}.$$

We have that v_ε satisfies the corresponding estimates (4.12) and (4.13) with a constant $c_\varepsilon = \max\{\tilde{c} + \varepsilon, \varepsilon^{-1}\}$ and further

$$\hat{v} \leq v_\varepsilon \leq \hat{v} + \varepsilon d_R^s \text{ in } \mathbb{R}^n, \quad v_\varepsilon - \hat{v} = 0 \text{ in } B_1(\bar{x}_R).$$

Therefore, using again Lemma C.5 together with (4.11) and Lemma A.2 we have

$$\begin{aligned}(-\Delta_g)^s v_\varepsilon &= (-\Delta_g)^s \hat{v} - 2\int_{B_{1/2}^c(\bar{x}_R)} \left[g\left(\frac{\hat{v}(x) - \hat{v}(y)}{|x-y|^s}\right) - g\left(\frac{\hat{v}(x) - v_\varepsilon(y)}{|x-y|^s}\right)\right] \frac{1}{|x-y|^{n+s}}\, dy \\ &\geq \frac{\eta}{2} - C\int_{B_1^c(\bar{x}_R)} \frac{\max\{\varepsilon d_R^s(x), g(\varepsilon d_R^s(x))\}}{|x-y|^{n+2s}}\, dy.\end{aligned}$$

Noticing that the second term is finite and vanishes as $\varepsilon \to 0^+$ independently of x, we can choose ε small enough so that

$$(-\Delta_g)^s v_\varepsilon \geq \frac{\eta}{4} \quad \text{ weakly in } B_{1/2}(\bar{x}_R) \setminus B_R(\tilde{x}_R).$$

Finally, up to proceeding as in the proof of Lemma 4.6 if needed, the function $w(x) := v_\varepsilon(\tilde{x}_R + Rx)$ fulfills the desired properties. □

Now we can prove the main result of this section:

Theorem 4.9. *Let $u \in W_0^{s,G}(\Omega)$ be a weak solution of*

$$|(-\Delta_g)^s u| \leq K \quad \text{in } \Omega$$

for some $K > 0$.

Then

$$|u| \leq C d^s \quad \text{a.e. in } \Omega \tag{4.14}$$

where C is a positive constant depending only on $s, n, \lambda, \Lambda, K, g$ and Ω.

Proof. Thanks to Proposition 4.7, and by taking a larger constant C if needed, it is enough to show (4.14) in a neighborhood of $\partial\Omega$. Let

$$U := \left\{ x \in \Omega : d(x) < \frac{R\rho}{2} \right\}$$

where R is given in Lemma 4.8 and ρ is given in Lemma 4.4. Let $\bar{x} \in U$ and $x_0 \in \partial\Omega$ at minimal distance from $\bar{x}$.

According to the referred lemmata, there exist two balls $B_{\rho/2}(x_1)$ and $B_\rho(x_2)$ which are tangent to $\partial\Omega$ and a function in $C^s(\mathbb{R}^n)$ such that

$$(-\Delta_g)^s w \geq \eta \quad \text{weakly in } B_{R\rho/2}(x_0) \setminus B_{\rho/2}(x_1) \tag{4.15}$$

and

$$c^{-1}\delta^s \leq w \leq c\delta^s \quad \text{in } \mathbb{R}^n \tag{4.16}$$

where $\delta(x) = \text{dist}(x, B^c_{\rho/2}(x_1))$. Recall that from Lemma 4.4 we also have

$$\delta(\bar{x}) = d(\bar{x}) = |\bar{x} - x_0| \tag{4.17}$$

and that further

$$\delta(x) \geq \theta > 0 \quad \text{in } B^c_\rho(x_2) \setminus B_{R\rho/2}(x_0)$$

for a constant θ depending only on Ω. This inequality, together with(4.16) and the fact that $\Omega \subset B^c_\rho(x_2)$ gives

$$w(x) \geq c^{-1}\theta^s \quad \text{in } \Omega \setminus B_{R\rho/2}(x_0)$$

and we may assume without loss of generality that $c^{-1}\theta^s < 1$.

We want to apply the comparison principle in the open set $V := \Omega \cap B_{R\rho/2}(x_0)$; set

$$M := \frac{c}{\theta^s} g^{-1}\left(\frac{C}{\eta}\right) \quad \text{and } \bar{w} = Mw$$

where C is the constant from (4.10).

Recall again that we can increase the constants if needed and get

$$(-\Delta_g)^s \bar{w} \geq (-\Delta_g)^s \quad \text{weakly in } V$$

and since, by construction, $\bar{w} \geq u$ in V^c the comparison principle and (4.16) give

$$u(x) \leq \bar{w}(x) \leq cM\delta^s(x) \quad \text{a.e. in } \mathbb{R}^n$$

so recalling (4.17) we have

$$u(\bar{x}) \leq cM\delta^s(\bar{x}) = cMd^s(\bar{x}) \quad \text{for any } \bar{x} = x_0 - t\nu_{x_0},\ t \in \left[0, \frac{R\rho}{2}\right]$$

where ν_{x_0} is the exterior unit normal to $\partial\Omega$ at x_0. A similar argument applied to $-u$ gives the other bound and the result is proven. □

5. Proof of Theorem 1.1

In this section we give the proof of our main result:

Proof of Theorem 1.1. We set $K = \|f\|_{L^\infty(\Omega)}$. By Proposition 4.7 we have that

$$\|u\|_{L^\infty(\Omega)} \leq C$$

where C is a positive constant depending only on $s, n, \lambda, \Lambda, K$ and $\operatorname{diam}(\Omega)$.

Let us deal with the Hölder seminorm. Let $\alpha \in (0, s]$ be the exponent given in Corollary 3.6. Through a covering argument, inequality (3.10) implies that $u \in C^\alpha_{loc}(\overline{\Omega'})$ for all Ω' compactly contained in Ω, with a bound of the form

$$\|u\|_{C^\alpha(\overline{\Omega'})} \leq C_{\Omega'} g^{-1}(K), \qquad C_{\Omega'} = C(n, s, \lambda, \Lambda, \Omega, \Omega').$$

Therefore, it suffices to prove (1.5) in the closure of a fixed $\rho-$neighborhood of $\partial\Omega$. Assume that $\rho = \rho(\Omega) > 0$ is small enough such that Lemma 4.4 holds, and thus the metric projection

$$\Pi\colon V \to \partial\Omega, \qquad \Pi(x) = \operatorname*{Argmin}_{y \in \Omega^c} |x - y|$$

is well defined on $V := \{x \in \overline{\Omega}\colon d(x) \leq \rho\}$. We claim that

$$[u]_{C^\alpha(B_{r/2})} \leq C_\Omega \quad \text{for all } x \in V \text{ and } r = d(x) \tag{5.1}$$

for some constant $C_\Omega = C(n, s, \lambda, \Lambda, \Omega, K)$, independent on $x \in V$. Recall that Corollary 3.6 states that

$$[u]_{C^\alpha(B_{r/2}(x))} \leq C(r^{s-\alpha} g^{-1}(K) + r^{-s}\|u\|^\beta_{L^\infty(B_r(x))} + r^{-s}\operatorname{Tail}_{p^+}(u; x, r)^\beta + r^{-s}\operatorname{Tail}_{p^-}(u; x, r)^\beta).$$

where C is a constant depending on n, s, λ, Λ, Ω. The first term in the right hand side of the previous inequality can be bounded as

$$g^{-1}(K) r^{s-\alpha} \leq g^{-1}(K)\rho^{s-\alpha} \leq C(K, \lambda, \Lambda)\rho^{s-\alpha}.$$

For the second one we use Theorem 4.9 and the fact that $\alpha \leq s$ to obtain

$$\|u\|_{L^\infty(B_r(x))} \leq C(d(x) + r)^s \leq C\rho^{s-\alpha} r^\alpha.$$

The third term can be bounded by using again Theorem 4.9 together with

$$d(x) \leq |y - \Pi(x)| \leq |y - x| + |x - \Pi(x)| \leq |y - x| + r \leq 2|x - y|, \quad \forall y \in B^c_r(x),$$

to obtain

$$\begin{aligned}\mathrm{Tail}_{p^+}(u;x,r)^{(p^+-1)\beta} &\leq r^{sp^+}C^{p^+-1}\int_{B_r^c}\frac{d^{s(p^+-1)(y)}}{|x-y|^{n+sp^+}}\,dy\\ &\leq r^{sp^+}C^{p^+-1}\int_{B_r^c}\frac{|x-y|^{s(p^+-1)(y)}}{|x-y|^{n+sp^+}}\,dy\\ &\leq r^{sp^+}C^{p^+-1}r^{s(p^+-1)}\end{aligned}$$

and the desired bound follows. The last term can be bounded analogously, and the proof of claim (5.1) is completed.

To prove the theorem, pick $x, y \in V$ and suppose without loss of generality that $|x-\Pi(x)| \geq |y-\Pi(y)|$. Two situations are possible: either $2|x-y| < |x-\Pi(x)|$, in which case we set $r = d(x)$ and apply (5.1) in $B_{r/2}(x)$ to get

$$|u(x)-u(y)| \leq C|x-y|^\alpha;$$

or $2|x-y| \geq |x-\Pi(x)| \geq |y-\Pi(y)|$, in which case Theorem 4.9 ensures that

$$\begin{aligned}|u(x)-u(y)| &\leq |u(x)|+|u(y)| \leq C(d^s(x)+d^s(y))\\ &\leq C(|x-\Pi(x)|^s+|y-\Pi(y)|^s)\\ &\leq C(|x-y|^s)\\ &\leq C\rho^{s-\alpha}(|x-y|^\alpha).\end{aligned}$$

Therefore, the $\alpha-$Hölder seminorm is bounded in V, which concludes the proof. □

Appendix A. Some inequalities for Young functions

In this appendix we gather for the reader's convenience some technical results that were used throughout the paper whose proofs are either simple or straightforward but we include for completeness.

Lemma A.1. *For every $a, b > 0$ it holds that*

$$g(a-b)-g(a) \leq -2^{1-\Lambda}g(b).$$

Proof. By using (Δ_2) and (g_4) we get

$$\begin{aligned}g(b) = g\left(2\frac{b-a+a}{2}\right) &\leq 2^\Lambda g\left(\frac{b-a+a}{2}\right)\\ &\leq 2^{\Lambda-1}(g(b-a)+g(a)) = 2^{\Lambda-1}(g(a)-g(a-b))\end{aligned}$$

where in the last inequality we have used that g can be extended as an odd function. □

Lemma A.2. *Fix $M > 0$. There exists $C = C_M > 0$ such that for every $|a| \leq M$ and $b > 0$ it holds that*

$$g(a)-g(a-b) \leq C_M \max\{b, g(b)\}.$$

Proof. We separate two cases: if $b \leq M$ we have
$$g(a) - g(a-b) \leq |g'(M)||b|$$
while if $b \geq M$ using (2.5) we get
$$g(a) - g(a-b) \leq g(M) + g(2M) \leq C_M g(b).$$
□

Lemma A.3. *For any $a, b \geq 0$ and $\theta \in (0,1)$*
$$g(a+b) \leq (1+\theta)^\Lambda g(a) + C_\theta g(b)$$
where $C_\theta \to \infty$ as $\theta \to 1^+$.

Proof. Given $\theta > 0$ and $a, b > 0$ (if either is equal to 0 the result is trivial).

If $b > \theta a$, due to the monotonicity of g and (Δ_2) we have, for $j_\theta \in \mathbb{N}$ large enough
$$g(a+b) \leq g\left(\left(\frac{1}{\theta}+1\right)b\right) \leq g(2^{j_\theta} b) \leq 2^{j_\theta \Lambda} g(b).$$
On the other hand, if $b \leq \theta a$
$$g(a+b) \leq g\left((1+\theta)a\right) \leq (1+\theta)^\Lambda g(a)$$
and the lemma is proved. □

Lemma A.4. *For all $a, b \geq 0$*
$$g^{-1}(a+b) \leq 2^{\frac{1}{\lambda}}(g^{-1}(a) + g^{-1}(b)).$$

Proof. It follows since it can be seen that
$$\frac{1}{\Lambda} \leq \frac{t(g^{-1})'(t)}{g^{-1}(t)} \leq \frac{1}{\lambda}.$$
□

Aside from the previous inequalities regarding Young functions, we will use a simple property of sets which are at positive distance from each other; recall that given $A, B \subset \mathbb{R}^n$ we define the distance between them as
$$\operatorname{dist}(A,B) := \inf_{x \in A, y \in B} |x-y|.$$

Lemma A.5. *If $A, B \subset \mathbb{R}^n$, with A bounded and dist$(A, B^c) = d > 0$, then*
$$|x-y| \geq C(A,B)(1+|y|), \qquad x \in A, y \in B^c.$$

Proof. Assume $A \subset B_R$ for some $R > 0$ and set
$$C = C(A,B) := \frac{1+R}{d}.$$
Now just compute
$$1+|y| \leq 1+|x|+|y-x| \leq 1+R+|x-y| = Cd + |x-y| \leq (1+C)|x-y|$$
which gives the result. □

Appendix B. Relation between weak, pointwise and strong solutions

In this section show the (expected) relation between weak, pointwise and strong solutions. The following lemma ensures that the definition of weak solution (Definition 2.2) makes sense:

Lemma B.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in \widetilde{W}^{s,G}(\Omega)$. Define*

$$\langle (-\Delta_g)^s u, \varphi \rangle := \iint_{\mathbb{R}^n \times \mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{(\varphi(x)-\varphi(y))}{|x-y|^s}\, d\mu$$

for $\varphi \in W_0^{s,G}(\Omega)$. Then the $\langle (-\Delta_g)^s u, \cdot \rangle \in W^{-s,\tilde{G}}(\Omega)$.

Proof. Let $U \supset\supset \Omega$ such that

$$\|u\|_{W^{s,G}(U)} + \int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right) \frac{dx}{(1+|x|)^{n+s}} < \infty.$$

Now, for $\varphi \in W_0^{s,G}(\Omega)$ we can write

$$\begin{aligned}\langle (-\Delta_g)^s u, \varphi \rangle &= \int\int_{U\times U} g\left(\frac{u(x)-u(y)}{|x-y|^s}|\right) \frac{(\varphi(x)-\varphi(y))}{|x-y|^{n+s}}\, dxdy \\ &\quad + 2\int\int_{\Omega\times U^c} g\left(\frac{u(x)-u(y)}{|x-y|^s}|\right) \varphi(x)\frac{1}{|x-y|^{n+s}}\, dxdy \\ &= I_1 + I_2.\end{aligned}$$

First, for I_1 we have

$$\begin{aligned}I_1 &\leq \int\int_{U\times U} \tilde{G}\left(g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right) \frac{dxdy}{|x-y|^n} + \int\int_{U\times U} G\left(\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n} \\ &\leq (p-1)\int\int_{U\times U} G\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} + \int\int_{U\times U} G\left(\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n}\end{aligned}$$

where in the last line we used Lemma 2.9 in [18]. Then, I_1 is finite and continuous with respect to the strong convergence.

As for I_2, we compute, using Lemmas A.3 and A.5 and the fact that g is increasing,

$$\begin{aligned}\int_{U^c} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{1}{|x-y|^{n+s}}\, dy &\leq \left(\frac{3}{2}\right)^{\Lambda} \int_{U^c} g\left(\frac{|u(x)|}{|x-y|^s}\right)\frac{1}{|x-y|^{n+s}}\, dy \\ &\quad + C\int_{U^c} g\left(\frac{|u(y)|}{|x-y|^s}\right)\frac{1}{|x-y|^{n+s}}\, dy \\ &\leq C\left(g(|u(x)|)\int_{U^c} \frac{\max\{|x-y|^{-s\lambda}, |x-y|^{-s\Lambda}\}}{|x-y|^{n+s}}\, dy\right. \\ &\quad \left. + \int_{\mathbb{R}^n} g\left(\frac{|u(y)|}{(1+|y|)^s}\right)\frac{1}{(1+|y|)^{n+s}}\, dy\right).\end{aligned}$$

Notice that $|x-y|$ on the first integral of the last term is bounded from below by $\mathrm{dist}(U^c, \Omega) > 0$, so both integrals are finite and we conclude the proof. □

The next lemma will be used to show that strong solutions are also weak solutions, but it will also be useful as stated below. First we need to recall the notion of Hausdorff distance between sets:

$$d_H(A,B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(A,y)\right\}.$$

Lemma B.2. *Let $u \in \widetilde{W}^{s,G}_{loc}(\Omega)$ and let $A_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}^n$ be a neighborhood of*

$$\mathbf{D} := \{x = y\}$$

such that

(i) *$(x,y) \in A_\varepsilon$ then $(y,x) \in A_\varepsilon$;*
(ii) *$d_H(A_\varepsilon, \mathbf{D}) \longrightarrow 0$ as $\varepsilon \to 0^+$.*

Consider for any $x \in \mathbb{R}^n$

$$h_\varepsilon(x) := \int_{A^c_\varepsilon(x)} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{dy}{|x-y|^{n+s}}$$

where

$$A^c_\varepsilon(x) := \{y \in \mathbb{R}^n : (x,y) \in A_\varepsilon\}.$$

If $2h_\varepsilon \longrightarrow f$ in $L^1_{loc}(\Omega)$ as $\varepsilon \to 0^+$ then $(-\Delta_g)^s u = f$ weakly in Ω.

Proof. We may assume Ω is bounded and that $U \supset\supset \Omega$ is such that

$$\|u\|_{W^{s,G}(U)} + \int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right) \frac{dx}{(1+|x|)^{n+s}} < \infty.$$

Further, by density it is enough to show that

$$\int\int g\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{\varphi(x)-\varphi(y)}{|x-y|^s}\, d\mu = \int_\Omega f\varphi\, dx$$

holds for any $\varphi \in C^\infty_c(\Omega)$. Let φ be such and denote its support by K.

Let us show first that $h_\varepsilon \in L^1(K)$. Notice that given $x \in K$ there exists $\rho > 0$ such that $B_\rho(x) \subset A_\varepsilon(x)$ and that such ρ can be taken independently of x (but not necessarily of ε) via a covering argument. We can compute, similarly to Lemma

B.1,

$$\begin{aligned}\int_K |h_\varepsilon(x)|\,dx &= \int_K \int_{A^c_\varepsilon(x)} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\\ &\leq C\left(\int_K \int_{A^c_\varepsilon(x)} g\left(\frac{|u(x)|}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\,dx\right.\\ &\quad\left.+\int_K \int_{A^c_\varepsilon(x)} g\left(\frac{|u(y)|}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\,dx\right)\\ &\leq C\left(\int_K \int_{B^c_\rho} g\left(\frac{|u(x)|}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\,dx\right.\\ &\quad\left.+|K|\int_{\mathbb{R}^n} g\left(\frac{|u(y)|}{(1+|y|)^s}\right)\frac{dy}{(1+|y|)^{n+s}}\,dx\right)<\infty.\end{aligned}$$

On the other hand, Lemma B.1 shows that

$$g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\in L^1(\mathbb{R}^n\times\mathbb{R}^n,d\mu)$$

and therefore, by using our hypothesis we get that

$$\begin{aligned}&\iint g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\frac{dxdy}{|x-y|^n}\\ &\qquad= \lim_{\varepsilon\to 0^+}\iint_{A^c_\varepsilon} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\frac{dxdy}{|x-y|^n}\\ &\qquad= \lim_{\varepsilon\to 0^+} 2\int_K\int_{A^c_\varepsilon(x)} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\varphi(x)\frac{dydx}{|x-y|^{n+s}}\\ &\qquad= \lim_{\varepsilon\to 0^+} 2\int_K h_\varepsilon(x)\varphi(x)\,dx\end{aligned}$$

and the result follows since $2h_\varepsilon(x)\longrightarrow f$ in $L^1(K)$. □

Corollary B.3. *Let $u\in \widetilde{W}^{s,G}_{loc}(\Omega)$ be a strong solution to $(-\Delta_g)^s u = f$ in Ω with $f\in L^1_{loc}(\Omega)$. Then u is also a weak solution.*

Proof. The proof is a direct application of Lemma B.2 with

$$A_\varepsilon=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |x-y|<\varepsilon\}.$$

□

Appendix C. Properties of $(-\Delta_g)^s$

In this last section we prove some properties of the operator $(-\Delta_g)^s$ that are used repeatedly in the paper. We start with two lemmata regarding the behavior under scaling and rotation:

Lemma C.1. *Let u be solution of*

$$\begin{cases}(-\Delta_g)^s u = f & \text{in } B_R\\ u = 0 & \text{in } B_R^c.\end{cases}$$

If we define, for $x \in \mathbb{R}^n$ and $t \geq 0$

$$u^R(x) = u\left(Rx\right), \quad \tilde{f}(x) = f\left(Rx\right), \quad g_R(t) = g\left(R^{-s}t\right),$$

then u_R solves

$$\begin{cases}(-\Delta_{g_R})^s u^R = \tilde{f} & \text{in } B_1\\ u^R = 0 & \text{in } B_1^c.\end{cases}$$

In particular, if $|(-\Delta_g)^s u| \leq K$ in B_R, then $|(-\Delta_{g_R})^s u^R| \leq K$ in B_1.

Proof. The proof that u^R solves the equation is a straightforward change of variables $(\tilde{x}, \tilde{y}) = (Rx, Rx)$ (recall the definition of μ)

$$\begin{aligned}\left\langle(-\Delta_g)^s u^R, \varphi\right\rangle &= \int\int g\left(R^{-s}\frac{|u\left(Rx\right) - u\left(Rx\right)|}{|x-y|^s}\right)\frac{u\left(Rx\right)-u\left(Rx\right)}{|u\left(Rx\right)-u\left(Rx\right)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu\\ &= R^{-n}\int\int g\left(\frac{|u(\tilde{x})-u(\tilde{y})|}{|\tilde{x}-\tilde{y}|^s}\right)\frac{u(\tilde{x})-u(\tilde{y})}{|u(\tilde{x})-u(\tilde{y})|}\frac{\varphi\left(\frac{\tilde{x}}{R}\right)-\varphi\left(\frac{\tilde{y}}{R}\right)}{|\tilde{x}-\tilde{y}|^s}\,d\mu\\ &= R^{-n}\left\langle(-\Delta_g)^s u, \varphi\left(\frac{\cdot}{R}\right)\right\rangle.\end{aligned}$$

On the other hand

$$\int \tilde{f}\varphi\,dx = \int f(Rx)\varphi(x)\,dx = R^{-n}\int f(\tilde{x})\varphi\left(\frac{\tilde{x}}{R}\right)\,d\tilde{x}$$

and we conclude. □

Remark C.2. Observe that our scaling preserves ellipticity; given g satisfying (2.1) and $R > 0$, the function $g_R(t) = g(R^{-s}t)$, $t \geq 0$ satisfies also (2.1). Indeed,

$$\frac{t(g_R(t))'}{g_R(t)} = \frac{\tau g'(\tau)}{g(\tau)}$$

where $\tau = R^{-s}t$. This simple remark is of paramount importance as it allows us to prove our estimates in, say, B_1 and obtain the general results by scaling.

Lemma C.3. *Let $u \in \widetilde{W}^{s,G}(\Omega)$ be a weak solution of $(-\Delta_g)^s u = f$ in Ω for some $f \in L^1_{loc}(\Omega)$.*

Then for any orthogonal matrix $O \in \mathbb{R}^{n\times n}$, $u_O(x) := u(Ox) \in \widetilde{W}^{s,G}(O^{-1}\Omega)$ and

$$(-\Delta_g)^s u_O = f_O \quad \text{weakly in } O^{-1}\Omega.$$

Proof. Let $(\tilde{x},\tilde{y})=(Ox,Ox)$ and change variables (recall orthogonal matrices preserve norms)

$$\begin{aligned}\langle(-\Delta_g)^s u_O,\varphi\rangle &= \int\int g\left(\frac{|u\left(Ox\right)-u\left(Ox\right)|}{|x-y|^s}\right)\frac{u\left(Ox\right)-u\left(Ox\right)}{|u\left(Ox\right)-u\left(Ox\right)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu\\ &= \int\int g\left(\frac{|u(\tilde{x})-u(\tilde{y})|}{|\tilde{x}-\tilde{y}|^s}\right)\frac{u(\tilde{x})-u(\tilde{y})}{|u(\tilde{x})-u(\tilde{y})|}\frac{\varphi\left(O^{-1}\tilde{x}\right)-\varphi\left(O^{-1}\tilde{y}\right)}{|O^{-1}\tilde{x}-O^{-1}\tilde{y}|^s}\,d\mu\\ &= \left\langle(-\Delta_g)^s u,\varphi\left(O^{-1}\cdot\right)\right\rangle.\end{aligned}$$

On the other hand

$$\int_\Omega f_O\varphi\,dx=\int f(Ox)\varphi(x)\,dx=\int_{O^{-1}\Omega} f(\tilde{x})\varphi\left(O^{-1}\tilde{x}\right)\,d\tilde{x}$$

and we conclude. □

Next, we prove a comparison principle for weak solutions, its proof follows the one given in [28]:

Proposition C.4 (Comparison principle)**.** *Let Ω be bounded, $u,v\in\widetilde{W}^{s,G}(\Omega)$ such that $u\le v$ in Ω^c and*

$$\langle u,\varphi\rangle\le\langle v,\varphi\rangle\qquad\forall\varphi\in W^{s,p}_0(\Omega),\quad\varphi\ge 0\ \textit{in}\ \Omega,$$

Then $u\le v$ in Ω.

Proof. By hypothesis we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu\le\iint_{\mathbb{R}^n\times\mathbb{R}^n} g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu$$

for any $\varphi\in W^{s,G}_0(\Omega)$.

Subtracting the left hand side of the previous inequality to the right hand side and using

$$g(b)-g(a)=(b-a)\int_0^1 g'(a+t(b-a))\,dt=(b-a)Q(x,y)$$

for $b=\frac{v(x)-v(y)}{|x-y|^s}$ and $a=\frac{u(x)-u(y)}{|x-y|^s}$

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} Q(x,y)\frac{((u(y)-v(y)-(u(x)-v(x))(\varphi(x)-\varphi(y))}{|x-y|^{2s}}\,d\mu\ge 0$$

with

$$Q(x,y):=\int_0^1 g'\left(\frac{u(x)-u(y)}{|x-y|^s}+t\left(\frac{v(x)-v(y)}{|x-y|^s}-\frac{u(x)-u(y)}{|x-y|^s}\right)\right)\,dt.$$

Notice that Q is nonnegative.

Now use $\varphi = (u-v)_+$ as a test function and note that, calling $w = u - v$,

$$\begin{aligned}((u(y)-v(y)-(u(x)-v(x))(\varphi(x)-\varphi(y)) &= -(w(x)-w(y))(w_+(x)-w_+(y)) \\ &= -(w_+(x)-w_+(y))^2 \\ &- w_-(x)w_+(y) - w_-(y)w_+(x) \\ &\leq 0.\end{aligned}$$

Therefore

$$Q(x,y)(w(x)-w(y))(\varphi(x)-\varphi(y)) = 0 \quad \forall x, y \in \mathbb{R}^n.$$

This can only happen if either of the terms vanishes, but in all three cases we get $(u-v)_+(x) = (u-v)_+(y)$, so this equality holds identically. Since outside Ω this gives 0 we must have $(u-v)_+ \equiv 0$ in Ω as desired. □

Finally, the next Lemma is instrumental in several parts of the rest of the paper and it is strongly nonlocal in character:

Lemma C.5. *Let $u \in \widetilde{W}^{s,G}_{loc}(\Omega)$ such that solves $(-\Delta_g)^s u = f$ (weakly, strongly, pointwisely) in Ω for some $f \in L^1_{loc}(\Omega)$. Let $v \in L^1_{loc}(\mathbb{R}^n)$ be such that*

$$dist(\operatorname{supp}(v), \Omega) > 0, \qquad H_{g,\Omega^c}(u) := \int_{\Omega^c} g\left(\frac{u(x)}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}}. < \infty,$$

and define for a.e. Lebesgue point $x \in \Omega$ of u

$$h(x) = 2\int_{\operatorname{supp}(v)}\left[g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s}\right) - g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right]\frac{1}{|x-y|^{n+s}}\,dy.$$

Then $u + v \in \widetilde{W}^{s,G}_{loc}(\Omega)$ and it solves $(-\Delta_g)^s(u+v) = f + h$ (weakly, strongly, pointwisely) in Ω.

Proof. It suffices to consider Ω bounded. Let us see that $u + v \in \widetilde{W}^{s,G}_{loc}(\Omega)$. Denote $K = \operatorname{supp}(v)$ and U such that

$$\|u\|_{s,G,U} + \int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} < \infty,$$

and without loss of generality that $\Omega \subset\subset U \subset\subset K^c$. It follows that $u + v = u$ in U and it belongs to $W^{s,G}(U)$. Moreover, in light of (Δ_2)

$$\begin{aligned}\int_{\mathbb{R}^n} g\left(\frac{|u(x)+v(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} \leq 2^\Lambda\Bigg(&\int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} \\ &+ \int_K g\left(\frac{|v(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}}\Bigg),\end{aligned}$$

which is finite due to the assumptions on u and v. Similarly, by using Lemma A.5 and the assumptions on u and v we get

$$h(x) \leq C((-\Delta_g)^s u(x) + H_{g,K}(v)) < \infty$$

for some constant $C > 0$ independent of u and v.

Assume that $(-\Delta_g)^s u = f$ weakly. Let $\varphi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\langle(-\Delta_g)^s(u+v),\varphi\rangle &= \iint_{\Omega\times\Omega} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu\\ &\quad+\iint_{\Omega\times\Omega^c} g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}\,d\mu\\ &\quad-\iint_{\Omega^c\times\Omega} \Phi_g\left(\frac{|u(x)+v(x)-u(y)|}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}\,d\mu.\end{aligned}$$

The last expression can be written as

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\,d\mu-\iint_{\Omega\times\Omega^c} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}\,d\mu\\ &-\iint_{\Omega^c\times\Omega} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}\,d\mu+2\iint_{\Omega\times\Omega^c} g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}\,d\mu,\end{aligned}$$

and we obtain that

$$\begin{aligned}\langle(-\Delta_g)^s(u+v),\varphi\rangle &= \int_\Omega f\varphi\,dx+\\ &\quad+2\iint_{\Omega\times\Omega^c}\left[g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s}\right)-g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right]\frac{\varphi(x)}{|x-y|^s}\,d\mu\\ &=\int_\Omega (f+h)\varphi\,dx\end{aligned}$$

where we have used Fubini's Theorem. Therefore, the result follows by a density argument.

If we have now $(-\Delta_g)^s u = f$ strongly or pointwisely in Ω. Let $x \in V \subset\subset \Omega$ and $\varepsilon <$dist(V, Ω^c), and consider

$$I_\varepsilon = \int_{B_\varepsilon^c(x)} g\left(\frac{u(x)+v(x)-u(y)-v(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}.$$

We want to take $\lim_{\varepsilon\to 0^+} I_\varepsilon$ to get the pointwise result. Since v vanishes inside Ω we have

$$\begin{aligned}I_\varepsilon &= \int_{\Omega\setminus B_\varepsilon(x)} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}+\int_{\Omega^c} g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\\ &=\int_{B_\varepsilon(x)} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\\ &\quad+\int_K\left(g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s}\right)-g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\right)\frac{dy}{|x-y|^{n+s}}\end{aligned}$$

so taking the limit gives the pointwise result. For the strong solution, we just need to be able to use Dominated Convergence Theorem; for that simply notice that

$$\int_K \left(g\left(\frac{u(x)-u(y)-v(y)}{|x-y|^s} \right) - g\left(\frac{u(x)-u(y)}{|x-y|^s} \right) \right) \frac{dy}{|x-y|^{n+s}} \in L^1(K)$$

by a similar reasoning to that of the proof of Lemma B.1. □

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(JFB and AS) Instituto de Matemática Luis A. Santaló (IMAS), CONICET, Departamento de Matemática, FCEN - Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, C1428EGA, Av. Cantilo s/n, Buenos Aires, Argentina

Email address, JFB: jfbonder@dm.uba.ar

Email address, AS: asalort@dm.uba.ar

(HV) Instituto de Matemática Luis A. Santaló (IMAS), CONICET, Centro Marplatense de Investigaciones matemáticas/Conicet, Dean Funes 3350, 7600 Mar del Plata, Argentina

Email address: havivas@mdp.edu.ar