THE BEST SOBOLEV TRACE CONSTANT IN DOMAINS WITH HOLES FOR CRITICAL OR SUBCRITICAL EXPONENTS

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Abstract. In this paper we study the best constant in the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ in a bounded smooth domain for critical or subcritical q, that is $1 < q \le 2_* = 2(N-1)/(N-2)$. First, we consider a domain with periodically distributed holes inside where we impose that the involved functions vanish. There exists a critical size of the holes for which the limit problem has an extra term. For sizes larger than critical the best trace constant diverges to infinity and for sizes smaller than critical it converges to the best constant in the domain without holes. Also, we study the problem with the holes located on the boundary of the domain. In this case other critical exists and its extra term appears on the boundary.

1. Introduction.

Sobolev inequalities have been studied by many authors and is by now a classical subject. It at least goes back to [2], for more references see [10]. Relevant for the study of boundary value problems for differential operators is the Sobolev trace inequality that has been intensively studied, see for example, [3, 11, 12, 13] and their references. Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$, we deal here with the best constant of the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ for critical or subcritical exponents, $1 < q \le 2_* := 2(N-1)/(N-2)$. When $q = 2$ this leads to an eigenvalue problem of the Steklov type (see [22]). When $q = 2_*$ this problem is related to the so called Yamabe problem for manifolds with boundaries, see [1, 3, 5, 11].

For subcritical q, $1 < q < 2_*$, the embedding is compact and hence there exists extremals [14]. When q is critical, $q = 2_*$, the embedding is continuous but no longer compact, hence the existence of extremals is more involved. In [1] it is proved that if the boundary of the domain contains a point with positive mean curvature then there is an extremal for the embedding. Hence, for any bounded smooth domain there exists en extremal even in the critical case, $q = 2_*$, see also [13].

Homogenization Theory was created to model and predict the behavior of inhomogeneous materials where inhomogeneities takes places on a small scale. The homogenization of solutions of boundary-value problems in perforated domains have

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attracted a lot of attention since the pioneering work [6]. In this paper we consider homogenization problems for the best Sobolev trace constant in perforated domains, following the approach developed in [6].

First, we consider a domain with holes periodically distributed inside the domain. That is, a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, perturbed periodically with holes located at the interior that decrease its size and increase its number as the homogenization parameter ε goes to zero, and find that there exists a critical size of the holes for which the limit problem has an extra term. For sizes larger than critical the best trace constant diverges to infinity and for sizes smaller than critical it converges to the best constant in the domain without holes. When we deal with a subcritical exponent we have compactness of the embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$, however to prove our result for $q = 2_*$ we need to impose a restriction on the involved domains in order to recover some compactness, in the spirit of [2] (see also [1, 13], etc.).

Next, we prove some homogenization results when the semi-holes are located on the boundary of the domain. In this case the critical size of the holes is different from the critical size for holes inside the domain and moreover for the critical size an extra term appears on the boundary.

1.1. **Holes in the interior.**

The Sobolev trace constant in domains with a hole (a subdomain of Ω where the functions are forced to vanish) was first studied in [14] where the authors show that there exists an optimal hole that minimizes the best constant among sets with given volume, in the class of measurable sets. On the other hand, it is also proved that a set A that maximizes S does not exist. In a subsequent paper, [15], the interior regularity of these optimal holes was studied.

For the study of the behavior of solutions of boundary value problems in domains perturbed periodically with holes complemented with homogeneous boundary conditions (either Dirichlet or Neumann) we refer, for example, to [6, 7, 8].

Let us first describe the domains that we are considering. Let $B(0, r(\varepsilon)) \subset \mathbb{R}^N$ be the ball of radius $r(\varepsilon)$, centered at 0. We assume that $r(\varepsilon) < \varepsilon$ for any $\varepsilon \leq 1$. For each ε and for any integer vector $n \in \mathbb{Z}^N$, we shall denote by B_n^ε the translated image of $B(0, r(\varepsilon))$ by the vector $2n\varepsilon$, i.e., $B_n^\varepsilon = B(0, r(\varepsilon)) + 2n\varepsilon$. Also, let us denote by $\mathbf{B}^\varepsilon$ the set of all the holes strictly contained in Ω, i.e.,

$$\mathbf{B}^\varepsilon = \bigcup \left\{ B_n^\varepsilon \mid \operatorname{dist}(2n\varepsilon, \partial\Omega) \geq \varepsilon, \quad n \in \mathbb{Z}^N \right\},$$

and we set

$$\Omega^\varepsilon = \Omega \setminus \mathbf{B}^\varepsilon.$$

Hence, Ω^ε is a periodically perforated domain with holes of size $r(\varepsilon)$. All of them have the same shape, the distance between two adjacent holes is of order ε and they do not overlap. Also, let us remark that the holes are located at a distance at least ε from the boundary, $\partial\Omega$. When the holes are allowed to touch the boundary the situation is different, see below and Section 3. Let us consider the space of functions, $H^1_\varepsilon(\Omega) = \{u \in H^1(\Omega) \ : \ u\,|_{\mathbf{B}^\varepsilon} \equiv 0\}$. The best Sobolev trace constant of the Sobolev embedding $H^1_\varepsilon(\Omega) \hookrightarrow L^q(\partial\Omega)$ is then given by

$$\lambda(\varepsilon) = \inf_{v \in H^1_\varepsilon(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 + v^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{2/q}}. \tag{1.1}$$

The extremals, normalized by

$$\int_{\partial\Omega} |u|^q\,dS = 1, \tag{1.2}$$

are positive in Ω^ε and weak solutions to

$$\begin{cases} \Delta u_\varepsilon = u_\varepsilon & \text{in } \Omega^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \lambda(\varepsilon)|u_\varepsilon|^{q-2}u_\varepsilon & \text{on } \partial\Omega, \\ u_\varepsilon = 0 & \text{in } \mathbf{B}^\varepsilon. \end{cases} \tag{1.3}$$

Our result for interior holes reads:

Theorem 1. *Let Ω_ε be a perforated domain with periodic interior holes of radius $r(\varepsilon) = c_0\varepsilon^a$, $c_0 > 0$. Let q be subcritical, $1 < q < 2_* := 2(N-1)/(N-2)$, or critical, $q = 2_*$. In the critical case we also assume that*

$$\big(\omega_N(N-2)(c_0/2)^N + 1\big)\frac{|\Omega|}{|\partial\Omega|^{2/2_*}} < \frac{1}{K(N)}, \tag{1.4}$$

where $K(N)$ is given by $K(N) := ((N-2)/2)(\omega_N)^{1/(N-1)}$ and ω_N is the volume of the unit sphere in $\mathbb{R}^N$. Then,

(1) *If $a = N/(N-2)$, then there exists a constant $\mu > 0$* (strange term) *such that the function $\lambda(\varepsilon)$ converges as $\varepsilon \to 0$ to λ_μ given by*

$$\lambda_\mu = \inf_{v \in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 + (1+\mu)v^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{2/q}}, \tag{1.5}$$

with $\mu = \omega_N(N-2)c_0^{N-2}/2^N$. Moreover, the normalized extremals u_ε converge weakly along subsequences to a normalized extremal of the limit problem (1.5)*.*

(2) *If $a > N/(N-2)$, the function $\lambda(\varepsilon)$ converges, as $\varepsilon \to 0$, to λ_0, the best Sobolev trace constant in the domain without holes, that is*

$$\lambda_0 = \inf_{v \in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 + v^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{2/q}}. \tag{1.6}$$

Moreover, the normalized extremals, u_ε, converge weakly along subsequences to a normalized extremal of the limit problem (1.6)*.*

(3) *If $a < N/(N-2)$, there holds $\lambda(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Moreover, we get a bound for the speed at which $\lambda(\varepsilon)$ goes to $+\infty$,*

$$\lambda(\varepsilon) \le C\varepsilon^{a(N-2)-N}. \tag{1.7}$$

Remark 1.1. *The constant $K(N)$ is the best Sobolev trace constant in the half-space,*

$$\frac{1}{K(N)} = \inf_{\nabla w \in L^2(\mathbb{R}^N_+),\ w \in L^{2_*}(\partial \mathbb{R}^N_+)} \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla w|^2 \, dx}{\left(\displaystyle\int_{\partial \mathbb{R}^N_+} |w|^{2_*} \, dx' \right)^{2/2_*}}. \tag{1.8}$$

The constant $K(N)$ is computed in [11].

Remark 1.2. *From the proof of Theorem 1, it can be checked that in the critical case, what is needed is that the best Sobolev trace constants, λ_ε, for the perforated domains, Ω_ε, to be bounded by $1/K(N)$ uniformly in ε, that is,*

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon) < \frac{1}{K(N)}. \tag{1.9}$$

Condition (1.4) *is the simplest condition that assures* (1.9).

We note that for every $\varepsilon > 0$ it holds $\lambda(\varepsilon) < 1/K(N)$ for every smooth bounded domain of $\mathbb{R}^N$ by [1]. *However, from their arguments, it is not obvious that this inequality can be made uniformly strict.*

Remark 1.3. *The extremals of* (1.5) *are weak solutions of*

$$\begin{cases} \Delta u = (1+\mu) u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda_\mu |u|^{q-2} u & \text{on } \partial\Omega. \end{cases}$$

Remark 1.4. *In the two-dimensional case, i.e., $\Omega \subset \mathbb{R}^2$, we have an analogous result. The critical radius is now*

$$r(\varepsilon) = \exp\left(-\frac{c_0}{\varepsilon^2} \right).$$

Thus, for radius larger than critical, the best Sobolev constant goes to ∞, for radius smaller than critical it goes to the best constant of the domain without holes and for the critical radius a strange term, μ, appears with $\mu = \pi/(2c_0)$. The proof of this fact is completely analogous to the case $N \geq 3$ with the choice of an appropriate test function, see [6].

Remark 1.5. *Theorem 1 can be generalized to other configuration of the holes. For example, we can consider non-spherical holes, cylinders, trusses, see* [6].

1.2. **Holes on the boundary.** We prove some homogenization results when the holes are located on the boundary of the domain. This problem is related with the study of the behavior of solutions of periodic mixed conditions on the boundary, Dirichlet and Neumann (see for example [9, 20]), and also, with vibration problems of systems with concentrated masses on the boundary (see for example [18, 19]).

To simplify the exposition we consider holes that are placed on a flat part of the boundary. That is, say, $\Omega \subseteq \{x_N \leq 0\}$, $\Gamma_1 := \partial\Omega \cap \{x_N = 0\}$ is the closure of a (nonempty) smooth open subset of $\mathbb{R}^{N-1}$. We consider periodically distributed semi-holes of size $r(\varepsilon) = c_0 \varepsilon^b$ located only on Γ_1. Assume that Γ_1 is divided by a reticula of size ε, $\Pi_\varepsilon = \{(2m\varepsilon, 0) \in \Gamma_1 | \text{ with } m \in \mathbb{Z}^{N-1}\}$, at each point of the reticula, $x_i \in \Pi_\varepsilon$, we take a semi-ball of size $r(\varepsilon)$,

$$S(x_i, r(\varepsilon)) = B(x_i, r(\varepsilon)) \cap \{x_N \le 0\} \subset \mathbb{R}^N,$$

and consider

$$\mathbf{S}^\varepsilon = \bigcup_{x\in\Pi_\varepsilon} S(x_i, r(\varepsilon)) \text{ such that } S(x_i, r(\varepsilon)) \cap (\partial\Omega \setminus \Gamma_1) = \emptyset,$$

and $\Omega^\varepsilon = \Omega \setminus \mathbf{S}^\varepsilon$. Thus, Ω^ε is a periodically perforated domain with holes of size $r(\varepsilon)$, the distance between two adjacent holes is of order ε and they do not overlap with $\partial\Omega \setminus \Gamma_1$. The holes are located on the boundary but they are "solid" since the semi-balls are considered in $\mathbb{R}^N$. We denote $\Gamma_1^\varepsilon = \Gamma_1 \cap \mathbf{S}^\varepsilon$. As above, we consider the space $H^1_\varepsilon(\Omega) = \{u \in H^1(\Omega) \ : \ u \,|_{\mathbf{S}^\varepsilon} \equiv 0\}$ and the best Sobolev trace constant associated to this space given by (1.1). The extremals normalized by (1.2) are weak solutions of

$$\begin{cases} \Delta u_\varepsilon = u_\varepsilon & \text{in } \Omega^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \lambda(\varepsilon)|u_\varepsilon|^{q-2}u_\varepsilon & \text{on } \partial\Omega \setminus \Gamma_1^\varepsilon, \\ u_\varepsilon = 0 & \text{in } \mathbf{S}^\varepsilon. \end{cases} \tag{1.10}$$

In this case we prove that the critical size is different from the critical size for holes inside the domain and for the critical size an extra term appears on the homogenized part of the boundary, Γ_1.

Theorem 2. *Let q subcritical, $1 < q < 2_*$ or critical, $q = 2_*$. Let Ω_ε be a perforated domain with periodic boundary holes of radius $r(\varepsilon) = c_0\varepsilon^b$.*

(1) *If $b = (N-1)/(N-2)$, the function $\lambda(\varepsilon)$ converges as $\varepsilon \to 0$ to λ_{μ_1} the best Sobolev trace constant in the domain with a weight on the boundary,*

$$\lambda_{\mu_1} = \inf_{v\in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 + v^2\,dx + \mu_1 \int_{\Gamma_1} v^2\,dS}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{2/q}}, \tag{1.11}$$

where $\mu_1 = \omega_N(N-2)c_0^{N-2}/2^N$. Moreover, the normalized extremals u_ε converge weakly along subsequences to a normalized extremal of the limit problem (1.11)*.*

(2) *If $b > (N-1)/(N-2)$, the function $\lambda(\varepsilon)$ converges as $\varepsilon \to 0$ to λ_0 the best Sobolev trace constant in the domain without holes defined in* (1.6)*. Moreover, the normalized extremals u_ε converge weakly along subsequences to a normalized extremal of the limit problem* (1.6)*.*

(3) *If $b < (N-1)/(N-2)$, the function $\lambda(\varepsilon)$ converges as $\varepsilon \to 0$ to λ_1 the best Sobolev trace constant in the Ω among functions that vanish on Γ_1,*

$$\lambda_{0,\Gamma_1} = \inf_{v\in H^1(\Omega)\setminus H^1_0(\Omega),\, v|_{\Gamma_1}\equiv 0} \frac{\displaystyle\int_\Omega |\nabla v|^2 + v^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{2/q}}. \tag{1.12}$$

Moreover, the normalized extremals u_ε converge weakly along subsequences to a normalized extremal of the limit problem (1.12)*.*

Remark 1.6. *For the existence of extremals in the problem of critical exponent, we need that $\lambda(\varepsilon)$ is uniformly bounded by $1/K(N)$ with respect to ε. However, under our geometric hypothesis, we always have a fixed point on $\partial\Omega - \Gamma_1$ (included in the boundary of Ω_ε) with positive main curvature and the distance to the holes is uniform (since we place the holes on a flat part of the boundary). So the result of* [1] *is applied and we have that* (1.9) *holds. Therefore, in Theorem 2, we do not need to impose any extra condition on the domain Ω for the critical case.*

Remark 1.7. *The extremals of* (1.11) *are weak solutions to*

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \mu_1 \chi_{\Gamma_1} u = \lambda_{\mu_1} |u|^{q-2} u & \text{on } \partial\Omega, \end{cases}$$

where χ_{Γ_1} is the characteristic function of Γ_1.

Remark 1.8. *If $\Omega \subset \mathbb{R}^2$, we have an analogous result. The critical radius is now*

$$r(\varepsilon) = \exp\left(-\frac{c_0}{\varepsilon}\right).$$

Thus, for the critical radius a strange term, μ, appears with $\mu_1 = \pi/(2c_0)$.

1.3. **Organization of the paper.** The rest of the paper is organized as follows: in Section 2 we deal with perforated domains with interior holes and finally in Section 3 we consider holes on the boundary.

2. Interior holes

In this section we prove that if we remove from Ω a periodic set of holes (where we impose that the considered functions vanish) there exists a limit of the best Sobolev trace constant if the size of the holes is not too large. We consider holes which are balls of radius $r(\varepsilon) = c_0\varepsilon^a$ (see Section 1.1).

First, we construct a sequence of appropriate test functions. Next, we prove Theorem 1 distinguishing three cases: $a = N/(N-2)$, $a > N/(N-2)$ and $1 \le a < N/(N-2)$.

2.1. **Construction of w_ε.** In this subsection, following [6], we show that there exists a sequence w_ε that verifies the following assumptions:

- (H1) $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$.
- (H2) $w_\varepsilon \equiv 0$ on the holes B_n^ε.
- (H3) There exists a distribution $\mu \in W^{-1,\infty}(\Omega)$ such that for every sequence v_ε that vanishes on the holes and converges weakly in $H^1(\Omega)$ to a limit v, one has

$$\langle -\Delta w_\varepsilon, \varphi v_\varepsilon \rangle \to \langle \mu, \varphi v \rangle, \qquad \text{for all } \varphi \in C^\infty(\overline{\Omega}), \tag{2.1}$$

where $\langle , \rangle$ is the duality pairing.

As in [6], the function w_ε is defined on each cube $P_n^\varepsilon = 2n\varepsilon + (-\varepsilon, \varepsilon)^N$, $n \in \mathbb{Z}^N$, by setting

$$\begin{cases} w_\varepsilon = 0 & \text{in } B_n^\varepsilon, \\ \Delta w_\varepsilon = 0 & \text{in } T_n^\varepsilon - B_n^\varepsilon, \\ w_\varepsilon = 1 & \text{in } P_n^\varepsilon - T_n^\varepsilon, \\ w_\varepsilon & \text{continuous at the interfaces,} \end{cases} \tag{2.2}$$

where $T_n^\varepsilon = 2n\varepsilon + B(0,\varepsilon)$, with $B(0,\varepsilon)$ the ball with radius ε and center 0. Now it is easy to compute w_ε in polar coordinates in the annulus $T_n^\varepsilon - B_n^\varepsilon$. One has

$$w_\varepsilon \,|_{T_n^\varepsilon - B_n^\varepsilon} = \frac{r(\varepsilon)^{-(N-2)} - r^{-(N-2)}}{r(\varepsilon)^{-(N-2)} - \varepsilon^{-(N-2)}} \qquad \text{if } N \geq 3, \tag{2.3}$$

where $r = |x - 2n\varepsilon|$. The result in [6] says:

Lemma 2.1. *If we choose $r(\varepsilon) = c_0 \varepsilon^{N/N-2}$ for $N \geq 3$ with c_0 a positive constant, then w_ε defined by* (2.2) *satisfies hypotheses* (H1)–(H3) *with*

$$\mu = \frac{\omega_N (N-2)}{2^N} c_0^{N-2}, \tag{2.4}$$

where ω_N is the surface of the unit sphere in $\mathbb{R}^N$.

In order to prove Theorem 1, we compute the L^2-norm of ∇w_ε. As the number $n(\varepsilon)$ of cubes P_n^ε with holes is about $|\Omega|/(2\varepsilon)^N$, we get

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)^N} \cong \frac{|\Omega|}{(2\varepsilon)^N} \int_{P_n^\varepsilon} |\nabla w_\varepsilon|^2 dx.$$

Thanks to (2.3) and $r(\varepsilon) < \varepsilon$, we obtain

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)^N} \cong \frac{|\Omega|}{(2\varepsilon)^N} \frac{\omega_N (N-2)}{r(\varepsilon)^{-(N-2)}}.$$

Considering that $r(\varepsilon) = c_0 \varepsilon^a$, we have

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)^N} = \begin{cases} \omega_N (N-2) (\frac{c_0}{2})^N |\Omega| \varepsilon^{a(N-2)-N} + o(\varepsilon^{a(N-2)-N}), & a \neq \frac{N}{N-2}, \\ \omega_N (N-2)(\frac{c_0}{2})^N |\Omega| + o(1), & a = \frac{N}{N-2}. \end{cases} \tag{2.5}$$

On the other hand, by the definition of B^ε, we have that every hole is contained in Ω and the distance to the boundary is larger than ε. Thus, we get

$$w_\varepsilon \equiv 1 \quad \text{and} \quad \frac{\partial w_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.6}$$

Hence,

$$\int_{\partial\Omega} |w_\varepsilon|^q \, dS = |\partial\Omega|. \tag{2.7}$$

2.2. **Case** $a = N/(N-2)$**.** Let us observe that, under the assumptions of Theorem 1, the best Sobolev trace constants of the perforated domains, $\lambda(\varepsilon)$, are bounded independently of ε. To this end, let us use w_ε as a test function in the infimum that defines $\lambda(\varepsilon)$, we get, using the estimates on w_ε (2.5) and (2.7) proved in the previous subsection,

$$\lambda(\varepsilon) \le \frac{\displaystyle\int_\Omega |\nabla w_\varepsilon|^2 + w_\varepsilon^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |w_\varepsilon|^q\,dS\right)^{2/q}} \le \frac{(\omega_N(N-2)(\frac{c_0}{2})^N|\Omega|+1)|\Omega|}{|\partial\Omega|^{2/q}} + o(1). \tag{2.8}$$

Hence, the extremals u_ε, the weak solutions of (1.3), are bounded in $H^1(\Omega)$ and we have, for a subsequence,

$$u_\varepsilon \rightharpoonup u \ge 0 \qquad \text{weakly in } H^1(\Omega),$$

As the extremals, u_ε, are weak solutions of (1.3) and since w_ε vanishes on the holes, $\psi = w_\varepsilon\varphi$ with $\varphi \in C^\infty(\overline{\Omega})$ is an admissible test function for the weak formulation of (1.3) and satisfies

$$\int_\Omega \nabla\varphi\nabla u_\varepsilon w_\varepsilon\,dx + \int_\Omega \varphi\nabla u_\varepsilon\nabla w_\varepsilon\,dx + \int_\Omega u_\varepsilon\varphi w_\varepsilon\,dx = \lambda(\varepsilon)\int_{\partial\Omega} u_\varepsilon^{q-1}\varphi w_\varepsilon\,dS. \tag{2.9}$$

Now we observe that, since $u_\varepsilon \to u$, $w_\varepsilon \to 1$ strongly in $L^2(\partial\Omega)$, we have

$$\int_{\partial\Omega} u_\varepsilon^{q-1}\varphi w_\varepsilon\,dS \to \int_{\partial\Omega} u^{q-1}\varphi\,dS.$$

Similarly, as $u_\varepsilon \to u$, $w_\varepsilon \to 1$ strongly in $L^2(\Omega)$,

$$\int_\Omega u_\varepsilon\varphi w_\varepsilon\,dx \to \int_\Omega u\varphi\,dx.$$

Moreover, since $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ and $w_\varepsilon \to 1$ strongly in $L^2(\Omega)$ we have

$$\int_\Omega \nabla\varphi\nabla u_\varepsilon w_\varepsilon\,dx \to \int_\Omega \nabla\varphi\nabla u\,dx.$$

To deal with the last term, we integrate by parts using that u_ε vanishes on $\mathbf{B}^\varepsilon$ and obtain

$$\int_{\Omega_\varepsilon} \varphi\nabla u_\varepsilon\nabla w_\varepsilon\,dx = -\int_{\Omega_\varepsilon} \nabla\varphi\nabla w_\varepsilon u_\varepsilon\,dx - \int_{\Omega_\varepsilon} \varphi u_\varepsilon\Delta w_\varepsilon\,dx + \int_{\partial\Omega} \varphi u_\varepsilon\frac{\partial w_\varepsilon}{\partial\nu}\,dS.$$

We have, by the properties (H1) and (H3) of w_ε,

$$\int_\Omega \nabla\varphi\nabla w_\varepsilon u_\varepsilon\,dx \to 0, \qquad \text{and} \qquad -\int_\Omega \varphi u_\varepsilon\Delta w_\varepsilon\,dx \to \int_\Omega \mu u\varphi\,dx.$$

Also, by (2.6),

$$\int_{\partial\Omega} \varphi u_\varepsilon\frac{\partial w_\varepsilon}{\partial\nu}\,dS = 0.$$

Now, if we assume that $\lambda(\varepsilon) \to \lambda$, we have

$$\int_\Omega \nabla\varphi\nabla u\,dx + \int_\Omega \mu u\varphi\,dx + \int_\Omega u\varphi\,dx = \lambda\int_{\partial\Omega} u^{q-1}\varphi\,dS. \tag{2.10}$$

Let us prove that $\lambda = \lambda_\mu$ defined in (1.5). We now distinguish between the subcritical and the critical cases.

Subcritical case, $1 < q < 2_*$. In this case, since the immersion $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact and $\|u_\varepsilon\|_{L^q(\partial\Omega)} = 1$, we have that $\|u\|_{L^q(\partial\Omega)} = 1$. Hence, taking $\varphi = u$ in (2.10) we get

$$\lambda = \int_\Omega |\nabla u|^2 + (1+\mu)u^2\,dx \geq \lambda_\mu.$$

Now, to prove $\lambda \leq \lambda_\mu$, let u_μ be an extremal of (1.5) and using $u_\mu w_\varepsilon$ as a test function in (1.1), we get

$$\lambda(\varepsilon) \leq \frac{\displaystyle\int_\Omega |\nabla(u_\mu w_\varepsilon)|^2 + (u_\mu w_\varepsilon)^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |u_\mu w_\varepsilon|^q\,dS\right)^{2/q}}. \tag{2.11}$$

By the results of [16] we obtain that $u_\mu \in C^\alpha(\overline{\Omega})$ and from the maximum principle and Hopf's Lemma we get that u_μ is strictly positive in $\overline{\Omega}$. Therefore the regularity results of [5] are applicable and we obtain that $u_\mu \in C^\infty(\overline{\Omega})$. Thus, by hypothesis (H1), we get

$$\int_\Omega (u_\mu w_\varepsilon)^2\,dx \to \int_\Omega u_\mu^2\,dx \quad \text{and} \quad \int_{\partial\Omega} |u_\mu w_\varepsilon|^q\,dS \to \int_{\partial\Omega} |u_\mu|^q\,dS.$$

On the other hand, we integrate by parts to obtain

$$\begin{aligned}
\int_\Omega |\nabla(u_\mu w_\varepsilon)|^2\,dx &= \int_\Omega w_\varepsilon^2 |\nabla u_\mu|^2\,dx + 2\int_\Omega u_\mu w_\varepsilon \nabla u_\mu \nabla w_\varepsilon\,dx \\
&\quad - \int_\Omega w_\varepsilon \operatorname{div}(u_\mu^2 \nabla w_\varepsilon)\,dx + \int_{\partial\Omega} w_\varepsilon u_\mu^2 \frac{\partial w_\varepsilon}{\partial\nu}\,dS \\
&= \int_\Omega w_\varepsilon^2 |\nabla u_\mu|^2\,dx - \int_\Omega u_\mu^2 w_\varepsilon \Delta w_\varepsilon dx + \int_{\partial\Omega} u_\mu^2 w_\varepsilon \frac{\partial w_\varepsilon}{\partial\nu} dS.
\end{aligned}$$

Thus, by the properties (H1) and (H3) of w_ε and as $u_\mu \in C^\infty(\overline{\Omega})$, we have

$$\int_\Omega w_\varepsilon^2 |\nabla u_\mu|^2\,dx \to \int_\Omega |\nabla u_\mu|^2\,dx \quad \text{and} \quad -\int_\Omega u_\mu^2 w_\varepsilon \Delta w_\varepsilon\,dx \to \int_\Omega \mu u_\mu^2\,dx.$$

Also, by (2.6),

$$\int_{\partial\Omega} u_\mu^2 w_\varepsilon \frac{\partial w_\varepsilon}{\partial\nu}\,dS = 0.$$

Finally, passing to the limit in (2.11), we get $\lambda \leq \lambda_\mu$ and we conclude the proof of the case $a = N/(N-2)$ for subcritical q.

Critical case, $q = 2_*$. The existence of extremals u_ε for (1.1) was proved in [1] (see also [13]). Let us prove that the weak limit of the extremals verifies $u \neq 0$. To this end we use the following theorem due to [17].

Theorem 3. *There exists a constant $B > 0$ such that,*

$$\left(\int_{\partial\Omega} v^{2_*}\,dS\right)^{2/2_*} \leq K(N)\int_\Omega |\nabla v|^2\,dx + B\int_\Omega v^2\,dx$$

for every $v \in H^1(\Omega)$. Here $K(N)$ is given by (1.8) and it is sharp.

Now, as $u_\varepsilon \geq 0$, it follows that $u \geq 0$ and, by classical regularity theory, u is smooth up to the boundary. By the strong maximum principle and Hopf's lemma, it follows that either $u > 0$ or $u \equiv 0$. In order to prove our claim, we have to rule out the possibility of $u \equiv 0$. To do this, we adapt the argument given in [13] (see also [2]) to show that $\|u\|_{L^2(\Omega)} \neq 0$. In fact, by Theorem 3, as u_ε are normalized such that $\|u_\varepsilon\|_{L^{2_*}(\partial\Omega)} = 1$, we have

$$1 = \left(\int_{\partial\Omega} u_\varepsilon^{2_*} \, d\sigma \right)^{2/2_*} \leq K(N) \int_\Omega |\nabla u_\varepsilon|^2 \, dx + B \int_\Omega u_\varepsilon^2 \, dx$$

and hence

$$1 \leq K(N)\lambda(\varepsilon) + (B - K(N)) \int_\Omega u_\varepsilon^2 \, dx. \tag{2.12}$$

From the estimate (2.8) and the hypothesis (1.4), we get (1.9). Passing to the limit $\varepsilon \to 0$ in (2.12) we arrive by (1.9) to

$$(B - K(N)) \int_\Omega u^2 \, dx > 0,$$

and the claim follows.

As before, the limit of the extremals u_ε, weak solutions of (1.3), satisfies (we assume, as before, that $\lambda(\varepsilon) \to \lambda$)

$$\int_\Omega \nabla\varphi \nabla u \, dx + \int_\Omega \mu u \varphi \, dx + \int_\Omega u\varphi \, dx = \lambda \int_{\partial\Omega} u^{2_*-1} \varphi \, dS. \tag{2.13}$$

Taking $\varphi = u$ in (2.13), we arrive at

$$\int_\Omega |\nabla u|^2 + (1+\mu)u^2 \, dx = \lambda \int_{\partial\Omega} u^{2_*} \, dS.$$

As $u \neq 0$, we have that $\lambda > 0$ and $\|u\|_{L^{2_*}(\partial\Omega)} \neq 0$. Therefore

$$\lambda_\mu \leq \frac{\displaystyle\int_\Omega |\nabla u|^2 + (1+\mu)u^2 \, dx}{\displaystyle\Big(\int_{\partial\Omega} u^{2_*} \, dS \Big)^{2/2_*}} = \lambda \Big(\int_{\partial\Omega} u^{2_*} \, dS \Big)^{1/(N-1)} \leq \lambda.$$

The reverse inequality follows exactly as in the subcritical case. ◻

2.3. **Case** $a > N/(N-2)$**.** Again, using as test functions w^ε, we get that the best Sobolev trace constants of the perforated domains, $\lambda(\varepsilon)$, are bounded independently of ε. In fact, by (2.5) and (2.7) we obtain

$$\lambda(\varepsilon) \leq \frac{((\frac{c_0}{2})^N \varepsilon^{a(N-2)-N} + 1)|\Omega|}{|\partial\Omega|^{2/q}} + o(\varepsilon^{a(N-2)-N}) \leq \frac{2|\Omega|}{|\partial\Omega|^{2/q}} + 1. \tag{2.14}$$

Thus, the extremals u_ε are weak solutions of (1.3) and are bounded in $H^1(\Omega)$. We consider a subsequence such that $\lambda(\varepsilon) \to \lambda$ and $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$.

As in the previous subsection, let us take $w_\varepsilon \varphi$ with $\varphi \in C^\infty(\overline{\Omega})$ as a test function and we get the weak formulation (2.9). We pass to the limit in this weak formulation and obtain

$$\int_\Omega \nabla\varphi \nabla u \, dx + \int_\Omega u\varphi \, dx = \lambda \int_{\partial\Omega} |u|^{q-2} u\varphi \, dS, \tag{2.15}$$

since w_ε converges strongly to 1 in $H^1(\Omega)$ thanks to (2.5) in the case $a > N/(N-2)$.

Now, as in the previous subsection, we divide the proof according the exponent q is subcritical or critical.

Subcritical case, $1 < q < 2_*$. In this case, as $\|u_\varepsilon\|_{L^q(\partial\Omega)} = 1$ and as the embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact, we conclude that $\|u\|_{L^q(\partial\Omega)} = 1$. Hence, taking $\varphi = u$ in (2.15), we get that $\lambda \geq \lambda_0$ by definition of λ_0 in (1.6).

To conclude the proof in this case let us prove that $\lambda \leq \lambda_0$. Let u_0 be an extremal of (1.6) and let us use $u_0 w_\varepsilon$ as a test function in (1.1). Thus,

$$\int_\Omega |\nabla(u_0 w_\varepsilon)|^2\,dx = \int_\Omega u_0^2|\nabla w_\varepsilon|^2\,dx + \int_\Omega w_\varepsilon^2|\nabla u_0|^2\,dx + 2\int_\Omega u_0 w_\varepsilon \nabla u_0 \nabla w_\varepsilon\,dx.$$

As $u_0 \in C^\infty(\overline{\Omega})$ (see [5]), and $\nabla w_\varepsilon \to 0$ strongly in $L^2(\Omega)$ (by (2.5)), we get

$$\int_\Omega |\nabla(u_0 w_\varepsilon)|^2\,dx \to \int_\Omega |\nabla u_0|^2\,dx.$$

Now, we pass to the limit in (2.11) to obtain

$$\lambda \leq \frac{\displaystyle\int_\Omega |\nabla u_0|^2 + u_0^2\,dx}{\displaystyle\left(\int_{\partial\Omega} |u_0|^q\,dS\right)^{2/q}} = \lambda_0.$$

This finishes the proof.

Critical case, $q = 2_*$. In this case, we need to check that $u \neq 0$, but this follows as in the previous subsection. In fact, by Theorem 3 we have

$$\begin{aligned} 1 = \left(\int_{\partial\Omega} |u_\varepsilon|^{2_*}\,dS\right)^{2/2_*} &\leq K(N)\int_\Omega |\nabla u_\varepsilon|^2\,dx + B\int_\Omega |u_\varepsilon|^2\,dx \\ &= K(N)\lambda(\varepsilon) + (B - K(N))\int_\Omega |u_\varepsilon|^2\,dx \\ &\to K(N)\lambda + (B - K(N))\int_\Omega |u|^2\,dx \end{aligned}$$

and, as from (1.9) it holds that $K(N)\lambda < 1$, the claim follows.

Finally, arguing exactly as before, we conclude that $\lambda = \lambda_0$ and that u is an extremal for λ_0. This finishes the proof. ☐

2.4. **Case** $1 \leq a < N/(N-2)$**.** In this case we have to prove $\lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Suppose, contrary to our claim, that there exists a sequence of $\varepsilon \to 0$ such that $\lambda(\varepsilon) \leq C$. Then, there exists a sequence of normalized functions $\{u_\varepsilon\}$ in the space $H^1_\varepsilon(\Omega)$ and satisfying

$$\int_\Omega |\nabla u_\varepsilon|^2 + u_\varepsilon^2\,dx \leq C. \tag{2.16}$$

Considering $n(\varepsilon)$, the number of cubes with holes P_n^ε contained in Ω, we get

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx \geq \sum_{i=1}^{n(\varepsilon)} \int_{P_i^\varepsilon} |\nabla u_\varepsilon|^2\,dx.$$

Let λ_1^ε be the Poincaré constant of Sobolev space

$$H := \{u \in H^1(B(0,\varepsilon) - B(0,r(\varepsilon))) \mid u = 0 \text{ on } \partial B(0,r(\varepsilon))\}.$$

It is shown in [21] that

$$\lambda_1^\varepsilon \geq C\frac{r(\varepsilon)^{N-2}}{\varepsilon^N} \qquad \text{for } N \geq 3. \tag{2.17}$$

Thus, we obtain

$$\sum_{i=1}^{n(\varepsilon)} \int_{P_i^\varepsilon} |\nabla u_\varepsilon|^2 \, dx \geq C\frac{r(\varepsilon)^{N-2}}{\varepsilon^N} \sum_{i=1}^{n(\varepsilon)} \int_{P_i^\varepsilon} |u_\varepsilon|^2 \, dx,$$

and, since $r(\varepsilon) = \varepsilon^a$ with $1 \leq a < N/(N-2)$, we obtain by (2.16) and passing to the limit

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{n(\varepsilon)} \int_{P_i^\varepsilon} |u_\varepsilon|^2 \, dx = 0.$$

Therefore, u_ε converges to 0 in $L^2(\Omega)$. This contradicts the normalization condition (1.2) of the sequence u_ε. Hence we obtain that $\lambda(\varepsilon) \to \infty$.

On the other hand, using w_ε as a test function in (1.1) we get, using (2.5), that $w_\varepsilon \to w$ in $L^2(\Omega)$ and (2.7),

$$\lambda(\varepsilon) \leq \frac{\displaystyle\int_\Omega |\nabla w_\varepsilon|^2 + w_\varepsilon^2 \, dx}{\left(\displaystyle\int_{\partial\Omega} |w_\varepsilon|^q \, dS\right)^{2/q}} \leq C\varepsilon^{a(N-2)-N}.$$

Thus, with these estimates we conclude (1.7). □

3. Holes on the boundary

In this Section we consider holes on the boundary. Recall that we assume that we are dealing with holes on a flat part of the boundary. We distinguish three cases: $b > (N-1)/(N-2)$, $b = (N-1)/(N-2)$ and $1 \leq b < (N-1)/(N-2)$. The proof of the critical exponent 2_* is the same that in the case of interior holes. Here, with our geometric hypothesis of the domain, (1.9) is held (see Remark 1.6).

3.1. Case $b > (N-1)/(N-2)$.

Using the same test function w_ε extended by $w_\varepsilon \equiv 1$ to the whole Ω we have

$$\|\nabla w_\varepsilon\|_{L^2(\Omega)^N} \leq \frac{Cr(\varepsilon)^{N-2}}{\varepsilon^{N-1}} = C\varepsilon^{b(N-2)-(N-1)} \to 0.$$

Hence $w_\varepsilon \to 1$ strongly in $H^1(\Omega)$. Using w_ε as a test function in the definition of $\lambda_1(\varepsilon)$ we obtain that there exists C independent of ε such that $\lambda_1(\varepsilon) \leq C$. From this point the proof follows exactly the same lines as the case $a > N/(N-2)$ in Section 2.3. However, since the holes are located on the boundary of the domain, (2.6) and (2.7) are not satisfied. We know that w_ε converges to 1 in $L^2(\partial\Omega)$. Now, we show that

$$\frac{\partial w_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.1}$$

This clearly holds on $\partial\Omega \setminus \Gamma_1$ by definition of ω_ε. Now, for any $\varphi \in L^2(\Gamma_1)$,we have

$$\int_{\Gamma_1} \varphi \frac{\partial w_\varepsilon}{\partial \nu} \, dS = \sum_{n=1}^{m(\varepsilon)} \int_{(T_n^\varepsilon - B_n^\varepsilon)\cap\Gamma_1} \varphi \frac{\partial w_\varepsilon}{\partial \nu} \, dS.$$

where $m(\varepsilon) \sim \varepsilon^{1-N}$ is the number of holes placed on Γ_1. Let us consider a single cell such that we may assume that the center of the cell is centered at $x = 0$. Since Γ_1 is considered flat, the normal unit vector is $-e_N = (0, ..., 0, -1)$. Therefore, such like w_ε is considered, we get $\langle -e_N, \nabla w_\varepsilon \rangle = 0$, and conclude (3.1).

3.2. **Case** $b = (N-1)/(N-2)$**.** As in the case of holes in the interior of Ω, the strange term comes from the term that involves Δw_ε.

We have that Δw_ε vanishes except on the spheres ∂T_i^ε and ∂B_i^ε. From the explicit form of w_ε we obtain,

$$-\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon.$$

The function γ_ε is supported on the spherical boundary of the semi-ball S_i^ε. We observe that, since $u_\varepsilon = 0$ on this region,

$$\langle \gamma_\varepsilon, u_\varepsilon \rangle = 0.$$

For other side, we know by [6] that in this case

$$\mu_\varepsilon = \sum_{i=1}^{m(\varepsilon)} \frac{\partial w_\varepsilon}{\partial n}\bigg|_{\partial T_i^\varepsilon} \delta_i^\varepsilon = \frac{(N-2)c_0^{N-2}}{1 - c_0^{N-2}\varepsilon} \sum_{i=1}^{m(\varepsilon)} \delta_i^\varepsilon, \qquad \text{in } \mathbb{R}^N,$$

where δ_i^ε are the Dirac masses supported by ∂T_i^ε for $i = 1, \dots, m(\varepsilon)$. Hence, we have

$$\int_\Omega \varphi u_\varepsilon \Delta w_\varepsilon \, dx = \sum_{i=1}^{m(\varepsilon)} \int_{\partial T_i^\varepsilon \cap \Omega} \frac{(N-2)c_0^{N-2}}{1 - c_0^{N-2}\varepsilon} \varphi u_\varepsilon \, dS.$$

Thanks to the strong convergence of u_ε to u we can pass to the limit and obtain since in [6]:

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi u_\varepsilon \Delta w_\varepsilon \, dx = \frac{\omega_N (N-2) c_0^{N-2}}{2^N} \int_{\Gamma_1} \varphi u \, dS = \mu_1 \int_{\Gamma_1} \varphi u \, dS.$$

This case is analogous to spherical holes periodically distributed on a hyperplane of $\mathbb{R}^N$ of [6]. We note that the capacity of a semi-sphere is a half of the one for the sphere. The rest of the terms can be handled as in the previous section.

3.3. **Case** $1 \le b < (N-1)/(N-2)$**.** Assume that there exists a sequence $\varepsilon \to 0$ with normalized extremals $\{u_\varepsilon\}$ in the space $H^1_\varepsilon(\Omega)$. By definitions of $\lambda(\varepsilon)$ and λ_{0,Γ_1}, we have $\lambda(\varepsilon) \le \lambda_{0,\Gamma_1}$. Therefore,

$$\limsup_{\varepsilon \to 0} \lambda(\varepsilon) \le \lambda_{0,\Gamma_1}, \tag{3.2}$$

and the extremals satisfy

$$\int_\Omega |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \, dx \le C. \tag{3.3}$$

Considering $m(\varepsilon) \sim \varepsilon^{1-N}$, the number of cells P_i^ε placed on Γ_1, we get

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx \ge \sum_{i=1}^{m(\varepsilon)} \int_{P_i^\varepsilon \cap \Omega} |\nabla u_\varepsilon|^2 \, dx \ge C \frac{\varepsilon^{b(N-2)}}{\varepsilon^N} \sum_{i=1}^{m(\varepsilon)} \int_{P_i^\varepsilon \cap \Omega} |u_\varepsilon|^2 \, dx, \tag{3.4}$$

by the Poincaré constant (2.17). Thus, we obtain

$$\sum_{i=1}^{m(\varepsilon)} \int_{P_i^\varepsilon \cap \Omega} |u_\varepsilon|^q \, dx \geq C \int_{\Gamma_1} \int_0^\varepsilon |u_\varepsilon|^2 \, dx_N \, dS_{x'}.$$

Considering the following change of variable $x_N = \varepsilon y_N$, we have

$$\int_{\Gamma_1} \int_0^\varepsilon |u_\varepsilon|^q \, dx_N dx' = \varepsilon \int_{\Gamma_1} \int_0^1 |u_\varepsilon(x', \varepsilon y_N)|^2 \, dy_N \, dS_{x'}.$$

Going back to (3.4), we get by (3.3) that

$$C \geq \frac{\varepsilon^{b(N-2)}}{\varepsilon^{N-1}} \int_{\Gamma_1} \int_0^1 |u_\varepsilon(x', \varepsilon y_N)|^2 \, dy_N \, dS_{x'}.$$

We pass to the limit as $\varepsilon \to 0$, using that $1 \leq b < (N-1)/(N-2)$, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1} \int_0^1 |u_\varepsilon(x', \varepsilon y_N)|^2 \, dy_N \, dS_{x'} = 0.$$

Hence, using the regularity of the extremals,

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1} |u_\varepsilon|^2 \, dS = 0.$$

Therefore, u_ε converges to 0 strongly in $L^2(\Gamma_1)$. This shows that every weak limit, u, of u_ε in $H^1(\Omega)$ verify $u \equiv 0$ on Γ_1. Therefore

$$\lambda_{0,\Gamma_1} \leq \|u\|^2_{H^1(\Omega)} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} = \liminf_{\varepsilon \to 0} \lambda(\varepsilon). \tag{3.5}$$

From (3.2) and (3.5) we obtain

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \lambda_{0,\Gamma_1}.$$

Moreover the above arguments show that the limit u is an extremal of (1.12). This finishes the proof. □

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