

MULTIPLE SOLUTIONS FOR THE p -LAPLACE OPERATOR WITH CRITICAL GROWTH

PABLO L. DE NÁPOLI, JULIÁN FERNÁNDEZ BONDER & ANALÍA SILVA

ABSTRACT. In this note we show the existence of at least three nontrivial solutions to the following quasilinear elliptic equation $-\Delta_p u = |u|^{p^*-2}u + \lambda f(x, u)$ in a smooth bounded domain Ω of \mathbb{R}^N with homogeneous Dirichlet boundary conditions on $\partial\Omega$, where $p^* = Np/(N-p)$ is the critical Sobolev exponent and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -laplacian. The proof is based on variational arguments and the classical concentrated compactness method.

1. INTRODUCTION.

Let us consider the following nonlinear elliptic problem:

$$(P) \quad \begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -laplacian, $1 < p < N$, $p^* = Np/(N-p)$ is the critical exponent in the Sobolev embedding and λ is a positive parameter.

Problems like (P) appears naturally in several branches of pure and applied mathematics, such as the theory of quasiregular and quasiconformal mappings in Riemannian manifolds (see [8, 19]), non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc. (see [1, 2, 3, 7]).

The purpose of this note, is to prove the existence of at least three nontrivial solutions for (P) under adequate assumptions on the source term f and the parameter λ . This result extends an old paper by Struwe [18]. A related result for the nonlinear boundary condition case can be found in [9].

Here, no oddness condition is imposed in f and a positive, a negative and a sign-changing solution are found. The proof relies on the variational principle of Ekeland (see [17]) complemented with the, by now, well known concentrated compactness method of P.L. Lions (see [15]).

One of the advantages in using Ekeland's variational principle is that it allows to split the geometry of the problem from the compactness aspect of it. This approach simplifies the one applied by M. Struwe in [18] for the subcritical case.

The use of the concentrated compactness method to deal with the p -laplacian has been used by several authors before. Probably the first result in this direction

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was obtained by J. García Azorero and I. Peral in [14]. Here, we borrow some ideas from that work.

For related results with subcritical growth, see the already mentioned [18] and more recently [4, 20].

Throughout this work, by (weak) solutions of (P) we understand critical points of the associated energy functional acting on the Sobolev space $W_0^{1,p}(\Omega)$:

$$(1) \quad \Phi(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} dx - \lambda \int_{\Omega} F(x, v) dx,$$

where $F(x, u) = \int_0^u f(x, z) dz$.

We will denote

$$(2) \quad \mathcal{F}_{\lambda}(v) = \int_{\Omega} \frac{1}{p^*} |v|^{p^*} + \lambda F(x, v) dx,$$

so the functional Φ can be rewritten as

$$\Phi(v) = \frac{1}{p} \|v\|_{W^{1,p}(\Omega)}^p - \mathcal{F}_{\lambda}(v).$$

2. ASSUMPTIONS AND STATEMENT OF THE RESULTS.

The precise assumptions on the source terms f are as follows:

- (F1) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x, 0) = 0$ for every $x \in \Omega$.
- (F2) There exist constants $p < q < p^* = Np/(N-p)$, $s > p^*/(p^* - q)$, $t = sq/(2 + (q-2)s) > p^*/(p^* - 2)$ and functions $a \in L^s(\Omega)$, $b \in L^t(\Omega)$, such that for $x \in \Omega$, $u, v \in \mathbb{R}$,

$$|f_u(x, u)| \leq a(x)|u|^{q-2} + b(x),$$

$$|(f_u(x, u) - f_u(x, v))u| \leq (a(x)(|u|^{q-2} + |v|^{q-2}) + b(x))|u - v|.$$

- (F3) There exist constants $c_1 \in (0, 1/(p^* - 1))$, $k_2 \in (p, p^*)$, $0 < c_3 < c_4$, such that for any $u \in L^q(\Omega)$ and $p < q < p^*$.

$$\begin{aligned} c_3 \|u\|_{L^q(\Omega)}^q &\leq k_2 \int_{\Omega} F(x, u) dx \leq \int_{\Omega} f(x, u)u dx \\ &\leq c_1 \int_{\Omega} f_u(x, u)u^2 dx \leq c_4 \|u\|_{L^q(\Omega)}^q \end{aligned}$$

So the main result of the paper reads:

Theorem 1. *Under assumptions (F1)–(F3), there exist $\lambda^* > 0$ depending only on n, p, q and the constant c_3 in (F3), such that for every $\lambda > \lambda^*$, there exists three different, nontrivial, (weak) solutions of problem (P). Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

3. PROOF OF THEOREM 1.

The proof uses the same approach as in [18]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that Φ has a critical point in K_i . These sets will be subsets of $C^{1,1}$ -manifolds $M_i \subset W^{1,p}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$\begin{aligned} M_1 &= \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} u_+ > 0 \text{ y } \int_{\Omega} |\nabla u_+|^p - |u_+|^{p^*} dx = \int_{\Omega} \lambda f(x, u) u_+ dx \right\}, \\ M_2 &= \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} u_- > 0 \text{ y } \int_{\Omega} |\nabla u_-|^p - |u_-|^{p^*} dx = \int_{\Omega} \lambda f(x, u) u_- dx \right\}, \\ M_3 &= M_1 \cap M_2. \end{aligned}$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of u , and $\langle \cdot, \cdot \rangle$ is the duality pairing of $W^{1,p}(\Omega)$.

Finally we define

$$\begin{aligned} K_1 &= \{u \in M_1 \mid u \geq 0\}, \\ K_2 &= \{u \in M_2 \mid u \leq 0\}, \\ K_3 &= M_3. \end{aligned}$$

First, we need a Lemma to show that these sets are nonempty and, moreover, give some properties that will be useful in the proof of the result.

Lemma 1. *For every $w_0 \in W_0^{1,p}(\Omega)$, $w_0 > 0$ ($w_0 < 0$), there exists $t_\lambda > 0$ such that $t_\lambda w_0 \in M_1$ ($\in M_2$). Moreover, $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$.*

As a consequence, given $w_0, w_1 \in W_0^{1,p}(\Omega)$, $w_0 > 0$, $w_1 < 0$, with disjoint supports, there exists $\bar{t}_\lambda, \underline{t}_\lambda > 0$ such that $\bar{t}_\lambda w_0 + \underline{t}_\lambda w_1 \in M_3$. Moreover $\bar{t}_\lambda, \underline{t}_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. We prove the lemma for M_1 , the other cases being similar.

For $w \in W_0^{1,p}(\Omega)$, $w \geq 0$, we consider the functional

$$\varphi_1(w) = \int_{\Omega} |\nabla w|^p - |w|^{p^*} - \lambda f(x, w) w dx.$$

Given $w_0 > 0$, in order to prove the lemma, we must show that $\varphi_1(t_\lambda w_0) = 0$ for some $t_\lambda > 0$. Using hypothesis (H3), we have that:

$$\varphi_1(tw_0) \geq At^p - Bt^{p^*} - \lambda c_4 C t^q$$

and

$$\varphi_1(tw_0) \leq At^p - Bt^{p^*} - \lambda c_3 C t^q,$$

where the coefficients A , B and C are given by:

$$A = \int_{\Omega} |\nabla w_0|^p dx, \quad B = \int_{\Omega} |w_0|^{p^*} dx, \quad C = \int_{\Omega} |w_0|^q dx.$$

Since $p < q < p^*$ it follows that $\varphi_1(tw_0)$ is positive for t small enough, and negative for t big enough. Hence, by Bolzano's theorem, there exists some $t = t_\lambda$ such that $\varphi_1(t_\lambda w_0) = 0$. (This t_λ needs not to be unique, but this does not matter for our purposes).

In order to give an upper bound for t_λ , it is enough to find some t_1 , such that $\varphi_1(t_1 w_0) < 0$. We observe that:

$$\varphi_1(tw_0) \leq At^p - \lambda c_3 C t^q.$$

so it is enough to choose t_1 such that $At_1^p - \lambda c_3 C t_1^q = 0$, i.e.,

$$t_1 = \left(\frac{A}{c_3 \lambda C} \right)^{1/(q-p)}.$$

Hence, again by Bolzano's theorem, we can choose $t_\lambda \in [0, t_1]$, which implies that $t_\lambda \rightarrow 0$ as $\lambda \rightarrow +\infty$. \square

For the proof of the Theorem, we need also the following Lemmas.

Lemma 2. *There exist $c_j > 0$ such that, for every $u \in K_i$, $i = 1, 2, 3$,*

$$\int_{\Omega} |\nabla u|^p dx \leq c_1 \left(\lambda \int_{\Omega} f(x, u)u dx + \int_{\Omega} |u|^{p^*} dx \right) \leq c_2 \Phi(u) \leq c_3 \left(\int_{\Omega} |\nabla u|^p dx \right).$$

Proof. As $u \in K_i$, we have that

$$\int_{\Omega} |\nabla u|^p dx = \int_{\Omega} \lambda f(x, u)u + |u|^{p^*} dx.$$

This proves the first inequality.

Now, by (F3)

$$\int_{\Omega} F(x, u) dx \leq \frac{1}{c_2} \int_{\Omega} f(x, u)u dx.$$

Furthermore,

$$\left| \lambda \int_{\Omega} F(x, u) dx \right| = \lambda \int_{\Omega} F(x, u) dx \leq \frac{1}{c_2} \int_{\Omega} \lambda f(x, u)u dx = \frac{1}{c_2} \left(\int_{\Omega} |\nabla u|^p - |u|^{p^*} dx \right),$$

so,

$$(3) \quad -\lambda \int_{\Omega} F(x, u) dx \leq \frac{1}{c_2} \left(\int_{\Omega} |\nabla u|^p - |u|^{p^*} dx \right).$$

By (3), we have:

$$\begin{aligned} \Phi(u) &= \int_{\Omega} \frac{1}{p} |\nabla u|^p - \frac{1}{p^*} |u|^{p^*} - \lambda F(x, u) dx \\ &\leq \int_{\Omega} \frac{1}{p} |\nabla u|^p - \frac{1}{p^*} |u|^{p^*} dx + \frac{1}{c_2} \left(\int_{\Omega} |\nabla u|^p - |u|^{p^*} dx \right) \\ &\leq \left(\frac{1}{c_2} + \frac{1}{p} \right) \int_{\Omega} |\nabla u|^p dx. \end{aligned}$$

This proves the third inequality.

To prove the middle inequality we proceed as follows:

$$\begin{aligned} \Phi(u) &= \int_{\Omega} \frac{1}{p} |\nabla u|^p - \frac{1}{p^*} |u|^{p^*} - \lambda F(x, u) dx \geq \int_{\Omega} \frac{1}{p} (|\nabla u|^p - |u|^{p^*}) - \lambda F(x, u) dx \\ &\geq \frac{1}{p} \int_{\Omega} \lambda f(x, u)u dx - \lambda \int_{\Omega} F(x, u) dx \geq \left(\frac{1}{p} - \frac{1}{c_2} \right) \lambda \int_{\Omega} f(x, u)u dx. \end{aligned}$$

This finishes the proof. \square

Lemma 3. *There exists $c > 0$ such that*

$$\begin{aligned} \|\nabla u_+\|_{L^p(\Omega)} &\geq c \quad \forall u \in K_1, \\ \|\nabla u_-\|_{L^p(\Omega)} &\geq c \quad \forall u \in K_2, \\ \|\nabla u_+\|_{L^p(\Omega)}, \|\nabla u_-\|_{L^p(\Omega)} &\geq c \quad \forall u \in K_3. \end{aligned}$$

Proof. By the definition of K_i , by (F3) and the Poincaré inequality we have that

$$\begin{aligned} \|\nabla u_{\pm}\|_{L^p(\Omega)}^p &= \int_{\Omega} \lambda f(x, u) u_{\pm} + |u_{\pm}|^{p^*} dx \leq C \|u_{\pm}\|_{L^q(\Omega)}^q + \|u_{\pm}\|_{L^{p^*}(\Omega)}^{p^*} \\ &\leq c_1 \|\nabla u_{\pm}\|_{L^p(\Omega)}^q + c_2 \|\nabla u_{\pm}\|_{L^p(\Omega)}^{p^*}. \end{aligned}$$

As $p < q < p^*$, this finishes the proof. \square

Lemma 4. *There exists $c > 0$ such that $\Phi(u) \geq c \|\nabla u\|_{L^p(\Omega)}^p$ for every $u \in W^{1,p}(\Omega)$ if $\|u\|_{W_0^{1,p}(\Omega)}$ is small enough.*

Proof. By (F3) and the Poincaré inequality we have

$$\begin{aligned} \Phi(u) &= \int_{\Omega} \frac{1}{p} |\nabla u|^p - \frac{1}{p^*} |u|^{p^*} - \lambda F(x, u) dx \\ &\geq \frac{1}{p} \|\nabla u\|_{L^p(\Omega)}^p - \frac{1}{p^*} \|u\|_{L^{p^*}(\Omega)}^{p^*} - C \|u\|_{L^q(\Omega)}^q \\ &\geq \frac{1}{p} \|\nabla u\|_{L^p(\Omega)}^p - C_1 (\|\nabla u\|_{L^p(\Omega)}^{p^*} + \|\nabla u\|_{L^p(\Omega)}^q) \\ &\geq C \|\nabla u\|_{L^p(\Omega)}, \end{aligned}$$

if $\|\nabla u\|_{L^p(\Omega)}$ is small enough, as $p < q < p^*$. \square

The following lemma describes the properties of the manifolds M_i .

Lemma 5. *M_i is a $C^{1,1}$ sub-manifold of $W_0^{1,p}(\Omega)$ of co-dimension 1 ($i = 1, 2$), 2 ($i = 3$) respectively. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_u W_0^{1,p}(\Omega) = T_u M_i \oplus \text{span}\{u_+, u_-\},$$

where $T_u M$ is the tangent space at u of the Banach manifold M . Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i .

Proof. Let us denote

$$\begin{aligned} \bar{M}_1 &= \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} u_+ dx > 0 \right\}, \\ \bar{M}_2 &= \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} u_- dx > 0 \right\}, \\ \bar{M}_3 &= \bar{M}_1 \cap \bar{M}_2. \end{aligned}$$

Observe that $M_i \subset \bar{M}_i$.

The set \bar{M}_i is open in $W^{1,p}(\Omega)$, therefore it is enough to prove that M_i is a smooth sub-manifold of \bar{M}_i . In order to do this, we will construct a $C^{1,1}$ function $\varphi_i : \bar{M}_i \rightarrow \mathbb{R}^d$ with $d = 1$ ($i = 1, 2$), $d = 2$ ($i = 3$) respectively and M_i will be the inverse image of a regular value of φ_i .

In fact, we define: For $u \in \bar{M}_1$,

$$\varphi_1(u) = \int_{\Omega} |\nabla u_+|^p - |u_+|^{p^*} - \lambda f(x, u) u_+ dx.$$

For $u \in \bar{M}_2$,

$$\varphi_2(u) = \int_{\Omega} |\nabla u_-|^p - |u_-|^{p^*} - \lambda f(x, u) u_- dx.$$

For $u \in \bar{M}_3$,

$$\varphi_3(u) = (\varphi_1(u), \varphi_2(u)).$$

Obviously, we have $M_i = \varphi_i^{-1}(0)$. We need to show that 0 is a regular value for φ_i . To this end we compute, for $u \in M_1$,

$$\begin{aligned} \langle \nabla \varphi_1(u), u_+ \rangle &= p \|\nabla u_+\|^p - p^* \|u_+\|^{p^*} - \lambda \int_{\Omega} f(x, u) u_+ - f_u(x, u) u_+^2 dx \\ &\leq p^* \left(\|\nabla u_+\|^p - \|u_+\|^{p^*} \right) - \lambda \int_{\Omega} f(x, u) u_+ - f_u(x, u) u_+^2 dx \\ &\leq (p^* \lambda - \lambda) \int_{\Omega} f(x, u) u_+ dx - \int_{\Omega} f_u(x, u) u_+^2 dx. \end{aligned}$$

By (F3) the last term is bounded by

$$\begin{aligned} (p^* \lambda - \lambda - \frac{\lambda}{c_1}) \int_{\Omega} f(x, u) u_+ dx &= \left(p^* - 1 - \frac{1}{c_1} \right) \left(\|\nabla u_+\|_{L^p(\Omega)}^p - \|u_+\|_{L^{p^*}(\Omega)}^{p^*} \right) \\ &\leq \left(p^* - 1 - \frac{1}{c_1} \right) \|\nabla u_+\|_{L^p(\Omega)}^p. \end{aligned}$$

Recall that $c_1 < 1/(p^* - 1)$. Now, the last term is strictly negative by Lemma 3. Therefore, M_1 is a $C^{1,1}$ sub-manifold of $W^{1,p}(\Omega)$. The exact same argument applies to M_2 . Since trivially

$$\langle \nabla \varphi_1(u), u_- \rangle = \langle \nabla \varphi_2(u), u_+ \rangle = 0$$

for $u \in M_3$, the same conclusion holds for M_3 .

To see that K_i is complete, let u_k be a Cauchy sequence in K_i , then $u_k \rightarrow u$ in $W^{1,p}(\Omega)$. Moreover, $(u_k)_{\pm} \rightarrow u_{\pm}$ in $W^{1,p}(\Omega)$. Now it is easy to see, by Lemma 3 and by continuity that $u \in K_i$.

Finally, by the first part of the proof we have the decomposition

$$T_u W^{1,p}(\Omega) = T_u M_i \oplus \text{span}\{u_+\}$$

Where $M_1 = \{u : \varphi_1(u) = 0\}$ and $T_u M_1 = \{v : \langle \nabla \varphi_1(u), v \rangle = 0\}$. Now let $v \in T_u W_0^{1,p}(\Omega)$ be a unit tangential vector, then $v = v_1 + v_2$ where $v_2 = \alpha u_+$ and $v_1 = v - v_2$. Let us take α as

$$\alpha = \frac{\langle \nabla \varphi_1(u), v \rangle}{\langle \nabla \varphi_1(u), u_+ \rangle}.$$

With this choice, we have that $v_1 \in T_u M_1$. Now

$$\langle \varphi_1(u), v_1 \rangle = 0.$$

The very same argument to show that $T_u W^{1,p}(\Omega) = T_u M_2 \oplus \langle u_- \rangle$ y $T_u W^{1,p}(\Omega) = T_u M_3 \oplus \langle u_+, u_- \rangle$.

From these formulas and from the estimates given in the first part of the proof, the uniform continuity of the projections onto $T_u M_i$ follows. \square

Now, we need to check the Palais-Smale condition for the functional Φ restricted to the manifold M_i . To this end, we need the following lemma from [14] which proves the Palais-Smale condition for the unrestricted functional below certain energy level.

Lemma 6 (J. García-Azorero, I. Peral, [14]). *Let S_p be the best Sobolev constant*

$$S_p := \inf_{\phi \in C_c^\infty(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^p dx}{\left(\int_{\mathbb{R}^N} |\phi|^{p^*} dx \right)^{p/p^*}}.$$

Then, the unrestricted functional Φ verifies the Palais-Smale condition for energy level c for every $c < \frac{1}{N} S_p^{N/2}$.

The proof of Lema 6 is based on the concentration compactness method. Now, we can prove the Palais-Smale condition for the restricted functional.

Lemma 7. *The functional $\Phi|_{K_i}$ satisfies the Palais-Smale condition for energy level c for every $c < \frac{1}{N} S_p^{N/2}$.*

Proof. Let $\{u_k\} \subset K_i$ be a Palais-Smale sequence, that is $\Phi(u_k)$ is uniformly bounded and $\nabla\Phi|_{K_i}(u_k) \rightarrow 0$ strongly. We need to show that there exists a subsequence u_{k_j} that converges strongly in K_i .

Let $v_j \in T_{u_j} W_0^{1,p}(\Omega)$ be a unit tangential vector such that

$$\langle \nabla\Phi(u_j), v_j \rangle = \|\nabla\Phi(u_j)\|_{W^{-1,p'}(\Omega)}.$$

Now, by Lemma 5, $v_j = w_j + z_j$ with $w_j \in T_{u_j} M_i$ and $z_j \in \text{span}\{(u_j)_+, (u_j)_-\}$.

Since $\Phi(u_j)$ is uniformly bounded, by Lemma 2, u_j is uniformly bounded in $W_0^{1,p}(\Omega)$ and hence w_j is uniformly bounded in $W_0^{1,p}(\Omega)$. Therefore

$$\|\nabla\Phi(u_j)\|_{W^{-1,p'}(\Omega)} = \langle \nabla\Phi(u_j), v_j \rangle = \langle \nabla\Phi|_{K_i}(u_j), v_j \rangle \rightarrow 0.$$

As w_j is uniformly bounded and $\nabla\Phi|_{K_i}(u_k) \rightarrow 0$ strongly, the inequality converges strongly to 0. Now the result follows by Lema 6. \square

We now immediately obtain

Lemma 8. *Let $u \in K_i$ be a critical point of the restricted functional $\Phi|_{K_i}$. Then u is also a critical point of the unrestricted functional Φ and hence a weak solution to (P).*

With all this preparatives, the proof of the Theorem follows easily.

Proof of Theorem 1. To prove the Theorem, we need to check that the functional $\Phi|_{K_i}$ verifies the hypotheses of the Ekeland's Variational Principle [5].

The fact that Φ is bounded below over K_i is a direct consequence of the construction of the manifold K_i .

Then, by Ekeland's Variational Principle, there existe $v_k \in K_i$ such that

$$\Phi(v_k) \rightarrow c_i := \inf_{K_i} \Phi \quad \text{and} \quad (\Phi|_{K_i})'(v_k) \rightarrow 0.$$

We have to check that if we choose λ large, we have that $c_i < \frac{1}{N} S_p^{N/2}$. This follows easily from Lemma 1. For instance, for c_1 , we have that choosing $w_0 \geq 0$,

$$c_1 \leq \Phi(t_\lambda w_0) \leq \frac{1}{p} t_\lambda^p \int_\Omega |\nabla w_0|^p dx$$

Hence $c_1 \rightarrow 0$ as $\lambda \rightarrow 0$. Moreover, it follows from the estimate of t_λ in Lemma 1, that $c_i < \frac{1}{N} S_p^{N/2}$ for $\lambda > \lambda^*(p, q, n, c_3)$. The other cases are similar.

From Lemma 6, it follows that v_k has a convergent subsequence, that we still call v_k . Therefore Φ has a critical point in K_i , $i = 1, 2, 3$ and, by construction, one of them is positive, other is negative and the last one changes sign. \square

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DEPARTAMENTO DE MATEMÁTICA, FCEYN

UBA (1428) BUENOS AIRES, ARGENTINA.

E-mail address: pdenapo@dm.uba.ar, jfbonder@dm.uba.ar, analia.silva82@gmail.com

Web-pages: <http://mate.dm.uba.ar/~pdenapo>, <http://mate.dm.uba.ar/~jfbonder>