ASYMPTOTIC BEHAVIOR FOR ANISOTROPIC FRACTIONAL ENERGIES

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Abstract. In this paper we investigate the asymptotic behavior of anisotropic fractional energies as the fractional parameter $s \in (0,1)$ approaches both $s \uparrow 1$ and $s \downarrow 0$ in the spirit of the celebrated papers of Bourgain-Brezis-Mironescu [6] and Maz'ya-Shaposhnikova [20].

Then, focusing con the case $s \uparrow 1$ we analyze the behavior of solutions to the corresponding minimization problems and finally, we also study the problem where a homogenization effect is combined with the localization phenomena that occurs when $s \uparrow 1$.

1. Introduction

The celebrated result by Bourgain, Brezis and Mironescu establishes the behavior of the so-called Gagliardo seminorm in fractional order Sobolev spaces of order s as $s \uparrow 1$, providing new characterizations for functions in the Sobolev space $W^{1,p}(\Omega)$. More precisely, given a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ and $p \in [1,\infty)$, for any $u \in W^{1,p}(\Omega)$ in [6] it is proved that

$$\lim_{s\uparrow 1}(1-s)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy = \mathcal{K}_{p,n}\|\nabla u\|_p^p$$

where the constant $\mathcal{K}_{p,n}$ is given by

$$\mathcal{K}_{p,n} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} |\omega_1|^p \, d\mathcal{H}^{n-1}.$$

Here $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ the Hausdorff $(n-1)-$dimensional measure.

The formula above was proved to hold with less assumptions on the domain. In fact, in [18] it is established the validity of the BBM-formula for any open set $\Omega \subset \mathbb{R}^n$, and recently, in [12], for any bounded domain. This analysis was completed in [9, 23], where it was proven that a similar formula holds for functions of bounded variation when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz set.

Motivated with the results in [6], Maz'ya and Shaposhnikova complemented the study by analyzing the behavior of the seminorm as $s \downarrow 0$. In fact, the authors proved in [20] that for any $n \geq 1$ and $p \in [1,\infty)$

$$\lim_{s\downarrow 0} s \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy = \mathcal{C}_{p,n}\|u\|_p^p$$

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whenever $u \in D^{s,p}(\mathbb{R}^n)$ for some $s \in (0,1)$ where $D^{s,p}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the Gagliardo seminorm. The constant $\mathcal{C}_{p,n}$ is given by

$$\mathcal{C}_{p,n} = \frac{4\pi^{\frac{n}{2}}}{p\Gamma(\frac{n}{2})},$$

where Γ denotes the Gamma function.

The singular limits mentioned above are natural and have a physical relevance in the framework of the theory of Lévy processes. This has led to the fact that in the last years, a huge effort in trying to extend the asymptotic results as $s \uparrow 1$ and $s \downarrow 0$ proved in [6, 20] to different contexts has been carried out. We mention just some examples: for the theory of fractional $s-$perimeters, the analysis of the asymptotic limits was addressed in [5, 11]; the extension to functions allowing a behavior more general than a power was done in [2, 3, 7, 15] in the context of fractional Orlicz-Sobolev spaces; in the magnetic setting, the behavior of the corresponding seminorms was studied [21, 22]; the extension of magnetic fractional Orlicz-Sobolev spaces was dealt in [17, 19].

The purpose of this paper is to study the asymptotic behavior as $s \uparrow 1$ and $s \downarrow 0$ of anisotropic Gagliardo seminorms, that is, the quantity

$$J_{m,s}(u) := \frac{1-s}{p} \iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x, x-y)\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy,$$

where m is a function bounded away from 0 and infinity satisfying some suitable conditions (see hypotheses (H_1)–(H_3) below).

Let us conclude this section by describing our main results. In Theorem 3.4 we prove that given $u \in L^p(\mathbb{R}^n)$ fixed,

$$\lim_{s\uparrow 1} J_{m,s}(u) = \int_{\mathbb{R}^n} \mathcal{A}(x, \nabla u)\,dx$$

where

$$\mathcal{A}(x,\xi) = \frac{1}{p}\int_{\mathbb{S}^{n-1}} a(x,w)|\xi\cdot w|^p\,d\mathcal{H}^{n-1}$$

and $a(x,\omega)$ is a *radial limit* of the weight function m (see (H_3)).

In Theorem 3.5 we also treat the case of a sequence, i.e., the behavior of $J_{m,s}(u_s)$ as $s \uparrow 1$, where $\{u_s\}_s$ is a sequence of functions in $L^p(\mathbb{R}^n)$ such that $(1-s)[u_s]_{s,p}^p + \|u_s\|_p^p$ is uniformly bounded.

When $u \in W^{s_0,p}(\mathbb{R}^n)$ for some $s_0 \in (0,1)$, then in Theorem 4.1 we prove that

$$\lim_{s\downarrow 0} sJ_{m,s}(u) = \int_{\mathbb{R}^n} |u|^p b(x)\,dx,$$

where $b(x) = \lim_{s\downarrow 0} b_s(x)$ a.e. $x \in \mathbb{R}^n$, and

$$b_s(x) := 2s\int_{\mathbb{S}^{n-1}}\int_{2|x|}^{\infty} \frac{m(x,r\omega)}{r^{sp+1}}\,dr\,d\mathcal{H}^{n-1}.$$

In the last part of the paper we analyze whether homogenization and localization processes can be interchanged. To be more precise, observe that the functional $J_{m,s}$ of a function $u_s \in W_0^{s,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, is related with weak solutions to

$$\mathcal{L}_{m,s}u_s = f \quad \text{ in } \Omega$$

where $f \in L^{p'}(\Omega)$ and $\mathcal{L}_{m,s}$ is the Fréchet derivative of $J_{m.s}$, i.e.,

$$\mathcal{L}_{m,s}(u) = p.v.(1-s)\int_{\mathbb{R}^n} m(x,x-y)\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,dy.$$

In Section 6 we consider solutions of a family of kernels $m_\varepsilon(x,x-y)$, $\varepsilon>0$ having the form $m_\varepsilon(x,x-y)=m(\frac{x}{\varepsilon},x-y)$, where $m(x,x-y)$ fullfills the previous assumptions and it is further a $Q-$periodic function in the first variable, being Q the unit cube in $\mathbb{R}^n$. Given a solution of $u_{s,\varepsilon}\in W^{1,p}_0(\Omega)$ to $\mathcal{L}_{m_\varepsilon,s}u_{\varepsilon,s}=f$ in Ω, in Proposition 6.1 we prove that

$$\lim_{\varepsilon\downarrow 0}\left(\lim_{s\uparrow 1} u_{s,\varepsilon}\right)=u^*,$$

(in the $L^p(\Omega)$ sense) where $u^*\in W^{1,p}_0(\Omega)$ is the weak solution of

$$-\operatorname{div}(\nabla_\xi\mathcal{A}^*(\nabla u^*))=f\quad\text{in }\Omega\qquad\text{with }\mathcal{A}^*(\xi)=\inf_{v\in W^{1,1}_{per}(Q)}\int_Q\mathcal{A}(y,\xi+\nabla v(y))\,dy.$$

On the other hand, in Propisition 6.2 we get that

$$\lim_{s\uparrow 1}\left(\lim_{\varepsilon\downarrow 0} u_{s,\varepsilon}\right)=\bar{u}$$

(in the $L^p(\Omega)$ sense), where $\bar{u}\in W^{1,p}_0(\Omega)$ is the solution to

$$-\operatorname{div}(\nabla_\xi\bar{\mathcal{A}}(\nabla\bar{u}))=f\quad\text{in }\Omega,\qquad\text{with }\bar{\mathcal{A}}(\xi)=\int_Q\mathcal{A}(y,\xi)\,dy.$$

This shows that in general, homogenization and localization do not commute.

Organization of the paper. After this introduction, in Section 2, we collect some preliminaries, and establish some notation that will be used in the sequel.

In Section 3, we analyze the problems for $s\uparrow 1$, the so-called BBM-type results in the spirit of Bourgains-Brezis-Mironescu's paper [6].

In Section 4, we analyze the problem $s\downarrow 0$, the MS-type results, in the spirit of Maz'ya-Shaposhnikova's paper [20].

In Section 5 we connect the BBM-type results of Section 3 with the asymptotic behavior of solutions to nonlocal problems and the transition to solutions to local ones.

Finally, in Section 6, we investigate the interplay between localization (i.e. $s\uparrow 1$) and homogenization.

2. Preliminaries

2.1. **Fractional order Sobolev spaces.** Throughout this article we will use the standard Gagliardo definition of fractional order Sobolev spaces. That is: Given a fractional parameter $s\in(0,1)$ and an integrable parameter $p\in[1,\infty)$, the fractional order Sobolev space, $W^{s,p}(\mathbb{R}^n)$ is defined as

$$W^{s,p}(\mathbb{R}^n):=\left\{u\in L^p(\mathbb{R}^n)\colon [u]^p_{s,p}<\infty\right\},$$

where $[\,\cdot\,]_{s,p}$ is the so-called Gagliardo seminorm that is defined as

$$[u]^p_{s,p}:=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dxdh.$$

This space is endowed with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + [u]_{s,p}^p\right)^{\frac{1}{p}}$$

and $(W^{s,p}(\mathbb{R}^n), \|\cdot\|_{s,p})$ is a separable Banach space, that is reflexive if $p > 1$.

When considering domains $\Omega \subset \mathbb{R}^n$ we will use the notation $W_0^{s,p}(\Omega)$ to denote the set of functions in $W^{s,p}(\mathbb{R}^n)$ that vanishes outside Ω, namely

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) \colon u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Observe that this space agrees with the closure of test functions in Ω if, for instance, Ω has Lipschitz boundary or if $s < \frac{1}{p}$.

For these spaces, the Rellich-Kondrashov compactness result holds true, i.e.

Theorem 2.1. *Assume that $s \in (0,1)$ and $p \in [1,\infty)$ and let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ be a bounded sequence. Then, there exists $u \in W_0^{s,p}(\Omega)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ such that*

$$u_{k_j} \to u \quad \text{in } L^p_{loc}(\Omega).$$

If Ω is bounded, the convergence is in $L^p(\Omega)$.

All of the above mentioned results are well known and can be found, for instance, in [10].

2.2. **Some notation.** In several places of the paper, the following notation will be used:

- The unit sphere in $\mathbb{R}^n$ will be denoted by $\mathbb{S}^{n-1}$.
- The $(n-1)-$dimensional Hausdorff measure will be denoted by $\mathcal{H}^{n-1}$.
- The volume of the unit ball in $\mathbb{R}^n$ will be denoted by ω_n.
- The volume of the $(n-1)-$dimensional unit sphere in $\mathbb{R}^n$ is then $n\omega_n$.

2.3. **Anisotropic fractional energies.** We consider a kernel function $m = m(x,h)$, $m \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and for each function m and each fractional parameter $s \in (0,1)$ we define the functional $J_{m,s} \colon W_0^{s,p}(\Omega) \to \mathbb{R}$,

$$J_{m,s}(u) := \frac{(1-s)}{p} \iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h) \frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\, dxdh,$$

where Ω is a domain in $\mathbb{R}^n$, not necessarily bounded.

If the kernel function m is bounded below away from 0, the functional $J_{m,s}$ is coercive, so we impose the following condition on m:

$$m_- \le m(x,h) \le m_+ \tag{H_1}$$

for some $0 < m_- \le m_+ < \infty$.

It is easy to see that $J_{m,s}$ is Fréchet differentiable. If we try to obtain an integral representation of the derivative $J'_{m,s}(u) \in W^{-s,p'}(\Omega)$, we need to impose some symmetry assumptions on the kernel m, namely,

$$m(x,h) = m(x-h,-h). \tag{H_2}$$

Under this condition, it is easy to see (see for instance [14]), that the derivative $J'_{m,s}(u)$ has the following integral representation,

$$\begin{aligned} J'_{m,s}(u) :=& \mathcal{L}_{m,s}(u) \\ =& p.v.(1-s) \int_{\mathbb{R}^n} m(x,h) \frac{|u(x)-u(x-h)|^{p-2}(u(x)-u(x-h))}{|h|^{n+sp}}\, dh, \end{aligned} \tag{2.1}$$

where *p.v.* stands for *in principal value.*

Observe that hypotheses (H_2) is by no means restrictive, since denoting

$$m_{\text{sym}}(x,h) = \frac{m(x,h)+m(x-h,-h)}{2}$$

we have that m_{sym} satisfies (H_2) and

$$J_{m,s} = J_{m_{\text{sym}},s}.$$

In order to analyze the case where $s \uparrow 1$ in our functionals $J_{m,s}$ we need to assume some asymptotic behavior on the kernel m. This condition, though it seems quite technical right now it will become apparent later on:

There exists a function $a : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that

$$(H_3) \qquad m(x, r\omega) = a(x,\omega) + O(r)$$

uniformly in $\omega \in \mathbb{S}^{n-1}$.

This condition is saying that m has some singular behavior on the diagonal that is determined by the angle in which one approaches the origin.

Observe that $m \in C^1$ implies that the limit function a in (H_3) is independent of ω. In fact, $a(x,\omega) = m(x,0)$ in this case and a typical nontrivial example to keep in mind is the following

$$m(x,h) = \left| M(x,h)\frac{h}{|h|}\right|^{\alpha},\ \alpha \neq 0$$

where $M(x,h) \in \mathbb{R}^{n\times n}$ is a symmetric uniformly elliptic matrix with the structural hypothesis

$$M(x,h) = M(x-h,-h).$$

In this case, the function $a(x,\omega)$ is given by

$$a(x,\omega) = |M(x,0)\omega|^{\alpha}.$$

3. Limit as $s \uparrow 1$ of $J_{m,s}$

The purpose of this section is to analyze the behavior as $s \uparrow 1$ of the functional $J_{m,s}$. This is the extension of the celebrated result of Bourgain-Brezis-Mironescu to the anisotropic case.

First we begin by studying the pointwise limit of the funcionales that is much simpler. Later on, we will deal with the Gamma-convergence of the funcional that is more subtle.

3.1. **Pointwise limit.**

The results in this subsection are inspired by [1] where the authors consider some particular case of weight function for $p = 2$.

To begin with, we cite a Lemma that can be found in [6].

Lemma 3.1. *Given $u \in W_0^{s,p}(\Omega)$ it holds that*

$$\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dh \leq m_+[u]_{s,p}^p \leq \frac{n\omega_n m_+}{p}\left(\frac{1}{1-s}\|\nabla u\|_p^p + \frac{2^p}{s}\|u\|_p^p\right).$$

Proof. Just combine (H_1) with [6, Theorem 1]. □

The following proposition is key in the proof of our main result.

Proposition 3.2. *Given $u \in C^2_c(\mathbb{R}^n)$ and a fixed $x \in \mathbb{R}^n$ we have that*

$$\lim_{s\uparrow 1}(1-s)\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dh = \mathcal{A}(x,\nabla u)$$

where $\mathcal{A}(x,\xi)$ is given by

$$\mathcal{A}(x,\xi) = \frac{1}{p}\int_{\mathbb{S}^{n-1}} a(x,w)|\xi\cdot w|^p\,d\mathcal{H}^{n-1}. \tag{3.1}$$

Remark 3.3. In the linear case, that is when $p=2$, the operator $\mathcal{A}(x,\xi)$ has a more explicit form,

$$\mathcal{A}(x,\xi) = A(x)\xi\cdot\xi,$$

where the matrix $A \in \mathbb{R}^{n\times n}$ is given by

$$a_{ij}(x) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} w_i w_j a(x,w)\,d\mathcal{H}^{n-1}.$$

Proof of Proposition 3.2. For each fixed $x\in\mathbb{R}^n$ we split the integral

$$\begin{aligned}\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dh &= \left(\int_{|h|\geq 1} + \int_{|h|<1}\right) m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dh\\ &= I_1 + I_2.\end{aligned}$$

Since (H_1) holds,

$$|I_1| \leq 2^{p-1}m_+\|u\|_\infty^p \int_{|h|\geq 1}\frac{1}{|h|^{n+sp}}\,dh < \infty,$$

and we focus only on I_2. Since $x\mapsto |x|^p$ is locally Lipschitz and $u\in C^2$, we have that

$$\left|\frac{|u(x)-u(x-h)|^p}{|h|^{sp}} - \frac{|\nabla u(x)\cdot h|^p}{|h|^{sp}}\right| \leq L\frac{|u(x)-u(x-h)-\nabla u(x)\cdot h|}{|h|^s} \leq C|h|^{2-s}$$

where C depends of the C^2-norm of u.

Since the following integral vanishes

$$\lim_{s\uparrow 1}(1-s)\int_{|h|\leq 1}|h|^{2-s-n}\,dh = \lim_{s\uparrow 1}(1-s)\frac{n\omega_n}{2-s} = 0,$$

it follows that

$$\lim_{s\uparrow 1}(1-s)I_2 = \lim_{s\uparrow 1}(1-s)\int_{|h|\leq 1} m(x,h)\frac{|\nabla u(x)\cdot h|^p}{|h|^{n+sp}}\,dh$$

Hence, by using polar coordinates we get

$$\begin{aligned}\int_{|h|\leq 1} m(x,h)\frac{|\nabla u(x)\cdot h|^p}{|h|^{n+sp}}\,dh &= \int_{|h|\leq 1} m(x,h)\frac{|\nabla u(x)\cdot \frac{h}{|h|}|^p}{|h|^{n+sp-p}}\,dh\\ &= \int_0^1\int_{\mathbb{S}^{n-1}} m(x,r\omega)|\nabla u(x)\cdot\omega|^p r^{p(1-s)-1}\,d\mathcal{H}^{n-1}\,dr\\ &= \int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^p\left(\int_0^1 m(x,r\omega)r^{p(1-s)-1}\,dr\right)d\mathcal{H}^{n-1}.\end{aligned}$$

Observe that from (H_3) we have that $m(x,r\omega)=a(x,\omega)+O(r)$, which implies that

$$\begin{aligned}\int_0^1 m(x,r\omega)r^{p(1-s)-1}\,dr &= \int_0^1 a(x,\omega)r^{p(1-s)-1}\,dr+\int_0^1 O(r^{p(1-s)})\,dr\\ &= a(x,\omega)\frac{1}{p(1-s)}+O(1)\end{aligned}$$

and consequently

$$\lim_{s\uparrow 1}(1-s)\int_0^1 m(x,r\omega)r^{p(1-s)-1}\,dr=\frac{1}{p}a(x,\omega).$$

Finally,

$$\begin{aligned}&\lim_{s\uparrow 1}(1-s)\int_{|h|\le 1} m(x,h)\frac{|\nabla u(x)\cdot h|^p}{|h|^{n+sp}}\,dh\\ &\qquad=\lim_{s\uparrow 1}(1-s)\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^p\left(\int_0^1 m(x,r\omega)r^{p(1-s)-1}\,dr\right)d\mathcal{H}^{n-1}\\ &\qquad=\frac{1}{p}\int_{\mathbb{S}^{n-1}}a(x,\omega)|\nabla u(x)\cdot\omega|^p\,d\mathcal{H}^{n-1},\end{aligned}$$

which concludes the proof. □

We are ready to state and proof our main result in this subsection.

Theorem 3.4. *Given $u\in L^p(\mathbb{R}^n)$ and a fixed $x\in\mathbb{R}^n$ we have that*

$$\lim_{s\uparrow 1}(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dxdh=\int_{\mathbb{R}^n}\mathcal{A}(x,\nabla u)\,dx$$

where $\mathcal{A}(x,\xi)$ is given in (3.1).

Proof. Given $u\in C_c^2(\mathbb{R}^n)$ with $\operatorname{supp}(u)\subset B_R(0)$, in view of Proposition 3.2, it only remains to show the existence of an integrable majorant for $(1-s)F_s$, where F_s is given by

$$F_s(x):=\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dh.$$

But, thanks to (H_1), this task is the same done in [6], more precisely,

$$(1-s)|F_s(x)|\le Cm_+\left(\chi_{B_R(0)}(x)+|x|^{-(n+\frac{1}{2})}\chi_{B_r(0)^c}(x)\right)\in L^1(\mathbb{R}^n).$$

Then, from Proposition 3.2 and the Dominated Convergence Theorem the result follows for any $u\in C_c^2(\mathbb{R}^n)$.

Using Lemma 3.1 and [6, Theorem 2], the result is extended to an arbitrary function $u\in W^{1,p}(\mathbb{R}^n)$.

Finally, arguing as in [6, Theorem 2] (see also [15]) it holds that if

$$\liminf_{s\uparrow 1}(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dxdh<\infty,$$

then $u\in W^{1,p}(\mathbb{R}^n)$ and the result follows. □

3.2. **The case of a sequence.** In this subsection we deal with the case of a sequence that will imply, among other things, the Gamma convergence of the functionals $J_{m,s}$.

Theorem 3.5. *Let* $0 \le s_k \uparrow 1$ *and* $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^n)$ *be such that*

$$\sup_{k\in\mathbb{N}}(1-s_k)[u_k]_{s_k,p}^p < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}} \|u_k\|_{L^p(\mathbb{R}^n)} < \infty.$$

Then there exists $u \in L^p(\mathbb{R}^n)$ *and a subsequence* $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ *such that* $u_{k_j} \to u$ *in* $L^p_{loc}(\mathbb{R}^n)$*. Moreover* $u \in W^{1,p}(\mathbb{R}^n)$ *and the following estimate holds*

$$\int_{\mathbb{R}^n} \mathcal{A}(x, \nabla u)\, dx \le \liminf_{k\to\infty} J_{m,s_k}(u_k).$$

The proof of the above result will be a direct consequence of the following useful estimate:

Theorem 3.6. *Let* $0 < s_1 < s_2 < 1$ *and* $u \in L^p(\mathbb{R}^n)$*. Then*

$$J_{m,s_1}(u) \le 2^{p(1-s_1)} J_{m,s_2}(u) + \frac{2^{p-1}m_+ n\omega_n(1-s_1)}{s_1}\|u\|_p^p.$$

The key point in proving Theorem 3.6 is the following lemma that is proved in [6].

Lemma 3.7 (Lemma 2, [6])**.** *Let* $g, h : (0,1) \to \mathbb{R}^+$ *measurable functions. Suppose that for some constant* $c > 0$ *it holds that* $g(t) \le cg(\frac{t}{2})$ *for* $t \in (0,1)$ *and that* h *is decreasing. Then, given* $r > -1$,

$$\int_0^1 t^r g(t)h(t)\, dt \ge \frac{r+1}{2^{r+1}} \int_0^1 t^r g(t)\, dt \int_0^1 t^r h(t)\, dt.$$

Actually, the proof in [6] is done with $c = 1$. The extension for general $c > 0$ is immediate.

Now we proceed with the proof of the estimate.

Proof of Theorem 3.6. The proof is very similar to that of [6, Theorem 4] (see also [15, Theorem 5.1]). We include some details in order to make the paper self contained.

Given $u \in L^p(\mathbb{R}^n)$, we define for $t > 0$ and $0 < s < 1$,

$$\begin{aligned} F(t) &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} m(x, tw)|u(x) - u(x - tw)|^p\, dx\, d\mathcal{H}^{n-1} \\ &= \frac{1}{t^{n-1}} \int_{|h|=t} \int_{\mathbb{R}^n} m(x, h)|u(x) - u(x - h)|^p\, dx\, d\mathcal{H}^{n-1} \end{aligned}$$

and $g(t) = \frac{F(t)}{t^p}$.

From [6, p. 13] and assumption (H_1) if follows that

$$g(2t) \le \frac{m_+}{m_-} g(t).$$

Then, observe that

$$
\begin{aligned}
&\int_{|h|<1}\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}dx\,dh \\
&\qquad = \int_0^1 \int_{|h|=t}\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{t^{n+sp}}dx\,d\mathcal{H}^{n-1}\,dt \\
&\qquad = \int_0^1 \frac{F(t)}{t^{1+sp}}\,dt = \int_0^1 \frac{g(t)}{t^{1-p(1-s)}}\,dt.
\end{aligned} \tag{3.2}
$$

Consider now $0 < s_1 < s_2 < 1$. Therefore,

$$
\int_0^1 \frac{1}{t^{1-p(1-s_2)}} g(t)\,dt = \int_0^1 \frac{1}{t^{1-p(1-s_1)}} g(t)\frac{1}{t^{p(s_2-s_1)}}\,dt.
$$

Now, from Lemma 3.7 with $r = p(1-s_1)-1$ and $h(t) = t^{-p(s_2-s_1)}$ we get

$$
\begin{aligned}
\int_0^1 \frac{1}{t^{1-p(1-s_2)}} g(t)\,dt \geq& \frac{p(1-s_1)}{2^{p(1-s_1)}}\int_0^1 \frac{1}{t^{1-p(1-s_1)}} g(t)\,dt \int_0^1 \frac{1}{t^{1-p(1-s_2)}}\,dt \\
=& \frac{1}{2^{p(1-s_1)}}\frac{1-s_1}{1-s_2}\int_0^1 \frac{1}{t^{1-p(1-s_1)}} g(t)\,dt.
\end{aligned} \tag{3.3}
$$

From (3.2) and (3.3) we deduce that

$$
\begin{aligned}
&\frac{(1-s_1)}{2^{p(1-s_1)}}\int_{|h|<1}\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+s_1p}}\,dxdh \\
&\qquad \leq (1-s_2)\int_{\{|h|<1\}}\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+s_2p}}\,dxdh.
\end{aligned} \tag{3.4}
$$

Finally we observe that

$$
\begin{aligned}
\int_{\{|h|\geq 1\}}\int_{\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dxdh &\leq 2^p m_+ n\omega_n \|u\|_p^p \int_1^\infty \frac{1}{t^{1+sp}}\,dt \\
&= \frac{2^p m_+ n\omega_n}{sp}\|u\|_p^p.
\end{aligned}
$$

The proof concludes combining this last inequality with (3.4). $\square$

Now we can proceed with the proof of Theorem 3.5.

Proof of Theorem 3.5. With the help of Theorem 3.6 the proof is the same as [6, Theorem 4] and [15, Theorem 5.1]

We include some details for the reader's convenience.

Let $0 < s_k \uparrow 1$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^n)$ such that

$$
\sup_{k\in\mathbb{N}}(1-s_k)[u_k]^p_{s_k,p} < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}}\|u_k\|_p < \infty.
$$

For a fixed $0 < t < 1$ using Theorem 3.6 we have that $\{u_k\}_{k\in\mathbb{N}} \subset W^{t,p}(\mathbb{R}^n)$ is bounded and so by the Rellich-Kondrashov compactness Theorem (Theorem 2.1), there exists a subsequence (still denoted by $\{u_k\}_{k\in\mathbb{N}}$) and a limit function $u \in L^p(\mathbb{R}^n)$ such that $u_k \to u$ in $L^p_{\text{loc}}(\mathbb{R}^n)$. We can also assume that $u_k \to u$ a.e. in $\mathbb{R}^n$.

Now, by Fatou's Lemma, we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+tp}}\,dxdy \\ \le \liminf_{k\to\infty}\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u_k(x)-u_k(x-h)|^p}{|h|^{n+tp}}\,dxdy$$

and by Theorem 3.6 we obtain

$$\frac{1-t}{2^{(1-t)p}}\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+tp}}\,dxdy \\ \le \liminf_{k\to\infty}(1-s_k)\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u_k(x)-u_k(x-h)|^p}{|h|^{n+s_kp}}\,dxdy \\ + \frac{n\omega_n 2^p(1-t)m_+}{tp}M$$

wher $M=\sup_{k\in\mathbb{N}}\|u_k\|_p^p$.

Finally the result follows taking the limit $t\uparrow 1$ and using Theorem 3.4. □

4. Limit as $s\downarrow 0$ of $J_{m,s}$

In this section we analyze the limit case where $s\downarrow 0$ of the functionals $J_{m,s}$. This is what is called a *Maz'ya-Shaposhnikova type result* after the results obtained in [20]. That is we are interested in studying the limit

$$\lim_{s\downarrow 0} sJ_{m,s}(u).$$

First we define the following weights depending on m and s,

$$b_s(x) := 2s\int_{\mathbb{S}^{n-1}}\int_{2|x|}^\infty \frac{m(x,r\omega)}{r^{sp+1}}\,dr\,d\mathcal{H}^{n-1}.$$

Observe that this weight has the following bounds

$$\frac{2^{1-sp}n\omega_n}{p}\frac{m_-}{|x|^{sp}} \le b_s(x) \le \frac{2^{1-sp}n\omega_n}{p}\frac{m_+}{|x|^{sp}},$$

where $m_\pm$ are given in (H_1).

We will assume that there exists the limit function

$$b(x) = \lim_{s\downarrow 0} b_s(x) \quad \text{a.e. } x\in\mathbb{R}^n.$$

Our main result in the section is

Theorem 4.1. *Under the above assumptions and notations, if $u\in W^{s_0,p}(\mathbb{R}^n)$ for some $s_0\in(0,1)$, then*

$$\lim_{s\downarrow 0} sJ_{m,s}(u) = \int_{\mathbb{R}^n}|u|^p b(x)\,dx.$$

The proof of this result follows the general strategy developed in [20] but also applies some ideas from [4].

The proof will be a direct consequence of the next two lemmas. The first one is a Hardy-type inequality with weights

Lemma 4.2. *Let $u\in W^{s_0,p}(\mathbb{R}^n)$ for some $s_0\in(0,1)$, then*

$$\liminf_{s\downarrow 0} sJ_{m,s}(u) \ge \int_{\mathbb{R}^n}|u|^p b(x)\,dx.$$

Proof. Let us call

$$I^p = \int_{\mathbb{R}^n} \int_{|h|>2|x|} m(x,h) \frac{|u(x)|^p}{|h|^{n+sp}}\, dhdx.$$

Then

$$\begin{aligned} I^p &= \int_{\mathbb{R}^n} \left(\int_{|h|\geq 2|x|} \frac{m(x,h)}{|h|^{n+sp}} dh \right) |u(x)|^p\, dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} \int_{2|x|}^{\infty} \frac{m(x,r\omega)}{r^{1+sp}}\, dr d\mathcal{H}^{n-1} \right) |u(x)|^p\, dx. \end{aligned}$$

Now, for any $\varepsilon > 0$,

$$\begin{aligned} I^p \leq& (1+\varepsilon)^{p-1} \int_{\mathbb{R}^n} \int_{|h|\geq 2|x|} m(x,h) \frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\, dhdx \\ &+ \left(\frac{1+\varepsilon}{\varepsilon} \right)^{p-1} \int_{\mathbb{R}^n} \int_{|h|\geq 2|x|} m(x,h) \frac{|u(x-h)|^p}{|h|^{n+sp}}\, dhdx \\ :=& (1+\varepsilon)^{p-1} (a)^p + \left(\frac{1+\varepsilon}{\varepsilon} \right)^{p-1} (b)^p. \end{aligned}$$

Let us first bound (b). As $|h| > 2|x|$ we have that $\frac{2}{3}|x-h| < |h| < 2|x-h|$. Hence

$$\begin{aligned} (b)^p &= \int_{\mathbb{R}^n} \int_{|h|\geq 2|x|} \frac{|u(x-h)|^p}{|h|^{n+sp}}\, dhdx \\ &\leq \left(\frac{3}{2}\right)^{n+sp} \int_{\mathbb{R}^n} \int_{\frac{2}{3}|y|<|x-y|<2|y|} \frac{|u(y)|^p}{|y|^{n+sp}}\, dydx \\ &= \left(\frac{3}{2}\right)^{n+sp} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{n+sp}} \left(\int_{\frac{2}{3}|y|<|x-y|<2|y|} dx \right) dy \\ &\leq \left(\frac{3}{2}\right)^{n+sp} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{n+sp}} n\omega_n 2^n |y|^n dy \\ &= n\omega_n \frac{3^{n+sp}}{2^{sp}} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{sp}} dy. \end{aligned}$$

Observe that this last quantity is finite by Hardy's inequality.

For (a), we observe that, changing variables,

$$\begin{aligned} (a)^p &= \int_{\mathbb{R}^n} \int_{|h|\geq 2|x|} m(x,h) \frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\, dhdx \\ &= \int_{\mathbb{R}^n} \int_{|h|\geq 2|x-h|} m(x,h) \frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\, dhdx := (\tilde{a})^p, \end{aligned}$$

where we have used the symmetry assumption (H_2).

Observe that the sets $\{|h| \geq 2|x-h|\}$ and $\{|h| \geq 2|x|\}$ are disjoints, so

$$2(a)^p = (a)^p + (\tilde{a})^p \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h) \frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\, dhdx.$$

Next, observe that

$$
\begin{aligned}
sJ_{m,s}(u) &\geq 2s(a)^p \geq 2s\left[\frac{1}{(1+\varepsilon)^{p-1}} I^p - \frac{1}{\varepsilon^{p-1}}(b)^p\right] \\
&\geq \frac{1}{(1+\varepsilon)^{p-1}} \int_{\mathbb{R}^n} |u(x)|^p b_s(x)\,dx - \frac{2s}{\varepsilon^{p-1}} n\omega_n \frac{3^{n+sp}}{2^{sp}} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{sp}} dy.
\end{aligned}
$$

Finally, using Fatou's Lemma,

$$
\liminf_{s\downarrow 0} sJ_{m,s}(u) \geq \frac{1}{(1+\varepsilon)^{p-1}} \int_{\mathbb{R}^n} |u(x)|^p b(x)\,dx,
$$

for any $\varepsilon > 0$ and the result follows. $\square$

The next lemma gives us the upper estimate.

Lemma 4.3 (Limsup estimate)**.** *For any $u \in W_0^{s_0,p}(\mathbb{R}^n)$ for some $s_0 \in (0,1)$, it holds that*

$$
\limsup_{s\downarrow 0} sJ_{m,s}(u) \leq \int_{\mathbb{R}^n} |u(x)|^p b(x)\,dx. \tag{4.1}
$$

Proof. Observe that our symmetry assumption (H_2) gives that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{|x-h|<|x|} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx \\
&\qquad = \int_{\mathbb{R}^n}\int_{|x-h|>|x|} m(x-h,-h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx \\
&\qquad = \int_{\mathbb{R}^n}\int_{|x-h|>|x|} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx \\
&\qquad = 2\int_{\mathbb{R}^n}\int_{|x-h|\geq|x|} m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx.
\end{aligned}
$$

Arguing as in the previous lemma, given $\varepsilon > 0$ we get

$$
\begin{aligned}
sJ_{m,s}(u) =& 2s\left(\int_{\mathbb{R}^n}\int_{|x-h|\geq 2|x|} + \int_{\mathbb{R}^n}\int_{|x|<|x-h|<2|x|}\right) m(x,h)\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx \\
\leq& (1+\varepsilon)^{p-1}\int_{\mathbb{R}^n}|u(x)|^p b_s(x)\,dx \\
&+ 2sm^+\Bigg[\left(\frac{1+\varepsilon}{\varepsilon}\right)^{p-1}\int_{\mathbb{R}^n}\int_{|x-h|\geq 2|x|}\frac{|u(x-h)|^p}{|h|^{n+sp}}\,dhdx \\
&\qquad + \int_{\mathbb{R}^n}\int_{|x|<|x-h|<2|x|}\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx\Bigg] \\
&:= (a)^p + 2sm^+[(b)^p + (c)^p].
\end{aligned}
$$

We need to get uniform, in s, bounds on (b) and (c). For (b), we use that $|x-h|\geq 2|x|$ implies that $|h|>\frac{1}{2}|x-h|$ together with Fubini's Theorem to obtain

$$\begin{aligned}(b)^p &\leq 2^{n+sp}\int_{\mathbb{R}^n}\left(\int_{|x-h|\geq|x|}\frac{|u(x-h)|^p}{|x-h|^{n+sp}}\,dh\right)dx\\ &= 2^{sp}n\omega_n\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx<\infty.\end{aligned}$$

Hardy's inequality gives us the desired uniform bound on (b) and therefore

$$\limsup_{s\to 0^+}\; s(b)^p=0.$$

It remains to bound (c). To begin with, we split the integral into two parts, one where $|h|$ is large and another one where $|h|$ is bounded.

$$\begin{aligned}(c)^p =&\int_{\mathbb{R}^n}\int_{\substack{|x|<|x-h|<2|x|\\|h|\leq N}}\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx\\ &+\int_{\mathbb{R}^n}\int_{\substack{|x|<|x-h|<2|x|\\|h|>N}}\frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,dhdx\\ :=&(c_1)^p+(c_2)^p.\end{aligned}$$

To bound (c_1) we proceed as follows

$$\begin{aligned}(c_1)^p &\leq N^{p(\tau-s)}\int_{\mathbb{R}^n}\int_{\substack{|x|<|x-h|<2|x|\\|h|\leq N}}\frac{|u(x)-u(x-h)|^p}{|h|^{n+\tau p}}\,dhdx\\ &\leq N^{p(\tau-s)}[u]_{\tau,p}^p,\end{aligned}$$

where $\tau>s$ is fixed. From this expression,

$$\limsup_{s\to 0^+}\; s(c_1)^p=0.$$

It remains to get a bound for (c_2). First, we observe that, as $|x|<|x-h|<2|x|$ and $|h|>N$, it follows that $|x|>\frac{N}{3}$ and $|x-h|>\frac{N}{3}$. Hence

$$\begin{aligned}(c_2)^p &\leq 2^{p-1}\int_{\mathbb{R}^n}\int_{\substack{|x|<|x-h|<2|x|\\|h|>N}}\frac{|u(x)|^p}{|h|^{n+sp}}\,dhdx+2^{p-1}\int_{\mathbb{R}^n}\int_{\substack{|x|<|x-h|<2|x|\\|h|>N}}\frac{|u(x-h)|^p}{|h|^{n+sp}}\,dhdx\\ &\leq 2^{p-1}\int_{\mathbb{R}^n}\int_{\substack{|x|>N/3\\|h|>N}}\frac{|u(x)|^p}{|h|^{n+sp}}\,dydx+2^{p-1}\int_{\mathbb{R}^n}\int_{\substack{|x-h|>N/3\\|h|>N}}\frac{|u(x-h)|^p}{|h|^{n+sp}}\,dhdx\\ &=2^p\int_{\mathbb{R}^n}\int_{\substack{|x|>N/3\\|h|>N}}\frac{|u(x)|^p}{|h|^{n+sp}}\,dydx\\ &=2^p\int_{|x|>\frac{N}{3}}|u(x)|^p\left(n\omega_n\int_N^\infty\frac{r^{n-1}}{r^{n+sp}}\,dr\right)dx\\ &=\frac{n\omega_n2^p}{sp}\frac{1}{N^{sp}}\int_{|x|>\frac{N}{3}}|u(x)|^pdx\end{aligned}$$

from where it follows that

$$\limsup_{s\to 0+}s(c_s)^p\leq\frac{n\omega_n2^p}{p}\int_{|x|>\frac{N}{3}}|u(x)|^pdx,$$

and this quantity is arbitrary small if N is large. $\square$

With the help of Lemmas 4.2 and 4.3 we can deduce the main result of the section.

Proof of Theorem 4.1. Immediate from Lemmas 4.2 and 4.3. □

5. Anisotropic nonlocal and local problems

One application of the results in Section 3 is to analyze the asymptotic behavior of the solutions to anisotropic nonlocal problems. That is, given a domain $\Omega \subset \mathbb{R}^n$ and a source term $f \in L^{p'}(\Omega)$ one wants to analyze the limit as $s \uparrow 1$ of the solutions to

$$\begin{cases} \mathcal{L}_{m,s} u_s = f & \text{in } \Omega \\ u_s = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.1}$$

where $\mathcal{L}_{m,s}$ is the Fréchet derivative of $J_{m,n}$ given by (2.1).

In the case where $m = 1$ this problem is well understood since the seminal works of [6] and for some recent results regarding this problem, even in the *semilinear-type* case (that is when $f = f(u)$) we refer to [15].

In this general case, the results in Section 3, suggest that the limit problem for (5.1) when $s \uparrow 1$ is

$$\begin{cases} \mathcal{L}_{\mathcal{A}} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $\mathcal{L}_{\mathcal{A}} u = -\operatorname{div}(\nabla_\xi \mathcal{A}(x, \nabla u))$.

Recall that $\mathcal{L}_{\mathcal{A}}$ is the Fréchet derivative of the functional

$$J(u) = \int_{\mathbb{R}^n} \mathcal{A}(x, \nabla u)\, dx.$$

The results of Section 3, immediately gives:

Theorem 5.1. *Assume that m verifies (H_1)–(H_3) and let $\mathcal{A}$ be defined by (3.1). Define the functionals $J_{m,s}, J \colon L^p(\Omega) \to \bar{\mathbb{R}}$ as*

$$J_{m,s}(u) = \begin{cases} \dfrac{1-s}{p} \displaystyle\iint_{\mathbb{R}^n \times \mathbb{R}^n} m(x,h) \frac{|u(x) - u(x-h)|^p}{|h|^{n+sp}}\, dh dx & \text{if } u \in W_0^{s,p}(\Omega) \\ \infty & \text{else} \end{cases}$$

$$J(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \mathcal{A}(x, \nabla u)\, dx & \text{if } u \in W_0^{1,p}(\Omega) \\ \infty & \text{else.} \end{cases}$$

Then $J_{m,s}$ Gamma-converges to J as $s \uparrow 1$.

The definition and properties of Gamma-convergence can be seen in [8], The proof of Theorem 5.1 is straightforward from Section 3 and the details are completely analogous as in [16].

The main feature of Gamma-convergence is that it implies the following result

Theorem 5.2. *Let $J_{m,s}$ and J be defined as in Theorem 5.1. Then, if $f \in L^{p'}(\Omega)$, there exists a unique minimum $u_s \in W_0^{s,p}(\Omega)$ of*

$$J_{m,s}(v) - \int_\Omega f v\, dx$$

a unique minimum $u \in W_0^{1,p}(\Omega)$ of

$$J(v) - \int_\Omega fv\,dx$$

and $u_s \to u$ in $L^p(\Omega)$.

Again, the details of Theorem 5.2 with the obvious modifications, can be found in [16].

As a corollary of Theorem 5.2 we get the connection between the solution to (5.1) with the solution to (5.2).

Corollary 5.3. *For each $s \in (0,1)$, there exists a unique solution $u_s \in W_0^{s,p}(\Omega)$ to (5.1). This sequence of solutions $\{u_s\}_{s\in(0,1)}$ converges, as $s \uparrow 1$, in $L^p(\Omega)$ to some function $u \in W_0^{1,p}(\Omega)$ and this function u is the unique solution to (5.2).*

6. Remarks on homogenization

The purpose of this section is to investigate the simultaneous effect that *localization* (i.e. $s \uparrow 1$) and homogenization can have in some problems.

To be precise, assume that now we have a family of kernels $m_\varepsilon(x,h)$ satisfying (H_1)–(H_3). Then, for each ε when we take $s \uparrow 1$, we obtain a limit function $\mathcal{A}_\varepsilon(x,\xi)$ as defined in (3.1).

The results of the previous section tell us that the solutions $u_{s,\varepsilon}$ of (5.1) with $m = m_\varepsilon$ are converging as $s \uparrow 1$ to u_ε, the solution to (5.2) with $\mathcal{A} = \mathcal{A}_\varepsilon$.

The problem that we want to address is what happens when $\varepsilon \downarrow 0$.

In order to understand this question, we focus on the model problem where $m_\varepsilon(x,h)$ is obtain from a single kernel m in the form

$$m_\varepsilon(x,h) = m(\tfrac{x}{\varepsilon}, h),$$

and $m(x,h)$ is a periodic function in x of period 1 in each x_i, $i = 1,\dots,n$.

To keep things even simpler, we start with the one dimensional problem.

6.1. The one-dimensional case.

In the 1−dimensional case explicit formulas describing the behavior of the limit problems are available. Indeed, consider a 1−periodic function in x, $m(x,h)$ satisfying (H_1)–(H_3). According to Theorem 3.4 we have that

$$\lim_{s\uparrow 1}(1-s)\iint_{\mathbb{R}\times\mathbb{R}} m(\tfrac{x}{\varepsilon},h)\frac{|u(x)-u(x-h)|^p}{|h|^{1+sp}}\,dxdh = \int_{\mathbb{R}^n}\mathcal{A}(\tfrac{x}{\varepsilon},|u'(x)|)\,dx =: J^\varepsilon(u).$$

In this one-dimensional case, the limit function $\mathcal{A}(y,\xi)$ can be easily computed as

$$\mathcal{A}(y,\xi) = \frac{1}{p}\left(a(y,-1)+a(y,1)\right)|\xi|^p =: A(y)|\xi|^p.$$

When ε vanishes, it is well known that (see [13, Proposition 3.7]) J^ε Gamma-converges to J^*, where

$$J^*(u) = \int_{\mathbb{R}} A^*|u'(x)|^p\,dx,$$

and A^* is a constant coefficient given by

$$A^* := \left(\int_0^1 A(t)^{-1/(p-1)}\,dt\right)^{-1/(p-1)}.$$

This fact gives as a result that minimizers $u_{s,\varepsilon}$ of (5.1) with coefficients m_ε verifiy that

$$\lim_{\varepsilon\downarrow 0}\left(\lim_{s\uparrow 1} u_{s,\varepsilon}\right) = u^*,$$

where u^* is the solution to

$$\begin{cases} -(A^*|u'|^{p-2}u')' = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

On the other hand, it is well-known (see, for instance [14]) that as $\varepsilon \downarrow 0$ it holds that solutions $u_{s,\varepsilon}$ of (5.1) with coefficients m_ε converge to the solution $\bar u_s$ of (5.1) with coefficient $\bar m(h)$ given by

$$\bar m(h) = \int_0^1 m(t,h)\,dt.$$

Finally, applying Theorem 3.4 we arrive at

$$\lim_{s\uparrow 1}\left(\lim_{\varepsilon\downarrow 0} u_{s,\varepsilon}\right) = \bar u,$$

where $\bar u$ is the solution to the problem

$$\begin{cases} -(\bar A|u'|^{p-2}u')' = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

In this case, $\bar A$ is given by

$$\bar A = \frac{1}{p}(\bar m(-1) + \bar m(1)) = \frac{1}{p}\int_0^1 m(t,-1) + m(t,1)\,dt.$$

From these simple formulas one can immediately see that the localization process and the homogenization process are not interchangeables.

6.2. **The general case.** The computations of the previous subsection can be extended with some care to the $n-$dimensional case.

Let now $m(x,h)$ be a $Q-$periodic function in its first variable, Q being the unit cube in $\mathbb{R}^n$, and satisfying hypothesis (H_1)–(H_3).

Let us now state the problem in a precise way.

A function $u_{s,\varepsilon} \in W_0^{s,p}(\Omega)$ is a weak solution of

$$\begin{cases} \mathcal{L}_{m_\varepsilon,s} u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

if

$$\langle \mathcal{L}_{m_\varepsilon,s} u_{s,\varepsilon}, v\rangle = \int_\Omega fv\;dx,$$

for every $v \in W_0^{s,p}(\Omega)$ where $f \in W^{-s,p'}(\Omega)$ and $\mathcal{L}_{m_\varepsilon,s}$ is the Fréchet derivative of $J_{m_\varepsilon,s}$ given in (2.1). Observe that it follows that $\langle \mathcal{L}_{m_\varepsilon,s} u, v\rangle$ is given by

$$(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} m(\tfrac{x}{\varepsilon},h)\frac{|u(x)-u(x-h)|^{p-2}(u(x)-u(x-h))(v(x)-v(x-h))}{|h|^{n+sp}}\,dhdx.$$

Our first result concerns with the problem of first localizing and then homogenizing, that is, we first take the limit as $s\uparrow 1$ and then the limit as $\varepsilon \downarrow 0$.

Proposition 6.1. *It holds that*

$$\lim_{\varepsilon\downarrow 0}\left(\lim_{s\uparrow 1} u_{s,\varepsilon}\right) = u^*,$$

(in the $L^p(\Omega)$ sense) where $u^* \in W^{1,p}_0(\Omega)$ is the weak solution of

$$\begin{cases} -\operatorname{div}(\nabla_\xi \mathcal{A}^*(\nabla u^*)) = f & \text{in } \Omega \\ u^* = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\mathcal{A}^*(\xi) = \inf_{v\in W^{1,1}_{per}(Q)} \int_Q \mathcal{A}(y, \xi + \nabla v(y))\, dy$$

Proof. First, we have to take the limit as $s \uparrow 1$ for fixed $\varepsilon > 0$, but this was carried out in Section 5, and it holds that

$$\lim_{s\uparrow 1} u_{s,\varepsilon} = u_\varepsilon,$$

where $u_\varepsilon \in W^{1,p}_0(\Omega)$ is the solution to

$$\begin{cases} -\operatorname{div}(\nabla_\xi \mathcal{A}_\varepsilon(x, \nabla u_\varepsilon)) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, we can apply the results of [8, Chapter 24] to conclude the desired result as $\varepsilon \downarrow 0$. □

To finish the section, we now deal with the case where first we homogenize and then localize, i.e. first take the limit $\varepsilon \downarrow 0$ and then the limit $s \uparrow 1$.

Proposition 6.2. *It holds that*

$$\lim_{s\uparrow 1}\left(\lim_{\varepsilon\downarrow 0} u_{s,\varepsilon}\right) = \bar{u}$$

(in the $L^p(\Omega)$ sense), where $\bar{u} \in W^{1,p}_0(\Omega)$ is the solution to

$$\begin{cases} -\operatorname{div}(\nabla_\xi \bar{\mathcal{A}}(\nabla \bar{u})) = f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\bar{\mathcal{A}}$ is given by

$$\bar{\mathcal{A}}(\xi) = \int_Q \mathcal{A}(y, \xi)\, dy,$$

and $\mathcal{A}$ is the one given by (3.1).

Proof. We first have to take the limit as $\varepsilon \downarrow 0$ for fixed $s \in (0, 1)$. But this problem was already solved in [14] and what is known is that

$$\lim_{\varepsilon\to 0} u_{s,\varepsilon} = \bar{u}_s,$$

where $\bar{u}_s$ is the solution to

$$\begin{cases} \mathcal{L}_{\bar{m},s}\bar{u}_s = f & \text{in } \Omega \\ \bar{u}_s = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and

$$\bar{m}(h) = \int_Q m(y, h)\, dy.$$

Now, we can take the limit as $s \uparrow 1$ using the results of Section 5 to conclude the desired result. $\square$

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