EIGENVALUE HOMOGENIZATION FOR QUASILINEAR ELLIPTIC OPERATORS IN ONE SPACE DIMENSION

JULIÁN FERNÁNDEZ BONDER, JUAN P. PINASCO, ARIEL M. SALORT

Abstract. In this paper we study the rate of convergence of the eigenvalues of 1-dimensional rapidly oscillating $p$-laplacian type problems and find explicit order of convergence both in $k$ and in $\varepsilon$. Moreover, explicit bounds on the constant entering in the estimate are also obtained.

1. Introduction

The objective of this paper is the study of the asymptotic behavior (as $\varepsilon \to 0$) of the eigenvalues of the following problem

\begin{equation}
\begin{cases}
-(a(\frac{x}{\varepsilon})|u_{\varepsilon}'|^{p-2}u_{\varepsilon}')' = \lambda \varepsilon \rho(\frac{x}{\varepsilon})|u_{\varepsilon}|^{p-2}u_{\varepsilon} & \text{in } I := (0,1) \\
u_{\varepsilon}(0) = u_{\varepsilon}(1) = 0,
\end{cases}
\end{equation}

where the diffusion coefficient $a(x)$ and the weight function $\rho(x)$ are 1-periodic functions, bounded away from zero and infinity and $\varepsilon > 0$ is a real parameter.

This type of problems have been considered extensively in the literature due to its many applications in different fields. The problem of finding the asymptotic behavior of the eigenvalues of (1.1) is an important part of what is called homogenization theory.

Homogenization of one-dimensional periodic linear problems was studied in the late 60’s by Spagnolo [24] and De Giorgi [10] and generalized to the linear multidimensional case in the mid-70’s by Sanchez-Palencia [22], Bensoussan, Lions and Papanicolaou [18] among others. Likewise, the study of eigenvalue problems with oscillating coefficients started with the works of Boccardo and Marcellini [4], and Kesavan [16, 17].

Problem (1.1) has a natural limit problem as $\varepsilon \to 0$ given by

\begin{equation}
\begin{cases}
-(a_{p}^*|u'|^{p-2}u')' = \lambda \bar{\rho}|u|^{p-2}u & \text{in } I \\
u(0) = u(1) = 0
\end{cases}
\end{equation}

where $\bar{\rho}$ is the average of $\rho$ in the interval $I$ and $a_{p}^*$ is given by

\[ a_{p}^* := \left( \frac{1}{\int_I \frac{1}{a(s)^{1/(p-1)}} \, ds} \right)^{-(p-1)}, \]

see Section 2 for a proof.

Now, what we are interested in is on the convergence of the eigenvalues of problem (1.1) to the ones of problem (1.2); more specifically, on the order of convergence
of the eigenvalues, i.e. we find explicit bounds on $\varepsilon$ and $k$ for the difference
\[ |\lambda_k^\varepsilon - \lambda_k| \]
where $\lambda_k^\varepsilon$ and $\lambda_k$ are the $k$-th eigenvalue of problem (1.1) and (1.2) respectively.

The first result in this problem, for the linear case $p = 2$ and in $N$-dimensional space, can be found in Chapter III, section 2 of [19]. By estimating the eigenvalues of the inverse operator, which is compact, and using tools from functional analysis in Hilbert spaces, they deduce that
\[ |\lambda_k^\varepsilon - \lambda_k| \leq \frac{C\lambda_k^\varepsilon (\lambda_k)^2}{1 - \lambda_k^\varepsilon \beta_k^\varepsilon} \varepsilon^{\frac{1}{2}}. \]
Here, $C$ is a positive constant, and $\beta_k^\varepsilon$ satisfies
\[ 0 \leq \beta_k^\varepsilon < \lambda_k^{-1}, \]
and
\[ \lim_{\varepsilon \to 0} \beta_k^\varepsilon = 0 \]
for each $k \geq 1$.

Then Santosa and Vogelius in [23], for smooth domains in $N$-dimensional space, improve this estimate and found
\[ |\lambda_k^\varepsilon - \lambda_k| \leq C \varepsilon \]
Recently, Kenig, Lin and Shen [15] studied the linear problem in any dimension and proved that for Lipschitz domains $\Omega$ one has
\[ |\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon |\log(\varepsilon)|^{\frac{1}{2} + \sigma} \]
for any $\sigma > 0$, $C$ depending on $k$ and $\sigma$.

Moreover, the authors show that if the domain $\Omega$ is more regular ($C^{1,1}$ is enough) they can get rid of the logarithmic term in the above estimate.

Let us stress the fact that no explicit dependance of $C$ on $k$ is obtained in these works.

More recently, in [14], in $N$-dimensional space and for the quasilinear problem with diffusion coefficients independent of $\varepsilon$, we obtain the bound
\[ |\lambda_k^\varepsilon - \lambda_k| \leq Ck^2 \varepsilon \]
with $C$ independent on $k$ and $\varepsilon$ for any Lipschitz domain. The constant $C$ is unknown.

It is expected that in the one dimensional case one can be more precise with the estimates. In fact, Castro and Zuazua in [5, 6], for the linear problem using the so-called WKB method which relays on asymptotic expansions of the solutions of the problem, and the explicit knowledge of the eigenfunctions and eigenvalues of the constant coefficient limit problem, proved
\[ |\lambda_k^\varepsilon - \lambda_k| \leq Ck^4 \varepsilon \]
and they also presented a variety of results on correctors for the eigenfunction approximation. Let us mention that their method needs higher regularity on the weight $\rho$ and on the diffusion $a$, which must belong at least to $C^2$ and that the
bound holds for $k \sim \varepsilon^{-1}$. Also, the value of the constant $C$ entering in the estimate is unknown.

The main result of our paper is the following Theorem:

**Theorem 1.1.** There exists a constant $C$ depending only on $p$, $a$ and $\rho$ such that

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{2p}\varepsilon.$$  
Moreover, $C$ can be estimated explicitly in terms of the functions $a$ and $\rho$, and $p$.

**Remark 1.2.** An explicit bound on the constant $C$ of Theorem 1.1 is given in Corollary 3.5.

A useful tool used in the proof of Theorem 1.1 is the variational characterization of the eigenvalues of (1.1) and (1.2). Also, it will be essential that the variational eigenvalues, for the one dimensional problem, exhaust the whole spectrum of (1.1) and (1.2). These facts are collected in Appendix A at the end of the article.

In the course of our arguments, a general result on the convergence of eigenvalues is proved. Namely, we prove that the eigenvalues of

$$
(1.3) \begin{cases}
-(a_\varepsilon(x)|u_\varepsilon'|^{p-2}u_\varepsilon')' = \lambda_\varepsilon \rho_\varepsilon(x)|u_\varepsilon'|^{p-2}u_\varepsilon & \text{in } I := (0,1) \\
u_\varepsilon(0) = u_\varepsilon(1) = 0,
\end{cases}
$$

converges to the ones of the limit problem

$$
(1.4) \begin{cases}
-(a_h(x)|u'_\varepsilon'|^{p-2}u'_\varepsilon)' = \lambda \rho_h(x)|u_\varepsilon'|^{p-2}u_\varepsilon & \text{in } I := (0,1) \\
u_\varepsilon(0) = u_\varepsilon(1) = 0,
\end{cases}
$$

where $\rho_h$ is the weak* limit of $\rho_\varepsilon$ and $a_h$ is the $G$–limit of $a_\varepsilon$. Here, the $G$–limit is understood in the sense of De Giorgi (see Section 2 for the precise definition).

In the linear case ($p = 2$), and in $N$–dimensional space, Kesavan in [16, 17] proved that if $a_\varepsilon G$–converges to $a_h$ and $\rho_\varepsilon \rightharpoonup \rho_h$ weakly* in $L^\infty$ then the sequence of the $k$–th eigenvalues of (1.1) converges to the $k$–th eigenvalue of (1.2).

In the general quasilinear setting, for $N$–dimensional space, the first result we are aware of is by Baffico, Conca and Rajesh, [3], where the authors prove that the limit of any convergent sequence of eigenvalues of (1.1) is an eigenvalue of (1.2) and, moreover, that the sequence of the first eigenvalues of (1.1) converges to the first eigenvalue of (1.2).

In a recent work, [14], we studied the same problem, again in $N$–dimensional space, and prove that the first and second eigenvalues of (1.1) converges to those of the limit operator (1.2). Moreover, when the diffusion coefficient $a_\varepsilon$ is independent of $\varepsilon$, we prove that the sequence of the $k$–th variational eigenvalues of (1.1) converges to the $k$–th variational eigenvalue of (1.2).

In one space dimension one can be more precise and we can prove the following

**Theorem 1.3.** Assume that $a_\varepsilon G$–converges to $a_h$ and that $\rho_\varepsilon \rightharpoonup \rho_h$ weakly* in $L^\infty(I)$. For each $k \geq 1$ let $\lambda_k^\varepsilon$ be the $k$–th eigenvalue of (1.1). Then we have that

$$\lim_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda_k,$$

where $\lambda_k$ the $k$–th eigenvalue of (1.2).
Moreover, up to a subsequence, an eigenfunction $u^\varepsilon_k$ associated to $\lambda^\varepsilon_k$ converges weakly in $W^{1,p}_0(I)$ to $u_k$, an eigenfunction associated to $\lambda_k$.

**Organization of the paper.** After finishing this introduction, in Section 2 we study the general convergence problem without any periodicity assumption and we prove Theorem 1.3.

Then, in Section 3, we specialize to the periodic case and we find explicit order of convergence (both in $\varepsilon$ and in $k$) together with an explicit estimate on the constants entering in the bounds.

In Section 4 we present some numerical experiments that illustrate the results of the paper and suggest some open questions about the convergence of the eigenvalues and eigenfunctions.

At the end of the paper, we include an appendix with some results on the eigenvalue problem (1.1) for fixed $\varepsilon$ that are needed in the course of our arguments.

**2. Convergence of eigenvalues**

In this Section we study the convergence problem for the eigenvalues of (1.3) to those of (1.4) where the weights $\rho^\varepsilon$ and the diffusion coefficients $a^\varepsilon$ are bounded uniformly away from zero and infinity, i.e.

\[
(2.1) \quad 0 < \rho_\varepsilon \leq \rho^\varepsilon(x) \leq \rho_+ < +\infty,
\]

\[
(2.2) \quad 0 < \alpha \leq a^\varepsilon(x) \leq \beta < +\infty.
\]

**2.1. Preliminaries on $G$–convergence.** The main tool that we use is the $G$–convergence that was introduced by Spagnolo and De Giorgi in the late 60s (see [10, 24]). Let us recall the definition restricted to our setting for the readers convenience.

**Definition 2.1.** We say that the functions $a^\varepsilon$ $G$–converges to $a_h$ if for every $f \in W^{-1,p'}(I)$ and for every $f^\varepsilon$ strongly convergent to $f$ in $W^{-1,p'}(I)$, the solutions $u^\varepsilon$ of the problem

\[
\begin{aligned}
-(a^\varepsilon(x)|u^\varepsilon'|^{p-2}u^\varepsilon)' &= f^\varepsilon & &\text{in } I \\
u^\varepsilon(0) &= u^\varepsilon(1) = 0
\end{aligned}
\]

satisfy the following conditions

\[
u^\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}_0(I)
\]

\[
a^\varepsilon(x)|u^\varepsilon'|^{p-2}u^\varepsilon' \rightharpoonup a_h(x)|u'|^{p-2}u' \text{ weakly in } L^{p'}(I)
\]

where $u$ is the solution of the equation

\[
\begin{aligned}
-(a_h(x)|u'|^{p-2}u')' &= f & &\text{in } I \\
u(0) &= u(1) = 0.
\end{aligned}
\]

In [7], Chiad` o Piat, Dal Maso and Defranceschi show that, up to a sequence $\varepsilon_j \to 0$, there exists a function $a_h(x)$ satisfying (2.2) such that $a^\varepsilon$ $G$–converges to $a_h$. Moreover, again up to a sequence, there exists a function $\rho_h(x)$ satisfying (2.1) such that $\rho^\varepsilon \rightharpoonup \rho_h$ weakly* in $L^\infty$. 

2.2. Convergence of eigenvalues. In order to prove Theorem 1.3 we need some preliminaries.

From Appendix A, according to Theorem A.4, we denote by \( \Sigma_\varepsilon := \{ \lambda_k^\varepsilon \}_{k \in \mathbb{N}} \) the full sequence of eigenvalues of problem (1.3) and by \( \Sigma_h := \{ \lambda_k \}_{k \in \mathbb{N}} \) those of its limit problem (1.4). They can be written as

\[
\lambda_k^\varepsilon = \inf_{C \in \Gamma_k} \sup_{u \in C} \int_I a_\varepsilon(x) |u'|^p, \quad \lambda_k = \inf_{C \in \Gamma_k} \sup_{u \in C} \int_I a_h(x) |u'|^p.
\]

We begin by stating a general result for bounded sequences of eigenvalues that can be found in [3] (see also [14] where a simplified proof of this result is given).

**Theorem 2.2.** Let \( \lambda^\varepsilon \in \Sigma_\varepsilon \) be a sequence of eigenvalues of the problem (1.3) with \( \{ u_\varepsilon \}_{\varepsilon > 0} \) associated normalized eigenfunctions. Assume that the sequence of eigenvalues is convergent

\[
\lim_{\varepsilon \to 0^+} \lambda^\varepsilon = \lambda.
\]

Then, \( \lambda \in \Sigma_h \) and there exists a sequence \( \varepsilon_j \to 0^+ \) such that

\[
u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}_0(I)
\]

with \( u \) a normalized eigenfunction associated to \( \lambda \).

Assume now that we take the family of the \( k \)-th eigenvalue of (1.1) \( \{ \lambda_k^\varepsilon \}_{\varepsilon > 0} \). It is not difficult to see that this family is bounded, in fact as

\[
\frac{\alpha}{\rho^+} \int_0^1 |v'|^p \leq \int_0^1 a_\varepsilon(x) |v'|^p \leq \frac{\beta}{\rho^-} \int_0^1 |v|^p,
\]

we have

\[
(2.5) \quad \frac{\alpha}{\rho^+} \mu_k \leq \lambda_k^\varepsilon \leq \frac{\beta}{\rho^-} \mu_k
\]

where \( \mu_k = \pi_p k^p \) is the \( k \)-th eigenvalue of the one dimensional \( p \)-Laplacian (see Appendix A, Theorem A.1).

Therefore, up to a subsequence, \( \lambda_k^\varepsilon \) converges to \( \lambda \in \Sigma_h \). The main tool that allows us to prove that \( \lambda = \lambda_k \) is Theorem A.3 that says that any eigenfunction associated to the \( k \)-th eigenvalue of (1.1) has exactly \( k \) nodal domains.

Moreover, we need a refinement of this result, namely an estimate on the measure of each nodal domain independent on \( \varepsilon \). This is the content of the next Lemma.

**Lemma 2.3.** Let \( \lambda_k^\varepsilon \) be an eigenvalue of (1.3) with corresponding eigenfunction \( u_\varepsilon^k \).

Let \( N = \mathcal{N}(k, \varepsilon) \) be a nodal domain of \( u_\varepsilon^k \). We have that

\[
|N| > C
\]

where \( C = C(k) \) is a positive constant independent of \( \varepsilon \).

**Proof.** We can write \( \lambda_k^\varepsilon \) as

\[
\lambda_k^\varepsilon(I) = \lambda_1^\varepsilon(N) = \inf_{u \in W^{1,p}_0(N)} \frac{\int_N a_\varepsilon(x) |u'|^p}{\int_N a_\varepsilon(x) |u|^p}.
\]
by our assumptions (2.1) we get
\[
\lambda^\varepsilon_k(I) \geq \frac{\alpha}{\rho_+} \rho_1(N) = \frac{\alpha}{\rho_+} \frac{\pi_p^p}{|N|^p}
\]
where \(\rho_1(N)\) is the first eigenvalue of the \(p\)-Laplacian on \(N\). Moreover,
\[
\lambda^\varepsilon_k(I) \leq \frac{\beta}{\rho_-} \mu_k(I) = \frac{\beta}{\rho_-} \pi_p^p k^p.
\]
Combining both inequalities we get
\[
|N| \geq \frac{\alpha}{\rho_+} \frac{\pi_p^p}{\lambda^\varepsilon_k(\Omega)} \geq \frac{\alpha}{\beta} \frac{\rho_+}{\rho_-} \frac{1}{k^p}
\]
and the result follows.

Now we are ready to establish the main result of this section:

**Proof of Theorem 1.3.** Let \(u_k\) be a normalized eigenfunction associated to \(\lambda_k\) and according to Theorem A.3, let \(I_i, i = 1, \ldots, k\) be the nodal domains of \(u_k\).

We denote by \(u^\varepsilon_i\) the first eigenfunction of (1.3) in \(I_i\) respectively. Extending \(u^\varepsilon_i\) to \(I\) by 0, these function have disjoint supports and therefore they are linearly independent in \(W^{1,p}_0(I)\).

Let \(S\) be the unit sphere in \(W^{1,p}_0(I)\) and we define the set \(C^\varepsilon_k\) as
\[
C^\varepsilon_k := \text{span}\{a^\varepsilon_1, \ldots, a^\varepsilon_k\} \cap S.
\]
Clearly \(C^\varepsilon_k\) is compact, symmetric and \(\gamma(C^\varepsilon_k) = k\). Hence,
\[
\lambda^\varepsilon_k = \inf_{C^\varepsilon_k} \sup_{v \in C^\varepsilon_k} \frac{\int_I a(x)|v'|^p}{\int_I \rho(x)|v|^p} \leq \sup_{v \in C^\varepsilon_k} \frac{\int_I a(x)|v'|^p}{\int_I \rho(x)|v|^p}
\]
As \(C^\varepsilon_k\) is compact, the supremum is achieved for some \(v^\varepsilon \in C^\varepsilon_k\) which can be written as
\[
v^\varepsilon = \sum_{i=1}^k a^\varepsilon_i u^\varepsilon_i
\]
with \(a^\varepsilon_i \in \mathbb{R}\) such that \(\sum_{i=1}^k |a^\varepsilon_i|^p = 1\). Since the functions \(u^\varepsilon_i\) have non-overlapping supports, we obtain
\[
\lambda^\varepsilon_k \leq \frac{\sum_{i=1}^k |a^\varepsilon_i|^p \int_{I_i} \rho(x)|u^\varepsilon_i|^p}{\int_I \rho(x)|v^\varepsilon|^p} \leq \max_{1 \leq i \leq k} \lambda^\varepsilon_{1,i}
\]
where \(\lambda^\varepsilon_{1,i}\) is the first eigenvalue of (1.1) in the nodal domain \(\Omega_i\) respectively.

Now, using that \(\lambda^\varepsilon_{1,i} \to \lambda_{1,i}\) respectively, where \(\lambda_{1,i}\) are the first eigenvalues of (1.2) in the domains \(I_i\) respectively (see Theorem 4.4, [14]). Moreover, we observe that these eigenvalues \(\lambda_{1,i}\) are all equal to the \(k\)-th eigenvalue \(\lambda_k\) in \(I\), therefore from (2.6), we get
\[
\lambda^\varepsilon_k \leq \lambda_k + \delta
\]
for $\delta$ arbitrarily small and $\varepsilon$ tending to zero. So
\begin{equation}
\limsup_{\varepsilon \to 0} \lambda_k^\varepsilon \leq \lambda_k.
\end{equation}
On the other hand, suppose that $\lim_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda$. By Lemma 2.3 the $k$ nodal domains of $u_k^\varepsilon$ have positive measure independent of $\varepsilon$. Then it must be $\lambda \geq \lambda_k$. It follows that
\begin{equation}
\lambda_k \leq \lambda = \lim_{\varepsilon \to 0} \lambda_k^\varepsilon.
\end{equation}
Combining (2.7) and (2.8) we obtain the desired result. \hfill \Box

2.3. An example of $G$–convergence. We finish this section by explicitly computing the $G$–limit operator in the periodic case.

In the periodic linear case (see for instance [1, 9]), it is known that the family $a(\frac{x}{\varepsilon})$ $G$–converges to $a^*_p \in \mathbb{R}$ given by
\begin{equation}
a^*_p = \left(\int_I a(x)^{-\frac{1}{p}} dx\right)^{(p-1)}.
\end{equation}

This result is a simple exercise but we include it here since we were unable to find it in the literature and it will be used in the next section.

**Proposition 2.4.** Let $a \in L^\infty(\mathbb{R})$ be $1$–periodic function such that satisfy (2.2). Then $a(x/\varepsilon) G$–converges to $a^*_p \in \mathbb{R}$ given by
\begin{equation}
a^*_p = \left(\int_I a(x)^{-\frac{1}{p+1}} dx\right)^{-\frac{p}{p+1}}.
\end{equation}

**Proof.** Let $f_\varepsilon \in W^{-1,p'}(I)$ be such that $f_\varepsilon \to f$ in $W^{-1,p'}(I)$.

Let $g_\varepsilon(x) := \langle f_\varepsilon, \chi(0,x) \rangle$, then $g_\varepsilon \in L^p(I)$, $g'_\varepsilon = f_\varepsilon$ and $g_\varepsilon \to g := \langle f, \chi(0,x) \rangle$ in $L^p(I)$.

Let $u_\varepsilon$ be the weak solution to
\begin{equation}
\begin{cases}
-(a(\frac{x}{\varepsilon}))|u'_\varepsilon|^{p-2}u'_\varepsilon = f_\varepsilon & \text{in } I, \\
u_\varepsilon(0) = u_\varepsilon(1) = 0
\end{cases}
\end{equation}
Then, there exists a constant $c_\varepsilon$ such that $a(x/\varepsilon)|u'_\varepsilon|^{p-2}u'_\varepsilon = c_\varepsilon - g_\varepsilon$.

Let $\varphi_p(x) = |x|^{p-2}x$. Then $\varphi_p$ is invertible and so
\begin{equation}
u'_\varepsilon = \varphi_p^{-1}(c_\varepsilon - g_\varepsilon)a(\frac{x}{\varepsilon})^{\frac{1}{p+1}}.
\end{equation}
Since $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,p}_0(I)$, we can assume that is weakly convergent to some $u \in W^{1,p}_0(I)$ and, since $a(\frac{\cdot}{\varepsilon})^{\frac{1}{p+1}} \to a^{\frac{1}{p+1}}$ weakly * in $L^\infty(I)$ and $g_\varepsilon \to g$ in $L^p(I)$, we can assume that there exists $c$ such that $c_\varepsilon \to c$.

Now we can pass to the limit in (2.9) and obtain
\begin{equation}
u' = \varphi_p^{-1}(c - g)a^{\frac{1}{p+1}}.
\end{equation}
The proof is now complete. \hfill \Box
3. Order of convergence. The periodic case

In this part of the article, we focus on the limit behavior of eigenvalues of (1.1). In fact, from the results of Section 2, it follows that the $k$–th eigenvalue of (1.1) converges to the $k$–th eigenvalue of the limit problem (1.2).

In order to clarify the statement of the main Theorem we introduce the following notation:

**Definition 3.1.** Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $0 < g^- \leq g \leq g^+ < \infty$. We define the oscillation of $g$ as

$$\omega_g = \frac{g^+ - g^-}{g^+ - g^-}.$$ 

The main result of this Section is the following Theorem:

**Theorem 3.2.** Let $\lambda^\varepsilon_k$ be the $k$–th eigenvalue of problem (1.1) and let $\lambda_k$ be the $k$–th eigenvalue of its limit problem (1.2). Then we have

$$|\lambda^\varepsilon_k - \lambda_k| \leq (a_p^*)^{p/(p-1)} \frac{\omega_p \omega_a^1/(p-1) - 1}{\rho^\alpha/(p-1)} \frac{\omega_p \omega_a^1/(p-1)}{\pi p^p k^{2p}} \left[ \frac{\rho^\alpha/(p-1) + \varepsilon^{p-1}}{\pi p^p} \right]$$

$$+ \frac{\beta}{\rho^\alpha/(p-1)} \frac{\varepsilon^{p-1}}{\pi p^p} k^p \varepsilon (1 + \varepsilon)^{p-1} C(\varepsilon),$$

where $\omega_p$ and $\omega_a$ are given by Definition 3.1 and

$$C(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon = 1/j, \ j \in \mathbb{N} \\
1 & \text{else.} \end{cases}$$

**Remark 3.3.** The main improvements of Theorem 3.2 from our results in [14] for the $N$–dimensional problem is that here we are allowing a rapidly oscillating diffusion coefficient in the problem where in [14] the diffusion coefficient must be fixed. Moreover, the constant in Theorem 3.2 is given explicitly in terms of $p$, $a$ and $\rho$ where in [14] is unknown.

**Remark 3.4.** As a consequence of Theorem 3.2 we obtain the rate of convergence of the nodal domains of the eigenfunctions of problem (1.1) to those of the limit problem, c.f. Theorem 3.17.

We state a simple corollary of Theorem 3.2.

**Corollary 3.5.** Let $C(p, a, \rho) = (a_p^*)^{p/(p-1)} \frac{\omega_p \omega_a^1/(p-1) - 1}{\rho^\alpha/(p-1)} \frac{\omega_p \omega_a^1/(p-1)}{\pi p^p} \rho^{p+1}$ then for $\varepsilon < \varepsilon_0$ we have

$$|\lambda^\varepsilon_k - \lambda_k| \leq 2C(p, a, \rho)\varepsilon k^{2p}. \tag{3.1}$$

**Remark 3.6.** The constant 2 in (3.1) can be replaced by any other constant greater than 1.

**Remark 3.7.** Corollary 3.5 is exactly Theorem 1.1.
3.1. Preliminary estimates. The following lemma is well known, but we included for the sake of completeness.

**Lemma 3.8.** Let $J = (0, \ell)$ be an interval in $\mathbb{R}$ and let $v \in W^{1,q}(J)$, $1 \leq q < \infty$, be such that $v(0) = 0$. Then

$$
\|v\|_{L^q(J)}^q \leq \frac{\ell^q}{q} \|v'\|_{L^q(J)}^q.
$$

**Proof.** We have

$$
v(x) = \int_0^x v' \leq \|v'\|_{L^q(J)} x^{1/q}.
$$

Integrating, we obtain

$$
\|v\|_{L^q(J)}^q \leq \|v'\|_{L^q(J)}^q \int_0^\ell x^{q/q'} = \|v'\|_{L^q(J)}^q \frac{\ell^q}{q},
$$
as we wanted to show. \hfill \Box

Now, we can show two immediate consequences of Lemma 3.8

**Corollary 3.9.** Let $v \in W^{1,q}_0(I)$. Then

$$
\int_0^\delta |v|^q \leq \frac{\delta^q}{q} \|v'\|_{L^q(J)}^q.
$$

**Proof.** Immediate from Lemma 3.8. \hfill \Box

**Corollary 3.10.** Let $J = (0, \ell)$ and let $v \in W^{1,q}(J)$, $1 \leq q < \infty$. Assume that there exists $x_0 \in I$ such that $u(x_0) = 0$. Then

$$
\|v\|_{L^q(J)}^q \leq \frac{\ell^q}{q} \|v'\|_{L^q(J)}^q.
$$

**Proof.** Let $J_1 = (0, x_0)$ and $J_2 = (x_0, \ell)$. Then, by Lemma 3.8 we have that

(3.2) \quad \|v\|_{L^q(J_1)}^q \leq \frac{x_0^p}{p} \|v'\|_{L^q(J_1)}^q \leq \frac{\ell^p}{p} \|v'\|_{L^q(J)}^q

and

(3.3) \quad \|v\|_{L^q(J_2)}^q \leq \frac{(\ell - x_0)^p}{p} \|v'\|_{L^q(J_2)}^q \leq \frac{\ell^p}{p} \|v'\|_{L^q(J)}^q.

Now, adding (3.2) and (3.3) we obtain the desired result. \hfill \Box

From Corollary 3.10 we can obtain the following

**Corollary 3.11.** Let $J = (0, \ell)$ and let $v \in W^{1,q}(J)$. Then

$$
\|v - \bar{v}\|_{L^q(J)}^q \leq \frac{\ell^q}{q} \|v'\|_{L^q(J)}^q,
$$

where $\bar{v}$ stands for the average of $v$ over $J$.

**Proof.** Just notice that for $w = v - \bar{v}$ there exists $x_0 \in J$ such that $w(x_0) = 0$. Then, use Corollary 3.10. \hfill \Box
Remark 3.12. Corollary 3.11 is the well known Poincaré inequality. As far as we know, the optimal constant in the Poincaré inequality is unknown even in this one dimensional setting. See [8] for a discussion on this. So, the purpose of Corollary 3.11 is to provide with a rough estimate on this constant and any improvements on the computation of the optimal constant will automatically give an improvement in the constant entering in our result.

Finally, we need a Lemma that controls the oscillating behavior of the weight function.

**Lemma 3.13.** Let \( v \in W^{1,1}(I) \) and let \( g \in L^\infty(\mathbb{R}) \) be a 1-periodic function such that \( \bar{g} = \int_I g = 0 \). Let \( \varepsilon > 0 \) and denote by \( m = \lfloor 1/\varepsilon \rfloor \) the integer part of \( 1/\varepsilon \). Then

\[
\left| \int_0^{m\varepsilon} g(\varepsilon x) v \right| \leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \|v'\|_{L^1(I)}.
\]

**Proof.** Let \( I_j^\varepsilon = ((j-1)\varepsilon, j\varepsilon] \) and \( J^\varepsilon = (0, m\varepsilon] = \bigcup_{j=1}^m I_j^\varepsilon \).

Let \( v_\varepsilon \) be defined as

\[
v_\varepsilon(x) = \frac{1}{\varepsilon} \int_{I_j^\varepsilon} v(y) \, dy \quad \text{for } x \in I_j^\varepsilon.
\]

Now, as \( \bar{g} = 0 \) and \( v_\varepsilon \) is constant on each \( I_j^\varepsilon \) we have that

\[
\int_{I_j^\varepsilon} g(\varepsilon x) v = \int_{I_j^\varepsilon} g(\varepsilon x)(v - v_\varepsilon),
\]

so, by Corollary 3.11,

\[
\left| \int_{I_j^\varepsilon} g(\varepsilon x) v \right| \leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \int_{I_j^\varepsilon} |v'|.
\]

Therefore

\[
\left| \int_0^{m\varepsilon} g(\varepsilon x) v \right| \leq \sum_{j=1}^m \left| \int_{I_j^\varepsilon} g(\varepsilon x) v \right| \leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \sum_{j=1}^m \int_{I_j^\varepsilon} |v'| \leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \int_I |v'|.
\]

The proof is now complete. \( \square \)

With these preliminaries, we arrive at the key estimate in our main result.

**Theorem 3.14.** Let \( g \in L^\infty(I) \) be a 1-periodic function such that \( \bar{g} = 0 \). Then, for every \( u \in W^{1,p}_0(I) \) we have that

\[
\left| \int_I g(\varepsilon x) |u|^p \right| \leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \|u'\|_{L^p(I)}^p \left[ \frac{p}{p-1} + \frac{p-1}{p} \right].
\]

**Proof.** For every \( \varepsilon > 0 \) we denote, as in the previous Lemma, \( I_j^\varepsilon = ((j-1)\varepsilon, j\varepsilon] \) and \( J^\varepsilon = \bigcup_{j=1}^m I_j^\varepsilon = (0, m\varepsilon] \) where \( m = \lfloor 1/\varepsilon \rfloor \) is the integer part of \( 1/\varepsilon \).
Theorem 3.15. show how the general case can be reduced to this one.

3.2. We first analyze the case where the diffusion coefficient is equal to 1 and then

\begin{align}
\int_I g(x) |u|^p &= \int_{G_\varepsilon} g(x) |u|^p + \int_{J_\varepsilon} g(x) |u|^p.
\end{align}

Now, by Corollary 3.9,

\begin{align}
\left| \int_{J_\varepsilon} g(x) |u|^p \right| &\leq \|g\|_{L^\infty(\mathbb{R})} \int_{1-\varepsilon}^1 |u|^p \leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \|u''\|_{L^p(I)}^p.
\end{align}

If we notice that \(|u|^p \in W^{1,1}_0(I)\), by Lemma 3.13 we get

\begin{align}
\left| \int_{J_\varepsilon} g(x) |u|^p \right| &\leq \|g\|_{L^\infty(\mathbb{R})} \varepsilon \int_I \left( |u|^p \right)'.
\end{align}

Finally, the proof is complete once we observe that

\begin{align}
\int_I \left( |u|^p \right)' &= p \int_I |u|^{p-1} |u'| \leq p \left( \int_I |u|^p \right) \left( \int_I |u'|^p \right)^{\frac{1}{p}}
&= p \|u''\|_{L^p(I)}^{p-1} \|u''\|_{L^p(I)} \leq p \pi_1^{-1} \|u''\|_{L^p(I)}^p.
\end{align}

The proof is finished. \( \square \)

3.2. Proof of Theorem 3.2. The case \( a = 1 \). In order to deal with Theorem 3.2 we first analyze the case where the diffusion coefficient is equal to 1 and then show how the general case can be reduced to this one.

Theorem 3.15. Let \( g \in L^\infty(\mathbb{R}) \) be a 1-periodic function such that

\begin{align}
0 < g^- \leq g \leq g^+ < \infty.
\end{align}

Consider the eigenvalue problem

\begin{align}
\begin{cases}
-((|u|^{p-2}u')') = g(x) \lambda |u|^{p-2}u & \text{in } I \\
u(0) = u(1) = 0
\end{cases}
\end{align}

and its limit problem

\begin{align}
\begin{cases}
-((|u|^{p-2}u')') = \tilde{g} \lambda |u|^{p-2}u & \text{in } I \\
u(0) = u(1) = 0.
\end{cases}
\end{align}

Let \( \{\lambda_k^\varepsilon\}_{k \geq 1} \) and \( \{\lambda_k\}_{k \geq 1} \) be the eigenvalues of (3.5) and (3.6) respectively.

Then, we have

\begin{align}
|\lambda_k^\varepsilon - \lambda_k| \leq \frac{g^+ - \tilde{g}}{\tilde{g}^+} \frac{g^+}{\tilde{g}} \pi_1^{2p} e^{k^2p} \left( \frac{p}{\pi_1^{p-1} + \varepsilon^{p-1}} \right).
\end{align}

Proof. The proof follows from Theorem 3.14. In fact, for every \( \delta > 0 \), let \( G_k^\varepsilon \in \Gamma_k \) be such that

\begin{align}
\lambda_k = \frac{\pi_1^{2p} k^p}{\tilde{g}} = \inf_{G \in \Gamma_k} \sup_{v \in G} \int_I |v'|^p = \sup_{v \in G_k^\varepsilon} \int_I |v'|^p + O(\delta).
\end{align}
We bound the eigenvalues of (3.5) as follows

\[
\lambda_k^{\varepsilon} = \inf_{G \in \Gamma_k} \sup_{v \in G} \frac{\int_I |v'|^p}{\int_I g(\varepsilon x) |v|^p} \int_I \bar{g}(x) |v|^p \\
\leq \sup_{v \in G_k^k} \frac{\int_I |v'|^p}{\int_I g(\varepsilon x) |v|^p} \int_I \bar{g}(x) |v|^p \\
\leq (\lambda_k + O(\delta)) \sup_{v \in G_k^k} \frac{\bar{g}}{\int_I g(\varepsilon x) |v|^p}.
\]

(3.8)

Now, by Theorem 3.14 we bound the quotient

\[
\frac{\bar{g}}{g} \int_I |v|^p \leq 1 + (g^+ - \bar{g})\varepsilon \left[ \frac{p}{\pi_p - 1} + \frac{\varepsilon^{p-1}}{p} \right] \frac{\int_I |v'|^p}{\int_I g(\varepsilon x) |v|^p}.
\]

(3.9)

By (3.4) we have, as \( v \in G^k_\delta \),

\[
\int_I |v'|^p \leq g \int_I |v|^p \leq \frac{\bar{g}}{g} (\lambda_k + O(\delta)).
\]

(3.10)

By using (3.10) we bound (3.9) as

\[
1 + (g^+ - \bar{g})\varepsilon \left[ \frac{p}{\pi_p - 1} + \frac{\varepsilon^{p-1}}{p} \right] (\lambda_k + O(\delta)).
\]

(3.11)

Finally, combining (3.8), (3.9) and (3.11) and letting \( \delta \to 0 \) we find that

\[
\lambda_k^{\varepsilon} - \lambda_k \leq \frac{g^+ - \bar{g}}{g} \varepsilon \lambda_k^2 \left[ \frac{p}{\pi_p - 1} + \frac{\varepsilon^{p-1}}{p} \right].
\]

(3.12)

Arguing in much the same way, we get the inequality

\[
\lambda_k^{\varepsilon} - \lambda_k \leq \frac{g^+ - \bar{g}}{g} \varepsilon \lambda_k^2 \left[ \frac{p}{\pi_p - 1} + \frac{\varepsilon^{p-1}}{p} \right].
\]

(3.13)

Now, from (3.4) and the variational characterization of the eigenvalues, we get the estimates

\[
\frac{\bar{g}}{g^+} \lambda_k \leq \lambda_k^{\varepsilon} \leq \frac{\bar{g}}{g} \lambda_k.
\]

(3.14)

Using (3.14) in (3.13), together with (3.12), we obtain

\[
|\lambda_k^{\varepsilon} - \lambda_k| \leq \frac{g^+ - \bar{g}}{g} \varepsilon \lambda_k^2 \left[ \frac{p}{\pi_p - 1} + \frac{\varepsilon^{p-1}}{p} \right]
\]

(3.15)

and from the explicit form of the eigenvalues

\[
\lambda_k = \frac{\pi_p k^p}{g},
\]

we arrive at

\[
|\lambda_k^{\varepsilon} - \lambda_k| \leq \frac{g^+ - \bar{g}}{g} \varepsilon \lambda_k^2 \left[ \frac{p}{\pi_p - 1} + \frac{\varepsilon^{p-1}}{p} \right],
\]

as we wanted to proved. \( \square \)
Remark 3.16. If we replace the unit interval $I = (0, 1)$ by $I_\ell = (0, \ell)$ by a simple change of variables, the estimates of Theorem 3.15 are modified as

\begin{equation}
|\lambda^\varepsilon_k(I_\ell) - \lambda_k(I_\ell)| = \ell^p|\lambda^\varepsilon_k(I) - \lambda_k(I)|.
\end{equation}

3.3. **Proof of Theorem 3.2. The general case.** Now we are ready to prove the main result of the section, namely Theorem 3.2

**Proof of Theorem 3.2.** The proof of the Theorem follows by converting problem (1.1) into (3.5) by a change of variables.

In fact, if we define

$$
P_\varepsilon(x) = \int_0^x \frac{1}{a_\varepsilon(s)^{1/(p-1)}} ds = \varepsilon \int_0^{x/\varepsilon} \frac{1}{a(s)^{1/(p-1)}} ds = \varepsilon P\left(\frac{x}{\varepsilon}\right)
$$

and perform the change of variables

\[(x, u) \to (y, v)\]

where

\[y = P_\varepsilon(x) = \varepsilon P\left(\frac{x}{\varepsilon}\right), \quad v(y) = u(x)\]  

By simple computations we get

\[
\begin{cases}
-\left(|\dot{v}|^{p-2}\dot{v}\right) = \lambda^\varepsilon \rho \|v\|^{p-2}v, \quad y \in [0, L_\varepsilon] \\
v(0) = v(L_\varepsilon) = 0
\end{cases}
\]

where

\[\cdot = d/dy,\]

with

\[L_\varepsilon = \int_0^1 \frac{1}{a_\varepsilon(s)^{1/(p-1)}} ds \to L = a_{p-1},\]

and

\[Q_\varepsilon(y) = a_\varepsilon(x)^{1/(p-1)} \rho_\varepsilon(x) = a(P^{-1}(\frac{y}{\varepsilon}))^{1/(p-1)} \rho(P^{-1}(\frac{y}{\varepsilon})) = Q\left(\frac{y}{\varepsilon}\right)\]

Observe that $Q$ is an $L$-periodic function.

Moreover, it is easy to see that

\begin{equation}
|L_\varepsilon - L| \leq \varepsilon L
\end{equation}

and that $L_\varepsilon = L$ if $\varepsilon = 1/j$ for some $j \in \mathbb{N}$.

In order to apply Theorem 3.15 we need to rescale to the unit interval. So we define

\[w(z) = v(L_\varepsilon z), \quad z \in I\]

and get

\[
\begin{cases}
-\left(|\dot{w}|^{p-2}\dot{w}\right) = L_\varepsilon^p \lambda^\varepsilon Q_\varepsilon(L_\varepsilon z)\|w\|^{p-2}w \quad \text{in } I \\
w(0) = w(1) = 0
\end{cases}
\]
So if we denote $\delta = \varepsilon L/L_{\varepsilon}$, $\mu^\delta = L_{\varepsilon}^\delta \lambda^\varepsilon$ and $g(z) = Q(Lz)$, we get that $g$ is a 1-periodic function and that $w$ verifies

\[
\begin{cases}
-(|\tilde{w}|^{p-2}\tilde{w}) = \mu^\delta g(\frac{z}{\varepsilon})|w|^{p-2}w & \text{in } I \\
w(0) = w(1) = 0
\end{cases}
\]

Now we can apply Theorem 3.15 to the eigenvalues $\mu_k$ to get

\[
|\mu_k^{\delta} - \mu_k| \leq \frac{g^+ - \tilde{g}}{g^- g^-} \pi^2 k^{2p} \left[ \frac{p}{\pi^p} + \frac{\varepsilon^{p-1}}{p} \right].
\]

In the case where $\varepsilon = 1/j$ with $j \in \mathbb{N}$ we directly obtain

\[
|\lambda^\varepsilon_k - \lambda_k| \leq \frac{1}{L^p} \left[ \frac{g^+ - \tilde{g}}{g^- g^-} \pi^2 k^{2p} \left[ \frac{p}{\pi^p} + \frac{\varepsilon^{p-1}}{p} \right] \right].
\]

Now we observe that $L^{-p} = (a^p_{\varepsilon})^{p/(p-1)}$ and that we have the bounds

\[
\rho^\varepsilon \frac{\alpha}{\pi^p} \leq g^- \leq g^+ \leq \rho^\varepsilon \frac{\beta}{\pi^p},
\]

\[
\alpha \pi^p \rho^\varepsilon \leq \tilde{g} \leq \beta \pi^p \rho^\varepsilon.
\]

Therefore, we get

\[
|\lambda^\varepsilon_k - \lambda_k| \leq (a^p_{\varepsilon})^{p/(p-1)} \frac{\alpha}{\pi^p} \left[ \frac{1}{\rho^\varepsilon} \left( \frac{\beta}{\alpha} \right)^{1/(p-1)} - \frac{\rho^\varepsilon}{\pi^p} \right] \pi^2 k^{2p} \left[ \frac{p}{\pi^p} + \frac{\varepsilon^{p-1}}{p} \right].
\]

In the general case, one has to measure the defect between $L$ and $L_{\varepsilon}$. So,

\[
|\lambda^\varepsilon_k - \lambda_k| \leq \frac{1}{L^p} (|\mu_k^{\delta} - \mu_k| + \lambda^\varepsilon_k |L_{\varepsilon}^p - L^p|)
\]

\[
\leq (a^p_{\varepsilon})^{p/(p-1)} (|\mu_k^{\delta} - \mu_k| + \beta \rho^- \pi^p k^{2p} |L_{\varepsilon}^p - L^p|).
\]

From (3.17) it is easy to see that

\[
|(L_{\varepsilon}^p)^p - 1| \leq p(1 + \varepsilon)^{p-1}\varepsilon.
\]

so

\[
|L_{\varepsilon}^p - L^p| = L^p |(L_{\varepsilon}^p)^p - 1| \leq (1 + \varepsilon)^{p-1} \frac{p}{(a^p_{\varepsilon})^{p/(p-1)} \varepsilon}.
\]

Finally, using (3.18), (3.19) and (3.20) we obtain the desired result. \hfill \Box

3.4. Convergence of nodal domains. To finish with this section, as a consequence of Theorem 1.1, we prove a result about the convergence of the nodal sets and of the zeroes of the eigenfunctions.

**Theorem 3.17.** Let $(\lambda^\varepsilon_k, u^\varepsilon_k)$ and $(\lambda_k, u_k)$ be eigenpairs associated to equations (1.1) and (1.2) respectively. We denote by $\mathcal{N}^\varepsilon_k$ and $\mathcal{N}_k$ to a nodal domains of $u^\varepsilon_k$ and $u_k$ respectively. Then

\[
|\mathcal{N}^\varepsilon_k| \rightarrow |\mathcal{N}_k| \quad \text{as } \varepsilon \rightarrow 0
\]

and we have the estimate

\[
||\mathcal{N}^\varepsilon_k|^{-p} - |\mathcal{N}_k|^{-p}| \leq c \varepsilon (k^{2p} + 1)
\]
Proof. By using Theorem 1.1, together with (3.16) and the explicit form of the eigenvalues of the limit problem we obtain that

\[ \lambda_k^p(I) = \lambda^p_k(N^\varepsilon_k) \leq \lambda_1(N^\varepsilon_k) + c|N^\varepsilon_k|^p - \frac{\pi_p^p}{|N^\varepsilon_k|^p} + c\varepsilon. \]

Also,

\[ \lambda_k^p(I) \geq \lambda_k(I) - ck^{2p}\varepsilon = \frac{k^p\pi_p^p}{p} - ck^{2p}\varepsilon. \]

As \( u_k(x) = \sin_p(k\pi_p x) \) (see Appendix A, Theorem A.1) has \( k \) nodal domain in \( I \) we must have \( |N_k| = k^{-1} \). Then by (3.21) and (3.22) we get

\[ \frac{\pi_p^p}{|N_k|^p} - ck^{2p}\varepsilon \leq \frac{1}{|N_k|^p} \frac{\pi_p^p}{p} + c\varepsilon \]

it follows that

\[ |N^\varepsilon_k|^{-p} - |N_k|^{-p} \leq c\varepsilon(k^{2p} + 1). \]

Similarly we obtain that

\[ \frac{\pi_p^p}{|N_k|^p} = \lambda_1(N_k) = \lambda_k(I) \geq \lambda_k^p(I) - c\varepsilon k^{2p} \geq \lambda_1^p(N^\varepsilon_k) - c\varepsilon k^{2p} \]

and using again Theorem 1.1 we get

\[ \lambda_1^p(N^\varepsilon_k) \geq \lambda_1(N^\varepsilon_k) - c\varepsilon = \frac{\pi_p^p}{|N_k|^p} - c\varepsilon \]

it follows that

\[ |N^\varepsilon_k|^{-p} - |N_k|^{-p} \leq c\varepsilon(k^{2p} + 1). \]

Combining (3.23) and (3.24) the result follows.

Finally, as a corollary of Theorem 3.17 we are able to prove the individual convergence of the zeroes of the eigenfunctions of (1.1) to those of the limit problem (1.2).

**Corollary 3.18.** Let \((\lambda^\varepsilon_k, u^\varepsilon_k)\) and \((\lambda_k, u_k)\) be eigenpairs associated to equations (1.1) and (1.2) respectively. Denote \( x^\varepsilon_j \) and \( x_j \), \( 0 \leq j \leq k \) its respective zeroes. Then for each \( 1 \leq j < k \)

\[ x^\varepsilon_j \to x_j \quad \text{when} \quad \varepsilon \to 0 \]

and

\[ |x^\varepsilon_j - x_j| \leq j \varepsilon(k^{2p} + 1). \]

In particular \( x^\varepsilon_0 = x_0 = 0 \) and \( x^\varepsilon_k = x_k = 1 \) by the boundary condition.

**Proof.** With the notation of Theorem 3.17 we have that \( |N^\varepsilon_k| \to |N_k| \). For the first pair of nodal domains we get

\[ |x^\varepsilon_0 - x_1| = |x^\varepsilon_1 - x_0 - x_1 + x_0| = ||N^\varepsilon_k| - |N_k|,1| \leq c\varepsilon(k^{2p} + 1) \]

for the second couple

\[ |(x^\varepsilon_2 - x_2) - (x^\varepsilon_1 - x_1)| = ||N^\varepsilon_k| - |N_k,2| \leq c\varepsilon(k^{2p} + 1) \]

then

\[ |x^\varepsilon_2 - x_2| \leq c\varepsilon(k^{2p} + 1) + |x^\varepsilon_1 - x_1| \leq 2c\varepsilon(k^{2p} + 1). \]
Inductively, for \( j < k \)
\[
|x_j^e - x_j| \leq j \varepsilon (k^{2p} + 1)
\]
and the proof is complete.

\[\square\]

4. SOME EXAMPLES AND NUMERICAL RESULTS

We define the following Prüfer transformation:

\[
\begin{align*}
\left( \frac{\lambda r(x)}{p-1} \right)^{1/p} u(x) &= \rho(x) S_p(\varphi(x)), \\
\rho'(x) &= \rho(x) C_p(\varphi(x))
\end{align*}
\]

As in [20], we can see show that \( \rho(x) \) and \( \varphi(x) \) are continuously differentiable functions satisfying

\[
\begin{align*}
\varphi'(x) &= \left( \frac{\lambda r(x)}{p-1} \right)^{1/p} + \frac{1}{p} \frac{r'(x)}{r(x)} \rho(x) |C_p(\varphi(x))|^{p-2} C_p(\varphi(x)) S_p(\varphi(x)) \\
\rho'(x) &= \frac{1}{p} \left( \frac{r'(x)}{r(x)} \rho(x) S_p(\varphi(x)) \right)^{p}
\end{align*}
\]

and we obtain that

\[
u_k(x) = \left( \frac{\lambda_k r(x)}{p-1} \right)^{-1/p} \rho_k(x) S_p(\varphi_k(x)), \quad k \geq 1
\]
is a eigenfunction of problem (1.1) corresponding to \( \lambda_k \) with zero Dirichlet boundary conditions. We propose the following algorithm to compute the eigenvalues of problem (1.1) based in the fact that the eigenfunction associate to \( \lambda_k \) has \( k \) nodal domain in \( I \), so the phase function \( \varphi \) must vary between 0 and \( k\pi \). It consists in a shooting method combined with a bisection algorithm (a Newton-Raphson version can be implemented too).

Let \( a < \lambda < b \) and let \( \tau \) be the tolerance

Solve the ODE 4.2 and obtain \( \varphi_\lambda \) and \( \rho_\lambda \)

Let \( w(x) = (p - 1)^{1/p} (\lambda r(x))^{-1/p} \rho_\lambda(x) S_p(\varphi_\lambda(x)) \)

Let \( a = w(1) \)

while \( (|a| \geq \tau) \)

\[
\lambda = (a + b)/2
\]

Solve the ODE 4.2 and obtain \( \varphi_\lambda \) and \( \rho_\lambda \)

Let \( w(x) = (p - 1)^{1/p} (\lambda r(x))^{-1/p} \rho_\lambda(x) S_p(\varphi_\lambda(x)) \)

Let \( \beta = w(1) \)

If \( (a \beta < 0) \)

\[
b = (a + b)/2
\]

else

\[
a = (a + b)/2
\]

end while

Then \( \lambda \) is the aproximation of eigenvalue with error \( \leq \tau \)
For example, let us consider \( r(x) = 2 + \sin(2\pi x) \). In this case we obtain that 
\[
\varphi = \int_0^1 2 + \sin(2\pi x) \, dx = 2,
\]
and the eigenvalues of the limit problem are given by 
\[
\lambda_k^{1/p} = \frac{k\pi}{2p}.
\]
When \( \epsilon \) tends to zero the value of \( \lambda^\epsilon \) tends to the limit value \( \lambda \) displaying oscillations.

When \( p = 2 \) the first limit eigenvalue is 
\[
\sqrt{\lambda_1} = \frac{\pi}{\sqrt{2}} \sim 2.2241441469.
\]
We see the oscillating behavior when plot \( \sqrt{\lambda_1} \) as function of \( \epsilon \) in Figure 1.

![Figure 1. The square root of the first eigenvalue as a function of \( \epsilon \).](image1)

A more complex behavior can be found in Figure 2, where we considered the weight \( r(x) = \frac{1}{2 + \sin 2\pi x} \). We observe that the sequence tends to 
\[
\lambda_1 = \frac{\pi^2}{2} \int_0^1 \frac{1}{2 + \sin 2\pi x} \, dx = \sqrt{3}\pi \sim 17.09465627.
\]

![Figure 2. The square root of the first eigenvalue as a function of \( \epsilon \).](image2)

It is not clear why the convergence of the first eigenvalue display the oscillations and the monotonicity observed (although the monotonicity is reversed for the weight \( r(x) = 2 - \sin 2\pi x \). We believe that some Sturmian type comparison theorem with
integral inequalities for the weights (instead of point-wise inequalities as usual) is involved. However, we are not able to prove it, and for higher eigenvalues it is not clear what happens.

Turning now to the eigenfunctions, with the weight \( r(x) = 2 + \sin(2\pi x) \), the normalized eigenfunction associated to the first eigenvalue of the limit problem is given by \( u_1(x) = \pi^{-1} \sin(\pi x) \). Applying the numerical algorithm we obtain that the graph of an eigenfunction associated to the first eigenvalue \( \lambda_1 \) intertwine with the graph of \( u_1(x) \). When \( \varepsilon \) decreases, the number of crosses increases, and the amplitude of the difference between them decreases. In Figure 3 we can observe this behavior and the difference between \( u_1 \) and \( u_1^\varepsilon \) for different values of \( \varepsilon \).

![Figure 3. The first eigenfunctions and the difference between them for different values of \( \varepsilon \).](image)

To our knowledge, it is not known any result about the number of the oscillations as \( \varepsilon \) decreases, nor it is known if those oscillations disappear for \( \varepsilon \) sufficiently small.

The same behavior seems to hold for the higher eigenfunctions, see in Figure 4 the behavior of the fourth eigenfunction \( u_4^\varepsilon \) when the parameter \( \varepsilon \) decrease.

Here, the convergence of the nodal domains and the fact that the restriction of an eigenfunction to one of its nodal domains \( \mathcal{N} \) coincides with the first eigenfunction of the problem in \( \mathcal{N} \), together with the continuous dependence of the eigenfunctions on the weight and the length of the domain, suggest that the presence or not of oscillations for the higher eigenfunctions must be the same as for the first one. However, the computations show very complex patterns in the oscillations.

**Appendix A. The nonlinear eigenvalue problem**

In this section we review some properties of the spectrum of (1.1) for fixed \( \varepsilon \). That is, we study

\[
\begin{aligned}
-(a(x)|u'|^{p-2}u')' &= \lambda \rho(x)|u|^{p-2}u \\
0 &< \rho_- \leq \rho(x) \leq \rho_+ \quad \text{and} \quad 0 < \alpha \leq a(x) \leq \beta
\end{aligned}
\]

where \( 0 < \rho_- \leq \rho(x) \leq \rho_+ \) and \( 0 < \alpha \leq a(x) \leq \beta \) for some constants \( \rho_-, \rho_+, \alpha \) and \( \beta \).
We denote by

\[ \Sigma := \{ \lambda \in \mathbb{R} : \text{there exists } u \in W_0^{1,p}(J), \text{ nontrivial solution to (A.1)} \} \]

the spectrum of (A.1).

By means of the critical point theory of Ljusternik–Schnirelmann it is straightforward to obtain a discrete sequence of variational eigenvalues \( \{ \lambda_k \}_{k \in \mathbb{N}} \) tending to \( +\infty \). We denote by \( \Sigma_{\text{var}} \) sequence of variational eigenvalues.

The \( k \)-th variational eigenvalue is given by

\[
\lambda_k = \inf_{C \in \Gamma_k} \sup_{v \in C} \frac{\int_J a(x)|v'|^p}{\int_J \rho(x)|v|^p}
\]

where

\[
\Gamma_k = \{ C \subset W_0^{1,p}(J) : C \text{ compact}, C = -C, \quad \gamma(C) \geq k \}
\]

and \( \gamma(C) \) is the Kranoselskii genus, see [21] for the definition and properties of \( \gamma \).

For the one dimensional \( p \)-Laplace operator in \( J \)

\[
-(|u'|^{p-2}u')' = \mu |u|^{p-2}u
\]

with zero Dirichlet boundary conditions, we have

\[
\mu_k = \inf_{C \in \Gamma_k} \sup_{u \in C} \frac{\int_J |u'|^p}{\int_J |u|^p},
\]

with \( u \in W_0^{1,p}(J) \).

Here, all the eigenvalues and eigenfunctions can be found explicitly:
Theorem A.1 (Del Pino, Drabek and Manasevich, [11]). The eigenvalues $\mu_k$ and eigenfunctions $u_k$ of equation (A.3) on the interval $J$ are given by

$$\mu_k = \frac{\pi_p k^p}{\ell^p},$$

$$u_k(x) = \sin_p(\pi_p kx/\ell).$$

Remark A.2. It was proved in [12] that they coincide with the variational eigenvalues given by equation (A.4). However, let us observe that the notation is different in both papers.

The function $\sin_p(x)$ is the solution of the initial value problem

$$\begin{cases}
-((|u'|^{p-2}u')') = |u|^{p-2}u \\
u(0) = 0, \quad u'(0) = 1,
\end{cases}$$

and is defined implicitly as

$$x = \int_0^{\sin_p(x)} \left(\frac{p-1}{1-t^p}\right)^{1/p} dt.$$ 

Moreover, its first zero is $\pi_p$, given by

$$\pi_p = 2 \int_0^1 \left(\frac{p-1}{1-t^p}\right)^{1/p} dt.$$ 

In [2], problem (A.1) with $a \equiv 1$ is studied and, among other things, it is proved that any eigenfunction associated to $\lambda_k$ has exactly $k$ nodal domains. As a consequence of this fact, in [2] it is obtained the simplicity of every variational eigenvalue.

The exact same proof of [2] works in our case, and so we obtain the following:

Theorem A.3. Every eigenfunction corresponding to the $k$-th eigenvalue $\lambda_k$ has exactly $k-1$ zeroes. Moreover, for every $k$, $\lambda_k$ is simple, consequently the eigenvalues are ordered as $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \rightarrow +\infty$.

Now, using the same ideas as in [13] is easy to prove that the spectrum of (A.1) coincides with the variational spectrum. In fact, we have:

Theorem A.4. $\Sigma = \Sigma_{var}$, i.e., every solution of problem (A.1) is given by (A.2).

Proof. The proof of this theorem is completely analogous to that of Theorem 1.1 in [13].

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