SPACES WHICH INVERT WEAK HOMOTOPY EQUIVALENCES

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Abstract. It is well known that if X is a CW-complex, then for every weak homotopy equivalence $f : A \to B$, the map $f_* : [X, A] \to [X, B]$ induced in homotopy classes is a bijection. For which spaces X is $f^* : [B, X] \to [A, X]$ a bijection for every weak equivalence f? This question was considered by J. Strom and T. Goodwillie. In this note we prove that a non-empty space inverts weak equivalences if and only if it is contractible.

We say that a space X *inverts weak homotopy equivalences* if the functor $[-, X]$ inverts weak equivalences, that is, for every weak homotopy equivalence $f : A \to B$, the induced map $f^* : [B, X] \to [A, X]$ is a bijection. As usual $[A, X]$ stands for the set of homotopy classes of maps from A to X. This property is clearly a homotopy invariant. In [1] Jeff Strom asked for the characterization of such spaces. Tom Goodwillie observed that if X inverts weak equivalences and is T_1 (i.e. its points are closed), then each path-component is weakly contractible (has trivial homotopy groups) and then contractible. His idea was to use finite spaces weak homotopy equivalent to spheres. A map from a connected finite space to a T_1-space has a connected and discrete image and is therefore constant. This is one of the many interesting applications of non-Hausdorff spaces to homotopy theory. Goodwillie also proved that if a space inverts weak equivalences, then it must be connected. In this note we follow his ideas and give a further application of non-Hausdorff spaces to obtain the expected characterization:

Theorem 1. *A non-empty space X inverts weak homotopy equivalences if and only if it is contractible.*

Lemma 2 (Goodwillie)**.** *Suppose that X inverts weak homotopy equivalences and is weakly contractible. Then it is contractible.*

Proof. Just take the weak homotopy equivalence $X \to *$. □

Proposition 3 (Goodwillie)**.** *Let X be a space which inverts weak homotopy equivalences. Then it is connected.*

Proof. We can assume X is non-empty. Suppose that X_0 and X_1 are two path-components of X. Let $x_0 \in X_0$ and $x_1 \in X_1$. Let $A = \mathbb{N}_0$ be the set of nonnegative integers with the discrete topology and $B = \{0\} \cup \{\frac{1}{n}\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ with the usual topology. The map $f : A \to B$ which maps 0 to 0 and n to $\frac{1}{n}$ for every n, is a weak homotopy equivalence. Take $g : A \to X$ defined by $g(0) = x_0$ and $g(n) = x_1$ for every $n \geq 1$. By hypothesis there exists a map $h : B \to X$ such that $h(0) \in X_0$ and $h(\frac{1}{n}) \in X_1$ for every $n \geq 1$. Since $\frac{1}{n} \to 0$, X_0 intersects the closure of X_1. Thus X_0 and X_1 are contained in the same component of X. □

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Lemma 4. *Let X be a space which inverts weak equivalences and let Y be a locally compact Hausdorff space. Then the mapping space X^Y, considered with the compact-open topology, also inverts weak equivalences.*

Proof. This follows from a direct application of the exponential law and the fact that a weak equivalence $f : A \to B$ induces a weak equivalence $f \times 1_Y : A \times Y \to B \times Y$. □

By Lemmas 2 and 4 it only remains to show that a map that inverts weak equivalences is path-connected. If we require a slightly different property, this is easy to prove using only Hausdorff spaces. The following result is not needed for the proof of Theorem 1.

Proposition 5. *Let (X, x_0) be a pointed space such that for every weak homotopy equivalence $f : A \to B$ between Hausdorff spaces and every $a_0 \in A$, the induced map*

$$f^* : [(B, f(a_0)), (X, x_0)] \to [(A, a_0), (X, x_0)]$$

is a bijection. Then X is path-connected.

Proof. Let X_0 be the path-component of x_0. Let X_1 be any path-component of X and let $x_1 \in X_1$. Let A, B, f, g be as in Proposition 3. Let $a_0 = 0 \in A$. By hypothesis there exists $h : (B, 0) \to (X, x_0)$ such that $hf \simeq g$ rel $\{0\}$. In particular $h(1) \in X_1$. Define $h' : B \to X$ by $h'(0) = x_0$ and $h'(\frac{1}{n}) = h(\frac{1}{n+1})$ for $n \geq 1$. The continuity of h' follows from that of h. Since $h'(\frac{1}{n}) = h(\frac{1}{n+1}) \in X_1$ and $h(\frac{1}{n}) \in X_1$ for every $n \geq 1$, there exists a homotopy $H : A \times I \to Y$ from $f^*(h')$ to $f^*(h)$. Moreover we can take H to be stationary on $0 \in A$. Since $f^* : [(B, 0), (X, x_0)] \to [(A, 0), (X, x_0)]$ is injective, there exists a homotopy $F : B \times I \to X$, $F : h' \simeq h$ rel $\{0\}$. The map F gives a collection of paths from $h(\frac{1}{n+1})$ to $h(\frac{1}{n})$. We glue all these paths to form a path from x_0 to $h(1)$. That is, define $\gamma : I \to X$ by $\gamma(0) = x_0$ and $\gamma(t) = F(\frac{1}{n}, (\frac{1}{n} - \frac{1}{n+1})^{-1}(t - \frac{1}{n+1}))$ if $t \in [\frac{1}{n+1}, \frac{1}{n}]$. Note that γ is continuous in $t = 0$ for if $U \subseteq X$ is a neighborhood of x_0, then $\{0\} \times I \subseteq F^{-1}(U)$, and by the tube lemma there exists $n_0 \geq 1$ such that $\{\frac{1}{n}\} \times I \subseteq F^{-1}(U)$ for every $n \geq n_0$. Then $[0, \frac{1}{n_0}] \subseteq \gamma^{-1}(U)$. Hence, x_0 and $h(1)$ lie in the same path-component, so $X_0 = X_1$. □

Note that if a contractible space X satisfies the hypothesis of Proposition 5 for some point x_0, then by taking $A = B = X$, $a_0 = x_0$ and f the constant map x_0, one obtains that $\{x_0\}$ is a strong deformation retract of X. Conversely, a based space (X, x_0) such that $\{x_0\}$ is a strong deformation retract of X, clearly satisfies the hypothesis of the proposition.

The following result is the key lemma for proving Theorem 1 and in contrast to the previous result, the proof provided uses non-Hausdorff spaces.

Lemma 6. *Let X be a space which inverts weak homotopy equivalences. Then X is path-connected.*

Proof. We can assume X is non-empty. Let X_0 and X_1 be path-components of X. Let B be a set with cardinality $\#B > \alpha = \max\{\#X, c\}$. Here c denotes the cardinality $\#\mathbb{R}$ of the continuum. Consider the following topology in B: a proper subset $F \subseteq B$ is closed if and only if $\#F \leq \alpha$. Note that the path-components of B are the singletons, for if $\gamma : I \to B$ is a path, then its image has cardinality at most α, so it is connected and discrete and then constant. Let A be the discretization of B, i.e. the same set with the discrete topology. Then the identity $id : A \to B$ is a weak homotopy equivalence. Let b_0

and b_1 be two different points of B. Define $g : A \to X$ in such a way that $g(b_0) \in X_0$ and $g(b_1) \in X_1$ (define g arbitrarily in the remaining points of A). Then g is continuous. Since the identity $id^* : [B, X] \to [A, X]$ is surjective, there exists a map $h : B \to X$ such that $h \circ id \simeq g$. In particular $h(b_0) \in X_0$ and $h(b_1) \in X_1$. Since $\#B > \alpha \geq \#X$ and $B = \bigcup_{x \in X} h^{-1}(x)$, there exists $x \in X$ such that $\#h^{-1}(x) > \alpha$. Let $U \subseteq X$ be an open neighborhood of $h(b_0)$. Then $h^{-1}(U^c) \subseteq B$ is a proper closed subset, so $\#h^{-1}(U^c) \leq \alpha$. Thus, $h^{-1}(x)$ is not contained in $h^{-1}(U^c)$ and then $x \in U$. Since every open neighborhood of $h(b_0)$ contains x, there is a continuous path from x to $h(b_0)$, namely $t \mapsto x$ for $t < 1$ and $1 \mapsto h(b_0)$. In particular $x \in X_0$. Symmetrically, $x \in X_1$. Therefore $X_0 = X_1$. □

Proof of Theorem 1. It is clear that a contractible space inverts weak equivalences. Suppose $X \neq \emptyset$ is a space which inverts weak equivalences. By Lemma 4, X^{S^n} inverts weak equivalences for every $n \geq 0$ and then it is path-connected. Therefore $\pi_n(X)$ is trivial for every $n \geq 0$ and by Lemma 2, X is contractible.

References

[1] J. Strom (https://mathoverflow.net/users/3634/jeff-strom), Spaces that invert weak homotopy equivalences., URL (version: 2010-11-23): https://mathoverflow.net/q/47042

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