

INVARIANTS FOR METABELIAN GROUPS OF PRIME POWER EXPONENT, COLORINGS AND STAIRS

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ABSTRACT. We study the free metabelian group $M(2, n)$ of prime power exponent n on two generators by means of invariants $M(2, n)' \rightarrow \mathbb{Z}_n$ that we construct from colorings of the squares in the integer grid $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$. In particular we improve bounds found by M.F. Newman for the order of $M(2, 2^k)$. We study identities in $M(2, n)$, which give information about identities in the Burnside group $B(2, n)$ and the restricted Burnside group $R(2, n)$.

1. CONTENT

In this paper we study the free metabelian group $M(2, n)$ of prime power exponent n in two generators, that is the quotient of the Burnside group $B(2, n)$ by its second derived subgroup. Contrary to Burnside groups, the groups $M(2, n)$ are known to be finite for every n . However their order has been determined in very few cases if n is not prime. We define invariants (group homomorphisms) $\Lambda : M(2, n)' \rightarrow \mathbb{Z}_n$ and use them to prove that certain identities do not hold in $M(2, n)$ and to improve bounds for its order obtained by M.F. Newman in [39]. Note that the abelianization of $M(2, n)$ has order n^2 , and since $M(2, n)'$ is finite abelian of exponent n , there is a monomorphism $M(2, n)' \rightarrow (\mathbb{Z}_n)^r$ (a product of invariants Λ as above) for r big enough, and the order of $M(2, n)'$ is then the order of the image of that map.

Algebraic tools have been used to study Engel identities and the nilpotence class of $M(2, n)$ in works by Bachmuth-Heilbronn-Mochizuki [7] and Dark-Newell [14]. Here we use instead combinatorial ideas based on a construction introduced in [9], called the winding invariant. To each word w in the derived subgroup F_2' of the free group $F_2 = F(x, y)$ of rank 2, we associate a Laurent polynomial $P_w \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$. This can be seen as a labeling of finitely many squares in the grid $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$ by non-zero integer numbers. We use different colorings of the squares in the grid to define the invariants $M(2, n)' \rightarrow \mathbb{Z}_n$.

Some results that we prove are: $[x, y]^{\frac{n}{2}}$ is not a product of n th powers in F_2 if $4|n$ (Proposition 7), $M(2, 2^k)$ satisfies the so called $(k + 3)$ th-Morse identity but not the $(k + 1)$ th (Proposition 15), $n^2 2^{\frac{5n^2}{16} - 1} \leq |M(2, n)| \leq n^{(n-1)^2}$ if $n \geq 8$ is a power of 2 (Theorem 18 and Proposition 22). We also give a lower bound for the order of the restricted Burnside group $R(2, 8)$ in Proposition 23 and obtain a complete invariant for $M(2, 4)$ in Examples 17 and 21.

This paper is the third in the winding invariant series, after [8] and [9] and it is a continuation of the results obtained in [9] Subsection 6.3. The combinatorial techniques

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introduced in this article can be used to study different equations over the group F_2 in the directions described in [9].

2. BACKGROUND ON THE RESTRICTED BURNSIDE PROBLEM AND METABELIAN BURNSIDE GROUPS

In 1902 William Burnside asked whether a finitely generated group of exponent n is necessarily finite [12]. Any such group G generated by d elements is a quotient of the free group of exponent n , $B(d, n) = \langle x_1, x_2, \dots, x_d | w^n, w \in F(x_1, x_2, \dots, x_d) \rangle$, also called the Burnside group of exponent n and d generators. Thus Burnside question is about finiteness of Burnside groups. Burnside proved that $B(d, n)$ is finite for $n = 3$ and arbitrary d . Sanov proved in [43] the finiteness of $B(d, 4)$ and M. Hall [23] did it for $B(d, 6)$. In 1968 Adian and Novikov proved that not all Burnside groups are finite. Indeed they proved in [4] that for $d \geq 2$ and n odd greater than or equal to 4381, $B(d, n)$ is infinite. Later Adian [2] was able to replace the number 4381 by 665. Ivanov proved in [28] that for $n \geq 2^{48}$ divisible by 2^9 , $B(d, n)$ is infinite. Later Lysënok improved this result. In particular there exist only finitely many exponents n for which $B(d, n)$ can be finite. The complete answer to Burnside's problem is still unknown. For instance, it has not been decided yet whether $B(2, 5)$ and $B(2, 8)$ are finite.

In 1950 Magnus [37] formulated a variant of Burnside's problem, known as the restricted Burnside problem: Given $d, n \geq 1$, is there a largest finite quotient $R(d, n)$ of $B(d, n)$? This is equivalent to the following question: are there only finitely many finite groups (up to isomorphism) generated by d elements and with exponent n ? Indeed if we define N to be the intersection of all the finite-index subgroups of $B(d, n)$, then N is normal and every d -generator finite group of exponent n is a homomorphic image of $B(d, n)/N$. The restricted Burnside problem asks whether $B(d, n)/N$ is finite. In that case, $R(d, n) = B(d, n)/N$ is called the *restricted Burnside group* or *universal finite d -generator group of exponent n* . In 1958 Kostrikin [29, 31, 33] proved that $R(d, p)$ exists for every prime p . This was previous to Adian-Novikov's proof of the existence of infinite Burnside groups and was key in the development of their result (see [3]). P. Hall and Higman [24] had already reduced the proof of the restricted Burnside problem to the proof of the prime power exponent case under the assumption that there are only finitely many finite simple groups of a given exponent and that the outer automorphism group of any finite simple group is solvable. In 1990 Zel'manov [54, 55] gave a positive answer to the restricted Burnside problem for prime power exponent and this, together with Hall-Higman's reduction and the classification of finite simple groups, provided a positive answer to the restricted Burnside problem for arbitrary exponent. One of the key elements in Kostrikin and Zel'manov's work is the relationship with Engel identities in Lie algebras. In fact, for prime exponent p , the restricted Burnside problem can be reduced to the proof that every Lie algebra over \mathbb{Z}_p which satisfies the so called Engel identity E_{p-1} is locally nilpotent.

Although $R(d, n)$ is known to exist for each d, n , little is known about its structure, in general. In particular the order of $R(d, n)$ is known in very few cases. For each d and each prime power n , the p -quotient algorithm can be used to obtain a consistent polycyclic/power-commutator presentation of $R(d, n)$ and then to compute its order (see [46, Section 11.7] and [42, 50]), however a direct implementation of this algorithm takes in general too much time. Havas, Wall and Wamsley [26] proved that $|R(2, 5)| = 5^{34}$, O'Brien and Vaughan-Lee [42] proved that $|R(2, 7)| = 7^{20416}$, Newman and O'Brien [41] proved that $|B(5, 4)| = 2^{2728}$, $|R(3, 5)| = 5^{2282}$. Also $|B(3, 4)| = 2^{69}$ ([10]) and $|B(4, 4)| = 2^{422}$

([5]). The order of $B(d, 4)$ for arbitrary d is still unknown. Upper and lower bounds for $|R(d, n)|$ have been given by Vaughan-Lee and Zel'manov [51, 52, 53] and by Groves and Vaughan-Lee [17].

It has long been known that there are connections between groups of prime power exponent and Engel conditions. The first Engel word is $e_1 = [y, x] = yxy^{-1}x^{-1} \in F_2$. The m th Engel word is defined by $e_m = [y, e_{m-1}] \in F_2$. A group G is said to be m -Engel if $e_m(a, b) = 1 \in G$ for every $a, b \in G^*$. Every group of exponent 3 is 2-Engel. This appears in [36] but is already implicit in Burnside's paper [12]. It is equivalent to saying that $e_2 \in F_2$ is a product of cubes. Every group of exponent 4 is 5-Engel (e_5 is a product of fourth powers). Kostrikin observed [30, 32] that if the 6th Engel word e_6 is not a product of 5th powers in F_2 , then $B(2, 5)$ is infinite.

In the quest for finding a solution to the restricted Burnside problem, before Zel'manov's proof, other relations apart from Engel conditions had been searched among groups of prime power exponent. Some papers have explored this path by imposing extra relations to the groups. In particular the metabelian law. Recall that a group G is termed metabelian if $G'' = 1$. Every d -generated metabelian group of exponent n is a quotient of the free metabelian group $M(d, n)$ of exponent n . If G is a group and $n \geq 1$, we denote by G^n the subgroup of G generated by the n th powers. If F_d denotes the free group of rank d , then $M(d, n) = \frac{F_d}{F_d^n F_d''}$. As explained by Newman in [39], one advantage of these groups is that they are always finite. Indeed, if we define $R = F_d^n F_d'$ and $S = R^n R'$, then $|F_d : R| = n^d$, so by Schreier formula $\text{rk}(R) = 1 + (d-1)n^d$. Similarly $|R : S| = n^{\text{rk}(R)} = n^{1+(d-1)n^d}$, so $|F_d/S| = n^{1+d+(d-1)n^d}$. On the other hand it is easy to see that $S \leq F_d^n F_d''$. Thus $|M(d, n)| \leq n^{1+d+(d-1)n^d}$. If $n = p^k$ for p a prime, Newman computes also a lower bound for $|M(d, n)|$, which is $p^{1+d(k-1)+(d-1)p^{(k-1)d}}$. This is explained later in this article. Since $M(d, n)$ is finite, its order gives a lower bound for $|R(d, n)|$. Moreover, if we know that a certain identity does not hold in $M(d, n)$ then it will not hold in $R(d, n)$ nor in $B(d, n)$. Recall that for a group G , the lower central series is defined by $\gamma_1(G) = G$ and $\gamma_{l+1}(G) = [G, \gamma_l(G)]$ for $l \geq 1$. A group G is nilpotent if there exists $l \geq 0$ such that $\gamma_{l+1}(G) = 1$, and the smallest l with this property is the nilpotency class of G . If n is a prime power, $M(d, n)$ is nilpotent and its class is known in many cases by results of Meier-Wunderli [38], Bachmuth-Heilbronn-Mochizuki [7], Dark-Newell [14], Gupta-Newman-Tobin [21], Newman [39]. In particular the class of $M(2, p^k)$ is $k(p-1)p^{k-1}$. The order of $M(d, n)$ is known when n is prime (see [38, 39]), but the general case is open. Even for $d = p = 2$, Newman's bounds only give $n^2 2^{\frac{n^2}{4}-1} \leq |M(2, n)| \leq n^{n^2+3}$ and it is not known, asymptotically, which of the two bounds is closer to reality. Given a positive integer d and a prime power n , we can compute $|M(d, n)|$ from any finite presentation using coset enumeration. Moreover, the nilpotent quotient algorithm computes a consistent power-commutator presentation (see [25, 39, 50]) for $M(d, n)$ from which its order can be directly read off. However, the number of operations needed to complete this task is unknown and $|M(d, n)|$ has been computed in very few cases. Once we have such a presentation it can be decided whether an arbitrary word $w \in F(x_1, x_2, \dots, x_d)$ is trivial in $M(d, n)$, but again, there is no general method known

*When the convention $(x, y) = x^{-1}y^{-1}xy$ is used for commutators, Engel words are defined by $\tilde{e}_1 = (x, y)$ and $\tilde{e}_m = (\tilde{e}_{m-1}, y)$. Both notions are related by the identity $\tilde{e}_m = e_m(x^{-1}, y^{-1})^{-1}$, so the definitions of m -Engel group coincide.

to do this for arbitrary w, d, n without implementation of this algorithm or other variants. For example, apart from the cases in which n is prime, the order of $M(d, n)$ is known in the following cases: $(d, n) \in \{(2, 4), (3, 4), (4, 4), (5, 4), (2, 8), (3, 8), (4, 8), (2, 9), (3, 9)\}$ (computed by Gupta-Tobin [20], Hermanns [27] and Newman [39]). The order of $M(2, 16)$ is known to be smaller than or equal to 2^{376} ([39]). For p prime and $n = p^2$, Skopin [47] has shown that $|M(2, n)| \leq p^{\frac{3}{2}p^4 - \frac{3}{2}p^3 - 2p^2 + p + 5}$, which improves Newman's general bound that gives here $|M(2, n)| \leq p^{2p^4 + 6}$. In [48] Skopin proves that the largest class 36 quotient $M(2, 27)/\gamma_{37}(M(2, 27))$ of $M(2, 27)$ has order less than or equal to 3^{648} .

3. HORIZONTAL INVARIANTS

We start by recalling the constructions and results of [9] that we will need. We will also recall some of the proofs for the sake of completeness and in order to make clear subsequent ideas.

An element $z \in F'_2$ in the derived subgroup of the free group F_2 generated by x, y is a word $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_l^{\epsilon_l}$ in the letters x, y with total exponents $\exp(x, z) = \exp(y, z) = 0$. One such word has an associated closed curve γ_z in the integer grid $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R} \subseteq \mathbb{R}^2$ which begins in the origin and is a concatenation of curves $\gamma_1, \gamma_2, \dots, \gamma_l$. The curve γ_i moves one unit horizontally if $x_i = x$ and vertically if $x_i = y$, to the right or upwards if $\epsilon_i = 1$ and to the left or downwards if $\epsilon_i = -1$. The winding invariant of z is the Laurent polynomial $P_z = \sum_{i, j \in \mathbb{Z}} a_{i, j} X^i Y^j \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$, where $a_{i, j}$ is the winding number of γ_z

around $(i + \frac{1}{2}, j + \frac{1}{2}) \in \mathbb{R}^2$. For instance, if $z = [x, y] = xyx^{-1}y^{-1}$, the curve γ_z traverses the boundary of the square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$ once counterclockwise, so $P_z = 1 \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$. The winding invariant map $W : F'_2 \rightarrow \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ which maps z to P_z has many different interpretations [9]. Clearly W is a group homomorphism. It is surjective and its kernel is F''_2 ([9, Theorem 14]) and thus, it induces an isomorphism $M'_2 \rightarrow \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ from the derived subgroup of the free metabelian group $M_2 = F_2/F''_2$ of rank 2. This map will also be denoted by W .

An elementary property which follows easily from the definition is:

Remark 1. If $z \in F'_2$ and $u \in F_2$, then $P_{uzu^{-1}} = X^i Y^j P_z$ where $(i, j) = (\exp(x, u), \exp(y, u))$. Therefore, $P_{[u, z]} = (X^i Y^j - 1)P_z$.

We recall the following result from [9]:

Proposition 2. [9, Proposition 8] *Let $n \geq 1$. If $z \in F'_2$ is a product of two n th powers, then P_z is a multiple of $1 + m + m^2 + \dots + m^{n-1}$ for some monomial $m = X^i Y^j$.*

Proof. If $z = a^n b^n = (ab)b^{-1}(ab)bb^{-2}(ab)b^2 \dots b^{1-n}(ab)b^{n-1}$, then $ab \in F'_2$ and by Remark 1, $P_z = P_{ab}(1 + m + m^2 + \dots + m^{n-1})$, for $m = X^{\exp(x, b^{-1})} Y^{\exp(y, b^{-1})}$. \square

In [9] we proved that the *area invariant* $A : F'_2 \rightarrow \mathbb{Z}$ defined by $A(z) = P_z(1, 1)$, maps a product of n th powers in F_2 to a multiple of $n/2$ when n is even and to a multiple of n when n is odd. In particular a product of 4th powers has even area. This is not a sufficient condition for an element $z \in M'_2$ to be a product of 4th powers in M_2 . There is another invariant $\kappa : F'_2 \rightarrow \mathbb{Z}$ which, instead of adding all the coefficients of P_z , adds some of them and subtracts others. A product of 4th powers must have $\kappa(z)$ divisible by 4. The main tools that we develop in this article are different generalizations of the invariant κ , which take into account higher powers of 2.

In Section 9 we give an example in which the invariant technique successfully describes a variety of groups. We will see that the area invariant completely characterizes the free nilpotent group $N(2, n)$ of class 2, rank 2 and exponent n for any $n \geq 1$, determining an isomorphism $\bar{A} : N(2, n)' \rightarrow \mathbb{Z}_l$ where $l = \frac{n}{2}$ if n is even and $l = n$ if n is odd. In particular the order $|N(2, n)|$ is $\frac{n^3}{2}$ or n^3 . For the free metabelian group $M(2, n)$ of exponent n the problem is much harder and other invariants will be introduced.

Let $k \geq 1, n = 2^k$. A *good coloring* of \mathbb{Z}_n is a function $c : \mathbb{Z}_n \rightarrow \{\text{black, white}\}$ which satisfies that for every $a \in \mathbb{Z}_n$, the color $c(a)$ of a is different from the color $c(a + \frac{n}{2})$ of $a + \frac{n}{2}$. A function $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ together with a good coloring c induces a coloring of the squares in the grid $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$: the square (whose lower left corner is) (i, j) is painted with color $c\varphi(i, j)$. The invariant $\Lambda = \Lambda_{\varphi, c} : \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}] \rightarrow \mathbb{Z}_n$ associated with φ and c is defined as follows. Given $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$, define

$$\Lambda(P) = \sum_{(i,j) \text{ is black}} P\{(i, j)\} - \sum_{(i,j) \text{ is white}} P\{(i, j)\} \in \mathbb{Z}_n.$$

Here $P\{(i, j)\}$ denotes the coefficient of the monomial $X^i Y^j$. The composition $\Lambda W : F_2' \rightarrow \mathbb{Z}_n$ will also be denoted by $\Lambda = \Lambda_{\varphi, c}$. That is, for $z \in F_2'$, $\Lambda(z) = \Lambda(P_z)$. Of course, $\Lambda : F_2' \rightarrow \mathbb{Z}_n$ is a group homomorphism. Since $\ker(W) = F_2''$, Λ induces a map $M_2' = F_2'/F_2'' \rightarrow \mathbb{Z}_n$ which is denoted Λ as well. Given a region A of the plane, which is a union of squares in the grid, we define the invariant $\Lambda(A)$ of A as the number of black squares in A minus the number of white squares, modulo n .

Example 3. Let $k = 2$, so $n = 4$, and let $c : \mathbb{Z}_4 \rightarrow \{\text{black, white}\}$ be the coloring that paints 0, 1 with black and 2, 3 with white. Let $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_4$ be defined by $\varphi(a, b) = b \in \mathbb{Z}_4$. The coloring associated to φ and c is illustrated in Figure 1, at the left. All the rows congruent with 0 or 1 modulo 4 are painted with black, while the remaining rows are painted with white.

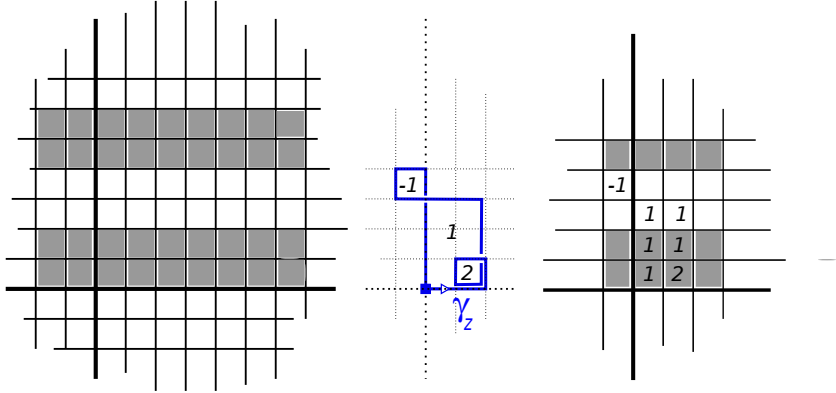


FIGURE 1. The coloring associated to φ and c , the curve γ_z with the winding numbers, and the superposition which computes $\Lambda(z)$.

Let $z = x^2 y x^{-1} y^{-1} x y^3 x^{-3} y x y^{-4} \in F_2'$. The curve γ_z is depicted in the center of Figure 1, together with the (non-zero) winding numbers of the curve around the squares. The winding invariant of z is thus $P_z = 1 + 2X + Y + XY + Y^2 + XY^2 - X^{-1}Y^3 \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$.

The invariant $\Lambda_{\varphi,c}(z)$ is just the sum of the coefficients in black squares minus the sum of the coefficients in white squares. That is, $\Lambda(z) = 1 + 2 + 1 + 1 - (1 + 1 - 1) = 0 \in \mathbb{Z}_4$.

Lemma 4. *Let n, d be positive integers such that $2d$ divides n . Then the elements of the integer interval $[n] = \{0, 1, 2, \dots, n-1\}$ can be matched in such a way that matched elements differ in d .*

Proof. For $0 \leq i \leq d-1$, match i with $i+d$. These pairs cover the set $[2d]$, and translations of them cover the whole set $[n]$. \square

Lemma 5. *Let $k \geq 1$, $n = 2^k$. Let $c : \mathbb{Z}_n \rightarrow \{\text{black}, \text{white}\}$ be a good coloring.*

(i) *Let $a, a+b, a+2b, \dots, a+(n-1)b$ be an arithmetic progression of length n in \mathbb{Z} . When we see this progression in \mathbb{Z}_n , the number of black terms minus the number of white terms is a multiple of n .*

(ii) *Let $a, a+b, a+2b, \dots, a+(\frac{n}{2}-1)b$ be an arithmetic progression of length $\frac{n}{2}$ in \mathbb{Z} with b even. When we see this progression in \mathbb{Z}_n the number of black terms minus the number of white terms is a multiple of $\frac{n}{2}$. Moreover, if n divides b , this number is $\frac{n}{2}$ modulo n and if n does not divide b , this number is 0 modulo n .*

Proof. We prove first part (i). If n divides b , the n numbers in the progression have the same color. Suppose then that $n \nmid b$. Let $d \in \mathbb{Z}$ be the greatest common divisor of n and b . By assumption $d < n$, so $\frac{n}{2d} \in \mathbb{Z}$ and $2\frac{n}{2d}$ divides n . By Lemma 4, $[n]$ admits a matching with matched elements of difference $\frac{n}{2d}$. Thus, the set $a + b[n] = \{a, a+b, a+2b, \dots, a+(n-1)b\}$ admits a matching with matched elements of difference $b\frac{n}{2d}$. But $b\frac{n}{2d} = \frac{b}{d}\frac{n}{2}$ is an odd multiple of $\frac{n}{2}$. Since c is good, matched elements have different colors, so the sequence has the same number of black terms and white terms.

For part (ii), if $n|b$, the $\frac{n}{2}$ terms of the progression have the same color. Otherwise $d = \gcd(n, b) < n$ and d is even by hypothesis. Then $2\frac{n}{2d}$ divides $\frac{n}{2}$ and by Lemma 4, $[\frac{n}{2}]$ can be matched with difference $\frac{n}{2d}$, so $a + b[\frac{n}{2}]$ admits a matching with difference $b\frac{n}{2d}$, and as in the first case matched elements receive opposite colors. \square

Let $n = 2^k$ for some $k \geq 2$. Let $c : \mathbb{Z}_n \rightarrow \{\text{black}, \text{white}\}$ be a good coloring. Define a group homomorphism $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ by $\varphi(1, 0) = 0$, $\varphi(0, 1) = 1$. Let $\mathbf{h} = \Lambda_{\varphi,c} : F_2' \rightarrow \mathbb{Z}_n$ be the associated invariant. It is called the horizontal invariant associated to c . In this case all the squares in a row are painted with the same color. If two rows are $\frac{n}{2}$ units away one from the other, then they are painted with opposite colors. This says in particular that among n consecutive squares in a column, half of them are black and the other half are white. If the distance between two rows is a multiple of n , they have the same color. The invariant in Example 3 is of this kind.

Theorem 6. *Let $n = 2^k$ for some $k \geq 2$. Let c be a good coloring of \mathbb{Z}_n . If $z \in F_2'$ lies in $F_2^n F_2''$, then $\mathbf{h}(z) = 0 \in \mathbb{Z}_n$. In particular, any element in F_2' which is a product of n th powers of elements in F_2 has trivial \mathbf{h} -invariant.*

Proof. The proof is similar to that of Theorem 59 in [9]. Let $z \in F_2' \cap F_2^n F_2''$. Since $F_2'' = \ker W \leq \ker \mathbf{h}$, we may assume $z \in F_2' \cap F_2^n$. Suppose then $z = u_1^n u_2^n \dots u_r^n$ for some $u_i \in F_2$.

Step 1: We may assume $r = 3$. Indeed, we can find $v_2, v_3, \dots, v_{r-2} \in F_2$ such that $u_1^n u_2^n v_2^n, v_2^{-n} u_3^n v_3^n, v_3^{-n} u_4^n v_4^n, \dots, v_{r-3}^{-n} u_{r-2}^n v_{r-2}^n, v_{r-2}^{-n} u_{r-1}^n u_r^n$ are all in F_2' . Since $\mathbf{h} : F_2' \rightarrow \mathbb{Z}_n$ is a group homomorphism, we may then assume that $r = 3$. Suppose then that $z = u^n v^n w^n$ for certain $u, v, w \in F_2$.

Step 2: We prove that $\mathbf{h}(z)$ depends only on the exponents of x, y in u, v, w . If we replace u by another element $\tilde{u} \in F_2$ with same exponents in x and y , then $\tilde{z} = \tilde{u}^n v^n w^n$ satisfies that $z^{-1}\tilde{z}$ is a conjugate of an element in F_2' which is product of two n th powers. Then by Proposition 2, $P_{z^{-1}\tilde{z}}$ is a multiple of $1 + m + m^2 + \dots + m^{n-1}$ for some monomial m . The same happens if we replace v or w by other elements with the same exponents. So, it suffices to check that for monomials $m = X^i Y^j, m' = X^{i'} Y^{j'}$, the polynomial $P = m'(1 + m + m^2 + \dots + m^{n-1})$ has invariant $\mathbf{h}(P) = 0$. By definition $\mathbf{h}(P)$ is (modulo n) the number of black elements minus the number of white elements in the sequence $j', j' + j, j' + 2j, \dots, j' + (n-1)j \in \mathbb{Z}_n$. By Lemma 5 this is 0.

Step 3: The stairs. Let $(a, b) = (\exp(x, u), \exp(y, u)), (c, d) = (\exp(x, uv), \exp(y, uv)) \in \mathbb{Z}^2$. Let R be the rectangle circumscribed about the triangle T with vertices $(0, 0), (a, b), (c, d)$. By the previous step we can assume $u = x^a y^b$ or $u = y^b x^a, v = x^{c-a} y^{d-b}$ or $v = y^{d-b} x^{c-a}$ and $w = x^{-c} y^{-d}$ or $w = y^{-d} x^{-c}$. We choose u, v, w in such a way that the curve γ_z associated to $z = u^n v^n w^n$ lies outside T (see Figure 2). We must show that the bounded region T' determined by γ_z has trivial horizontal invariant, because up to sign this is $\mathbf{h}(z)$. Since the height of R is a multiple of n , $\mathbf{h}(R) = 0 \in \mathbb{Z}_n$. Therefore it suffices to prove that the region $R - T'$ has trivial invariant. Now, $R - T'$ consists of three stair-like regions and possibly a rectangular region.

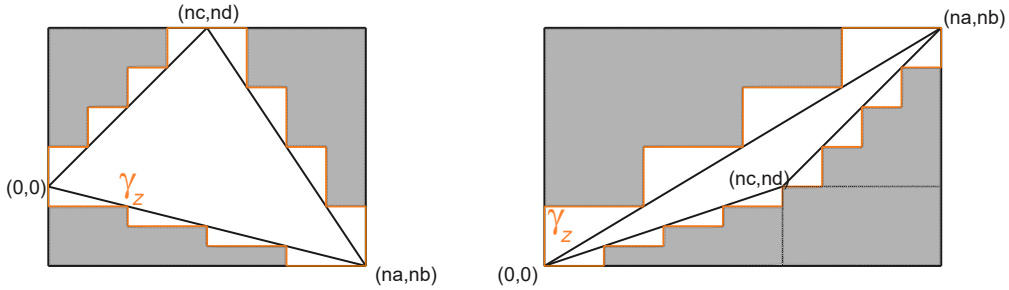


FIGURE 2. The rectangle R circumscribed about T . In this picture $n = 4$. The curve γ_z appears with orange and the region $R - T'$ is shaded.

Therefore the proof reduces to showing that in any rectangular region with sides divisible by n and in stair-like regions with $n - 1$ steps, the number of black squares minus the number of white squares is a multiple of n . For a rectangle we have the same argument we used for R .

Consider a stair-like region with $n - 1$ steps as in Figure 3. We must prove that the number of black squares minus the number of white squares is a multiple of n , independently of the position of the stair in the integer lattice, the height and the length of the steps.

Let $h, l \geq 1$ be the height and the length of the steps in the stair. We divide the stairs in rectangles as shown in Figure 3. There are 2^i rectangles of size $2^{k-i-1}l \times 2^{k-i-1}h$ for every $0 \leq i \leq k - 1$. We call the $2^{k-i-1}l \times 2^{k-i-1}h$ rectangles, rectangles of type i . We prove that for each i , among all the rectangles of type i , the number of black squares minus the number of white squares is a multiple of 2^k . The first row of one rectangle of type i and the first row of the next rectangle of that type are $2^{k-i}h$ rows apart. Therefore, if $2^i | h$, all the rectangles of type i are painted exactly in the same way. Moreover, in this

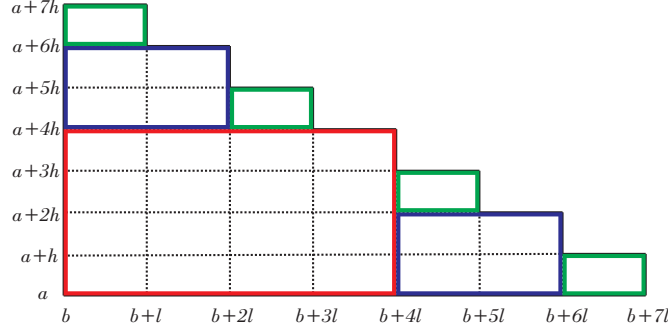


FIGURE 3. A seven-step stair. Here $k = 3$. The rectangle of type 0 appears with red in the picture, those of type 1 are blue, and those of type 2 are green.

case each of the columns in these rectangles has height $2^{k-i-1}h$, which is a multiple of 2^{k-1} , so in particular it is even. It is easy to see then, that in each column the number of black squares minus the number of white squares is even. But there are $2^{k-1}l$ of these columns, so the number of black squares minus the number of white squares among all the rectangles of type i is a multiple of 2^k . It remains to analyze the case that 2^i does not divide h . In this case not all the rectangles of type i are painted in the same way. In fact if 2^r is the maximum power of 2 dividing h (so in particular $r \leq i - 1$) then the j th rectangle and the $(j + 2^{i-r-1})$ th rectangle of type i are $2^{i-r-1}2^{k-i}h = 2^{k-r-1}h$ rows apart. But $2^{k-r-1}h$ is a multiple of 2^{k-1} and not a multiple of 2^k . This implies that the two rectangles are painted exactly with opposite colors. Thus the rectangles of type i can be paired in such a way that paired rectangles are painted with opposite colors ($[2^i]$ can be matched with matched elements of difference 2^{i-r-1} by Lemma 4). Therefore among rectangles of type i there is the same number of black and white squares. \square

In [49], Struik proves that for any even positive integer n , $[x, y]^{\frac{n}{2}} \in \gamma_3(F_2)F_2^n$. This is not hard to prove using the area invariant (see Section 9 below). This result is closely related to a result of Sanov [44] and a conjecture by Bruck [13].

Our horizontal invariants can be used to prove that the result is false when we replace $\gamma_3(F_2)$ by the smaller subgroup F_2'' .

Proposition 7. *Let n be a positive integer divisible by 4. Then $[x, y]^{\frac{n}{2}} \notin F_2''F_2^n$. In particular, $[x, y]^{\frac{n}{2}}$ is not a product of n th powers in F_2 .*

Proof. Suppose $n = 2^k q$ for q odd, $k \geq 2$. Let $c : \mathbb{Z}_{2^k} \rightarrow \{\text{black, white}\}$ be any good coloring of \mathbb{Z}_{2^k} . Let $\mathbf{h} : F_2' \rightarrow \mathbb{Z}_{2^k}$ be the horizontal invariant associated with c . Then $P_{[x, y]^{\frac{n}{2}}} = \frac{n}{2} \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$, so $\mathbf{h}([x, y]^{\frac{n}{2}}) = \frac{n}{2} = 2^{k-1} \in \mathbb{Z}_{2^k}$. Since $\mathbf{h}([x, y]^{\frac{n}{2}})$ is non-trivial then by Theorem 6, $[x, y]^{\frac{n}{2}}$ is not in $F_2''F_2^{2^k}$, and then not in $F_2''F_2^n$. \square

From the previous result we deduce that if n is divisible by 4, the exponent of $M(2, n)'$ is n , and not less. Indeed if l is the order of $[x, y]$ in $M(2, n)'$, $[x, y]^l \in F_2''F_2^n$ so the area $l = A([x, y]^l)$ is a multiple of $\frac{n}{2}$. By Proposition 7, $l \neq \frac{n}{2}$, so $l = n$.

4. ENGEL AND MORSE IDENTITIES

Given a word w in the free group $F = F(x_1, x_2, \dots, x_s)$ and a group G , we say that G satisfies the identity w if for every homomorphism $\phi : F \rightarrow G$ we have $\phi(w) = 1$. If G is a quotient of a group H and G does not satisfy identity w , then neither does H . For instance, from Proposition 7 we deduce that the restricted Burnside group $R(2, n)$ does not satisfy the identity $[x, y]^{\frac{n}{2}}$ for any n divisible by 4. A group is m -Engel if it satisfies the identity e_m (defined in Section 2). Note that $B(2, n)$ is m -Engel if and only if $e_m \in F_2$ is a product of n th powers. It is known that $R(2, 5)$ is 6-Engel, so if e_6 is not a product of fifth powers, $B(2, 5)$ is not finite. Of course, a nilpotent group of class m is m -Engel. Conversely, Zorn's Theorem establishes that a finite m -Engel group is nilpotent. Not every Burnside group satisfies an Engel identity: $B(2, 6)$ is not nilpotent and finite. It is an open problem whether every finitely generated m -Engel group is nilpotent (see [34, 14.70]). Metabelian groups of prime power exponent are (finite and) nilpotent but if n is not a prime power and $d \geq 2$, $M(d, n)$ is (finite and) not nilpotent ([7]). By results of Bachmuth-Heilbronn-Mochizuki [7] and of Dark-Newell [14] it is known that the class of $M(2, p^k)$ is $m = k(p-1)p^{k-1}$ and $M(2, p^k)$ does not satisfy the $(m-1)$ -Engel identity. In order to prove that $e_{m-1} \neq 1 \in M(2, p^k)$, Bachmuth, Heilbronn and Mochizuki find a representation of $M(2, p^k)$ by 2×2 matrices using the Magnus embedding. Since Engel identities are completely understood for metabelian groups of prime power exponent, our results are not new and we only exhibit examples that illustrate how to use the invariants introduced in the previous section. In the next section however, we will study basic commutator identities, which generalize Engel identities.

Engel identities (or *congruences* which are a variant for non-nilpotent groups) have been studied for Burnside (non-metabelian) groups. Here the problem becomes harder and there is a still open conjecture by Bruck (see [13, 16, 18, 19, 44, 35]).

Since our invariant \mathbf{h} is trivial in $F_2''F_2^n$, it can be used to detect non-trivial elements in $M(2, n)$ and to prove that certain identities do not hold in that group. Using Remark 1 it is easy to prove that the winding invariant of e_m is $P_{e_m} = -(Y-1)^{m-1}$.

Lemma 8. *Let $k \geq 2$. Then $\binom{2^k}{2^{k-1}} - 2$ is not a multiple of 8.*

Proof. There is an action of the cyclic group \mathbb{Z}_{2^k} on the set $\binom{\mathbb{Z}_{2^k}}{2^{k-1}}$ of subsets of \mathbb{Z}_{2^k} of cardinality 2^{k-1} , induced by the natural action of \mathbb{Z}_{2^k} on itself. The orbits of this action have order a power of 2. There is no orbit of cardinality 1. There is exactly one orbit of cardinality 2, given by the subset of even numbers in \mathbb{Z}_{2^k} and the subset of odd numbers. There is exactly one orbit of cardinality 4, it is the orbit of the subset of numbers congruent to 0 and 1 modulo 4 in \mathbb{Z}_{2^k} . All the other orbits have cardinality divisible by 8. Thus, $\binom{2^k}{2^{k-1}} \equiv 6 \pmod{8}$. \square

Example 9. Let $n = 2^k$ for some $k \geq 3$. Then the $(n+1)$ -Engel word e_{n+1} is not a product of n th powers. Moreover, $M(2, n)$ is not $(n+1)$ -Engel. This follows immediately from Bachmuth, Heilbronn and Mochizuki results: $M(2, n)$ is not $(m-1)$ -Engel for $m = k2^{k-1}$, and $m-1 \geq n+1$. However we give a short proof of this using our invariant \mathbf{h} . Moreover, the advantage of our approach is that in Proposition 19 we will be able to extend this example to all the basic commutators of weight $\leq n+2$, using the notion of *diagonal invariant*.

The winding invariant of e_{n+1} is $-(Y-1)^n$. Take the coloring $\mathbb{Z}_n \rightarrow \{\text{black, white}\}$ which paints $0, 1, \dots, n/2-1$ with black and $n/2, n/2+1, \dots, n-1$ with white. We will prove that $\mathbf{h}(e_{n+1}) \in \mathbb{Z}_n$ is not a multiple of 8. By definition $\mathbf{h}(e_{n+1}) = \sum_{i=0}^{n/2-1} (-1)^{i+1} \binom{n}{i} - \sum_{i=n/2}^{n-1} (-1)^{i+1} \binom{n}{i} - 1$. Since $\binom{n}{i} = \binom{n}{n-i}$ for every i , most of the terms in this expression cancel and we obtain $\mathbf{h}(e_{n+1}) = \binom{n}{n/2} - 2$. The result follows then from Lemma 8 and Theorem 6.

Example 10. $M(2, 16)$ is not a 25-Engel group. In fact Bachmuth-Heilbronn-Mochizuki's result says that it is not 31-Engel. Our proof is easy by considering the invariant \mathbf{h} as above. We only have to check that $\mathbf{h}(e_{25}) \neq 0$. The winding invariant of e_{25} is $-(Y-1)^{24} = \sum_{i=0}^{24} (-1)^{i+1} \binom{24}{i} Y^i$. Modulo 16 the coefficients of this polynomial are: $-1, 8, -4, 8, -2, 8, -4, 8, 1, 0, 8, 0, 4, 0, 8, 0, 1, 8, -4, 8, -2, 8, -4, 8, -1$. Thus, $\mathbf{h}(e_{25}) = (-1+8-4+8-2+8-4+8)-(1+0+8+0+4+0+8+0)+(1+8-4+8-2+8-4+8)-(-1) = 5-5+7+1 = 8 \in \mathbb{Z}_{16}$.

We define the following $n/2$ colorings of \mathbb{Z}_n , $c_0, c_1, \dots, c_{n/2-1} : \mathbb{Z}_n \rightarrow \{\text{black, white}\}$. The coloring c_i paints $i, i+1, \dots, i+n/2-1$ with black and the remaining elements with white. The associated horizontal invariants are $\mathbf{h}^0, \mathbf{h}^1, \dots, \mathbf{h}^{n/2-1}$. Similarly we have the vertical invariants $\mathbf{v}^0, \mathbf{v}^1, \dots, \mathbf{v}^{n/2-1}$. The invariant \mathbf{v}^i is associated to the coloring of the squares in the grid which paints the columns congruent to $i, i+1, \dots, i+n/2-1$ with black and the remaining columns with white. That is $\mathbf{v}^i(z) = \mathbf{h}^i(P_z(Y, X))$. By Theorem 6, the map $\Omega : F_2' \rightarrow (\mathbb{Z}_n)^n = \mathbb{Z}_n \times \dots \times \mathbb{Z}_n$ defined by

$$\Omega(z) = (\mathbf{h}^0(z), \mathbf{h}^1(z), \dots, \mathbf{h}^{n/2-1}(z), \mathbf{v}^0(z), \mathbf{v}^1(z), \dots, \mathbf{v}^{n/2-1}(z))$$

is trivial in $F_2'' F_2^n$ so it induces a well-defined homomorphism $\Omega : M(2, n)' \rightarrow (\mathbb{Z}_n)^n$.

Lemma 11. Ω is trivial in $\gamma_{j+1}(F_2)$ if and only if $\Omega(e_j) = 0$.

Proof. Since $e_j \in \gamma_{j+1}(F_2)$, one implication is trivial. Suppose now that $\Omega(e_j) = 0$. Then by symmetry $\Omega(e_j(y, x)) = 0$. Let $z \in \gamma_{j+1}(F_2)$. Using Remark 1 it is easy to see that P_z is a sum of multiples of polynomials of the form $(m_1-1)(m_2-1)\dots(m_{j-1}-1)$ where the m_i are monomials (see also [9, Proposition 23]). If $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ and m is a monomial, $\Omega(P)$ and $\Omega(mP)$ have the same coordinates up to a shift and change of sign. It suffices to prove then that for a polynomial $P = (m_1-1)(m_2-1)\dots(m_{j-1}-1)$, where the m_i are monomials, Ω vanishes. Now, we can consider only the invariants $\mathbf{h}^0, \mathbf{h}^1, \dots, \mathbf{h}^{n/2-1}$ since for the \mathbf{v}^i the proof is symmetrical. Since the colors of the squares in the integer lattice used to compute \mathbf{h}^i depend only on the row, we may also assume that each monomial m_i is a power of Y . Also, since $Y-1$ divides Y^l-1 for every $l \in \mathbb{Z}$, P is a multiple of $(Y-1)^{j-1} = -P_{e_j}$, so as Ω vanishes at e_j , it also vanishes at z . \square

We consider the map $\bar{\Omega} : F_2' \rightarrow \mathbb{Z}_{n/2} \times (\mathbb{Z}_n)^n$ defined by $\bar{\Omega}(z) = (A(z), \Omega(z))$. Recall that A denotes the area invariant $A : F_2' \rightarrow \mathbb{Z}_{n/2}$ defined by $A(z) = P_z(1, 1)$, which is trivial in $F_2'' F_2^n$ by Theorem 59 of [9].

Proposition 12. The image $Im(\Omega)$ of $\Omega : F_2' \rightarrow (\mathbb{Z}_n)^n$ is a subgroup of order $2(n/2)^n$ and $Im(\bar{\Omega})$ has order $(n/2)^{n+1}$.

Proof. Let $z \in F'_2$ be an element with winding invariant $P_z = 1 - Y^{-1}$. It is easy to see that $\Omega(z) = (2, 0, 0, \dots, 0) \in \text{Im}(\Omega)$. For $1 \leq i \leq n/2 - 1$, the elements $y^i z y^{-i}$ give permutations of the coordinates of this vector with the non-trivial coordinate in any of the first $n/2$ positions. Changing Y by X we obtain the remaining permutations of the coordinates. This already proves that $\text{Im}(\Omega)$ contains a subgroup D of order $(n/2)^n$. The element $z = 1$ has $\Omega(z) \notin D$. On the other hand it is easy to see that for any $z \in F'_2$, all the coordinates of $\Omega(z)$ have the same parity, so $|D| < |\text{Im}(\Omega)| \leq 2(n/2)^n$ and the result follows.

The area of an element $z \in F'_2$ with winding invariant $P_z = 1 + X^{n/2} Y^{n/2}$ is 2, while $\Omega(z) = 0$. Thus, $|\text{Im}(\bar{\Omega})| \geq 2(n/2)^n n/4 = (n/2)^{n+1}$. The parity of the area is also determined by the value of Ω , so the other inequality holds as well. \square

Recall that $M(2, 16)$ is nilpotent of class $m = 32$. In Example 10, we proved that $\mathbf{h}^0(e_{25}) \neq 0 \in \mathbb{Z}_{16}$. It is easy to prove that $\mathbf{h}^0(e_{26}) = 0$ and moreover that in fact $\Omega(e_{26}) = 0$. By Lemma 11, Ω is trivial in $\gamma_{27}(F_2)$. The area invariant is already trivial in $\gamma_3(F_2)$. Thus $\bar{\Omega}$ induces a map $\bar{\Omega} : M(2, 16)' / \gamma_{27}(M(2, 16)) \rightarrow \mathbb{Z}_8 \times (\mathbb{Z}_{16})^{16}$ and from Proposition 12 we deduce that $|\frac{M(2, 16)'}{\gamma_{27}(M(2, 16))}| \geq 2^{51}$. On the other hand $|\frac{M(2, 16)}{M(2, 16)'}| = 16^2 = 2^8$. Therefore we deduce the following

Proposition 13. *The order $|M(2, 16) / \gamma_{27}(M(2, 16))|$ of the largest class 26 quotient of $M(2, 16)$ is greater than or equal to 2^{59} .*

A similar result can be obtained for $n = 8$. Here $\Omega(e_{10}) = 0 \in \mathbb{Z}_8$, so $\bar{\Omega}$ is trivial in $\gamma_{11}(F_2)$ and we deduce from Proposition 12:

Remark 14. The order of $M(2, 8) / \gamma_{11}(M(2, 8))$ is greater than or equal to 2^{24} .

In fact something stronger is already known. $M(2, 8)$ has nilpotency class 12 and its order has been computed by Hermanns [27]: $|M(2, 8)| = 2^{63}$. Skopin computed in [47] the orders of all the factors $\gamma_l(M(2, 8)) / \gamma_{l+1}(M(2, 8))$, from which it can be seen that $M(2, 8) / \gamma_{11}(M(2, 8))$ has order 2^{54} . As far as we know the order of $M(2, 16)$ has not been determined, but $2^{87} \leq |M(2, 16)| \leq 2^{376}$ (see [39] and next section). A GAP [15] program using the package NQ [40] returns after 120 hours that $|M(2, 16) / \gamma_{17}(M(2, 16))| = 2^{185}$, which is of course much better than our result in Proposition 13, and improves the lower bound given above. The computation that $|M(2, 16) / \gamma_9(M(2, 16))| = 2^{70}$ takes seconds.

For $n = 4$, Proposition 12 implies that $|M(2, 4)| \geq 2^9$. In fact $|M(2, 4)| = 2^{10}$, so Ω (and $\bar{\Omega}$) is very close to a complete invariant that could be used to solve the word problem in $M(2, 4)$ in an explicit way. In the next section we will introduce invariants more general than the horizontal and vertical invariants, though based on the same coloring ideas, and we will use them to obtain a complete invariant for $M(2, 4)$.

A *semigroup identity* is an identity of the form uv^{-1} , where $u, v \in F(x_1, x_2, \dots, x_s)$ are positive words (no letter appears with negative exponent). The Burnside identity x^n is an example of this type. Another is the (m th) *Morse identity* $u_m v_m^{-1} \in F_2$, where $u_1 = x$, $v_1 = y$ and $u_{i+1} = u_i v_i$, $v_{i+1} = v_i u_i$ for every $i \geq 1$. In some sense Burnside identities and Morse identities generate all the semigroup identities (see [45, Theorem 5.7]). Finite groups satisfying a Morse identity are completely characterized by a theorem of Boffa and Point [11, Corollaire]: They are the extensions of a nilpotent group by a 2-group. The Morse identities are related to the original proof of Novikov and Adian that there are infinite Burnside groups (see [6, Section 6]).

It is easy to see that the total exponents of x and of y in both u_m and v_m is 2^{m-2} if $m \geq 2$. Let $m \geq 3$. Since $u_m v_m^{-1} = u_{m-1} v_{m-1} u_{m-1}^{-1} v_{m-1}^{-1}$ and $v_{m-1} u_{m-1}^{-1} \in F'_2$, then the winding invariant $P_{u_m v_m^{-1}} = P_{u_{m-1} v_{m-1}^{-1}} + X^{2^{m-3}} Y^{2^{m-3}} P_{v_{m-1} u_{m-1}^{-1}} = (1 - X^{2^{m-3}} Y^{2^{m-3}}) P_{u_{m-1} v_{m-1}^{-1}}$. By induction we obtain then

$$P_{u_m v_m^{-1}} = (1 - X^{2^{m-3}} Y^{2^{m-3}})(1 - X^{2^{m-4}} Y^{2^{m-4}}) \dots (1 - XY).$$

Proposition 15. *Let $k \geq 2$, $n = 2^k$. Then $M(2, n)$ satisfies the $(k+3)$ th Morse identity but it does not satisfy the $(k+1)$ th Morse identity. In particular, the restricted Burnside group $R(2, n)$ does not satisfy the $(k+1)$ th Morse identity, and neither does $B(2, n)$.*

Proof. In order to check that $M(2, n)$ satisfies the $(k+3)$ th Morse identity we only have to show that $u_{k+3} v_{k+3}^{-1} = 1 \in M(2, n)$. The winding invariant of $u_{k+3} v_{k+3}^{-1}$ is $P_{u_{k+3} v_{k+3}^{-1}} = (1 - X^n Y^n) Q$ for some polynomial Q . Let $w \in F'_2$ be such that $P_w = Q$. Then $P_{[w, (xy)^n]} = (1 - X^n Y^n) Q$. Thus, $u_{k+3} v_{k+3}^{-1} = [w, (xy)^n] \pmod{\ker(W)} = F''_2$, and since $[w, (xy)^n]$ is a product of n th powers, $u_{k+3} v_{k+3}^{-1} \in F''_2 F_2^n$, as we wanted to show.

To prove that $M(2, n)$ does not satisfy the $(k+1)$ th Morse identity it suffices to prove that $\Omega(u_{k+1} v_{k+1}^{-1}) \neq 1$, where $\Omega : F'_2 \rightarrow (\mathbb{Z}_n)^n$ is the invariant defined above. Now, $P_{u_{k+1} v_{k+1}^{-1}} = 1 + R$, where R is a polynomial with monomials $X^l Y^l$ for $1 \leq l \leq 2^{k-1} - 1 = n/2 - 1$. Thus the difference between $\mathbf{h}^0(u_{k+1} v_{k+1}^{-1})$ and $\mathbf{h}^1(u_{k+1} v_{k+1}^{-1})$ is $2 \in \mathbb{Z}_n$. In particular one of them has to be non-zero, so $u_{k+1} v_{k+1}^{-1}$ is non-trivial in $M(2, n)$. \square

5. MORE INVARIANTS

In the previous section we have deduced some lower bounds for the order of $M(2, n)$ via the invariant $\bar{\Omega}$ which is composed by n invariants $M(2, n)' \rightarrow \mathbb{Z}_n$ and the area invariant $M(2, n)' \rightarrow \mathbb{Z}_{\frac{n}{2}}$. We proved (a refinement of) the inequality $|M(2, n)| \geq n^2 (\frac{n}{2})^{n+1}$. On the other hand, Newman's remarks in [39] show that $n^2 2^{\frac{n^2}{4}-1} \leq |M(2, n)| \leq n^{n^2+3}$. This suggests that we should be able to find quadratically many invariants $M(2, n)' \rightarrow \mathbb{Z}_n$ and not just linearly many. As explained in Section 1, there exists a monomorphism $M(2, n)' \rightarrow (\mathbb{Z}_n)^r$ for r big enough, so it makes sense to look for more invariants of this type. The aim of this section is to find and make explicit some of those maps.

Theorem 16. *Let $k \geq 2$, $n = 2^k$. Let c be a good coloring of \mathbb{Z}_n and let $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ be a homomorphism such that $\varphi(1, 0), \varphi(0, 1) \in \mathbb{Z}_n$ are odd. Let $t : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be a translation by any vector $(i_0, j_0) \in \mathbb{Z} \times \mathbb{Z}$. Let $\Lambda = \Lambda_{\varphi t, c}$ be the associated invariant. Then for any $z \in F'_2$ which is a product of n th powers in F_2 , $\Lambda(z) = 0 \in \mathbb{Z}_n$.*

Proof. Note that $\Lambda_{\varphi t, c}(z) = \Lambda_{\varphi t, c}(P_z) = \Lambda_{\varphi, c}(X^{i_0} Y^{j_0} P_z) = \Lambda_{\varphi, c}(x^{i_0} y^{j_0} z y^{-j_0} x^{-i_0})$. Thus, we may assume $i_0 = j_0 = 0$.

Just as in Theorem 6 we can assume z is a product of three n th powers. Then we must show that for any polynomial of the form $P = m'(1 + m + m^2 + \dots + m^{n-1})$, the invariant Λ is trivial. But if $m' = X^{i'} Y^{j'}$ and $m = X^i Y^j$, then $\Lambda(P)$ is the number of black terms minus the number of white terms in the progression $\varphi(i', j'), \varphi(i', j') + \varphi(i, j), \varphi(i', j') + 2\varphi(i, j), \dots, \varphi(i', j') + (n-1)\varphi(i, j)$, which is 0 modulo n by Lemma 5.

We move to step 3 in the proof of Theorem 6. Note that in any n consecutive squares in a row or a column, there are exactly $\frac{n}{2}$ which are black and $\frac{n}{2}$ which are white. This is because $\varphi(1, 0)$ and $\varphi(0, 1)$ are odd. Thus, for any rectangle with side lengths multiple

of n the number of black squares is the same as white squares. So we consider once again the stair-like regions.

Because this coloring is not symmetric with respect to horizontal and vertical reflections, we need to consider in principle four types of stairs: one that we walk up from right to left and the three rotations of these, by $\pi/2, \pi$ and $3\pi/2$. Again we must show that the number of black squares minus the number of white squares is a multiple of n . The proof will be by induction on the length l and the height h of the steps.

Case 1: h is odd and l is even. Assume first that the stair is in the position of Figure 4, so one climbs up from right to left. We divide the stair in four regions, $A, -A, B, C$. Region A consists of the top $n/2 - 1$ levels of the stair, while $-A$ is region A translated downwards $\frac{n}{2}h$ units. Since h is odd, $\varphi(0, 1)$ is odd and c is good, then A and $-A$ are painted with opposite colors. Region B consists of the squares below A which are not in $-A$. Region B' is outside the stairs and is the translation of B by $\frac{n}{2}l$ units to the right. Since l is even, B and B' are painted identically, so the invariant Λ of the stair is the invariant of the rectangle $C \cup B'$, where C are the squares of the stair which are not in $A, -A, B$. But this rectangle has one side of length $\frac{n}{2}l$, which is a multiple of n , so its invariant $\Lambda(C \cup B')$ is trivial.

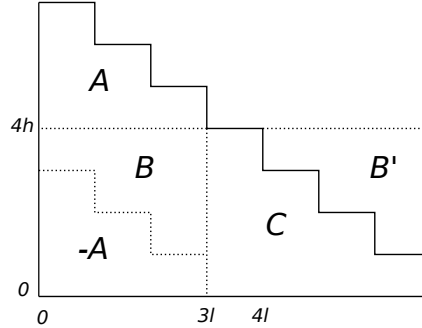


FIGURE 4. An $(n - 1)$ -step stair for $n = 8$, h odd, l even.

If the stair is in a different position (climbing from left to right, or the corresponding upside-down stairs), the same proof works. We have not used an induction argument for this case.

Case 2: h even, l odd, is symmetric to case 1.

Case 3: h and l even. In this case we consider four $(n - 1)$ -step stairs contained in our big stair (see Figure 5). The height of the steps in these four small stairs is $h/2$ and the length is $l/2$. Region A is disjoint from B, C, D , and D is also disjoint from B and C . But B and C overlap in $n - 1$ steps of size $l/2 \times h/2$. They are marked with the sign $-$ in the figure. There are $n - 1$ rectangles of size $l/2 \times h/2$ in the big stair which are not in $A \cup B \cup C \cup D$. They are marked with sign $+$.

By induction the invariants $\Lambda(A), \Lambda(B), \Lambda(C)$ and $\Lambda(D)$ are zero, so the invariant in the big stair is the invariant in the union of the $+$ regions minus the invariant in the $-$ regions. We add two $l/2 \times h/2$ rectangles to the picture, those marked with $+'$ and $-'$. Since they are $\frac{n}{2}h$ squares apart and h is even, they are painted with the same color. Now, the $+$ rectangles together with the $+'$ rectangle are n rectangles which are aligned and the distance between two consecutive is constant. Thus the characteristic polynomial of this

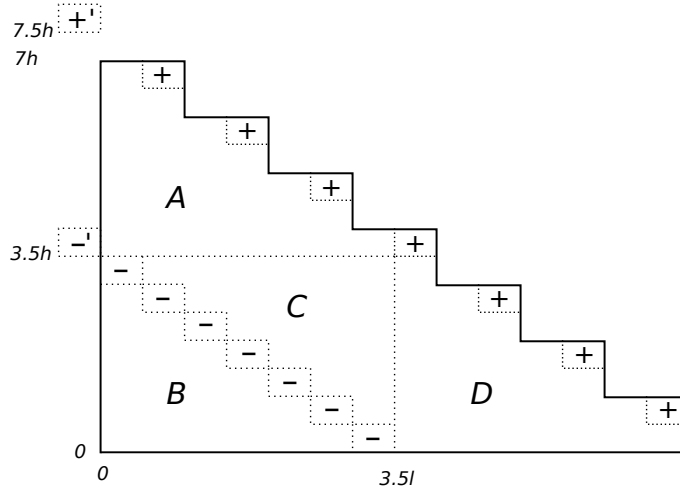


FIGURE 5. An $(n - 1)$ -step stair for $n = 8$, h and l even.

region (the sum of the monomials $X^i Y^j$ for squares (i, j) of the region) is a multiple of a polynomial of the form $1 + m + m^2 + \dots + m^{n-1}$, so we already know that the invariant is trivial. The same happens with the region formed by the $-$ rectangles together with the $-'$ rectangle. Thus, the invariant of the big stair is also trivial.

Case 4: h and l odd. Just as in case 1, we consider the regions A and $-A$ inside the stair (see Figure 6). Since h is odd, they are painted with opposite colors, and we only care about the region B which consists of the squares which are not in A nor in $-A$.

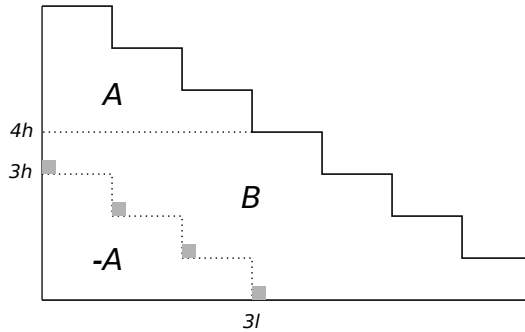


FIGURE 6. An $(n - 1)$ -step stair for $n = 8$, h and l odd.

Consider the region C formed by the squares of B which satisfy that the neighboring squares to the left and below are not in B (they appear with grey in Figure 6). The characteristic polynomial of C is $P = m'(1 + m + m^2 + \dots + m^{\frac{n}{2}-1})$ for $m = X^l Y^{-h}$ and some monomial m' . In order to understand the invariant $\Lambda(P)$ we should analyze the colors of the numbers $a, a + b, a + 2b, \dots, a + (\frac{n}{2} - 1)b \in \mathbb{Z}_n$, where a depends on m' and $b = \varphi(l, -h)$. By assumption, b is even, so Lemma 5 says that $\Lambda(C)$ is a multiple of $\frac{n}{2}$. Moreover, $\Lambda(C) = \frac{n}{2} \in \mathbb{Z}_n$ if $n|b$ and $\Lambda(C) = 0$ if $n \nmid b$. Region B can be covered by $hl\frac{n}{2}$

translates of region C . Each of them has the same invariant as C . Since $k \geq 2$, $hl\frac{n}{2}$ is even and $\Lambda(B) = hl\frac{n}{2}\Lambda(C) = 0 \in \mathbb{Z}_n$. \square

If $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$, c and (i_0, j_0) are in the hypotheses of Theorem 16, then $\Lambda_{\varphi,t,c} : F_2' \rightarrow \mathbb{Z}_n$ induces a well defined group homomorphism $\Lambda_{\varphi,t,c} : M(2, n)' \rightarrow \mathbb{Z}_n$.

Example 17. In this example we exhibit a complete invariant for $M(2, 4)$, that is a concrete solution to the word problem. We describe a method which decides for any $u \in F_2$, whether u is trivial in $M(2, 4)$. Of course, this can be obtained from any finite presentation and coset enumeration or a consistent power-commutator presentation, but our methods are elementary and very easy to carry out by hand in short time.

Remember that we constructed the map $\Omega : M(2, 4)' \rightarrow (\mathbb{Z}_4)^4$ whose image has order 2^5 . Consider the map $\tilde{\Omega} : M(2, 4)' \rightarrow (\mathbb{Z}_4)^5$ defined by $\tilde{\Omega}(w) = (\Omega(w), \Lambda(w))$, where $\Lambda : M(2, 4)' \rightarrow \mathbb{Z}_4$ is the *diagonal invariant* defined as in the statement of Theorem 16 for $\varphi(1, 0) = \varphi(0, 1) = 1$, $(i_0, j_0) = (0, 0)$, and $c : \mathbb{Z}_4 \rightarrow \{\text{black, white}\}$ the coloring that paints 0, 1 with black and 2, 3 with white. Figure 7 shows the five invariants that compose $\tilde{\Omega}$. If $w \in F_2'$ is such that $P_w = 1 + X^2Y^2$, then $\tilde{\Omega}(w) = (0, 0, 0, 0, 2)$. This already implies that the order $|\text{Im}(\tilde{\Omega})| \geq 2|\text{Im}(\Omega)| = 2^6$. On the other hand, since $|M(2, 4)| = 2^{10}$, then $|M(2, 4)'| = 2^6$, so $\tilde{\Omega}$ is a monomorphism.

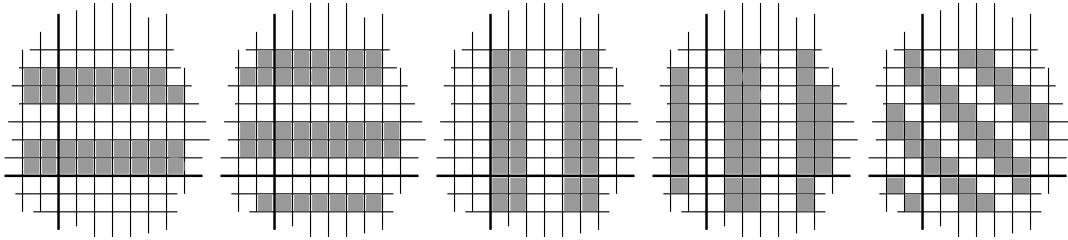


FIGURE 7. The five colorings that form a complete invariant for $M(2, 4)$: two horizontal invariants, two vertical invariants and a diagonal invariant.

The algorithm to decide whether a given element $u \in F_2$ is trivial in $M(2, 4)$ is the following: compute the total exponents $a = \exp(x, u)$, $b = \exp(y, u)$. If any of them is not divisible by 4, u is non-trivial in $M(2, 4)$. Otherwise, let $v = x^{-a}y^{-b}u \in F_2'$. Compute $\tilde{\Omega}(v)$. If this is non-trivial, v and then u are non-trivial in $M(2, 4)$. If it is trivial, then $u = v = 1 \in M(2, 4)$.

In the previous example we have used that $|M(2, 4)| = 2^{10}$. In fact the inequality $|M(2, 4)| \geq 2^{10}$ follows from the fact that $|\text{Im}(\tilde{\Omega})| \geq 2^6$. In Example 21 we will prove the inequality $|M(2, 4)| \leq 2^{10}$ by different methods, thus providing an alternative proof that $|M(2, 4)| = 2^{10}$.

Let $n = 2^k$. As we recalled in Section 2 and the beginning of this section, Newman proves in [39] that $|M(2, n)| \geq n^2 2^{\frac{n^2}{4} - 1}$. His short argument is as follows. Take $R = F_2^{\frac{n}{2}} F_2' \leq F_2$ and $S = R^2 R' \leq F_2$. Then $F_2' \leq R$, so $F_2'' \leq R' \leq S$ and $F_2^n \leq R^2 \leq S$. Thus, $F_2^n F_2'' \leq S$, which shows that the order of $M(2, n) = F_2 / F_2^n F_2''$ is greater than or equal to the order of F_2 / S . But the latter is easy to compute. Note that F_2 / R is an abelian group of order $(\frac{n}{2})^2$, so by Schreier formula R is a free group of rank $1 + (\frac{n}{2})^2$. Once again, R / S is abelian of order $2^{\text{rk}(R)} = 2^{1 + (\frac{n}{2})^2}$ and the order of F_2 / S is $|F_2 / R| |R / S| = (\frac{n}{2})^2 2^{\frac{n^2}{4} + 1}$.

Our next result improves Newman's lower bound just a little bit.

Theorem 18. *Let $k \geq 3$, $n = 2^k$. Then $|M(2, n)| \geq n^2 2^{\frac{5n^2}{16} - 1}$.*

Proof. Let $S = R^2 R'$ be as above. It suffices to prove then that $|S/F_2^n F_2''| \geq 2^{\frac{n^2}{16}}$. We will show that $|\frac{S \cap F_2'}{F_2^n F_2'' \cap F_2'}| \geq 2^{\frac{n^2}{16}}$. Given any Laurent polynomial $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$, there exists $z \in F_2'$ with $P_z = P$. Thus, $z^2 \in S \cap F_2'$ and $2P \in W(S \cap F_2') \leq \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$.

Let $O = \{1, 9, 17, \dots, n-7\}$ be the set of integers congruent to 1 modulo 8 between 0 and $n-1$. Let $c_0, c_1 : \mathbb{Z}_n \rightarrow \{\text{black}, \text{white}\}$ be two good colorings which differ only in 0 and $n/2$.

Consider the action of $\Gamma = \mathbb{Z}_8 \times \mathbb{Z}_2$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ where the generator of \mathbb{Z}_8 maps a pair $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n$ to $(i - n/8, j + n/8)$ and the generator of \mathbb{Z}_2 maps (i, j) to $(i + n/2, j)$. Let Ξ be a set of representatives of the orbits of this action. This set has $n^2/16$ elements.

Let $\phi : M(2, n)' \rightarrow \mathbb{Z}_n^{O \times \{c_0, c_1\} \times \Xi}$ be the group homomorphism defined by $\phi(z)_{b,c,(i_0,j_0)} = \Lambda_{\varphi_{1,b} t_{-(i_0,j_0),c}}(z)$, where $\varphi_{1,b} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ is the homomorphism determined by $\varphi_{1,b}(i, j) = i + bj$ and $t_{-(i_0,j_0)}$ is the translation by the vector $-(i_0, j_0)$ (any vector in the fiber of $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n \times \mathbb{Z}_n$ over $-(i_0, j_0)$). By Theorem 16, ϕ is a well-defined group homomorphism. To prove the result it suffices to show that the order of $\phi(S \cap F_2'/F_2^n F_2'' \cap F_2')$ is greater than or equal to $2^{\frac{n^2}{16}}$. Let $\bar{\phi} : \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}] \rightarrow \mathbb{Z}_n^{O \times \{c_0, c_1\} \times \Xi}$ be defined by $\bar{\phi}(P)_{b,c,(i_0,j_0)} = \Lambda_{\varphi_{1,b} t_{-(i_0,j_0),c}}(P)$, so we have $\bar{\phi}W = \phi q$, where $q : F_2' \rightarrow M(2, n)'$ is the projection. Then $\phi(S \cap F_2'/F_2^n F_2'' \cap F_2') = \bar{\phi}W(S \cap F_2') \geq \bar{\phi}(2\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}])$. The map $\bar{\phi}$ factorizes as $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}] \rightarrow \mathbb{Z}_n[X^{\pm 1}, Y^{\pm 1}] \rightarrow \mathbb{Z}_n^{O \times \{c_0, c_1\} \times \Xi}$ and we are going to study the restriction ψ of the map $\mathbb{Z}_n[X^{\pm 1}, Y^{\pm 1}] \rightarrow \mathbb{Z}_n^{O \times \{c_0, c_1\} \times \Xi}$ to the subgroup H of Laurent polynomials with coefficients in \mathbb{Z}_n supported in the square $[n]^2$ and with all coefficients even, i.e. $P\{(i, j)\}$ is even for every $i, j \in \mathbb{Z}$ and $P\{(i, j)\} \neq 0$ implies that $0 \leq i, j \leq n-1$. Note that the order of H is $(\frac{n}{2})^{n^2}$. We want to prove that $|\text{Im}(\psi)| \geq 2^{\frac{n^2}{16}}$.

We have a sequence

$$\ker(\psi) \hookrightarrow H \xrightarrow{\psi} \mathbb{Z}_n^{O \times \{c_0, c_1\} \times \Xi}.$$

Let $P \in \ker(\psi)$. This means that for every $b \in O$, $c = c_0, c_1$ and $(i_0, j_0) \in \Xi$,

$$\sum_{c(i-i_0+b(j-j_0)) \text{ is black}} P\{(i, j)\} - \sum_{c(i-i_0+b(j-j_0)) \text{ is white}} P\{(i, j)\} = 0 \in \mathbb{Z}_n.$$

If we subtract the equations corresponding to c_0 and to c_1 we obtain

$$2 \sum_{i-i_0+b(j-j_0)=0} P\{(i, j)\} - 2 \sum_{i-i_0+b(j-j_0)=n/2} P\{(i, j)\} = 0 \in \mathbb{Z}_n.$$

Here the equalities $i - i_0 + b(j - j_0) = 0$ and $i - i_0 + b(j - j_0) = n/2$ are considered modulo n .

We make b vary among all the elements in O and add all these equations to obtain

$$2 \sum_{b \in O} \sum_{i-i_0+b(j-j_0)=0} P\{(i, j)\} - 2 \sum_{b \in O} \sum_{i-i_0+b(j-j_0)=n/2} P\{(i, j)\} = 0 \in \mathbb{Z}_n. \quad (1)$$

Given $0 \leq i, j \leq n-1$, the coefficient $P\{(i, j)\}$ appears in the left term of Equation 1 if and only if $i - i_0 + b(j - j_0) \equiv 0 \pmod n$ for some $b \equiv 1 \pmod 8$. This is equivalent to $d = \gcd(8(j - j_0), n) \mid i - i_0 + j - j_0$. In this case, the number of times that $P\{(i, j)\}$ appears in the left term is $d/8$ (multiplied by the coefficient 2). On the other hand, $P\{(i, j)\}$ appears in the right term of Equation 1 if and only if $i - i_0 + b(j - j_0) \equiv n/2 \pmod n$ for some $b \in O$. This is equivalent to $d \mid i - i_0 + j - j_0 - n/2$, and the number of times $P\{(i, j)\}$ appears is also $d/8$. Therefore, if $n/8$ does not divide $j - j_0$, the coefficient $P\{(i, j)\}$ in Equation 1 cancels. The unique coefficients that survive in the equation are those $P\{(i, j)\}$ in the left term with $j \equiv j_0 \pmod{n/8}$ and $i \equiv i_0 + j_0 - j \pmod n$, and those $P\{(i, j)\}$ in the right with $j \equiv j_0 \pmod{n/8}$ and $i \equiv i_0 + j_0 - j + n/2 \pmod n$. Each of them appears $d/8 = n/8$ times, multiplied by the coefficient 2. The coefficients in the left have positive sign and those in the right negative. But since $P \in H$, $P\{(i, j)\}$ is even for all (i, j) , so $\frac{n}{4}P\{(i, j)\} = -\frac{n}{4}P\{(i, j)\} \in \mathbb{Z}_n$. In conclusion

$$\frac{n}{4} \sum_{(i,j) \in \Gamma \cdot (i_0, j_0)} P\{(i, j)\} = 0 \in \mathbb{Z}_n,$$

where $\Gamma \cdot (i_0, j_0)$ is the orbit of (i_0, j_0) under $\Gamma = \mathbb{Z}_8 \times \mathbb{Z}_2$. In other words,

$$\sum_{(i,j) \in \Gamma \cdot (i_0, j_0)} P\{(i, j)\} = 0 \in \mathbb{Z}_4.$$

This means that if $P \in \ker(\psi)$, the congruence of $P\{(i_0, j_0)\}$ modulo 4 is determined by the other coefficients $P\{(i, j)\}$ with (i, j) in the orbit of (i_0, j_0) . Thus, the order of $\ker(\psi)$ is smaller than or equal to $\left(\binom{n}{2}^{15} \frac{n}{4}\right)^{\frac{n^2}{16}}$ (there are $\frac{n^2}{16}$ orbits and there are only $\frac{n}{4}$ choices for $P\{(i_0, j_0)\}$ once we have chosen the other 15 values of $P\{(i, j)\}$ in the same orbit). Finally,

$$|\text{Im}(\psi)| = \frac{|H|}{|\ker(\psi)|} \geq \frac{\left(\frac{n}{2}\right)^{n^2}}{\left(\frac{n}{2}\right)^{n^2} \frac{1}{2^{\frac{n^2}{16}}}} = 2^{\frac{n^2}{16}}.$$

□

Recall that for every $m \geq 1$, $\gamma_m(F_2)/\gamma_{m+1}(F_2)$ is a free abelian group of finite rank and a basis is given by the basic commutators of weight m . The basic commutators of weight 1 are y and x . We set $y < x$. Once we have defined the basic commutators of weight smaller than m and we have ordered them, we define those basic commutators of weight m : they are commutators $[c_j, c_i]$ where c_j and c_i are basic commutators of weight j and i , $j + i = m$, $c_j < c_i$, and, if $i \geq 2$ and $c_i = [c_t, c_s]$ then $c_t \leq c_j$. We set an arbitrary order among the basic commutators of weight m . Each of them is greater than any commutator of weight $< m$. Our basic commutators are the inverses of the classical ones, which are defined using the other convention for commutators $(a, b) = a^{-1}b^{-1}ab$ (see [22, Chapter 11]). If $[c_j, c_i]$ is a basic commutator constructed as above, which is non-trivial in the free metabelian group $M_2 = F_2/F_2''$ of rank 2, then c_j has to be a basic commutator of weight 1 (i.e. x or y), and c_i has to be non-trivial in M_2 . If $c_i = [c_t, c_s]$, then c_t is also of weight 1 and $c_t \leq c_j$. Thus, for $m \geq 2$, $\gamma_m(M_2)/\gamma_{m+1}(M_2)$ is generated by the basic commutators $e_{i, m-i-1} = \underbrace{[x, x, \dots, x]_i}_{i} \underbrace{[y, y, \dots, y]_{m-i-1}}_{m-i-1} [x] = [x, [x, \dots, [x, [y, [y, \dots, [y, x]] \dots]]]$, where $0 \leq i \leq m-2$. For $i = 0$, $e_{0, m-1}$ this is just the $(m-1)$ -Engel word e_{m-1} .

Proposition 19. *Let $n = 2^k$ for some $k \geq 3$. Then $e_{i,m-i-1}$ is non-trivial in the class m factor $\gamma_m(M(2,n))/\gamma_{m+1}(M(2,n))$ for every $2 \leq m \leq n+2$ and every $0 \leq i \leq m-2$.*

Proof. It suffices to prove the statement for $m = n+2$. The winding invariant of $e_{i,n-i+1}$ is $-(X-1)^i(Y-1)^{n-i}$. Let $\Lambda : M(2,n)' \rightarrow \mathbb{Z}_n$ be the diagonal invariant associated to $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$, where $\varphi(1,0) = \varphi(0,1) = 1$, t the identity of $\mathbb{Z} \times \mathbb{Z}$ and $c : \mathbb{Z}_n \rightarrow \{\text{black, white}\}$ the good coloring which paints $0, 1, \dots, \frac{n}{2} - 1$ with black. Since $\Lambda(X^r Y^s) = \Lambda(Y^{r+s}) = \mathbf{h}^0(Y^{r+s})$, then $\Lambda(e_{i,n-i+1}) = \mathbf{h}^0(-(Y-1)^n) = \mathbf{h}^0(e_{n+1}) = \binom{n}{n/2} - 2 \in \mathbb{Z}_n$ is not a multiple of 8 by Example 9 and Lemma 8. On the other hand we will see that $\Lambda(z)$ is a multiple of 8 for every $z \in \gamma_{n+3}(F_2)$. The proof is similar to that of Lemma 11. If $z \in \gamma_{n+3}(F_2)$, then P_z is a sum of multiples of polynomials of the form $P = (m_1 - 1)(m_2 - 1) \dots (m_{n+1} - 1)$ where the m_i are monomials. Thus, it suffices to prove that for any good coloring $c' : \mathbb{Z}_n \rightarrow \{\text{black, white}\}$ the diagonal invariant $\Lambda' : F_2' \rightarrow \mathbb{Z}_n$ associated to φ , t and c' , satisfies that $\Lambda'(P)$ is a multiple of 8. Now, $\Lambda'(P(X,Y)) = \Lambda'(P(Y,Y))$, and $P(Y,Y)$ is a multiple of $Q = (Y-1)^{n+1}$. Therefore, it suffices to prove that $\Lambda'(Q)$ is a multiple of 8 for each good coloring c' .

Suppose first that $c' = c$, so $\Lambda' = \Lambda$. Then

$$\Lambda(Q) = \sum_{i=0}^{\frac{n}{2}-1} (-1)^{i+1} \binom{n+1}{i} - \sum_{i=\frac{n}{2}}^{n-1} (-1)^{i+1} \binom{n+1}{i} + \sum_{i=n}^{n+1} (-1)^{i+1} \binom{n+1}{i}.$$

Using that $\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}$ for $i \geq 1$, we obtain that $\Lambda(Q) = 2\binom{n}{n/2-1} - 2n$. Now, the action of \mathbb{Z}_n on the set $\binom{\mathbb{Z}_n}{n/2-1}$ is free: if $S \subseteq \mathbb{Z}_n$ is a subset of cardinality $n/2 - 1$ and $\frac{n}{2} \cdot S = S$, then $|S| = 2|S \cap [n/2]|$ is even, a contradiction. Thus, every orbit has cardinality n and then $\binom{n}{n/2-1}$ is a multiple of n . This proves that $\Lambda(Q) = 0 \in \mathbb{Z}_n$, so in particular it is a multiple of 8.

Suppose now that $c' : \mathbb{Z}_n \rightarrow \{\text{black, white}\}$ is any good coloring. Then c' can be obtained from c by a sequence of changes of colors of pairs $j, j + \frac{n}{2}$ for $0 \leq j \leq \frac{n}{2} - 1$. Thus, it suffices to prove that if two good colorings c', c'' of \mathbb{Z}_n differ only in j and $j + \frac{n}{2}$, then $\Lambda''(Q) - \Lambda'(Q)$ is a multiple of 8 for the invariants Λ', Λ'' associated to c', c'' .

If $j = 0$, then $\Lambda''(Q) - \Lambda'(Q) = 2(\binom{n+1}{0} - \binom{n+1}{n/2} + \binom{n+1}{n}) \equiv 2(2 - \binom{n+1}{n/2}) \pmod{8}$. Again, $\binom{n+1}{n/2} = \binom{n}{n/2-1} + \binom{n}{n/2}$ and the first term is a multiple of n . By the argument in Lemma 8, $\binom{n}{n/2} \equiv 2 \pmod{4}$, so $\Lambda''(Q) - \Lambda'(Q)$ is a multiple of 8.

If $j = 1$, $\Lambda''(Q) - \Lambda'(Q) = 2(-\binom{n+1}{1} + \binom{n+1}{n/2+1} - \binom{n+1}{n+1}) \equiv 2(\binom{n+1}{n/2} - 2) \equiv 0 \pmod{8}$ by the previous paragraph.

Suppose now that $2 \leq j \leq \frac{n}{2} - 1$. Then $\Lambda''(Q) - \Lambda'(Q) = (-1)^j 2(\binom{n+1}{j} - \binom{n+1}{n/2+j}) = (-1)^j 2(\binom{n}{j-1} + \binom{n}{j} - \binom{n}{n/2+j-1} - \binom{n}{n/2+j})$. If $0 \leq r \leq n$ is different from $0, \frac{n}{2}, n$, then the action of \mathbb{Z}_n on $\binom{\mathbb{Z}_n}{r}$ has no orbits of cardinality 1 or 2. In this case, $4 \mid \binom{n}{r}$. Since $j-1, j, \frac{n}{2} + j - 1$ and $\frac{n}{2} + j$ are different from $0, \frac{n}{2}, n$, then $\Lambda''(Q) - \Lambda'(Q)$ is a multiple of 8. \square

Proposition 19 shows that the order of each of the $m-1$ generators $e_{i,m-i-1}$ of the class m factor $\gamma_m(M(2,n))/\gamma_{m+1}(M(2,n))$ is at least 2. If moreover one could prove that the order of this factor is greater than or equal to 2^{m-1} , for $2 \leq m \leq n+2$, then

$|M(2, n)'| \geq 2^{\frac{(n+1)(n+2)}{2}}$, and this would provide a better bound than that given in Theorem 18.

6. UPPER BOUNDS FOR $|M(2, 2^k)|$

As explained in Section 2, Newman proved [39] that $|M(2, 2^k)| \leq 2^{3k+k2^{2k}}$. In general this bound is far from optimal. In this section we use the winding invariant map to find better upper bounds. We will focus on the special case $k = 2$, where the bounds found are sharp.

We recall from [9] the construction of the winding invariant map associated to a group with a *cocommutative* presentation. A Laurent polynomial $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ can be represented graphically by writing each non-zero coefficient $P\{(i, j)\}$ in the square (i, j) of the grid. The graphical representation of a polynomial is called a *piece*, so we will also say that a polynomial *is* a piece. The piece $P_w = W(w)$ associated to an element $w \in F_2'$ has inside each square, the winding number of the curve γ_w around the center of that square.

In [9, Proposition 18] it is proved that for a group G presented by $\langle x, y | S \rangle$ where S is a subset of F_2' , $W : F_2' \rightarrow \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ induces an epimorphism $\overline{W} : G' \rightarrow \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$ where I is the ideal of $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ generated by the polynomials P_s with $s \in S$. Moreover $\ker(\overline{W}) = G''$, so there is an induced isomorphism $G'/G'' \rightarrow \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$, which we also denote by \overline{W} . We apply this to the group G presented by $\langle x, y | F_2' \cap F_2^n \rangle$. We have then that G'/G'' is isomorphic to $M(2, n)'$, so there is an isomorphism $\overline{W} : M(2, n)' \rightarrow \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$, where I is the ideal generated by (consisting of) the polynomials P_s with $s \in S = F_2' \cap F_2^n$. The obvious observation is then:

Remark 20. The order of $M(2, n)'$ is smaller than or equal to the cardinality of any set $A \subseteq \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ which contains a representative of each equivalence class of $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$.

We will understand the quotient $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$ graphically. The sum of two pieces $P, Q \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ is the piece which is obtained by adding the two numbers of each square. The product of a piece P by a monomial $X^i Y^j$ is the translation of the piece by the vector $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. All the computations that follow can be done algebraically, but we believe the graphical representation makes proofs simpler.

Example 21. Gupta and Tobin [20] proved that $|M(2, 4)| = 2^{10}$. Since $M(2, 4)/M(2, 4)' = \mathbb{Z}_4 \times \mathbb{Z}_4$, this is equivalent to proving that $|M(2, 4)'| = 2^6$. In Example 17 we gave an alternative proof that $|M(2, 4)'| \geq 2^6$. We will use the representation by pieces together with Remark 20 to give an alternative proof that $|M(2, 4)'| \leq 2^6$, thus providing a different proof of Gupta and Tobin's result. In turn, this will give a new algorithm to decide whether an element in $M(2, 4)$ is trivial (different from the one given in Example 17).

Step 1. Suppose first that k is an arbitrary positive integer and $n = 2^k$. Let $[x, y] = xyx^{-1}y^{-1} \in F_2$. Then $w = [x, y]^n \in S = F_2' \cap F_2^n$ and $W(w) = n \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$. Therefore, the piece that has the number n in the square $(0, 0)$ and all the other numbers equal to zero, is in I . Hence, all the pieces whose coefficients are divisible by n are also in I . This proves that any element in $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$ has a representative with all the coefficients in the interval $\{0, 1, 2, \dots, n-1\}$.

Step 2. Let $u = ([x, y]y[y, x])^n y^{-n} \in S$. Then $W(u) = 1 - Y^n \in I$. Thus, the second piece in Figure 8 lies in I . For any piece in $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ we can add and subtract translates of u to obtain a representative whose only non-zero coefficients lie in the rows $0, 1, \dots, n-1$. A symmetric argument shows that in fact there is a representative with all the non-zero

coefficients in the square $[0, n] \times [0, n]$ of rows 0 to $n - 1$ and columns 0 to $n - 1$. We say that the representative is *concentrated* in the square $[0, n] \times [0, n]$.

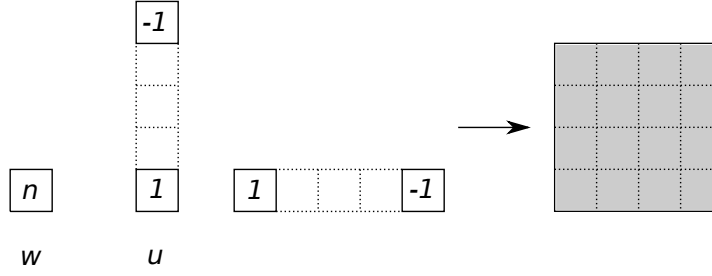


FIGURE 8. The pieces associated to $w = [x, y]^n$, $u = ([x, y]y[y, x])^n y^{-n}$ and $([x, y]x[y, x])^n x^{-n}$ turn any element of $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ to a representative concentrated in $[0, n] \times [0, n]$ with all the coefficients in the interval $[0, n - 1]$. In this picture $n = 4$.

This argument shows that $|M(2, n)'|$ is smaller than or equal to $n^{n^2} = 2^{k2^{2k}}$, so $|M(2, n)| \leq n^{n^2+2}$.

Step 3. Let $v = (xyx^{-1})^n y^{-n} \in S$, so $W(v) = 1 + Y + Y^2 + \dots + Y^{n-1} \in I$ is the piece which appears in Figure 9. Adding multiples of horizontal translates of this piece we can turn any piece concentrated in the square $[0, n] \times [0, n]$ to another piece concentrated in the rectangle $[0, n] \times [0, n - 1]$. And by a symmetric argument, for any element in $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ there is a representative concentrated in the square $[0, n - 1] \times [0, n - 1]$ with all the coefficients between 0 and $n - 1$. Therefore $|M(2, n)'| \leq n^{(n-1)^2}$ (so $|M(2, n)| \leq n^{(n-1)^2+2}$).

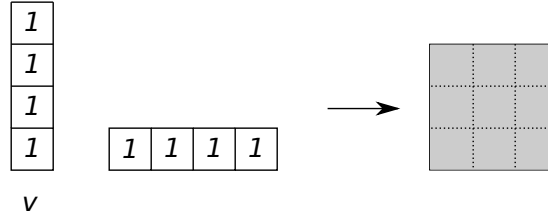


FIGURE 9. The pieces associated to $v = (xyx^{-1})^n y^{-n}$ and $x^n (yx^{-1}y^{-1})^n$ turn any element of $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ concentrated in $[0, n] \times [0, n]$ to a representative concentrated in $[0, n - 1] \times [0, n - 1]$. Together with piece w , we can assume each coefficient is in the interval $[0, n - 1]$.

Before going further, let us summarize some simple yet useful remarks. If $Q \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ is a piece in I , then any translate of Q is in I and any multiple as well. In particular $-Q \in I$, which is the piece that contains in each square of the grid the opposite of the coefficient Q has. Also, if $s = s(x, y) \in S = F_2' \cap F_2^n$, then $s(x^{-1}, y)$ also lies in S . Thus if $Q = Q(X, Y)$ is a piece in I , the symmetric piece $Q(X^{-1}, Y)$ with respect to the y -axis also lies in I . A similar argument shows that rotations of Q by $\pi/2, \pi$ and $3\pi/2$ are also in I . Therefore I is invariant by translations, rotations of angle multiple of $\pi/2$ and symmetries about the axes and the lines $y = x$ and $y = -x$.

Remark. Suppose A is a region (a union of squares of the grid), and that a piece $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ is concentrated in A . That is, all the coefficients of P out of A are zero. If Q is a piece in I which is also concentrated in A , and if (i, j) is a square of Q with coefficient 1 or -1 , then there is a representative of P in $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$ concentrated in the region B , obtained from A by removing the square (i, j) . This representative is just P plus a multiple of Q . This is the idea we already used in step 3 (and step 2 as well).

Step 4. The first three steps worked for arbitrary k . From now on we will assume $k = 2, n = 4$. Let $t = x^4(x^{-1}y)^4y^{-4} \in S$. The piece corresponding to t appears in Figure 10, together with a translate T of a symmetric. The difference between the two pieces is $Q_1 \in I$. Subtracting a translate of Q_1 to itself we obtain the piece $Q_2 \in I$.

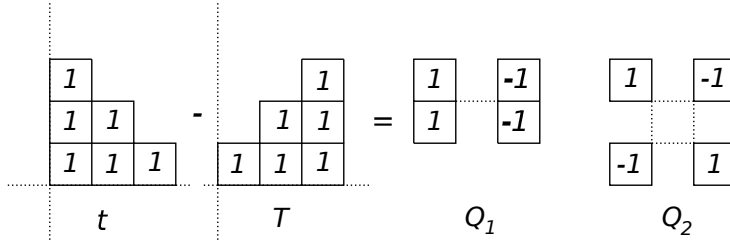


FIGURE 10. The piece associated to $t = x^4(x^{-1}y)^4y^{-4}$ and a symmetric T . At the right, the pieces Q_1 and Q_2 .

We apply the idea of the Remark to the piece $Q_1 \in I$. This can be used to transform an element $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ concentrated in the square $[0, 3] \times [0, 3]$ into a representative concentrated in a region with one square less (see Figure 11). We do this two more times with Q_1 and a rotation of Q_1 to find a representative concentrated in the stair of three steps.

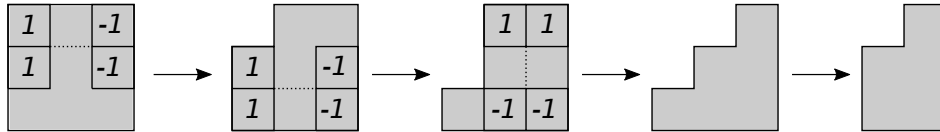


FIGURE 11. An element concentrated in $[0, 3] \times [0, 3]$ has a representative concentrated in the region of five squares.

We then use the piece T in Figure 10 to obtain a representative concentrated in the last region of Figure 11, which consists of just five squares.

Step 5. Let $t' = x^8(x^{-2}y)^4y^{-4} \in S$. The piece associated to t' appears in Figure 12. If we subtract the piece associated to t , then we subtract (a translate of) the horizontal piece in Figure 9, then piece Q_2 , and, finally piece T , we obtain a piece with only two non-zero coefficients, both equal to -2 . This is piece $D \in I$.

Step 6. From previous steps we know that any element $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ has a representative in $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$ concentrated in the region of five squares in Figure 11. Adding multiples of piece D and of its vertical version, we can find a representative of P concentrated in the same five-square region, and where four of the five coefficients inside are 0 or 1. Finally we can use the piece associated with w (step 1), to make the last coefficient

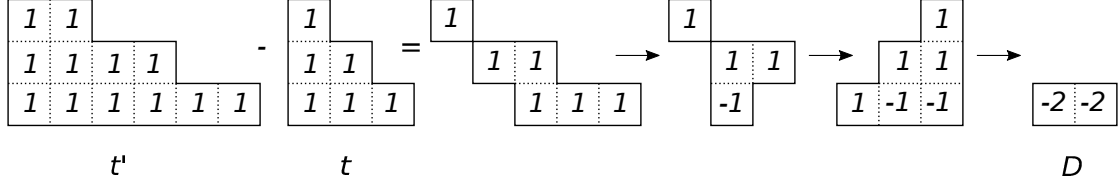


FIGURE 12. Using pieces from I we obtain piece $D \in I$.

lie in the interval $\{0, 1, 2, 3\}$. There are only 2^6 pieces satisfying these conditions, so the order of $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]/I$ is at most this number.

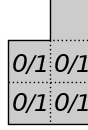


FIGURE 13. The representative of P is one among 64 pieces.

This finishes the proof that $|M(2, 4)| \leq 2^{10}$.

In the previous example we have proved the following bound for arbitrary $k \geq 1$ and $n = 2^k$: $|M(2, n)| \leq n^{(n-1)^2+2}$. In fact, using the piece corresponding to $t = x^n(x^{-1}y)^n y^{-n}$ twice, we can obtain a representative concentrated in the region $[0, n-1] \times [0, n-1]$ with two squares removed. Note that this part of the proof does not use the fact that n is a power of 2. It works for an arbitrary integer $n \geq 3$. Thus we have proved the following

Proposition 22. *Let $n \geq 3$ be an integer. Then $|M(2, n)| \leq n^{(n-1)^2}$.*

7. A LOWER BOUND FOR THE ORDER OF THE RESTRICTED BURNSIDE GROUP $R(2, 8)$

Although Zel'manov [54, 55] proved that the restricted Burnside group $R(d, m)$ exists for all d, m , its order is known in very few cases (see Section 2 above). As far as we know, the best lower bound for $|R(2, 8)|$ available in the literature is given in [18]: 2^{4109} . In fact, this is the order of the group $F_2/(F_2^4)^2$. Indeed, $|F_2/F_2^4| = |B(2, 4)| = 2^{12}$, so by Schreier's formula, $\text{rk}(F_2^4) = 1 + 2^{12} = 4097$. Now $(F_2^4)' \leq (F_2^4)^2$, so $F_2^4/(F_2^4)^2$ is abelian of order $2^{\text{rk}F_2^4}$. Thus, $|F_2/(F_2^4)^2| = |F_2/F_2^4| |F_2^4/(F_2^4)^2| = 2^{4109}$. We will use our invariants to show that this bound is not sharp. Moreover we will prove the following

Proposition 23. $|R(2, 8)| \geq 2^{4115}$.

It is not hard to see that $F_2^8 F_2'''$ is a finite-index subgroup of F_2 . To prove this we define $R = F_2^8 F_2' \leq F_2$, $S = R^8 R' \leq F_2$ and $T = S^8 S' \leq F_2$. Then $|F_2 : R|$ is finite, so R has finite rank. Similarly $|R : S|$ is finite and $|S : T|$ is finite. Then F_2/T is finite. In fact we can compute its order explicitly, but we will not need that. Also, it is easy to prove that $T \leq F_2^8 F_2'''$. It suffices to show that $S \leq F_2^8 F_2''$, and similarly this follows from $R \leq F_2^8 F_2'$. Thus, $F_2^8 F_2'''$ is a finite-index subgroup of F_2 .

Since both $F_2^8 F_2'''$ and $(F_2^4)^2$ are finite-index subgroups of F_2 , $\frac{F_2}{(F_2^8 F_2''') \cap (F_2^4)^2}$ is a finite group of exponent 8, and its order gives a lower bound for $|R(2, 8)|$.

$$\left| \frac{F_2}{(F_2^8 F_2''') \cap (F_2^4)^2} \right| = \left| \frac{F_2}{(F_2^4)^2} \right| \left| \frac{(F_2^4)^2}{(F_2^8 F_2''') \cap (F_2^4)^2} \right|.$$

Let $\phi : \frac{(F_2^4)^2 \cap F_2'}{(F_2^8 F_2''') \cap (F_2^4)^2 \cap F_2'} \rightarrow (\mathbb{Z}_8)^8$ be the map induced by the restriction of $\Omega : F_2' \rightarrow (\mathbb{Z}_8)^8$. Note that ϕ is well-defined since $F_2^8 F_2''' \cap F_2' \leq F_2^8 F_2'' \cap F_2'$. Let $z = (x^4(x^{-1}y)^4y^{-4})^2 \in (F_2^4)^2 \cap F_2'$. It is easy to see that $\phi(z) = (4, 0, 0, 4, 4, 0, 0, 4)$. Moreover, considering the conjugates $x^i y^j z y^{-j} x^{-i}$ with $i, j = 0, 1, 2$, we obtain the nine 8-tuples $(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$ where $(a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) \in \{(4, 0, 0, 4), (4, 4, 0, 0), (0, 4, 4, 0)\}$. These 8-tuples generate a subgroup of $(\mathbb{Z}_8)^8$ of order 2^6 . Thus $\left| \frac{(F_2^4)^2 \cap F_2'}{(F_2^8 F_2''') \cap (F_2^4)^2 \cap F_2'} \right| \geq 2^6$. In particular $|R(2, 8)| \geq 2^{4109} 2^6 = 2^{4115}$.

8. OTHER PRIMES, OTHER COLORS

Up to this point we have only studied the case that the exponent n is a power of 2. Many of the techniques we developed, naturally extend to powers of different primes. In this section we will generalize the notion of good coloring. In fact this is a generalization of a variant of the notion we know already. For the case $p = 2$ this new notion would have led to the same results already proved.

Let p be a positive prime number. Given a subset S of the integers and d a positive integer, a p -matching of S of difference d is a partition of S in which every part is an arithmetic progression of p terms and common difference d . We have the following version of Lemma 4:

Lemma 24. *Let n, d be positive integers. If pd divides n , then there is a p -matching of $[n] = \{0, 1, \dots, n-1\}$ of difference d .*

Let k be a positive integer and let $n = p^k$. A p -good coloring of \mathbb{Z}_n is a function $c : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ such that for every arithmetic progression $a, a + \frac{n}{p}, a + 2\frac{n}{p}, \dots, a + (p-1)\frac{n}{p}$ of p terms and difference $\frac{n}{p}$, we have that the sum $c(a) + c(a + \frac{n}{p}) + c(a + 2\frac{n}{p}) + c(a + (p-1)\frac{n}{p})$ of its colors is zero in \mathbb{Z}_n . Note that a 2-good coloring is not exactly the same as a good coloring in the sense of previous sections. We have the following lemma.

Lemma 25. *Let $k \geq 1, n = p^k$. Let $c : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be a p -good coloring.*

(i) *Let $a, a + b, a + 2b, \dots, a + (n-1)b$ be an arithmetic progression of length n in \mathbb{Z} . Then we have $c(a) + c(a + b) + c(a + 2b) + \dots + c(a + (n-1)b) = 0 \in \mathbb{Z}_n$.*

(ii) *Let $a, a + b, a + 2b, \dots, a + (\frac{n}{p} - 1)b$ be an arithmetic progression of length $\frac{n}{p}$ in \mathbb{Z} with b a multiple of p . Then the sum of the colors of its terms is a multiple of $\frac{n}{p}$.*

The proofs of these lemmas are similar to the proofs of Lemmas 4 and 5, so we omit them.

Given a function $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ and a p -good coloring c of \mathbb{Z}_n , define the associated invariant $\Lambda = \Lambda_{\varphi, c} : \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}] \rightarrow \mathbb{Z}_n$ by

$$\Lambda(P) = \sum_{(i, j) \in \mathbb{Z} \times \mathbb{Z}} c\varphi(i, j) P\{(i, j)\} \in \mathbb{Z}_n.$$

For $z \in F_2'$ we define $\Lambda(z) = \Lambda(P_z)$.

Theorem 26. *Let p be a positive prime, $k \geq 2$, $n = p^k$. Let c be a p -good coloring of \mathbb{Z}_n and let $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ be a homomorphism such that $\varphi(1, 0), \varphi(0, 1) \in \mathbb{Z}_n$ are invertible. Let $t : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be a translation by any vector $(i_0, j_0) \in \mathbb{Z} \times \mathbb{Z}$. Let $\Lambda = \Lambda_{\varphi t, c}$ be the associated invariant. Then for any $z \in F_2'$ which is a product of n th powers in F_2 , $\Lambda(z) \in \mathbb{Z}_n$ is trivial in $\mathbb{Z}_{\frac{n}{p}}$.*

Proof. The proof is almost identical to the proof of Theorem 16. The difference is perhaps in case 4. The cost that we pay for using $p \neq 2$ and the general notion of p -good coloring is that $\Lambda(z)$ may be non-trivial in \mathbb{Z}_n , so we have to descend to $\mathbb{Z}_{\frac{n}{p}}$.

We may assume $(i_0, j_0) = (0, 0)$ and that z is a product of three n th powers. By Lemma 25 we only have to study the invariant for stair-like regions.

Case 1: $p \nmid h, p|l$. Inside the stair of $n - 1$ steps, consider the region A_0 formed by the top $\frac{n}{p} - 1$ levels (see Figure 14). We define the regions A_1, A_2, \dots, A_{p-1} , where A_{i+1} is the translation of A_i by $\frac{hn}{p}$ units downwards.

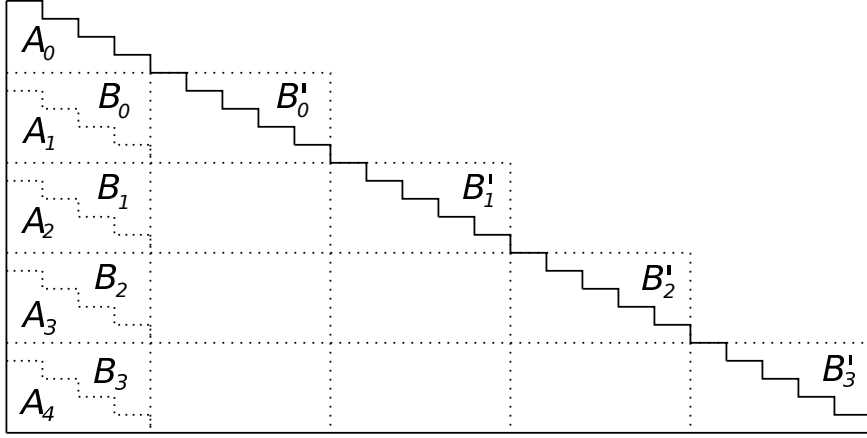


FIGURE 14. Here $p = 5$, $n = 5^2$, $p \nmid h$ and $p|l$.

Since $p \nmid \varphi(0, 1)$ and c is p -good, the sum of the invariants of the regions A_i is trivial in \mathbb{Z}_n . For $0 \leq i \leq p - 2$ we define B_i as the region between A_i and A_{i+1} . B'_i is the translation of B_i by $(i + 1)\frac{hn}{p}$ units to the right, so it is outside the original stairs. Since $n | \frac{(i+1)hn}{p}$, B'_i and B_i are painted in the same way. Therefore the invariant Λ of the original stairs is a sum of invariants of rectangles with one side of length divisible by n , and thus, trivial in \mathbb{Z}_n .

Case 2: $p|h, p \nmid l$ is symmetric to case 1.

Case 3: $p|h, p|l$. We must show that the stairs with $n - 1$ steps have trivial invariant. We will change here the proof we did in Theorem 16 just a bit to make the argument simpler. We can add one more step to the stairs without changing the value of Λ , since the l new columns that we added have height hn , a multiple of n .

The n -step stair appears in Figure 15. We consider region A_0 inside the top n/p levels of the stairs. A_0 is an n -step stair itself but with height h/p and length l/p . We define A_1, A_2, \dots, A_{p-1} where A_{i+1} is obtained from A_i by a translation of $\frac{n}{p}(l, -h)$. By induction each stair A_i has invariant Λ trivial in $\mathbb{Z}_{\frac{n}{p}}$. In fact this holds when we remove one step

from A_i , but again, removing columns of height $\frac{hn}{p}$ does not change Λ since this is a multiple of n . The region below the A_i 's is a union of rectangles of size $\frac{ln}{p} \times \frac{hn}{p}$, so its invariant is trivial in \mathbb{Z}_n . The region above the A_i 's appears in Figure 15 with the sign $+$. This region is a union of n regions of the form $C, C + v, C + 2v, \dots, C + (n - 1)v$ for certain vector v and certain region C . By Lemma 25, its invariant is trivial in \mathbb{Z}_n .

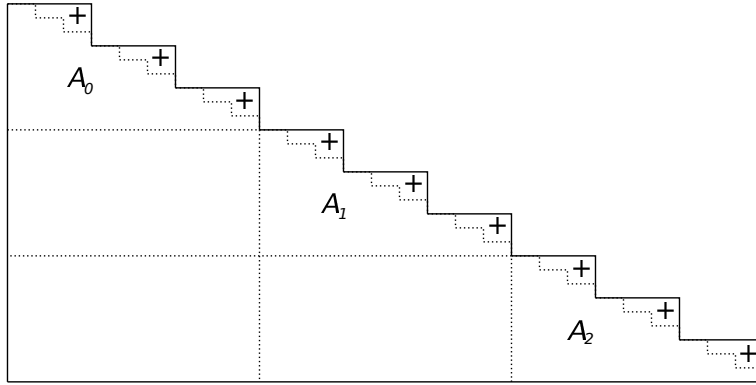


FIGURE 15. Here $p = 3, n = 3^2, p|h$ and $p|l$.

Case 4: $p \nmid h, p \nmid l$. We have arrived to the crux of the proof, where we have to pay the price that Λ is trivial in $\mathbb{Z}_{\frac{n}{p}}$ and maybe not in \mathbb{Z}_n . Consider the regions A_0, A_1, \dots, A_{p-1} defined as in case 1 (see Figure 16).

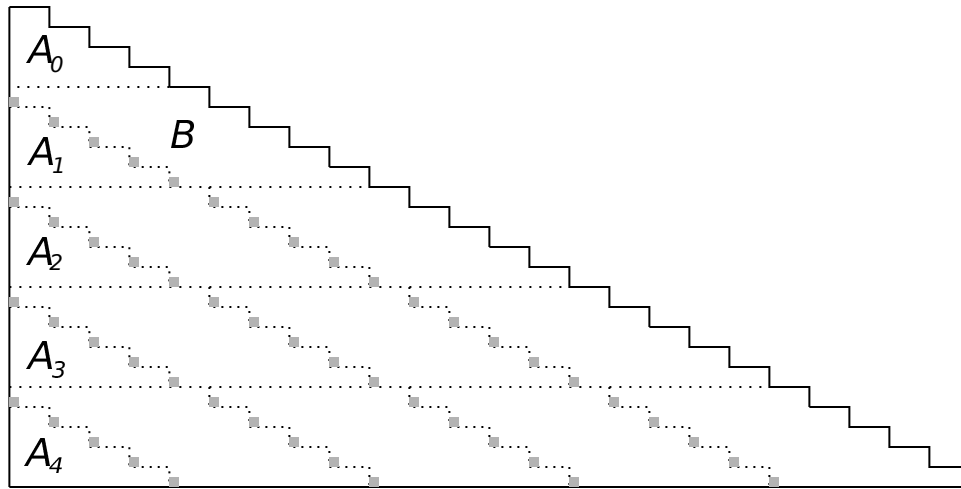


FIGURE 16. Here $p = 5, n = 5^2, p \nmid h$ and $p \nmid l$.

Let B be the region inside the stairs which is at the right of A_1 . The complement C of the region formed by the A_i 's is a union of $\frac{p(p-1)}{2}$ translates of B . Take the square Q in B which is a neighbor of A_1 and is in the left margin of the stairs. Let $v = (l, -h)$,

$v' = (0, -h)$. Consider the $\frac{(p-1)n}{2}$ translates of Q which appear with gray in Figure 16. They are

$$\text{(line 0) } Q, Q + v, Q + 2v, \dots, S + \left(\frac{(p-1)n}{p} - 1\right)v,$$

$$\text{(line 1) } Q + \frac{n}{p}v', Q + \frac{n}{p}v' + v, Q + \frac{n}{p}v' + 2v, \dots, Q + \frac{n}{p}v' + \left(\frac{(p-2)n}{p} - 1\right)v,$$

$$\text{(line 2) } Q + \frac{2n}{p}v', Q + \frac{2n}{p}v' + v, Q + \frac{2n}{p}v' + 2v, \dots, Q + \frac{2n}{p}v' + \left(\frac{(p-3)n}{p} - 1\right)v,$$

\dots

$$\text{(line } p-2) \text{ } Q + \frac{(p-2)n}{p}v', Q + \frac{(p-2)n}{p}v' + v, Q + \frac{(p-2)n}{p}v' + 2v, \dots, Q + \frac{(p-2)n}{p}v' + \left(\frac{n}{p} - 1\right)v.$$

If $b = \varphi(v) \in \mathbb{Z}_n$ is not invertible, then by Lemma 25 (ii), the sum of the invariants of the translates in each of this lines is trivial in \mathbb{Z}_n , so the invariant of the grey region is trivial in \mathbb{Z}_n .

If $b = \varphi(v)$ is invertible in \mathbb{Z}_n , then we claim that we can divide the $p-1$ lines above in groups, in such a way that for each group, the sequences obtained after applying φ can be concatenated to form an arithmetic progression in \mathbb{Z}_n of length divisible by n .

Let $a = \varphi(Q)$, $b' = \varphi(v')$. When we apply φ to line 0 above, we obtain an arithmetic progression of difference b which begins in a and finishes in $a + \left(\frac{(p-1)n}{p} - 1\right)b$. So, the next term of this sequence should be $a + \frac{(p-1)n}{p}b$.

After applying φ , line 1 begins with $a + \frac{n}{p}b'$ and finishes with $a + \frac{n}{p}b' + \left(\frac{(p-2)n}{p} - 1\right)b$, so the next term should be $a + \frac{n}{p}b' + \frac{(p-2)n}{p}b$.

We do this for each line. Line i begins with $a + i\frac{n}{p}b'$ and the next term after the last one should be $a + i\frac{n}{p}b' + \frac{(p-(i+1))n}{p}b$. In order to prove that these lines can be glued to form longer arithmetic progressions, we will prove the following Claim: the subsets $\{a + i\frac{n}{p}b'\}_{0 \leq i \leq p-2}$ and $\{a + i\frac{n}{p}b' + \frac{(p-(i+1))n}{p}b\}_{0 \leq i \leq p-2}$ of \mathbb{Z}_n are equal and each of them has $p-1$ elements.

This is equivalent to proving that the subsets $\{ib'\}_{0 \leq i \leq p-2}$ and $\{ib' - (i+1)b\}_{0 \leq i \leq p-2}$ of \mathbb{Z}_p are equal and that each has $p-1$ elements. That their cardinality is $p-1$ follows from the fact that b' and $b' - b$ are invertible in \mathbb{Z}_p and this, in turn, follows from the hypothesis that $\varphi(1, 0)$ and $\varphi(0, 1)$ are invertible and that $p \nmid hl$. In order to see that both sets are equal, it suffices to observe that the missing element $(p-1)b'$ of the first set is equal to the missing element $(p-1)b' - pb$ in the second.

The Claim says that we can glue the $p-1$ arithmetic progressions to form arithmetic cycles in \mathbb{Z}_n of difference b . By arithmetic cycle of difference b we mean an arithmetic progression $e, e + b, e + 2b, \dots, e + (t-1)b \in \mathbb{Z}_n$ such that $e + tb = e \in \mathbb{Z}_n$. Since $b \in \mathbb{Z}_n$ is invertible, the length t of one such cycle must be a multiple of n . Thus, the invariant Λ of the grey region is the sum of the colors of arithmetic progressions of lengths divisible by n . By Lemma 25 (i), the invariant of the grey region is trivial in \mathbb{Z}_n .

Now, region C is a union of translates of the grey region, so we have that $\Lambda(C) = 0 \in \mathbb{Z}_n$ and then the invariant of the stairs is also trivial in \mathbb{Z}_n . \square

In the hypotheses of Theorem 26, $\Lambda(z)$ need not be trivial in \mathbb{Z}_n . This fails even for $p = 2$. Let $n = 4$. Consider the 2-good coloring $c : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$ defined by $c(0) = c(2) = 0$, $c(1) = 1$, $c(3) = 3$. Let $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_n$ be defined by $\varphi(1, 0) = \varphi(0, 1) = 1$. Then $\Lambda_{\varphi, c}(x^4(x^{-1}y)^4y^{-4}) = 2 \neq 0 \in \mathbb{Z}_4$.

9. THE FREE NILPOTENT GROUP OF CLASS 2 AND EXPONENT n

In the previous sections we tried to understand some features of the free metabelian group $M(2, n)$ of rank 2 and exponent n . The free nilpotent group of class 2, rank 2 and exponent n is the group $N(2, n) = \frac{F_2}{\gamma_3(F_2)F_2^n}$, where $\gamma_3(F_2) = [F_2, F_2']$ denotes the third term in the lower central series of F_2 . Since $F_2'' \leq \gamma_3(F_2)$, $N(2, n)$ is a quotient of $M(2, n)$. This group is much easier to understand, even for arbitrary $n \geq 1$, using invariants. In fact the area invariant $A : F_2' \rightarrow \mathbb{Z}$, $A(z) = P_z(1, 1)$ describes $N(2, n)$ completely in the following sense.

We already know by Remark 1 that if $z \in \gamma_3(F_2)$, then P_z is a sum of multiples of polynomials of the form $m - 1$ where m is a monomial, so the area $A(z) = 0$. Conversely, each $P \in \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$ with $P(1, 1) = 0$ is a linear combination of $(X - 1)$ and $(Y - 1)$ so, $A(z) = 0$ implies that $z \in \gamma_3(F_2)$. If n is even, denote by $\bar{A} : F_2' \rightarrow \mathbb{Z}_{\frac{n}{2}}$ the composition of A with the projection. If n is odd, take $\bar{A} : F_2' \rightarrow \mathbb{Z}_n$, instead. Theorem 59 in [9] implies that $\gamma_3(F_2)F_2^n \cap F_2' \leq \ker(\bar{A})$. Conversely, if $z \in \ker(\bar{A})$, there exists $w \in F_2^n$ with the same area: indeed $A(x^n(x^{-1}y)^ny^{-n}) = \frac{n(n-1)}{2}$ and $A([x, y]^n) = n$, so a product of powers of those two elements has area $\gcd(\frac{n(n-1)}{2}, n)$, which is $\frac{n}{2}$ or n , depending on the parity of n . Thus $A(w^{-1}z) = 0$, which shows that $w^{-1}z \in \gamma_3(F_2)$, so $z \in \gamma_3(F_2)F_2^n \cap F_2'$. Since \bar{A} is surjective, we deduce that $N(2, n)'$ is isomorphic to $\mathbb{Z}_{\frac{n}{2}}$ for even n and to \mathbb{Z}_n for odd n . Since the abelianization of $N(2, n)$ has order n^2 , we conclude that $|N(2, n)| = \frac{n^3}{2}$ for even n and $|N(2, n)| = n^3$ for odd n .

So, this is a case where the idea of invariants works pretty well. This case is of course much simpler than the metabelian.

In [44], Sanov proved that for any prime p and $1 \leq i \leq k$, the Engel congruence $e_{ip^{i-1}}(x, y)^{p^{k-i}} \in \gamma_{ip^{i+1}}(F_2)F_2^{p^k}$ holds. When $p = 2$ and $i = 1$, this reduces to $[x, y]^{2^{k-1}} \in \gamma_3(F_2)F_2^{2^k}$. An alternative proof of this particular case has been given by Struik in [49, Theorem 1] in which she proves that $[x, y]^{\frac{n}{2}} \in \gamma_3(F_2)F_2^n$ for each even positive integer n . Note that this follows immediately from the exposition above: $A([x, y]^{\frac{n}{2}}) = \frac{n}{2}$, so $[x, y]^{\frac{n}{2}} \in \ker(\bar{A}) = \gamma_3(F_2)F_2^n \cap F_2'$. In fact, the area invariant gives a complete description of all the elements in $\gamma_3(F_2)F_2^n$.

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