

CONLEY INDEX FOR MULTIVALUED MAPS ON FINITE TOPOLOGICAL SPACES

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ABSTRACT. We develop Conley's theory for multivalued maps on finite topological spaces. More precisely, for discrete-time dynamical systems generated by the iteration of a multivalued map which satisfies appropriate regularity conditions, we establish the notions of isolated invariant sets and index pairs, and use them to introduce a well-defined Conley index. In addition, we verify some of its fundamental properties such as the Ważewski property and continuation.

1. INTRODUCTION

Topological methods have always been at the heart of the qualitative study of dynamical systems. For example, topological fixed point theorems can establish the existence of stationary states based purely on topological properties of the underlying system and the space it is acting on. But even more complicated dynamical behavior can be studied in this way, for example recurrent and chaotic dynamics. One of the central tools in this context was developed by Charles Conley in [8]. He realized that rather than focusing on the qualitative study of arbitrary invariant sets, it is advantageous to restrict one's attention to isolated invariant sets. Broadly speaking, such sets are more robust to continuous perturbations than general invariant sets. This insight allowed Conley to associate an index to isolated invariant sets S , which encodes some of their dynamical properties. The Conley index of S can be determined without explicit knowledge of the specific isolated invariant set through associated index pairs, which provide rough topological enclosures of S . In the case of classical continuous-time dynamical systems the Conley index can either be defined as a pointed topological space, or in a more computationally friendly version, as a homology module.

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While Conley's theory originally considered continuous-time dynamical systems, it has since been extended to the case of iterated maps, i.e., to discrete-time dynamical systems. As it turns out, its definition is more elaborate in this situation, since the use of the underlying flow for the construction of homotopies and other auxiliary techniques are no longer available. In the discrete-time setting, in its most general form, the Conley index is the shift equivalence class of the homotopy type of the so-called index map, which is defined on a topological space constructed via index pairs ([19], see also [36]). On the homology level, it is the Leray reduction of the homology of the index map [29]. For more details, we refer the reader to [26]. Moreover, Conley's theory has successfully been extended to the case of multivalued discrete-time dynamics, see for example [5, 21, 22, 35] and the references therein.

All of the results mentioned so far assume that the underlying phase space has nice topological properties, in particular, that it is at least a Hausdorff space. This is not a problem in classical applications, where the dynamical system is typically given by a vector field or a homeomorphism in the euclidean space. With the advent of modern data sciences, however, there is a growing number of dynamical systems known only from data gathered from experiments, observations or computations. This constitutes a challenge, because for such systems the phase space is known only via a sample, in the form of a point cloud. Contemporary topological data analysis (TDA, [7, 14]) provides methods to approximate the phase space by a simplicial complex or a more general cell complex. But, such a space is not convenient to reconstruct the dynamical system itself. An option is to embed the cell complex in a euclidean space and interpolate the data in order to obtain a formula for a vector field or a generator of a dynamical system. The problem is then reduced to a classical one, which, typically, is studied by numerical experiments. Unfortunately, this means a re-discretization with no direct connection to the original discrete data.

Another option is to eliminate the classical dynamical system and to construct a combinatorial analogue of a dynamical system directly on the cell complex or, more abstractly, a Lefschetz complex or a finite topological space. Dynamics on such spaces were studied by Forman in [17, 18] using the concept of combinatorial vector fields. While these papers primarily served to extend Morse theory to the case of cell complexes, they also addressed some more general dynamical concepts. It was shown in [23] that the notion of isolated invariant set does indeed have an analogue in the setting of combinatorial vector fields, and that one can define a Conley index. This was later extended to the case of combinatorial multivector fields on Lefschetz complexes in [32], and on general finite topological spaces in [24]. For related results on Conley theory for combinatorial vector fields we refer the reader to [4, 33, 34]. What is important, such a combinatorial approach to dynamics combines well with TDA methods [10, 12, 13, 11].

Common to all of these results is that the underlying notion of dynamics is created through a combinatorialized version of a vector field, i.e., through the generator of a dynamical system which is reminiscent of continuous-time dynamics. In terms of a topological approach to general sampled dynamics a study of the persistence of homological eigenvalues of a self map presented in [15, 6] may be considered a first step towards the persistence of the Conley index, but in the very special setting of the whole space as an isolated invariant set.

In the present paper, we aim to demonstrate that Conley's theory can be extended to the case of general dynamical systems on finite topological spaces. As we will see in more detail in Section 3, actual dynamical systems on such combinatorial objects necessarily have to be multivalued and time-discrete. Thus, we consider the iteration of multivalued maps on finite topological spaces and define the notions of isolated invariant sets and their Conley index. We prove that the index is well-defined, and establish some of its basic properties. While our approach is modeled after previous results [29, 5], the involved proof techniques are significantly different. This is due to the lack of sufficient separation in finite topological spaces, and will be addressed in more detail later.

The remainder of this paper is organized as follows. In Section 2 we recall basic definitions concerning finite topological spaces and continuity properties of multivalued maps. This is followed in Section 3 by a brief discussion of combinatorial topological dynamics, which specifically demonstrates that on finite topological spaces interesting dynamics can only be observed in the context of iterating a multivalued map. In addition, we introduce the central notion of solution in this context. We then turn our attention to Conley theory. Section 4 is devoted to isolated invariant sets and Morse decompositions, while Section 5 is concerned with index pairs and their properties. Using these results, we can define the Conley index in Section 6, and derive some of its fundamental properties in Section 7. Finally, Section 8 addresses some future work and open problems.

2. PRELIMINARIES

We begin by recalling basic concepts and definitions for finite topological spaces, as well as for multivalued maps between them. While we focus only on the essentials, additional material can be found in [1, 2, 3].

Given a finite topological space X and a subspace A , we denote by $\text{opn } A$ the open hull of A , that is, the smallest open set containing A . When A consists of a unique point a we also write $\text{opn } A = \text{opn } a$. Note that $\text{opn } A = \bigcup_{a \in A} \text{opn } a$. The closure of A is denoted by $\text{cl } A$. Notice that for arbitrary elements $x, y \in X$ the inclusion $x \in \text{opn } y$ is satisfied if and only if $y \in \text{cl } x$. Every finite space has an associated preorder \leq (i.e., a reflexive and transitive relation) given by $x \leq y$

if $x \in \text{cl } y$.¹ Conversely every finite set with a preorder \leq has a corresponding topology with the up-sets as the open sets. Recall that a subset $A \subseteq X$ is an up-set, if $a \leq x$ for some $a \in A$ implies $x \in A$. Then, the dually defined down-sets correspond to the closed sets in this topology. A finite space X is T_0 if and only if the preorder is an order (i.e., antisymmetric). A map $f : X \rightarrow Y$ between finite spaces is continuous if and only if it is order preserving, that is, if the inequality $x \leq x'$ always implies $f(x) \leq f(x')$. Although this correspondence is very useful to understand finite spaces from a combinatorial perspective, we have chosen to use the topological notation $\text{cl } A$ instead of $X_{\leq A} = \{x \in X \mid \exists a \in A \text{ with } x \leq a\}$ and $\text{opn } A$ instead of $X_{\geq A} = \{x \in X \mid \exists a \in A \text{ with } a \leq x\}$ in order to make more evident the connection between this theory and the classical one.

We say that a multivalued map $F : X \multimap Y$ between two topological spaces has *closed values*, if $F(x) \subseteq Y$ is closed for every $x \in X$. Furthermore, the map F is called *lower semicontinuous* if the small preimage $F^{-1}(H) = \{x \in X \mid F(x) \subseteq H\}$ is closed for every closed subset $H \subseteq Y$. For a multivalued map $F : X \multimap Y$ with closed values between finite spaces, one can easily verify that being lower semicontinuous is equivalent to the condition that $x' \leq x$ implies $F(x') \subseteq F(x)$, or, in other words, $x' \in \text{cl } x$ implies $F(x') \subseteq F(x)$, see also [3, Lemma 3.5]. Finally, we say that F has *acyclic values*, if for every $x \in X$ the subspace $F(x) \subseteq Y$ is acyclic.

For the majority of the paper, we consider multivalued maps $F : X \multimap Y$ which are lower semicontinuous and have closed values. If we assume in addition that the map has acyclic values, then the projection $p_1 : F \rightarrow X$ from the graph

$$F = \{(x, y) \in X \times Y \mid y \in F(x)\} \subseteq X \times Y$$

into X induces isomorphisms in all homology groups. This in turn implies that for such multivalued maps there is an induced homomorphism $F_* : H_*(X) \rightarrow H_*(Y)$ given by $F_* = (p_2)_*(p_1)_*^{-1}$ where $p_2 : F \rightarrow Y$ stands for the other projection (see [3, Proposition 4.7]).

3. COMBINATORIAL TOPOLOGICAL DYNAMICS

In this brief section we introduce the notion of a combinatorial dynamical system on a finite topological space, as well as the assumed topological properties of its multivalued generator $F : X \multimap X$. Moreover, we indicate why in the setting of a finite topological space only discrete-time dynamics is of interest. We would like to point out, however, that through the notion of combinatorial vector fields on finite topological spaces, one can in fact arrive at a notion of dynamics which

¹Note that this convention is the one used in [1], and it is the most appropriate one for the setting of dynamics. We would like to point out, however, that alternatively the preorder could be defined by letting $x \leq y$ if $x \in \text{opn } y$. This definition is also extensively used in the literature, see for example the discussion in [3].

is similar in spirit to the continuous-time case, albeit not the same. Finally, we introduce the notion of solution, which is central to this paper.

3.1. Multivalued dynamics on finite topological spaces. Classical dynamical systems can broadly be divided into two categories — discrete-time and continuous-time dynamical systems. In the former case, one is interested in the evolution of a system state at discrete points in time, and this is usually modeled by the iteration of a continuous map $F : X \rightarrow X$. Unfortunately, in the context of a finite topological space this leads to trivial dynamical behavior, with every orbit of the system eventually becoming periodic. Thus, in order to capture interesting dynamics, one is forced to consider multivalued maps $F : X \multimap X$. While this has already been described in [4, 23, 24, 32, 33, 34], these papers consider very specific multivalued maps generated by an underlying combinatorial vector field or combinatorial multivector field — and this approach is more in the spirit of the continuous-time case. See also our comments below.

In contrast, the present paper is devoted to the study of general multivalued discrete-time dynamical systems on a finite topological space X . Since such general systems cannot rely on any supporting underlying structure such as a combinatorial multivector field, we need to impose certain regularity assumptions on the map F . Throughout this paper, we assume that $F : X \multimap X$ is a lower semicontinuous multivalued map with closed values. These assumptions are inspired by the case of classical multivalued dynamics [9, 20], and they have also been used recently in the proof of a Lefschetz fixed point theorem for multivalued maps on finite spaces [3]. We think of the map F as a *combinatorial dynamical system*, which is obtained by iterations of the map, and which naturally leads to the concept of a *solution* — as described in more detail in the following section. For now we would like to point out that a combinatorial dynamical system may also be viewed as a finite directed graph whose set of vertices is the topological space X , and with F interpreted as the map sending a vertex to the collection of its neighbors connected via an outgoing directed edge. This so-called *F -digraph* encodes the dynamics of F on a purely combinatorial level. However, for the derivation of more advanced concepts such as isolated invariant sets and their Conley index the topological properties of X and F are essential.

In view of our focus on the discrete-time case, it is natural to wonder why we exclude the continuous-time case. As the following result shows, the semigroup property of a multivalued continuous-time dynamical system immediately forces the dynamics to be trivial. In fact, every orbit of the system has to be constant.

Theorem 3.1 (Triviality of continuous-time dynamics). *Let X be a finite set and let $F : X \times \mathbb{R}_{\geq 0} \multimap X$ denote a multivalued map which satisfies the semigroup property $F(x, t + s) = F(F(x, t), s)$ for every $t, s \geq 0$. Then $F(x, -) : \mathbb{R}_{> 0} \multimap X$, given by $t \mapsto F(x, t)$, is constant for every $x \in X$.*

Proof: The map F induces a singlevalued map $F : \mathcal{P}(X) \times \mathbb{R}_{\geq 0} \rightarrow \mathcal{P}(X)$ given by $(A, t) \mapsto F(A, t)$. Here $\mathcal{P}(X)$ denotes the power set of X . Note that the identity $F(A, t + s) = F(F(A, t), s)$ holds for every $t, s \geq 0$. Since $\mathcal{P}(X)$ is finite, it suffices to prove the following assertion:

- If Y is a finite set and $F : Y \times \mathbb{R}_{> 0} \rightarrow Y$ is a singlevalued map satisfying the semigroup property $F(y, t + s) = F(F(y, t), s)$ for every $t, s \geq 0$, then the map $F(y, -)$ is constant on $\mathbb{R}_{> 0}$ for each $y \in Y$.

To show this, note that if $f : Y \rightarrow Y$ is any map, then the sequence $(f^n(Y))_{n \in \mathbb{N}}$ is decreasing. We call $f^\infty(Y) \subseteq Y$ its eventual value. It is clear that $f^\infty(Y) = f^n(Y)$ for every n greater than or equal to the cardinality N of Y . The map f induces a bijection from the eventual value $f^\infty(Y)$ to itself, and since the group of bijections has order dividing $N!$, the map $f^{N!} : f^\infty(Y) \rightarrow f^\infty(Y)$ is the identity. This in turn shows that $f^{N!} : Y \rightarrow f^\infty(Y)$ is a retraction, i.e., it is the identity when restricted to its codomain.

Every $t \geq 0$ induces a map $F_t : Y \rightarrow Y, y \mapsto F(y, t)$. Denote $R_t = F_t^\infty(Y) \subseteq Y$. By the comments above, the iterate $F_t^{N!} : Y \rightarrow R_t$ is a retraction. Furthermore, the set R_t is the set of fixed points of $F_t^{N!}$. Now let $n \in \mathbb{N}$ and $t \geq 0$. Then we have $F_{nt} = F_t^n$ in view of our hypothesis on F . Thus $F_{nt}^{N!} = F_t^{nN!}$ fixes every point of R_t and does not fix any point outside R_t . This proves that $R_{nt} = R_t$. We deduce that for $t > 0$, the set R_t depends only on the class of t modulo $\mathbb{Q}_{> 0}$, i.e., we have $R_t = R_s$ if $t^{-1}s \in \mathbb{Q}$. In particular, this implies $R_{t/N!} = R_t$, and thus R_t is the image of F_t , and F_t is the identity on R_t .

Finally, let $t, s > 0$. Since $F_{t+s} = F_s F_t$, the image of F_{t+s} is contained in the image of F_s , i.e., $R_{t+s} \subseteq R_s$. Thus $R_t \subseteq R_s$ for every $t \geq s$. Since $R_s = R_{s/n}$ for every $n \in \mathbb{N}$, one also obtains $R_t \subseteq R_s$ for every $t > 0$. This in turn establishes the identity $R_t = R_s$ for every $t, s > 0$. Suppose $s > t > 0$. Then F_{s-t} is the identity on $R_{s-t} = R_s = R_t$. Since $F_s = F_{s-t} F_t$ and F_{s-t} is the identity on the image of F_t , then $F_s = F_t$. This proves the assertion. \square

The above result shows that it is the semigroup property alone which is incompatible with nonconstant dynamics if the underlying phase space is finite. As the reader undoubtedly noticed, we did not make use of any topological structure on X . Note, however, that one can mimic the behavior of a continuous-time dynamical system even on finite topological spaces by restricting dynamical transitions between subsets to shared boundaries. This is precisely what Forman had in mind with his combinatorial vector fields, and also lies at the center of the theory of multivector fields. In contrast, the discrete-time dynamics studied in the present paper does not have these restrictions, as it allows for transitions between states without topological closeness.

3.2. Solutions and invariant sets. Our study of the dynamics of discrete-time multivalued dynamical systems is based on the notions of solution and invariant set. These are defined just as in the classical situation.

Consider a multivalued map $F : X \multimap X$. Then a *solution* of F in $A \subseteq X$ is a partial map $\sigma : \mathbb{Z} \multimap A$ whose *domain*, denoted $\text{dom } \sigma$, is an interval of integers, and for any $i, i + 1 \in \text{dom } \sigma$ the inclusion $\sigma(i + 1) \in F(\sigma(i))$ is satisfied. The solution σ is called a *full solution* if $\text{dom } \sigma = \mathbb{Z}$, otherwise it is a *partial solution*. A partial solution whose domain is bounded is referred to as a *path*. We denote the set of all paths with values in $A \subseteq X$ by $\text{Path}(A)$. Given a path σ with domain $\text{dom } \sigma = \mathbb{Z} \cap [m, n]$ for some $m, n \in \mathbb{Z}$, we call $\sigma(m)$ and $\sigma(n)$, respectively, the left and right *endpoint* of σ . We denote these endpoints by the symbols σ^\square and σ^\triangleright , respectively.

If τ is another path with $\text{dom } \tau = \mathbb{Z} \cap [m', n']$ and such that $\tau^\square \in F(\sigma^\triangleright)$ holds, then we define the concatenation of the paths σ and τ , denoted by $\sigma.\tau$, as the path with domain $\text{dom } \sigma.\tau := \mathbb{Z} \cap [m, n + n' - m' + 1]$ and defined by

$$(\sigma.\tau)(k) := \begin{cases} \sigma(k) & \text{if } k \in \mathbb{Z} \cap [m, n], \\ \tau(k + m' - n - 1) & \text{if } k \in \mathbb{Z} \cap [n + 1, n + 1 + n' - m']. \end{cases}$$

It is straightforward to verify that $\sigma.\tau$ is indeed a path.

We now recall the definition of invariance. For this, we say that a solution σ *passes* through $x \in X$ if $x = \sigma(i)$ for some $i \in \text{dom } \sigma$. Moreover, a set $A \subseteq X$ is called *invariant* if for every $x \in A$ there exists a full solution in A which passes through x . Thus, A is invariant if $A \subseteq F(A)$ and for each $a \in A$, $F(a) \cap A \neq \emptyset$.

4. ISOLATED INVARIANT SETS AND MORSE DECOMPOSITIONS

The concept of isolated invariant set lies at the heart of Conley theory. In the classical situation, an isolated invariant set S is characterized by the property that it is the largest invariant set in some neighborhood of S . Unfortunately, it is not possible to define isolated invariant sets in an analogous way in the context of finite topological spaces due to the lack of sufficient separation. Therefore, in this section we introduce an appropriate notion for our setting and derive some first properties of such isolated invariant sets. We also show how they form the building blocks for Morse decompositions of phase space. Throughout this section, we assume that X is a finite T_0 topological space and that the multivalued map $F : X \multimap X$ is lower semicontinuous with closed values.

4.1. Isolated invariant sets. We begin by introducing the notion of isolating invariant set, which in turn is based on an isolating set. The latter set is the analogue of the isolating neighborhood in classical Conley theory, but its topological properties are weaker to account for the poor separation in finite spaces.

Definition 4.1 (Isolated invariant set, isolating set). *A closed set $N \subseteq X$ is called an isolating set for an invariant set S if the following two conditions are satisfied:*

- (IS1) *Every path in N with endpoints in S has all its values in S .*
- (IS2) *We have the equality $S \cap \text{cl}(F(S) \setminus N) = \emptyset$, i.e., the set S and $\text{cl}(F(S) \setminus N)$ are disjoint.*

If such an isolating set for S exists, we say that S is an isolated invariant set.

Notice that condition (IS2) is satisfied if and only if $\text{opn } S \cap (F(S) \setminus N) = \emptyset$. Thus, it is equivalent to assuming the inclusion

$$(IS2') \quad \text{opn } S \cap F(S) \subseteq N.$$

Note also that since S is invariant, $S \subseteq F(S)$, and hence (IS2') implies $S \subseteq N$.

Establishing condition (IS2), or its equivalent reformulation (IS2'), is the less intuitive aspect of verifying an invariant set as an isolated invariant set. It is therefore useful to also have sufficient conditions for its validity. Two of these are the subject of the following remark.

Remark 4.2 (Sufficient conditions for (IS2)). *Assume that S is an invariant set and that N is closed. Then any of the following two conditions imply (IS2):*

- (i) *We have $S \subseteq \text{int } N$, where $\text{int } N$ denotes the interior of N , or*
- (ii) *the inclusion $F(S) \subseteq N$ is satisfied.*

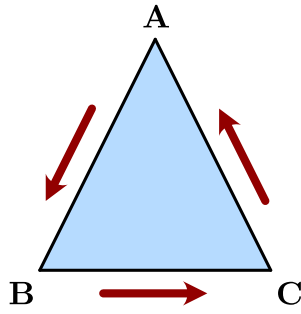
Indeed, the first condition is equivalent to $\text{opn } S \subseteq N$, and therefore either of the above two conditions implies (IS2'), and thus (IS2).

As we mentioned earlier, in classical Conley theory, the isolated invariant set is uniquely determined by its isolating neighborhood N . In fact, it is the largest invariant subset of N . In contrast, in the above setting the same set N may be an isolating set for more than one isolated invariant set. This is illustrated in the following two examples.

Example 4.3 (A rotational multivalued map). We begin with a simple example that rotates an equilateral triangle. In the left part of Figure 1 we indicate the action of the map on a simplicial complex, which is just a two-dimensional simplex. More precisely, the map rotates the triangle in a counterclockwise fashion by 120° . This example is inspired by a combinatorial vector field in the sense of Forman, which contains the three vectors $\{A, AB\}$, $\{B, BC\}$, and $\{C, AC\}$ along the boundary, as well as the critical cell $\{ABC\}$. While we refer the reader to [17, 18, 23, 34] for more details on the general definition of a combinatorial vector field and its relation to classical dynamics, it is intuitively clear that in the situation of Figure 1 one can observe both an unstable fixed point at the triangle, as well as periodic motion along its simplicial boundary.

In order to formulate this dynamical behavior via a multivalued map on a finite topological space, we use the standard construction given by the face poset, that is the poset X of simplices where $x \leq y$ if x is a face of y . In other words, the topology is given by $x \in \text{cl } y$ if and only if x is a face of y . The associated multivalued map $F : X \multimap X$ is defined in the table in Figure 1. One can easily verify that F has closed values, and that it is lower semicontinuous. Iteration of the map F leads for example to the following three isolated invariant sets:

$$S_1 = \{A, B, C\}, \quad S_2 = \{AB, BC, AC\}, \quad \text{and} \quad S_3 = \{ABC\}.$$



x	Elements of $F(x)$
A	B
B	C
C	A
AB	B, C, BC
BC	A, C, AC
AC	A, B, AB
ABC	A, B, C, AB, BC, AC, ABC

FIGURE 1. A simple rotation on a simplicial complex given by one triangle, as well as three edges and three vertices. The table on the right defines the associated multivalued map $F : X \multimap X$ on the finite topological space consisting of all seven simplices, and equipped with the closure operation induced by the face relationship.

If we then define the closed sets

$$N_1 = \{A, B, C\}, \quad N_2 = \{A, B, C, AB, BC, AC\}, \quad \text{and} \quad N_3 = X,$$

then one can easily verify that N_3 is an isolating set for all three of the above isolated invariant sets, the set N_2 isolates both S_1 and S_2 , and the set N_1 is an isolating set for S_1 only. Finally, we note that also the unions $S_1 \cup S_2$ and $S_2 \cup S_3$ are isolated invariant sets, with isolating sets N_2 and N_3 , respectively.

On the other hand, while the union $S_1 \cup S_3$ is invariant, it is not an isolated invariant set. To see this, note that any isolating set for $S_1 \cup S_3$ has to contain the closure of $S_1 \cup S_3$, and therefore $N = X$ would be the only possibility. Yet, one can easily see that (IS1) is not satisfied for this choice.

Example 4.4 (A reflection-based multivalued map). Our second example is similar to the previous one but it is induced by the reflection of the triangle about the vertical line through A , as depicted in the left panel of Figure 2. The corresponding multivalued map $G : X \multimap X$ is defined in the table on the right. Notice that also G has closed values and is lower semicontinuous.

Iteration of the map G leads to new isolated invariant sets. For example, both the singleton $R_1 = \{A\}$ and the doubleton $R_2 = \{B, C\}$ are examples, and they have associated isolating sets $M_1 = R_1$ and $M_2 = R_2$, respectively. Notice, however, that both sets are also isolated by $M = X$. In addition, we have the isolated invariant sets

$$R_3 = \{BC\}, \quad R_4 = \{AB, AC\}, \quad \text{and} \quad R_5 = \{ABC\}.$$

If we then define the closed sets

$$M_3 = \{B, C, BC\}, \quad M_4 = \{A, B, C, AB, AC\}, \quad \text{and} \quad M_5 = X,$$

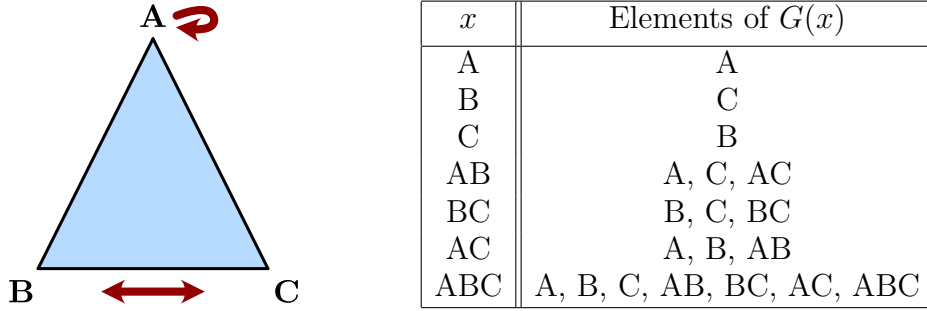


FIGURE 2. The map $G : X \multimap X$ defined on the right induced by a reflection about the vertical line through A indicated on the left.

then one can easily verify that M_k is an isolating set for R_k for $k = 3, 4, 5$. Furthermore, the set M_5 isolates both R_3 and R_4 as well. We leave it to the reader to find additional isolated invariant sets.

The examples above will be analyzed along the paper. We have chosen finite spaces associated with simplicial complexes because the geometric interpretation they have make notions simpler to visualize. However, we want to stress that the theory we develop here can be applied to any finite T_0 space.

While at first glance the nonuniqueness of the isolating set seems strange, it is necessary in finite topological space due to the lack of sufficient separation. Nevertheless, the following remark sheds more light on this issue.

Remark 4.5 (The smallest isolating set). *It is clear that there is a smallest closed set N satisfying condition (IS2'), which is the set*

$$(1) \quad N = \text{cl}(\text{opn } S \cap F(S)) .$$

On the other hand, one can easily see that condition (IS1) is preserved by taking subsets: If $N' \subseteq N$ and N satisfies this condition, then so does N' . In conclusion, the invariant set S is an isolated invariant set if and only if the set N defined in (1) satisfies condition (IS1). We will see, however, that it is frequently useful to work with different isolating sets for the same isolated invariant set.

We leave it to the reader to illustrate the above remark in the context of Examples 4.3 and 4.4, and close this section with the following simple result.

Proposition 4.6. *Assume M and N are two isolating sets for an isolated invariant set S . Then their intersection $M \cap N$ is also an isolating set for S .*

Proof: Clearly the intersection $M \cap N$ is closed. Since every path in $M \cap N$ is also a path in M , property (IS1) for $M \cap N$ follows from the validity of property (IS1) for M . Finally, it is clear that (IS2') for M and N implies that (IS2') also holds for $M \cap N$. \square

4.2. Morse decompositions. Isolated invariant sets as defined in the last section are the fundamental building blocks for analyzing the global dynamics of a dynamical system. In general, they can be used to divide phase space into regions of recurrent and gradient-like behavior. This leads to the notion of a Morse decomposition.

Definition 4.7 (Morse decomposition). *Consider a lower semicontinuous multivalued map $F : X \multimap X$ with closed values, on a finite T_0 topological space X . A family $\{M_p\}_{p \in P}$ of mutually disjoint, non-empty, isolated invariant sets indexed by a poset P is called a Morse decomposition of X if for every full solution γ either all values of γ are contained in the same set M_p , or there exist indices $q > r$ in P and $t_q, t_r \in \mathbb{Z}$ such that $\gamma(t) \in M_q$ for $t \leq t_q$ and $\gamma(t) \in M_r$ for $t \geq t_r$. In the latter case, the solution γ is called a connection from M_q to M_r . Furthermore, the sets M_p are called the Morse sets of the Morse decomposition.*

In the context of classical dynamics, Morse decompositions are a fairly difficult object of study, since it is possible for a dynamical system to have infinitely many different Morse decompositions. Of course, this cannot happen in the setting of a finite topological space. In fact, there is always a finest Morse decomposition which can easily be determined using graph theoretic methods.

To see this, recall that the dynamics of a multivalued map $F : X \multimap X$ can be encoded via its F -digraph G_F , whose vertices are given by the elements of X , and such that there is a directed edge from x to y if and only if $y \in F(x)$. On X , we can define an equivalence relation by saying that $x \sim y$ if and only if there is both a directed path in G_F from x to y , and one from y to x .² This equivalence relation partitions X into equivalence classes which are called the *strongly connected components* of G_F . Such a component is called *trivial*, if it consists of a single vertex which is not connected to itself with an edge, otherwise it is *non-trivial*. Moreover, if each strongly connected component (along with all the edges which begin and finish in the component) is contracted to a single vertex, the resulting graph is a directed acyclic graph, called the *condensation* of G_F . After these preparations, one obtains the following result.

Proposition 4.8 (Morse decomposition via strongly connected components). *Consider a lower semicontinuous multivalued map $F : X \multimap X$ with closed values on a finite T_0 topological space X . Denote the non-trivial strongly connected components of the associated F -digraph G_F by $\{M_p\}_{p \in P}$. Furthermore, let $q > r$ if there exists a directed path in G_F from M_q to M_r . Then each of the sets M_p is an isolated invariant set for F , and $\{M_p\}_{p \in P}$ is a Morse decomposition of X .*

Proof: We begin by showing that any path which starts and ends in M_p has to be completely contained in M_p .

²Notice that we have $x \sim x$ for every $x \in X$, since there always exists a path of length zero from x to x , i.e., a path without edges.

To see this, let $p \in P$ be fixed, let $x, y \in M_p$, let γ denote any path from x to y , and let z denote any point on the path γ . Then γ clearly can be restricted to a path from z to y . Furthermore, since M_p is a non-trivial strongly connected component of G_F , there exists a path from y to x . Concatenation of this path with the part of γ from x to z gives a path in G_F from y to z . This immediately implies that $y \sim z$, and therefore we have $z \in M_p$, and the above statement follows.

We now turn to the verification of the proposition. It is easy to see that $>$ is indeed a (strict) partial order on P , since the condensation of G_F is acyclic. Moreover, for any $x \in M_p$ one can easily construct a full solution through x in M_p , by infinite concatenations of the paths from x to y and from y to x , for some $y \in M_p$, again using the above observation. Thus, every set M_p is invariant. These sets are also isolated invariant sets, since the whole space X is an isolating set for each M_p . For this, note that (IS2) follows trivially from Remark 4.2, and (IS1) from our above observation. Finally, if γ denotes an arbitrary full solution, then the acyclicity of the condensation of G_F together with the finiteness of X immediately implies the existence of $t_q, t_r \in \mathbb{Z}$ such that $\gamma(t) \in M_q$ for $t \leq t_q$ and $\gamma(t) \in M_r$ for $t \geq t_r$, for some $q \geq r$. If $q = r$, then our above observation implies that γ is contained in M_q , and this completes the proof of the proposition. \square

Finding strongly connected components in digraphs can be done efficiently, and thus the problem of decomposing the dynamics of a multi-valued map F into *recurrent dynamics*, given by the Morse sets M_p , and *gradient-like dynamics*, encoded in the condensation of G_F , is inherently computable. Furthermore, one can easily see that the above result does in fact produce the finest Morse decomposition of X . It is customary to represent the information about this Morse decomposition in the form of its *Morse graph*. This graph consists of the Hasse diagram of the poset P with vertices representing the individual Morse sets M_p . In other words, it is the subgraph of the condensation induced by the non-trivial strongly connected components.

Example 4.9 (Morse decompositions for Examples 4.3 and 4.4). We return to the two examples introduced earlier in this section. Recall that these examples introduced two multivalued maps $F, G : X \multimap X$ on the finite topological space

$$X = \{A, B, C, AB, AC, BC, ABC\}$$

induced by a two-dimensional simplex. In Example 4.3 we identified the three isolated invariant sets

$$S_1 = \{A, B, C\}, \quad S_2 = \{AB, BC, AC\}, \quad \text{and} \quad S_3 = \{ABC\},$$

and one can easily see that they are all strongly connected components of the F -digraph. Similarly, in Example 4.4 we found the isolated invariant sets

$$R_1 = \{A\}, \quad R_2 = \{B, C\}, \quad R_3 = \{BC\}, \quad R_4 = \{AB, AC\}, \quad R_5 = \{ABC\},$$

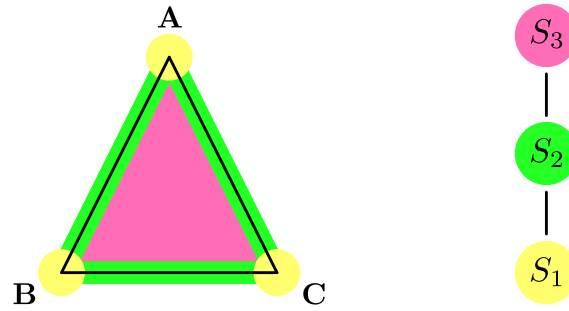


FIGURE 3. Finest Morse decomposition for the map $F : X \multimap X$ from Example 4.3. The Morse graph is shown on the right, the Morse sets are indicated on the left.

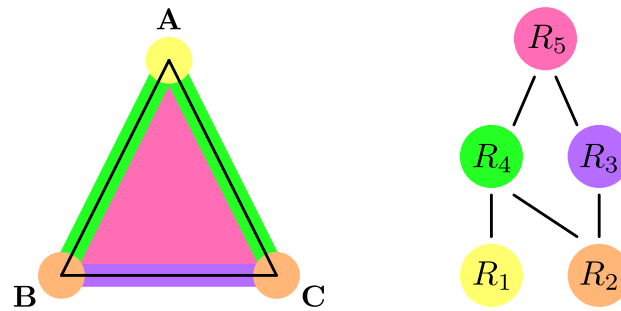


FIGURE 4. Finest Morse decomposition for the map $G : X \multimap X$ from Example 4.4. The Morse graph is shown on the right, the Morse sets are indicated on the left.

which also partition the space X and are again strongly connected components. Thus, in both of these examples all strongly connected components are non-trivial, and one obtains the Morse graphs shown in Figures 3 and 4, respectively.

We would like to point out that the Morse sets involved in these example exhibit different types of recurrent dynamics. While the sets S_3 , R_1 , R_3 , and R_5 are all equilibria of the dynamics, the remaining Morse sets are periodic orbits. More precisely, the sets S_1 and S_2 are periodic orbits of period 3, while the two sets R_2 and R_4 have period 2.

5. INDEX PAIRS

While isolated invariant sets S are the fundamental objects of study in Conley theory, it is their Conley index that provides algebraic information about the dynamics inside of S . In classical dynamics, this index information can be computed easily from certain isolating neighborhoods called isolating blocks, but these are often difficult to find. For this reason, one usually uses a different object for the index computation, called an *index pair*. In the present section, we transfer this

concept to the setting of multivalued maps. Throughout, we again assume that X is a finite T_0 topological space and that the multivalued map $F : X \multimap X$ is lower semicontinuous with closed values.

5.1. Definition and existence of index pairs. For the definition of an index pair, we need to recall two important concepts. On the one hand, if $A \subseteq X$ is any subset, then the *invariant part* $\text{Inv}(A)$ of F in A is the set of all points $x \in A$ for which there exists a full solution in A which passes through x . On the other hand, a *topological pair* in X is a pair P of subsets $P = (P_1, P_2)$ which satisfy the inclusion $P_2 \subseteq P_1$. With this, we have the following central definition.

Definition 5.1 (Index pair). *We say that a topological pair $P = (P_1, P_2)$ of closed subsets of an isolating set N for an isolated invariant set S is an index pair for S in N if the following three conditions are satisfied:*

- (IP1) $F(P_i) \cap N \subseteq P_i$ for $i = 1, 2$,
- (IP2) $P_1 \cap \text{cl}(F(P_1) \setminus N) \subseteq P_2$,
- (IP3) $S = \text{Inv}(P_1 \setminus P_2)$.

In addition, we say that index pair $P = (P_1, P_2)$ is saturated if $S = P_1 \setminus P_2$.

We would like to point out that condition (IP1) implies $F(P_i) \cap N = F(P_i) \cap P_i$, and therefore (IP2) could also be replaced by the inclusion $P_1 \cap \text{cl}(F(P_1) \setminus P_1) \subseteq P_2$ to obtain an equivalent definition.

In the remainder of this section, we establish some basic properties of index pairs. In addition, we show that every isolated invariant set S with isolating set N does indeed have an associated index pair. For this we need another definition. For subsets $S \subseteq N \subseteq X$ we define

$$\begin{aligned} \text{Inv}^-(N, S) &:= \{ y \in N \mid \exists \sigma \in \text{Path}(N) \text{ with } \sigma^\sqsubset \in S, \sigma^\sqsupset = y \}, \\ \text{Inv}^+(N, S) &:= \{ y \in N \mid \exists \sigma \in \text{Path}(N) \text{ with } \sigma^\sqsubset = y, \sigma^\sqsupset \in S \}. \end{aligned}$$

In other words, the set $\text{Inv}^+(N, S)$ consists of all points in N from which one can reach S in forward time with a path in N , and $\text{Inv}^-(N, S)$ is the analogous set in backwards time. The following proposition follows immediately from the definition of $\text{Inv}^\pm(N, S)$.

Proposition 5.2 (Inclusion properties). *Assume that $M \subseteq N$ are two isolating sets for an isolated invariant set S . Then $\text{Inv}^\pm(M, S) \subseteq \text{Inv}^\pm(N, S)$. \square*

In addition, the above two sets have interesting topological properties, and they can be used to reconstruct an isolated invariant set S , as the next result shows.

Proposition 5.3 (Topological properties). *Assume that $S \subseteq N \subseteq X$, that S is an invariant set, and that N is closed. Then the set $\text{Inv}^-(N, S)$ is closed and $\text{Inv}^+(N, S)$ is open in N . If in addition N isolates S , then one also has*

$$(2) \quad \text{Inv}^-(N, S) \cap \text{Inv}^+(N, S) = S,$$

and the isolated invariant set S is locally closed in X , that is S is a difference of two closed sets in X (see [16, Problem 2.7.1]).

Proof: Denote $N^- := \text{Inv}^-(N, S)$ and $N^+ := \text{Inv}^+(N, S)$. In order to prove that N^- is closed take a $y \in \text{cl} N^-$. Then $y \in \text{cl} y'$ for some $y' \in N^-$. Hence, we may take a path $\sigma \in \text{Path}(N)$ from a point in S to y' . Since S is invariant, without loss of generality we may assume that $|\sigma| \geq 2$. Since F has closed values, replacing y' by y in σ one obtains a new path, so $y \in N^-$. This proves that N^- is indeed closed.

To see that the set N^+ is open in N , choose any $x \in \text{opn}_N N^+ = N \cap \text{opn} N^+$. Then $x \in \text{opn} x'$ for some $x' \in N^+$. Let $\sigma \in \text{Path}(N)$ be a path from x' to some point in S . Since F is lower semicontinuous with closed values, $F(x') \subseteq F(x)$. Thus, replacing x' by x in σ gives another path, so $x \in N^+$. Therefore $N \cap \text{opn} N^+ \subseteq N^+$, so N^+ is open in N .

Finally, the inclusion $S \subseteq N^- \cap N^+$ is obvious. Suppose now that N isolates S . To see the opposite inclusion, let $x \in N^- \cap N^+$ be arbitrary. Then there exist a path in N from a point in S to x and a path in N from x to a point in S . Concatenation of these gives a path in N through x , and with endpoints in S . Hence, since N isolates S , we obtain $x \in S$ and (2) holds. Moreover, the representation (2) shows that S can be written as

$$S = N^- \setminus (N \setminus N^+).$$

Since N^- and $N \setminus N^+$ are closed, S is locally closed in X . □

The above result shows that also in the multivalued map case, isolated invariant sets necessarily have to be locally closed. This is reminiscent of the situation in the multivector case [24], and it provides a sufficient condition for recognizing invariant sets which are not isolated invariant. In fact, this criterion does not make any reference to an associated isolating set N . For example, one can easily see that the set $S_1 \cup S_3$ in Example 4.3 is not locally closed, and therefore it cannot be an isolated invariant set.

We now turn our attention to the existence of index pairs for isolated invariant sets. For this, we need the following definition, as well as the subsequent result.

Definition 5.4 (Standard index pair). *Given an isolating set N for an isolated invariant set S , we define the standard index pair $P^N = (P_1^N, P_2^N)$ by*

$$P_1^N := \text{Inv}^-(N, S) \quad \text{and} \quad P_2^N := P_1^N \setminus \text{Inv}^+(N, S).$$

If we want to explicitly emphasize the dependence of the index pair on the isolated invariant set S , we also write $P^{S,N} = (P_1^{S,N}, P_2^{S,N})$ instead of $P^N = (P_1^N, P_2^N)$.

Theorem 5.5 (Existence of saturated index pair). *Assume that $N \subseteq X$ is an isolating set for an isolated invariant set S . Then P^N is a saturated index pair for S in N .*

Proof: It follows from Proposition 5.3 that the sets P_1^N and P_2^N are both closed. Moreover, property (IP1) is a straightforward consequence of the definition of the sets $\text{Inv}^-(N, S)$ and $\text{Inv}^+(N, S)$. From (2) we obtain $S = P_1^N \setminus P_2^N$, which establishes both (IP3) and the fact that P^N is saturated, once condition (IP2) has been proved.

Thus, it remains to verify that property (IP2) is satisfied. For this, assume to the contrary that there exists an element $y \in (P_1^N \cap \text{cl}(F(P_1^N) \setminus N)) \setminus P_2^N$. This implies that $y \in P_1^N \setminus P_2^N = S$, and there exists $y' \in F(P_1^N) \setminus N$ such that $y \in \text{cl } y'$. Let $x \in P_1^N$ be such that $y' \in F(x)$. Then $y \in \text{cl } y' \subseteq \text{cl } F(x) = F(x)$, since F has closed values. In view of $x \in P_1^N = \text{Inv}^-(N, S)$, there exists a path $\sigma \in \text{Path}(N)$ such that $\sigma^\square \in S$ and $\sigma^\square = x$. It follows that $\sigma \cdot y$ is a path in N with endpoints in S , and therefore (IS1) yields $x \in S$. This in turn implies $y' \in F(x) \subseteq F(S)$. Thus, one obtains $y \in S \cap \text{cl } y' \subseteq S \cap \text{cl}(F(S) \setminus N)$, which contradicts (IS2). \square

The standard index pair P^N that can be associated with every isolated invariant S with isolating set N will be important for our further considerations. Yet, as we pointed out earlier, this is only one possible choice among many. In particular, although the standard index pair is sufficient to define the Conley index, the flexibility in choosing index pairs matters when addressing properties of the index, for instance continuation (see Sec. 7.2). While the collection of index pairs will be further studied in the next section, we close this one with a simple observation.

Proposition 5.6 (Inclusion property of standard index pairs). *Assume $M \subseteq N$ are two isolating sets for an isolated invariant set S . Then the associated standard index pairs satisfy $P_i^M \subseteq P_i^N$ for $i = 1, 2$.*

Proof: The inclusion $P_1^M \subseteq P_1^N$ follows immediately from Proposition 5.2. On the other hand, in view of (2) we have $P_2^M = P_1^M \setminus S \subseteq P_1^N \setminus S = P_2^N$. \square

To close this section, we briefly return to our previous two examples and present the standard index pairs for selected isolated invariant sets.

Example 5.7 (Sample standard index pairs). For the two simple multivalued maps $F : X \multimap X$ and $G : X \multimap X$ from Examples 4.3 and 4.4, respectively, on the finite topological space $X = \{A, B, C, AB, AC, BC, ABC\}$, one can easily determine the associated standard index pairs. Recall that in Example 4.3 we used the closed sets $N_1 = \{A, B, C\}$, $N_2 = \{A, B, C, AB, BC, AC\}$, and $N_3 = X$ as respective isolating sets for the three isolated invariant sets S_1, S_2 , and S_3 given below. This leads to the standard index pairs

$$(3) \quad \begin{array}{llll} P_1^{S_1, N_1} = N_1, & P_2^{S_1, N_1} = \emptyset & \text{for } S_1 = \{A, B, C\} & \text{in } N_1, \\ P_1^{S_2, N_2} = N_2, & P_2^{S_2, N_2} = N_1 & \text{for } S_2 = \{AB, BC, AC\} & \text{in } N_2, \\ P_1^{S_3, N_3} = N_3, & P_2^{S_3, N_3} = N_2 & \text{for } S_3 = \{ABC\} & \text{in } N_3. \end{array}$$

For example, in order to establish the second standard index pair in this list, note that $\text{Inv}^-(N_2, S_2) = \{A, B, C, AB, BC, AC\}$ and $\text{Inv}^+(N_2, S_2) = \{AB, BC, AC\}$, which immediately yields the above form for P^{S_2, N_2} .

We now turn our attention to Example 4.4. In this case, we defined the closed sets $M_1 = \{A\}$, $M_2 = \{B, C\}$, $M_3 = \{B, C, BC\}$, $M_4 = \{A, B, C, AB, AC\}$, as well as $M_5 = X$, as respective isolating sets for the isolated invariant sets R_k given below. More precisely, one obtains the standard index pairs

$$(4) \quad \begin{array}{llllll} P_1^{R_1, M_1} = M_1, & P_2^{R_1, M_1} = \emptyset & \text{for } R_1 = \{A\} & \text{in } M_1, \\ P_1^{R_2, M_2} = M_2, & P_2^{R_2, M_2} = \emptyset & \text{for } R_2 = \{B, C\} & \text{in } M_2, \\ P_1^{R_3, M_3} = M_3, & P_2^{R_3, M_3} = M_2 & \text{for } R_3 = \{BC\} & \text{in } M_3, \\ P_1^{R_4, M_4} = M_4, & P_2^{R_4, M_4} = M_1 \cup M_2 & \text{for } R_4 = \{AB, AC\} & \text{in } M_4, \\ P_1^{R_5, M_5} = M_5, & P_2^{R_5, M_5} = M_3 \cup M_4 & \text{for } R_5 = \{ABC\} & \text{in } M_5. \end{array}$$

Thus, we have identified the standard index pairs for all isolated invariant sets contained in the Morse decompositions shown in Figures 3 and 4.

5.2. Properties of index pairs. In the last section, we introduced the notion of an index pair $P = (P_1, P_2)$ associated with an isolated invariant set S and its isolating set N . These index pairs will prove to be central for the definition of the Conley index. Yet, as we already mentioned several times, index pairs are not unique, and the present section collects results on the construction of a variety of index pairs. These results will be crucial for the next section, which introduces the Conley index.

In the following, we assume that N is an isolating set for the isolated invariant set S . If $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ are two topological pairs, we use the abbreviation $P \subseteq Q$ for the validity of the two inclusions $P_1 \subseteq Q_1$ and $P_2 \subseteq Q_2$. Furthermore, by $P \cap Q$ we denote the pair $(P_1 \cap Q_1, P_2 \cap Q_2)$. We begin by showing that index pairs are closed under intersection.

Lemma 5.8 (Intersection preserves index pairs). *If P and Q are two index pairs for an isolated invariant set S in an isolating set N , then so is $P \cap Q$.*

Proof: Applying property (IP1) of P we get $F(P_i \cap Q_i) \cap N \subseteq F(P_i) \cap N \subseteq P_i$. Similarly, we obtain $F(P_i \cap Q_i) \cap N \subseteq Q_i$. Therefore, $F(P_i \cap Q_i) \cap N \subseteq P_i \cap Q_i$ for $i = 1, 2$, which proves the inclusions in (IP1) for $P \cap Q$.

As for the second property (IP2) of an index pairs, we observe that since both P and Q satisfy it, one obtains the inclusions

$$\begin{aligned} P_1 \cap Q_1 \cap \text{cl}(F(P_1 \cap Q_1) \setminus N) &\subseteq P_1 \cap \text{cl}(F(P_1) \setminus N) \cap Q_1 \cap \text{cl}(F(Q_1) \setminus N) \\ &\subseteq P_2 \cap Q_2. \end{aligned}$$

It remains to establish (IP3). First observe that in view of (IP3) for both P and Q we have

$$S \subseteq (P_1 \setminus P_2) \cap (Q_1 \setminus Q_2) \subseteq (P_1 \cap Q_1) \setminus (P_2 \cap Q_2).$$

Therefore, $S = \text{Inv } S \subseteq \text{Inv}((P_1 \cap Q_1) \setminus (P_2 \cap Q_2))$. To prove the opposite inclusion, assume to the contrary that there exists $y \in \text{Inv}((P_1 \cap Q_1) \setminus (P_2 \cap Q_2)) \setminus S$. Moreover, let $\sigma : \mathbb{Z} \rightarrow (P_1 \cap Q_1) \setminus (P_2 \cap Q_2)$ be a full solution through y . Then there has

to exist an index $p \in \mathbb{Z}$ such that $\sigma(p) \in P_2$, because otherwise we obtain the inclusion $\text{im } \sigma \subseteq \text{Inv}(P_1 \setminus P_2) = S$ in view of (IP3) for P . In addition, due to (IP1) for P , together with $\text{im } \sigma \subseteq P_1 \subseteq N$, one has to have $\sigma(r) \in P_2$ for every $r \geq p$. Symmetrically, there exists an index $q \in \mathbb{Z}$ such that $\sigma(r) \in Q_2$ for all $r \geq q$. In particular, this implies $\sigma(\max\{p, q\}) \in P_2 \cap Q_2$, a contradiction. \square

The next two results introduce a few ways for constructing new index pairs from two given nested ones.

Lemma 5.9 (New index pairs from nested ones). *If $P \subseteq Q$ are index pairs in N for an isolated invariant set S , then so are $(P_1, P_1 \cap Q_2)$ and $(P_1 \cup Q_2, Q_2)$.*

Proof: Let us start with the first pair $(P_1, P_1 \cap Q_2)$. The verification of property (IP1) is straightforward. Observe that in view of (IP2) for the index pair P we get $P_1 \cap \text{cl}(F(P_1) \setminus N) \subseteq P_2 \subseteq P_1 \cap Q_2$, and therefore (IP2) holds. To establish (IP3), we observe that due to (IP3) for both P and Q one has

$$\begin{aligned} S &= \text{Inv } S \subseteq \text{Inv}((P_1 \setminus P_2) \cap (Q_1 \setminus Q_2)) \subseteq \text{Inv}(P_1 \setminus Q_2) \\ &= \text{Inv}(P_1 \setminus (P_1 \cap Q_2)) \subseteq \text{Inv}(P_1 \setminus P_2) = S. \end{aligned}$$

Hence, $\text{Inv}(P_1 \setminus (P_1 \cap Q_2)) = S$, which completes the proof that $(P_1, P_1 \cap Q_2)$ is indeed an index pair.

Consider now the second pair $(P_1 \cup Q_2, Q_2)$. As before, the verification of (IP1) is straightforward. In order to establish (IP3) we observe that as seen above

$$S = \text{Inv}(P_1 \setminus Q_2) = \text{Inv}((P_1 \cup Q_2) \setminus Q_2).$$

Finally, in order to verify (IP2) for $(P_1 \cup Q_2, Q_2)$ we note that

$$(P_1 \cup Q_2) \cap \text{cl}(F(P_1 \cup Q_2) \setminus N) \subseteq Q_1 \cap \text{cl}(F(Q_1) \setminus N) \subseteq Q_2,$$

which yields (IP2) for the second pair and completes the proof. \square

The second lemma is concerned with a useful construction of new index pairs which includes the action of F itself. For this, suppose we are given two index pairs P and Q for an isolated invariant set S in N , and such that $P \subseteq Q$. We then define a topological pair of sets $G(P, Q) = (G_1(P, Q), G_2(P, Q))$ by

$$G_i(P, Q) = P_i \cup (F(Q_i) \cap N) \quad \text{for } i = 1, 2.$$

Note that we always have $G_2(P, Q) \subseteq G_1(P, Q) \subseteq N$, as required by a topological pair, and that $G_i(P, Q)$ is closed for $i = 1, 2$. The latter fact is due to the closedness of the values of F . While in general the pair $G(P, Q)$ is not an index pair for S in N , the following result gives sufficient conditions, as well as a number of other useful properties.

Lemma 5.10 (Properties of the pair $G(P, Q)$). *Let $P \subseteq Q$ be two index pairs for the isolated invariant set S in N , and let $G = G(P, Q)$ be defined as above. Then we have the following properties.*

- (i) $P \subseteq G \subseteq Q$.

- (ii) $P_i = Q_i$ implies $P_i = G_i = Q_i$, for $i = 1, 2$.
- (iii) If $P_1 = Q_1$ or $P_2 = Q_2$ then G is an index pair in N .
- (iv) $F(Q_i) \cap N \subseteq G_i$, for $i = 1, 2$.
- (v) If $P_{3-i} = Q_{3-i}$ and $G_i = Q_i$, then $P_i = Q_i$ for $i = 1, 2$.

Proof: The first inclusion in (i) is obvious. The second one follows from (IP1) for Q . Moreover, property (ii) is an immediate consequence of (i).

In order to prove property (iii), let us begin with property (IP1). Its verification does not require the hypothesis of (iii), since we have

$$F(G_i) \cap N = (F(P_i) \cap N) \cup (F(F(Q_i) \cap N) \cap N) \subseteq P_i \cup (F(Q_i) \cap N) = G_i$$

in view of (IP1) applied to P and Q .

If $P_1 = Q_1$, then we have $G_1 \cap \text{cl}(F(G_1) \setminus N) = P_1 \cap \text{cl}(F(P_1) \setminus N) \subseteq P_2 \subseteq G_2$ by (i), (ii), and (IP2) for P , and this establishes (IP2) for G in this case. On the other hand, if the equality $P_2 = Q_2$ holds, then

$$G_1 \cap \text{cl}(F(G_1) \setminus N) \subseteq Q_1 \cap \text{cl}(F(Q_1) \setminus N) \subseteq Q_2 = G_2,$$

in view of (IP2) for Q , (i), and (ii). This proves (IP2) for G also in this case.

In order to verify property (IP3), observe that by (IP3) applied to P and Q one obtains $S \subseteq (P_1 \setminus P_2) \cap (Q_1 \setminus Q_2) = P_1 \setminus Q_2 \subseteq G_1 \setminus G_2$, and this in turn immediately yields $S = \text{Inv } S \subseteq \text{Inv}(G_1 \setminus G_2)$. According to property (i) we have the inclusion $G_1 \setminus G_2 \subseteq Q_1 \setminus P_2$. Hence, if $P_1 = Q_1$, we obtain $G_1 \setminus G_2 \subseteq P_1 \setminus P_2$, and (IP3) applied to P further implies $\text{Inv}(G_1 \setminus G_2) \subseteq \text{Inv}(P_1 \setminus P_2) = S$. Similarly, if instead the equality $P_2 = Q_2$ holds, then one obtains $G_1 \setminus G_2 \subseteq Q_1 \setminus Q_2$, and (IP3) applied to Q furnishes $\text{Inv}(G_1 \setminus G_2) \subseteq \text{Inv}(Q_1 \setminus Q_2) = S$. Altogether, we get the inclusion $\text{Inv}(G_1 \setminus G_2) \subseteq S$, which completes the proof of (IP3) for G , and establishes the latter as an index pair, as claimed in (iii).

Since property (iv) is obvious, it remains to establish (v). For this, fix $i \in \{1, 2\}$ and assume that the identities $P_{3-i} = Q_{3-i}$ and $G_i = Q_i$ are satisfied. We want to show that $P_i = Q_i$. Since $P_i \subseteq Q_i$ by assumption, we only need to verify the inclusion $Q_i \subseteq P_i$.

Thus, take an arbitrary point $y \in Q_i$. We will begin by constructing recursively a function $\sigma : \mathbb{Z}_- \rightarrow Q_i$ as follows, where \mathbb{Z}_- denotes the set of all nonpositive integers. We set $\sigma(0) := y \in Q_i$. Assuming $\sigma(-k) \in Q_i$ has already been defined for $k \in \mathbb{N}_0$, we consider two cases to define $\sigma(-k-1)$. If we have $\sigma(-k) \in P_i$, then we define $\sigma(-k-1) := \sigma(-k)$. If instead we have $\sigma(-k) \notin P_i$, then one obtains from the assumption $Q_i = G_i$ and the above definition $G_i = P_i \cup (F(Q_i) \cap N)$ that $\sigma(-k) \in F(Q_i)$, and we can select an element $\sigma(-k-1) \in Q_i$ which satisfies the inclusion $\sigma(-k) \in F(\sigma(-k-1))$.

We claim that $\text{im } \sigma \cap P_i \neq \emptyset$. Assume the contrary. Then $\sigma : \mathbb{Z}_- \rightarrow Q_i \setminus P_i$ is a solution. Since the space X is finite, we can therefore find indices $m, n \in \mathbb{Z}_-$ such that $m < n$ and $\sigma(m) = \sigma(n)$. Thus, the point $\sigma(m)$ lies on a periodic solution in the set difference $Q_i \setminus P_i$. But then we have $\sigma(m) \in \text{Inv}(Q_i \setminus P_i)$. Consider now

first the case $i = 1$. Then we have $P_2 = Q_2$ and $\sigma : \mathbb{Z}_- \rightarrow Q_1 \setminus P_1$, as well as the inclusion $Q_1 \setminus P_1 \subseteq Q_1 \setminus P_2 = Q_1 \setminus Q_2$. Hence, using property (IP3) applied to Q one obtains that $\sigma(m) \in \text{Inv}(Q_1 \setminus Q_2) = S \subseteq P_1$, which contradicts our assumption that $\text{im } \sigma \cap P_1 = \emptyset$. Consider now the second case $i = 2$. Then one has $P_1 = Q_1$ and $\sigma : \mathbb{Z}_- \rightarrow Q_2 \setminus P_2$, as well as $Q_2 \setminus P_2 \subseteq Q_1 \setminus P_2 = P_1 \setminus P_2$. Hence, we get from (IP3) applied to P that $\sigma(m) \in \text{Inv}(P_1 \setminus P_2) = S \subseteq Q_1 \setminus Q_2$. Therefore, $\sigma(m) \notin Q_2$, again a contradiction. Thus, we established $\text{im } \sigma \cap P_i \neq \emptyset$.

With this we can immediately complete the proof of (v). According to the last paragraph, the index $m := \max \{ k \in \mathbb{Z}_- \mid \sigma(k) \in P_i \}$ is well defined. We cannot have $m < 0$, because in that case one obtains $\sigma(m+1) \in F(\sigma(m)) \subseteq F(P_i)$, and due to (IP1) applied to P one further gets $\sigma(m+1) \in Q_i \cap F(P_i) \subseteq N \cap F(P_i) \subseteq P_i$, which is a contradiction. Hence, we have to have $m = 0$, and thus $y = \sigma(0) \in P_i$. This completes the proof of the lemma. \square

The next result shows that for nested index pairs $P \subseteq Q$ which satisfy $P_1 = Q_1$ or $P_2 = Q_2$, it is always possible to construct a sequence of index pairs between them with certain mapping properties. While the specifics of this lemma might seem strange at first sight, it is essential for proving that the Conley index computation is independent of the underlying index pair.

Lemma 5.11 (Interpolating between nested index pairs). *Let $P \subseteq Q$ be index pairs for an isolated invariant set S in N such that either $P_1 = Q_1$ or $P_2 = Q_2$. Then there exists a sequence of index pairs for S in N*

$$P = Q^n \subseteq Q^{n-1} \subseteq \dots \subseteq Q^1 \subseteq Q^0 = Q$$

which satisfy the following:

- (a) $P_i = Q_i$ implies $Q_i^k = P_i = Q_i$ for all $k = 1, 2, \dots, n-1$ and $i = 1, 2$,
- (b) $F(Q_i^k) \cap N \subseteq Q_i^{k+1}$ for all $k = 0, 1, \dots, n-1$ and $i = 1, 2$.

Proof: Define the index pairs Q^k recursively by $Q^0 := Q$ and $Q^{k+1} := G(P, Q^k)$ for $k \in \mathbb{N}$. Using Lemma 5.10(i), (ii) and (iii), together with induction on k , one can easily show that the family $\{Q^k\}$ forms a decreasing sequence of index pairs with respect to k which satisfies property (a). In addition, Lemma 5.10(iv) implies that they also satisfy property (b) for all $k \in \mathbb{N}_0$. Finally, since X is finite, there has to be an $n \in \mathbb{N}_0$ such that $Q^n = Q^{n+1} = G(P, Q^n)$, and an application of Lemma 5.10(v) shows that then $Q^n = P$. \square

For the remainder of this section, we briefly introduce and study a topological pair which can be associated with an index pair, and which plays a crucial role for the definition of the *index map* in the next section.

To define this topological pair, we again let $P = (P_1, P_2)$ denote an index pair for an isolated invariant set S in the isolating set N . Then we define $\bar{P} := (\bar{P}_1, \bar{P}_2)$ via

$$\bar{P}_i := P_i \cup \text{cl}(F(P_1) \setminus N) \quad \text{for } i = 1, 2.$$

Notice that the new topological pair \bar{P} extends the index pair P by adding the closure of the images of P_1 under F that lie outside of N . The resulting pair \bar{P} still consists of closed sets, but in general it is no longer an index pair. Nevertheless, it will allow us to study the action of F on P on the homological level in the next section. For now, we note the following proposition.

Proposition 5.12 (The extended topological pair \bar{P}). *Assume that $P = (P_1, P_2)$ is an index pair for the isolated invariant set S in an isolating set N . Then the following hold for the extended topological pair \bar{P} defined above:*

- (5) $P \subseteq \bar{P}$,
- (6) $F(P) = (F(P_1), F(P_2)) \subseteq \bar{P}$,
- (7) $\bar{P}_1 \setminus \bar{P}_2 = P_1 \setminus P_2$.

Proof: As we already mentioned, property (5) follows directly from the definition of \bar{P} . To see (6), note that in view of (IP1) we have $F(P_i) \setminus P_i \subseteq F(P_i) \setminus N$, and therefore $F(P_i) \subseteq P_i \cup (F(P_i) \setminus P_i) \subseteq P_i \cup (F(P_i) \setminus N) \subseteq \bar{P}_i$. Finally, observe that property (IP2) implies

$$\begin{aligned} \bar{P}_1 \setminus \bar{P}_2 &= (P_1 \cup \text{cl}(F(P_1) \setminus N)) \setminus (P_2 \cup \text{cl}(F(P_1) \setminus N)) \\ &= P_1 \setminus P_2 \setminus \text{cl}(F(P_1) \setminus N) = P_1 \setminus P_2, \end{aligned}$$

and this completes the proof of the proposition. \square

As our final result of this section, we consider the extended topological pairs of the standard index pairs introduced in Definition 5.4. More precisely, we consider the situation of nested isolating sets for the same isolated invariant set S .

Proposition 5.13 (The extended topological pair for standard index pairs). *Assume that the closed sets $M \subseteq N$ are two isolating sets for the same isolated invariant set S . Then the inclusion $\overline{P^M} \subseteq \overline{P^N}$ holds.*

Proof: We first establish the validity of $\overline{P_1^M} \subseteq \overline{P_1^N}$. Using Proposition 5.6 one obtains the inclusion

$$\begin{aligned} \overline{P_1^M} &= P_1^M \cup \text{cl}(F(P_1^M) \setminus M) \subseteq P_1^N \cup \text{cl}(F(P_1^N) \setminus M) \\ &= P_1^N \cup \text{cl}(F(P_1^N) \setminus N) \cup \text{cl}((F(P_1^N) \cap N) \setminus M) \\ &\subseteq P_1^N \cup \text{cl}(F(P_1^N) \setminus N) = \overline{P_1^N}, \end{aligned}$$

where the last inclusion follows from (IP1) and the fact that P_1^N is closed.

It remains to show that $\overline{P_2^M} \subseteq \overline{P_2^N}$. For this, let $y \in \overline{P_2^M} = P_2^M \cup \text{cl}(F(P_1^M) \setminus M)$. If in fact we have $y \in P_2^M$, then an application of Proposition 5.6 immediately implies that $y \in P_2^N \subseteq \overline{P_2^N}$. Suppose therefore that we have $y \in \text{cl}(F(P_1^M) \setminus M)$ and $y \notin P_2^N$. Furthermore, let $y' \in F(P_1^M) \setminus M$ be such that $y \in \text{cl } y'$.

We now claim that $y' \notin N$. To verify this, assume to the contrary that $y' \in N$. Let $x \in P_1^M$ be such that $y' \in F(x)$. If $x \in P_2^M$, then $y' \in F(P_2^M) \subseteq F(P_2^N)$, and

therefore $y' \in F(P_2^N) \cap N \subseteq P_2^N$ by (IP1), as well as $y \in \text{cl } y' \subseteq \text{cl } P_2^N = P_2^N$, a contradiction. Thus we have to have $x \notin P_2^M$, and this yields $x \in P_1^M \setminus P_2^M = S$ by Theorem 5.5. Hence, $y' \in F(S) \setminus M$ and $y \in \text{cl}(F(S) \setminus M)$.

Since we assumed $y' \in N$, one obtains $y' \in F(P_1^M) \cap N \subseteq F(P_1^N) \cap N \subseteq P_1^N$, and together with the closedness of P_1^N this further implies $y \in P_1^N$. This in turn shows that the inclusion $y \in P_1^N \setminus P_2^N = S$ holds, by Theorem 5.5. However, this finally furnishes $y \in S \cap \text{cl}(F(S) \setminus M)$, which contradicts (IS2). Thus, we deduce that our assumption on y' was wrong and we actually have $y' \notin N$.

With this in hand the proof of the second inclusion can easily be finished. We now have $y' \in F(P_1^M) \setminus N \subseteq F(P_1^N) \setminus N$, as well as $y \in \text{cl}(F(P_1^N) \setminus N) \subseteq \overline{P_2^N}$. \square

We close this section by deriving the extended topological pairs for the index pairs in Example 5.7.

Example 5.14 (Sample extended topological pairs). We leave it to the reader to verify that all eight standard index pairs given in (3) and (4) are in fact equal to their extended topological pairs as defined above. In every one of these cases, if S is an isolated invariant set in an isolating set N , we have both $P_1^{S,N} = N$, as well as either $F(N) \subseteq N$ or $G(N) \subseteq N$, respectively, depending on which multivalued map is considered. From this our claim follows immediately.

6. DEFINITION OF THE CONLEY INDEX

With this section, we finally turn our attention to the Conley index for isolated invariant sets. For this, we first introduce the index map based on an index pair in Section 6.1, which transfers the action of the multivalued map F restricted to the index pair to the algebraic level in terms of homology. Clearly, this map will depend on the chosen index pair, and the remainder of the section is aimed at deriving an index definition from the index map which only depends on the isolated invariant set. Our approach relies on the notion of normal functor, which is introduced in Section 6.2. Finally, Section 6.3 combines both notions to define the Conley index and prove that it is well-defined. In addition, we compute the Conley indices for the isolated invariant sets in our earlier examples. In contrast to the previous section, we need to impose an additional condition on the underlying multivalued map. In this section the multivalued map $F : X \multimap X$ will be assumed to be lower semicontinuous with closed and *acyclic* values. The additional acyclicity assumption is needed in order to obtain induced maps in homology. All the homology groups are considered with coefficients in a fixed ring R .

6.1. The index map. The basic idea of the Conley index in this paper is to lift information from the multivalued map $F : X \multimap X$ close to an isolated invariant set S to the setting of homology. On the level of the phase space, this is accomplished by considering the relative homology $H_*(P) = H_*(P_1, P_2)$ of an index pair for S , and on the level of the map F by the associated *index map* I_P ,

which is an endomorphism of $H_*(P)$. The latter map should of course in some way lift the dynamics of F to the homology level.

Passing from a multivalued map to a map induced in homology is slightly more involved than the classical map setting, and we begin by reviewing the necessary approach. As it was already said in Section 2, every lower semicontinuous map with closed values is in fact *strongly lower semicontinuous (slsc)*. Recall that a multivalued map $F : X \multimap Y$ between finite T_0 spaces is called strongly lower semicontinuous, if $x \in \text{cl } x'$ implies $F(x) \subseteq F(x')$. If in addition $F : X \multimap Y$ has acyclic values, then it induces a homomorphism $F_* : H_*(X) \rightarrow H_*(Y)$ in homology. More precisely, in [3, Proposition 4.7] it is shown that if F is identified with its graph $F \subseteq X \times Y$, then the restriction $p_1 : F \rightarrow X$ of the projection onto the first coordinate is an isomorphism in homology in every degree, and therefore one can define the induced map in homology as $F_* = (p_2)_*(p_1)_*^{-1}$ with $p_2 : F \rightarrow Y$ denoting the restriction of the other projection. Although the results in [3] are stated only for integer coefficients, it is easy to see that the same results hold for homology with coefficients in an arbitrary ring (see [3, Theorem 2.2]).

As we mentioned earlier, the index map will be a homological version of the action of F on a given index pair, and it is therefore not surprising that we have to recall a few notions about maps of pairs. A multivalued map $F : (X, A) \multimap (Y, B)$ between pairs of finite T_0 spaces is a multivalued map $F : X \multimap Y$ which satisfies the inclusion $F(a) \subseteq B$ for every $a \in A$. We say that $F : (X, A) \multimap (Y, B)$ is slsc (or with closed values, or with acyclic values) if $F : X \multimap Y$ has the respective property. Suppose that $F : (X, A) \multimap (Y, B)$ is slsc with acyclic values. Then the restriction $F|_A^B : A \multimap B$ is also slsc with acyclic values, and its graph is a subspace of F . Since the projections $F \rightarrow X$ and $F|_A^B \rightarrow A$ induce isomorphisms in homology, by the long exact sequence of homology and the five lemma, so does the projection of pairs $p_1 : (F, F|_A^B) \rightarrow (X, A)$. Finally, in view of these preparations we can define the homomorphisms $F_* : H_*(X, A) \rightarrow H_*(Y, B)$ by letting $F_* = (p_2)_*(p_1)_*^{-1}$ as before.

Before moving on to the definition of the index map, we need the following two auxiliary results concerning maps in homology induced by compositions.

Lemma 6.1 (Homology map of compositions). *Let $F : X \multimap Y$ and $G : Z \multimap Y$ be slsc multivalued maps with acyclic values, and suppose that $f : X \rightarrow Z$ is a continuous map such that $Gf = F$. Then we have $G_*f_* = F_* : H_*(X) \rightarrow H_*(Y)$. The same result holds more generally, for pairs.*

Proof: Consider the following two commutative diagrams:

$$\begin{array}{ccc}
 X & \xrightarrow{F} \circ & Y \\
 \downarrow f & \nearrow G & \\
 Z & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 X & \xleftarrow{p_1} & F & \xrightarrow{p_2} & Y \\
 \downarrow f & & \downarrow f \times 1_Y & & \nearrow p_2 \\
 Z & \xleftarrow{p_1} & G & &
 \end{array}$$

The commutativity of the first diagram implies that $f \times 1_Y : F \rightarrow G$ is well defined, and this immediately leads to the second commutative diagram. The result then follows by definition. For pairs we have the exact same proof. \square

Lemma 6.2 (Homology map of compositions). *Let $F : Z \multimap X$ and $G : Z \multimap Y$ be slsc multivalued maps with acyclic values, and let $f : Y \rightarrow X$ be a continuous map such that $fG = F$. Then $f_*G_* = F_* : H_*(Z) \rightarrow H_*(X)$. The same result holds more generally, for pairs.*

Proof: Similar to the last proof, consider the following commutative diagrams:

$$\begin{array}{ccc}
 & Y & \\
 G \nearrow & \circ & \downarrow f \\
 Z & \xrightarrow{F} & X
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & G & \xrightarrow{p_2} & Y & \\
 p_1 \nearrow & \downarrow 1_Z \times f & & \downarrow f & \\
 Z & \xleftarrow{p_1} & F & \xrightarrow{p_2} & X
 \end{array}$$

The commutativity of the first diagram implies that $1_Z \times f : Z \times Y \rightarrow Z \times X$ is well defined. This leads to the second commutative diagram, and the result then follows by definition. For pairs we have the exact same proof. \square

As our last preparation we turn our attention briefly to the *strong excision* property. For this, let (Y_1, Y_2) and (Z_1, Z_2) denote two topological pairs of closed subspaces of a finite T_0 space X such that the inclusions $Y_i \subseteq Z_i$ hold for $i = 1, 2$, and that $Y_1 \setminus Y_2 = Z_1 \setminus Z_2$. Then the inclusion $i : (Y_1, Y_2) \rightarrow (Z_1, Z_2)$ induces a homomorphism i_* between the relative homology groups $H_*(Y_1, Y_2)$ and $H_*(Z_1, Z_2)$. In fact, the strong excision property states that $i_* : H_*(Y_1, Y_2) \rightarrow H_*(Z_1, Z_2)$ is an isomorphism. This result follows directly from the pair of McCord isomorphisms $H_*(\mathcal{K}(Y_1), \mathcal{K}(Y_2)) \rightarrow H_*(Y_1, Y_2)$ and $H_*(\mathcal{K}(Z_1), \mathcal{K}(Z_2)) \rightarrow H_*(Z_1, Z_2)$, where \mathcal{K} is the functor which associates to each poset its order complex ([25, Corollary 1]). The hypotheses imply that the chains in Y_1 which are not in Y_2 are the same as the chains in Z_1 not in Z_2 , and thus $i_* : C_*(\mathcal{K}(Y_1), \mathcal{K}(Y_2)) \rightarrow C_*(\mathcal{K}(Z_1), \mathcal{K}(Z_2))$ is already an isomorphism of chain complexes, and in particular an isomorphism in homology.

After these preparations we can finally introduce the index map. In the rest of the paper X will be a finite T_0 topological space and $F : X \multimap X$ will be lower semicontinuous with closed and acyclic values. The index map lifts the action of the multivalued map F on an index pair P to the homological level. This has to be done with care, since we usually do not have $F(P) \subseteq P$. In fact, we will make use of the extended pair \bar{P} whose properties were established in Proposition 5.12. More precisely, let P be an index pair for an isolated invariant set S in the isolating set N . By applying Proposition 5.12, we then immediately obtain both an inclusion induced isomorphism $(\iota_P)_* : H_*(P) \rightarrow H_*(\bar{P})$ and a homomorphism $(F_P)_* : H_*(P) \rightarrow H_*(\bar{P})$, where the latter is induced by the multivalued map $F_P = F|_{\bar{P}_1} : P \multimap \bar{P}$. This leads to the following definition.

Definition 6.3 (Index map). *Let P be an index pair for an isolated invariant set S in the isolating set N . Then the associated index map is the endomorphism*

$$I_P : H_*(P_1, P_2) \rightarrow H_*(P_1, P_2) \quad \text{given by} \quad I_P := (\iota_P)_*^{-1} \circ (F_P)_*,$$

where we use the maps induced in homology by the restriction $F_P = F|_{P_1}^{\bar{P}_1} : P \multimap \bar{P}$ and the inclusion $\iota_P : P \rightarrow \bar{P}$.

Remark 6.4. *Note that the definition of \bar{P} makes sense and the conclusion of Proposition 5.12 remains true even if $P = (P_1, P_2)$ is merely a pair of closed subspaces of X , and if N is a closed subspace of X such that $P_2 \subseteq P_1 \subseteq N$ and conditions (IP1) and (IP2) hold. In other words, the isolated invariant set S , and the conditions (IS1), (IS2), and (IP3) are not needed for the above. Thus, the index map $I_P : H_*(P_1, P_2) \rightarrow H_*(P_1, P_2)$ can still be defined as in Definition 6.3.*

6.2. Normal functors. Next we need to recall some definitions and results from category theory, in particular centered around the notion of normal functors. For this, let \mathcal{E} denote a category. We define the category of endomorphisms of \mathcal{E} , denoted by $\text{Endo}(\mathcal{E})$ as follows:

- The objects of $\text{Endo}(\mathcal{E})$ are pairs (A, a) , where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$ is an endomorphism of A .
- The set of morphisms from $(A, a) \in \text{Endo}(\mathcal{E})$ to $(B, b) \in \text{Endo}(\mathcal{E})$ is the subset of $\mathcal{E}(A, B)$ consisting of exactly those morphisms $\varphi \in \mathcal{E}(A, B)$ for which $\varphi a = b\varphi$.

We write $\varphi : (A, a) \rightarrow (B, b)$ to denote that φ is a morphism from (A, a) to (B, b) in $\text{Endo}(\mathcal{E})$. It is easy to see that if $\varphi : (A, a) \rightarrow (B, b)$ is a morphism in $\text{Endo}(\mathcal{E})$ which is an isomorphism in \mathcal{E} , then it is also an isomorphism in $\text{Endo}(\mathcal{E})$. Note that any endomorphism $a \in \mathcal{E}(A, A)$ is in particular a morphism $a : (A, a) \rightarrow (A, a)$ in $\text{Endo}(\mathcal{E})$. Such morphisms of $\text{Endo}(\mathcal{E})$ are called *induced*.

Now let $L : \text{Endo}(\mathcal{E}) \rightarrow \mathcal{C}$ be a functor. We say that L is *normal* if $L(a)$ is an isomorphism in \mathcal{C} for every induced morphism $a : (A, a) \rightarrow (A, a)$ in $\text{Endo}(\mathcal{E})$. Then we have the following result.

Proposition 6.5 (Isomorphism inducing property of normal functors). *In the situation above, let $L : \text{Endo}(\mathcal{E}) \rightarrow \mathcal{C}$ denote a normal functor, and let $\varphi : A \rightarrow B$ and $\psi : B \rightarrow A$ be morphisms in \mathcal{E} . Then $\varphi : (A, \psi\varphi) \rightarrow (B, \varphi\psi)$ is a morphism in the category $\text{Endo}(\mathcal{E})$, and $L(\varphi)$ is an isomorphism in \mathcal{C} .*

Proof: Clearly we have that φ is a morphism from $(A, \psi\varphi)$ to $(B, \varphi\psi)$ in $\text{Endo}(\mathcal{E})$ and ψ is a morphism from $(B, \varphi\psi)$ to $(A, \psi\varphi)$. In addition, one obtains

the commutative diagram

$$\begin{array}{ccc}
 (B, \varphi\psi) & \xrightarrow{\varphi\psi} & (B, \varphi\psi) \\
 \psi \downarrow & \nearrow \varphi & \downarrow \psi \\
 (A, \psi\varphi) & \xrightarrow{\psi\varphi} & (A, \psi\varphi)
 \end{array}$$

If we now apply the functor L to this diagram, then the horizontal morphisms become isomorphisms in \mathcal{C} . Thus, the image $L(\varphi)$ has both a left and a right inverse, and therefore it is also an isomorphism. \square

We would like to point out that if W denotes the class of induced morphisms in $\text{Endo}(\mathcal{E})$, then the natural functor $\text{Endo}(\mathcal{E}) \rightarrow \text{Endo}(\mathcal{E})[W^{-1}]$ to the localization is universal in the sense that any other normal functor $\text{Endo}(\mathcal{E}) \rightarrow \mathcal{C}$ factorizes through it, see also [31, 36]. We close this section with one specific example of a normal functor. For further examples we refer the reader to the paper [30].

Example 6.6 (The Leray functor). For the example computations of this paper, we make use of the specific normal functor introduced in [29], the *Leray functor*. For this, let Mod denote the category of graded moduli over the ring R together with homomorphisms of degree zero. Using the setting for the definition of normal functors from above, we consider the categories

$$\mathcal{E} = \text{Mod} \quad \text{and} \quad \mathcal{C} = \text{Auto}(\text{Mod}),$$

where $\text{Auto}(\text{Mod}) \subseteq \text{Endo}(\text{Mod})$ is the subcategory of automorphisms of Mod . Then the *Leray functor* $L_{\text{Leray}} : \text{Endo}(\text{Mod}) \rightarrow \text{Auto}(\text{Mod})$ can be defined as the composition of the following maps:

- Let $(H, h) \in \text{Endo}(\text{Mod})$ be arbitrary. Then the *generalized kernel* of h can be defined as

$$\text{gker}(h) := \bigcup_{n \in \mathbb{N}} h^{-n}(0),$$

and one can easily see that the map $h : H \rightarrow H$ induces a well-defined map $h' : H/\text{gker}(h) \rightarrow H/\text{gker}(h)$. Thus, the definition

$$L'(H, h) := (H/\text{gker}(h), h') \in \text{Mono}(\text{Mod}) \subseteq \text{Endo}(\text{Mod})$$

gives an object in the category $\text{Mono}(\text{Mod})$ of monomorphisms of Mod . Furthermore, it is straightforward to define $L'(\varphi)$ also for morphisms φ in $\text{Endo}(\text{Mod})$, and to show that in this way one obtains a well-defined contravariant functor $L' : \text{Endo}(\text{Mod}) \rightarrow \text{Mono}(\text{Mod})$.

- Now let $(H, h) \in \text{Mono}(\text{Mod})$ be arbitrary. Then the *generalized image* of h can be defined as

$$\text{gim}(h) := \bigcap_{n \in \mathbb{N}} h^n(H),$$

and it is not difficult to verify that the map $h : H \rightarrow H$ induces a well-defined map $h'' : \text{gim}(h) \rightarrow \text{gim}(h)$. Thus, the definition

$$L''(H, h) := (\text{gim}(h), h'') \in \text{Auto}(\text{Mod}) \subseteq \text{Endo}(\text{Mod})$$

gives an object in the category $\text{Auto}(\text{Mod})$ of automorphisms of Mod . In addition, it is again straightforward to define $L''(\varphi)$ also for morphisms φ in $\text{Mono}(\text{Mod})$, and to show that this time one obtains a well-defined contravariant functor $L'' : \text{Mono}(\text{Mod}) \rightarrow \text{Auto}(\text{Mod})$.

- Finally, the Leray functor is defined as $L_{\text{Leray}} := L'' \circ L'$.

For more details on the above construction, as well as the proof that the Leray functor is indeed a normal functor, we refer the reader to [29, Section 4]. For our applications below, we note that by the construction of L_{Leray} we have the implication

$$(8) \quad (H, h) \in \text{Auto}(\text{Mod}) \subseteq \text{Endo}(\text{Mod}) \implies L_{\text{Leray}}(H, h) = (H, h),$$

i.e., the Leray functor is the identity on $\text{Auto}(\text{Mod}) \subseteq \text{Endo}(\text{Mod})$. This fact will enable us to determine the Conley index of isolated invariant sets in many situations.

6.3. The Conley index. After these preparations we can finally define the Conley index. A first attempt would be to use the index map $I_P : H_*(P) \rightarrow H_*(P)$ introduced in Definition 6.3. Unfortunately, however, this would mean that the index depends on the chosen index pair of the isolated invariant set.

This issue can be addressed by using the concept of normal functors from the last section. More precisely, let Mod denote as before the category of graded moduli over the ring R and let $L : \text{Endo}(\text{Mod}) \rightarrow \text{Auto}(\text{Mod})$ be a fixed normal functor. Note that if P is an index pair for an isolated invariant set S in an isolating set N , then one obtains $(H_*(P), I_P) \in \text{Endo}(\text{Mod})$. Thus, the L -reduction $L(H_*(P), I_P)$ is an automorphism of a graded module over R , and we have the following crucial result.

Theorem 6.7 (Well-definedness of the Conley index). *In the situation described above, the isomorphism type of $L(H_*(P), I_P) \in \text{Auto}(\text{Mod})$ does not depend on the choice of the isolating set N for the isolated invariant set S , or on the chosen index pair P in N .*

Proof: To begin, let M and N be two isolating sets for S , and let P and Q denote two index pairs in N and M , respectively. Our goal is to establish the equivalence $L(H_*(P), I_P) \cong L(H_*(Q), I_Q)$. This is accomplished in five steps.

Step 1. We first consider the special case

- (i) $M = N$,
- (ii) $P \subseteq Q$,
- (iii) $P_1 = Q_1$ or $P_2 = Q_2$,
- (iv) $F(Q) \cap N \subseteq P$.

Let $D = (D_1, D_2)$ be the pair of closed sets defined by $D_i = P_i \cup \text{cl}(F(Q_1) \setminus N)$ for $i = 1, 2$. By (iv) we may treat F as a map of pairs $F_{QD} = F|_Q^D : Q \rightarrow D$. In view of (i) and (ii), we also have $\overline{P} \subseteq D \subseteq \overline{Q}$. This gives the following commutative diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{F_P} & \overline{P} & \xleftarrow{\iota_P} & P \\
 \downarrow j & & \downarrow k & & \downarrow j \\
 & & D & & \\
 & \nearrow F_{QD} & \downarrow l & & \\
 Q & \xrightarrow{F_Q} & \overline{Q} & \xleftarrow{\iota_Q} & Q
 \end{array}$$

in which vertical arrows denote inclusions. Since F induces a map $F_{PD} = F|_P^D$, by Lemmas 6.1 and 6.2 we have $k_*(F_P)_* = (F_{PD})_* = (F_{QD})_*j_*$ and $l_*(F_{QD})_* = (F_Q)_*$. We then obtain a commutative diagram

$$\begin{array}{ccccc}
 H_*(P) & \xrightarrow{(F_P)_*} & H_*(\overline{P}) & \xleftarrow{(\iota_P)_*} & H_*(P) \\
 \downarrow j_* & & \downarrow k_* & & \downarrow j_* \\
 & & H_*(D) & & \\
 & \nearrow (F_{QD})_* & \downarrow l_* & & \\
 H_*(Q) & \xrightarrow{(F_Q)_*} & H_*(\overline{Q}) & \xleftarrow{(\iota_Q)_*} & H_*(Q)
 \end{array}$$

We claim that k induces an isomorphism in homology. Indeed, if $P_1 = Q_1$, k is the identity. Otherwise, by (iii) we have $P_2 = Q_2$. In this case we claim that k fulfills the hypothesis of strong excision, namely,

$$P_1 \setminus (P_2 \cup \text{cl}(F(P_1) \setminus N)) = P_1 \setminus (P_2 \cup \text{cl}(F(Q_1) \setminus N)).$$

Inclusion of the second subspace in the first is trivial, and their difference is

$$P_1 \cap \text{cl}(F(Q_1) \setminus N) \setminus (P_2 \cup \text{cl}(F(P_1) \setminus N)) \subseteq Q_1 \cap \text{cl}(F(Q_1) \setminus N) \setminus P_2,$$

which is equal to $Q_1 \cap \text{cl}(F(Q_1) \setminus N) \setminus Q_2$, and this is empty by (IP2).

If one defines $I_{QP} := (\iota_P)_*^{-1} k_*^{-1} (F_{QD})_*$, then we get the commutative diagram in Mod given by

$$\begin{array}{ccc}
 H_*(P) & \xrightarrow{I_P} & H_*(P) \\
 \downarrow j_* & \nearrow I_{QP} & \downarrow j_* \\
 H_*(Q) & \xrightarrow{I_Q} & H_*(Q)
 \end{array} ,$$

and $L(j_*) : L(H_*(P), I_{QP} j_*) = L(H_*(P), I_P) \rightarrow L(H_*(Q), j_* I_{QP}) = L(H_*(Q), I_Q)$ is an isomorphism in view of Proposition 6.5.

Step 2. Next we drop assumption (iv). According to Lemma 5.11 we can find a sequence Q^0, Q^1, \dots, Q^n of index pairs such that $Q^0 = Q$ and $Q^n = P$, and such that each pair (Q^{k+1}, Q^k) satisfies assumptions (i)–(iv). Due to *Step 1* the L -reductions $L(H_*(Q^k, I_{Q^k}))$ and $L(H_*(Q^{k+1}, I_{Q^{k+1}}))$ are isomorphic, and the conclusion follows.

Step 3. We now drop assumptions (iii) and (iv). For this, notice that in view of Lemma 5.9 the pairs $R = (P_1, P_1 \cap Q_2)$ and $T = (P_1 \cup Q_2, Q_2)$ are index pairs.

The pairs P and R satisfy assumptions (ii) and (iii), and therefore they have isomorphic L -reductions. The same holds for T and Q . On the other hand, the inclusion $j : R \hookrightarrow T$ induces an isomorphism $j_* : H_*(R) \rightarrow H_*(T)$ by strong excision. Since $R \subseteq T$, we have an inclusion $\bar{j} : \bar{R} \hookrightarrow \bar{T}$, as well as the commutative diagram

$$\begin{array}{ccccc}
 H_*(R) & \xrightarrow{(F_R)_*} & H_*(\bar{R}) & \xleftarrow{(\iota_R)_*} & H_*(R) \\
 \downarrow j_* & & \downarrow \bar{j}_* & & \downarrow j_* \\
 H_*(T) & \xrightarrow{(F_T)_*} & H_*(\bar{T}) & \xleftarrow{(\iota_T)_*} & H_*(T)
 \end{array}$$

Thus, one obtains $j_* I_R = j_* (\iota_R)_*^{-1} (F_R)_* = (\iota_T)_*^{-1} \bar{j}_* (F_R)_* = (\iota_T)_*^{-1} (F_T)_* j_* = I_T j_*$. This shows that $j_* \in \text{Endo}(\text{Mod})((H_*(R), I_R), (H_*(T), I_T))$, and since j_* is an isomorphism in Mod, it also is an isomorphism in $\text{Endo}(\text{Mod})$. This in turn implies that $L(j_*) : L(H_*(R), I_R) \rightarrow L(H_*(T), I_T)$ is an isomorphism, and that P and Q indeed have isomorphic L -reductions.

Step 4. Now we only assume (i). By Lemma 5.8 the pair $P \cap Q$ is an index pair. Hence, the claim follows from *Step 3* applied to $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$.

Step 5. Finally, we drop all auxiliary assumptions. We have already proved that the isomorphism type of the L -reduction depends only on the isolating set for S . Moreover, since by Proposition 4.6, the intersection of two isolating sets is again an isolating set, we may assume $M \subseteq N$.

Consider the index pairs P^M for S in M and P^N for S in N . In view of Proposition 5.6 and Proposition 5.13 we then have the commutative diagram

$$\begin{array}{ccccc}
 P^M & \xrightarrow{F_{P^M}} & \overline{P^M} & \xleftarrow{\iota_{P^M}} & P^M \\
 \downarrow j & & \downarrow k & & \downarrow j \\
 P^N & \xrightarrow{F_{P^N}} & \overline{P^N} & \xleftarrow{\iota_{P^N}} & P^N
 \end{array}$$

in which vertical arrows denote inclusions. Then $I_{P^N}j_* = j_*I_{P^M}$, which implies that $j_* : (H_*(P^M), I_{P^M}) \rightarrow (H_*(P^N), I_{P^N})$ is a morphism in $\text{Endo}(\text{Mod})$. On the other hand, since P^M and P^N are saturated by Theorem 5.5, strong excision shows that $j_* : H_*(P^M) \rightarrow H_*(P^N)$ is an isomorphism in Mod . Thus, the map j_* is an isomorphism in $\text{Endo}(\text{Mod})$, and then so is $L(j_*)$. \square

Based on the above result, the Conley index can now be defined as follows. We would like to point out that the functor L in the definition could be, for example, the computationally convenient Leray functor of Example 6.6.

Definition 6.8 (The Conley index). *The L -reduction $L(H_*(P), I_P)$ will be called the homological Conley index of S , and be denoted by $C(S, F)$, or simply $C(S)$ if F is clear from context. Due to Theorem 6.7 the Conley index $C(S) \in \text{Auto}(\text{Mod})$ is well-defined up to isomorphism.*

In order to illustrate the above abstract definition of the Conley index, we now briefly return to our earlier two examples and determine the Conley indices of all the Morse sets shown in Figures 1 and 2.

Example 6.9 (Sample Conley index computations). We return one last time to the two simple multivalued maps $F : X \multimap X$ and $G : X \multimap X$ from Examples 4.3 and 4.4, respectively. We have already seen that these maps give rise to associated Morse decompositions with three and five isolated invariant sets, which themselves are subsets of the finite topological space $X = \{A, B, C, AB, AC, BC, ABC\}$. Notice that in view of Example 5.14 in all of these cases the extended topological pair \overline{P} equals the index pair P that was chosen for each isolated invariant set. Thus, the index map I_P is simply given by $I_P = (F_P)_* : H_*(P) \rightarrow H_*(P)$ for the sets in (3), and similarly for the isolated invariant sets in (4).

Consider now the multivalued map $F : X \multimap X$ from Example 4.3. For the sake of simplicity, we compute the Conley index for the ring $R = \mathbb{Z}$ and with respect to the Leray functor. Then for the isolated invariant set $S_1 = \{A, B, C\}$ one can easily see that $H_0(P^{S_1, N_1}) \simeq \mathbb{Z}^3$. Moreover, the index map $I_{P^{S_1, N_1}}$ maps the generators in a cyclic fashion, i.e., it is an automorphism. Based on (8), this shows that the Conley index with respect to L_{Leray} is just $(H_*(P^{S_1, N_1}), I_{P^{S_1, N_1}})$. In a similar way, one can determine the Conley index for all the isolated invariant

sets in Figure 1 as

$$\begin{aligned} S_1 = \{A, B, C\} & : H_0(P^{S_1, N_1}) \simeq \mathbb{Z}^3 & \text{with } I_{P^{S_1, N_1}}(e_i) = e_{(i+1) \bmod 3}, \\ S_2 = \{AB, BC, AC\} & : H_1(P^{S_2, N_2}) \simeq \mathbb{Z}^3 & \text{with } I_{P^{S_2, N_2}}(e_i) = e_{(i+1) \bmod 3}, \\ S_3 = \{ABC\} & : H_2(P^{S_3, N_3}) \simeq \mathbb{Z} & \text{with } I_{P^{S_3, N_3}}(e_i) = e_i, \end{aligned}$$

where in each case all unlisted homology groups are trivial, and the listed group \mathbb{Z}^k has a suitable basis $\{e_0, e_1, \dots, e_{k-1}\}$. Similarly, for the multivalued map G from Example 4.4 and the isolated invariant sets in Figure 2 one obtains

$$\begin{aligned} R_1 = \{A\} & : H_0(P^{R_1, M_1}) \simeq \mathbb{Z} & \text{with } I_{P^{R_1, M_1}}(e_i) = e_i, \\ R_2 = \{B, C\} & : H_0(P^{R_2, M_2}) \simeq \mathbb{Z}^2 & \text{with } I_{P^{R_2, M_2}}(e_i) = e_{(i+1) \bmod 2}, \\ R_3 = \{BC\} & : H_1(P^{R_3, M_3}) \simeq \mathbb{Z} & \text{with } I_{P^{R_3, M_3}}(e_i) = -e_i, \\ R_4 = \{AB, AC\} & : H_1(P^{R_4, M_4}) \simeq \mathbb{Z}^2 & \text{with } I_{P^{R_4, M_4}}(e_i) = e_{(i+1) \bmod 2}, \\ R_5 = \{ABC\} & : H_2(P^{R_5, M_5}) \simeq \mathbb{Z} & \text{with } I_{P^{R_5, M_5}}(e_i) = -e_i, \end{aligned}$$

where we use the same conventions as above. We leave the details of these straightforward computations to the reader.

7. PROPERTIES OF THE CONLEY INDEX

In this section, we present first properties of the Conley index for multivalued maps defined in the last section. In addition to the Ważewski property, we also briefly address continuation.

7.1. The Ważewski property. In classical Conley theory, the Ważewski property is central, as it allows one to deduce the existence of a nontrivial isolated invariant set S from a nontrivial index, and the latter can be computed from an index pair without explicit knowledge of S .

In order to show that the same result still holds in the multivalued context of the present paper, let $P = (P_1, P_2)$ denote a topological pair of closed subspaces of X . Suppose further that $N = P_1$ satisfies conditions (IP1) and (IP2), i.e., we have the inclusion $P_1 \cap (\text{cl}(F(P_1) \setminus P_1) \cup F(P_2)) \subseteq P_2$. In view of Remark 6.4, the index map $I_P : H_*(P) \rightarrow H_*(P)$ is defined in this situation. Then we have the following result.

Proposition 7.1 (Ważewski property). *Suppose that X is a finite T_0 topological space and that the multivalued map $F : X \multimap X$ is lower semicontinuous with closed and acyclic values. Moreover, let $P = (P_1, P_2)$ be a pair of closed subspaces of X such that*

$$P_1 \cap (\text{cl}(F(P_1) \setminus P_1) \cup F(P_2)) \subseteq P_2.$$

If one further has $L(H_(P), I_P) \neq 0 \in \text{Auto}(\text{Mod})$, then $\text{Inv}(P_1 \setminus P_2) \neq \emptyset$.*

Proof: Suppose $\text{Inv}(P_1 \setminus P_2) = \emptyset$. Then $N = P_1$ is an isolating set for the invariant set $S = \emptyset$, and P is an index pair for S in N . According to our hypothesis, we have $C(S) \neq 0$. But this is absurd since S admits $N' = \emptyset$ as

isolating set and $P' = (\emptyset, \emptyset)$ is an index pair for S in N' . Thus, we have the equality $H_*(P') = 0$, as well as $C(S) = L(H_*(P'), I_{P'}) = 0$. \square

7.2. Homotopies and continuation. As our second property of the Conley index we address the fundamental concept of continuation. For this, we first need to review some results on homotopies in finite topological spaces.

Let X and Y be two finite T_0 spaces. Two lower semicontinuous multivalued maps $F, G : X \multimap Y$ with closed and acyclic values are called *homotopic* if there exists a lower semicontinuous map $H : X \times [0, 1] \multimap Y$ with closed and acyclic values such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$ for every $x \in X$. This definition extends in a natural way to maps $(X, A) \multimap (Y, B)$ between pairs of finite T_0 spaces by requiring that $H : X \times [0, 1] \rightarrow Y$ maps (a, t) to $H(a, t) \subseteq B$ for every $a \in A$ and $t \in [0, 1]$.

General homotopies in the setting of finite topological spaces can be more succinctly described as follows. Define an order on the set of all lower semicontinuous multivalued maps $X \multimap Y$ with closed and acyclic values by letting $F \leq G$ if we have $F(x) \subseteq G(x)$ for all $x \in X$. A sequence $F = F_0 \leq F_1 \geq F_2 \leq \dots \leq F_k = G$ is called a *fence* from F to G . Then the proof of the following result is essentially the same as the proof of [3, Proposition 8.1], and therefore we omit it.

Proposition 7.2 (Homotopy characterization via fences). *Let X and Y be two finite T_0 spaces and let $F, G : X \multimap Y$ be two lower semicontinuous multivalued maps with closed and acyclic values. Then the maps F and G are homotopic if and only if there exists a fence $F = F_0 \leq F_1 \geq F_2 \leq \dots \leq F_k = G$ of lower semicontinuous multivalued maps $X \multimap Y$ with closed and acyclic values.*

Furthermore, if the maps $F, G : (X, A) \multimap (Y, B)$ are maps of pairs of finite T_0 spaces, then they are homotopic if and only if there exists a fence as above in which the maps are maps of pairs $(X, A) \multimap (Y, B)$.

In terms of the associated maps in homology we have the following result, which is in the spirit of [3, Corollary 8.2].

Lemma 7.3 (Homotopic maps induce the same map in homology). *Let X, Y be finite T_0 spaces, and let $F, G : X \multimap Y$ be two homotopic lower semicontinuous multivalued maps with closed and acyclic values. Then $F_* = G_* : H_*(X) \rightarrow H_*(Y)$ for the maps induced in homology. The same result holds more generally for pairs.*

Proof: We may assume $F \leq G$. Consider the following commutative diagram

$$\begin{array}{ccc}
 & F & \\
 p_1 \swarrow & & \searrow p_2 \\
 X & & Y \\
 \tilde{p}_1 \swarrow & \downarrow j & \searrow \tilde{p}_2 \\
 & G &
 \end{array}$$

in which j denotes the inclusion between the graphs, and the other maps are the projections to the first or second coordinate. Since p_1 and \tilde{p}_1 induce isomorphisms in homology, so does j . This immediately implies

$$G_* = (\tilde{p}_2)_*(\tilde{p}_1)_*^{-1} = (p_2)_*(j_*)^{-1}j_*(p_1)_*^{-1} = F_* : H_*(X) \rightarrow H_*(Y).$$

The result for pairs follows with the exact same proof. \square

The following definition introduces the notion of *continuation* for the setting of multivalued maps in finite topological spaces.

Definition 7.4 (Continuation of isolated invariant sets). *Let X be a finite T_0 space and let $F, G : X \multimap X$ be two lower semicontinuous multivalued maps with closed and acyclic values such that $F \leq G$ or $F \geq G$. Moreover, let $S_F, S_G \subseteq X$ be isolated invariant sets for F and G , respectively. We say that (S_F, F) and (S_G, G) (or just S_F and S_G) are related by an elementary continuation if there exist isolating sets N_F and N_G for S_F and S_G with respect to F and G , respectively, as well as a pair $P = (P_1, P_2)$ which is both*

- an index pair for S_F in N_F with respect to F , and
- an index pair for S_G in N_G with respect to G .

More generally, let $F, G : X \multimap X$ denote two homotopic lower semicontinuous multivalued maps with closed and acyclic values. We say that isolated invariant sets S_F and S_G for F and G , respectively, are related by continuation, if there exists a fence $F = F_0 \leq F_1 \geq F_2 \leq \dots \leq F_k = G$ of lower semicontinuous multivalued maps $X \multimap X$ with closed and acyclic values, as well as isolated invariant sets S_i for F_i , for $0 \leq i \leq k$, such that $S_0 = S_F$, $S_k = S_G$, and (S_i, F_i) , (S_{i+1}, F_{i+1}) are related by an elementary continuation for each $0 \leq i < k$.

As in the classical case, we then have the following central result.

Proposition 7.5 (Continuation). *Let $F, G : X \multimap X$ be homotopic lower semicontinuous multivalued maps with closed and acyclic values, and let S_F and S_G be isolated invariant sets for F and G , respectively, which are related by continuation. Then the Conley index $C(S_F, F)$ is isomorphic to the Conley index $C(S_G, G)$.*

Proof: We can assume without loss of generality that $F \leq G$, and that S_F and S_G are related by an elementary continuation. Let N_F, N_G , and P be as in Definition 7.4. Since we have $F \leq G$, one obtains the inclusion

$$\begin{aligned} \overline{P}_i^F &= P_i \cup \text{cl}(F(P_1) \setminus N_F) = P_i \cup \text{cl}(F(P_1) \setminus P_1) \\ &\subseteq P_i \cup \text{cl}(G(P_1) \setminus P_1) = P_i \cup \text{cl}(G(P_1) \setminus N_G) = \overline{P}_i^G. \end{aligned}$$

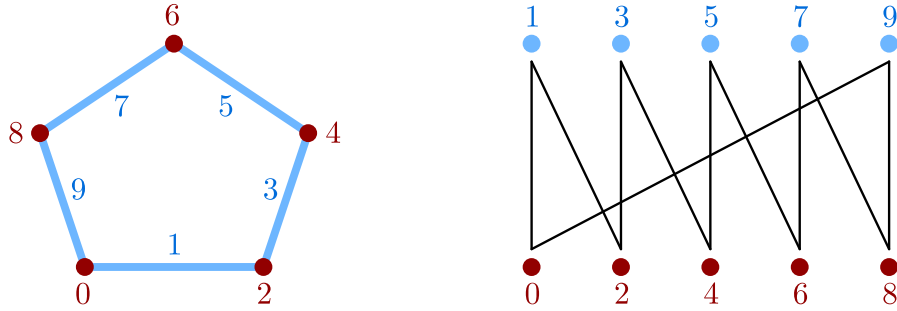


FIGURE 5. The finite T_0 topological space X used in Example 7.6. The left panel shows a simplicial complex in the form of a pentagon, given by five vertices and five edges. Using the order given by the face relationship, one obtains the ten-point finite topological space X , which is shown in the right panel via its poset representation.

Thus we have a (non-commutative) diagram

$$\begin{array}{ccc}
 & \overline{P}^F & \\
 F_P \circlearrowleft & \downarrow j & \circlearrowright \iota_{P,F} \\
 P & & P \\
 G_P \circlearrowleft & \downarrow \iota_{P,G} & \circlearrowright \\
 & \overline{P}^G &
 \end{array}$$

in which j denotes inclusion. According to Lemma 6.2 one has $j_*(F_P)_* = (jF_P)_*$ as a map from $H_*(P)$ to $H_*(\overline{P}^G)$. Moreover, our assumption $F \leq G$ immediately implies $jF_P \leq G_P$, and therefore Lemma 7.3 yields $(jF_P)_* = (G_P)_*$. Since the right triangle is in fact commutative, the map j_* is an isomorphism. Thus the index map $I_{P,F}$ of P with respect to F is given by

$$(\iota_{P,F})_*^{-1}(F_P)_* = (\iota_{P,F})_*^{-1}j_*^{-1}j_*(F_P)_* = (\iota_{P,G})_*^{-1}(G_P)_* = I_{P,G},$$

and this furnishes in particular $L(H_*(P), I_{P,F}) = L(H_*(P), I_{P,G})$. In other words, the Conley indices $C(S_F, F)$ and $C(S_G, G)$ are isomorphic. \square

To close this section, we present a detailed example which illustrates the concept of continuation, and also provides further insight into isolated invariant sets and their Conley indices.

Example 7.6 (Continuation of isolated invariant sets). For this example, we let X denote the finite topological space which is generated by a simplicial representation of a pentagon, as shown in Figure 5. Using the Alexandrov topology induced by the face relation, one obtains the ten-point topological space X indicated in the

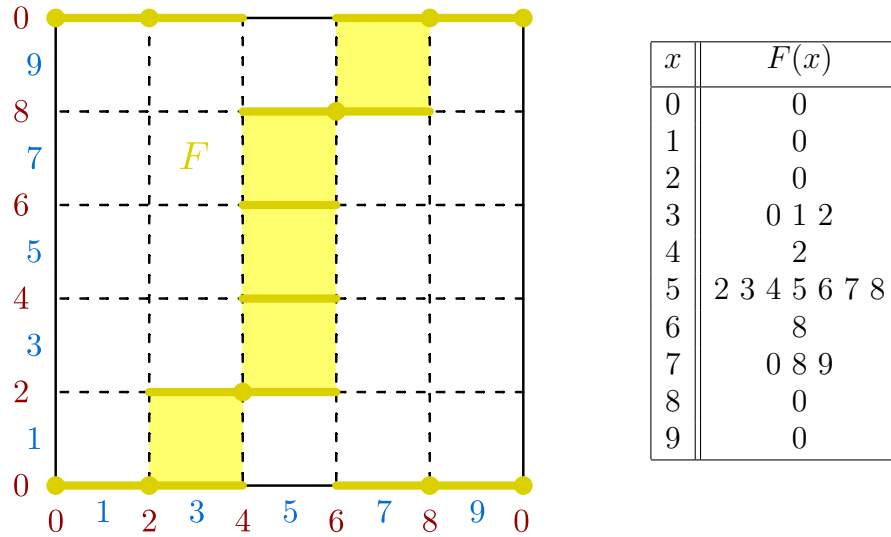


FIGURE 6. Definition of the multivalued map $F : X \multimap X$. The left image shows the graph of F . For this, we represent the pentagon from Figure 5 as a line segment, whose end points are identified. The table on the right lists all function values $F(x)$.

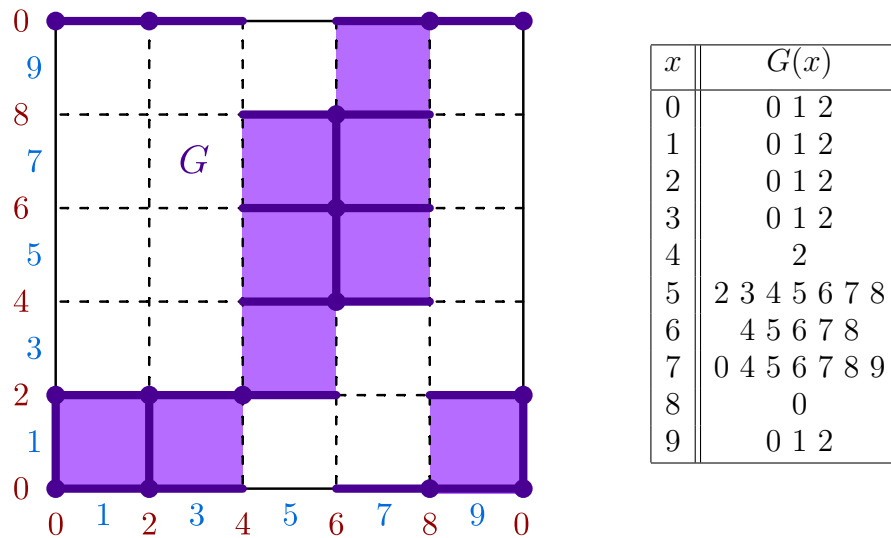


FIGURE 7. Definition of the multivalued map $G : X \multimap X$. The left image shows the graph of G . As before, the pentagon from Figure 5 is represented by a line segment with identified end points. The table on the right lists all function values $G(x)$.

right panel of the figure as a poset. Note that we can identify X with the set \mathbb{Z}_{10} , where the topology is given as in the poset.

On the topological space X , we consider the two multivalued maps $F : X \multimap X$ and $G : X \multimap X$ which are defined in the tables in Figures 6 and 7, respectively. In addition, these two figures show the graphs of these maps, where we represent the pentagon from Figure 5 as a line segment, whose end points correspond to 0 and are identified. Both maps are lower semicontinuous and have closed and acyclic values. In addition, one can easily see that both maps give rise to a Morse decomposition with two isolated invariant sets, namely

$$\begin{aligned} S_F &= \{0\} & \text{and} & & R_F &= \{5\} & \text{for } F, \text{ and} \\ S_G &= \{0, 1, 2\} & \text{and} & & R_G &= \{5, 6, 7\} & \text{for } G. \end{aligned}$$

We claim that the isolated invariant sets (S_F, F) and (S_G, G) are related by an elementary continuation. For this, we use the isolating sets $N_F = N_G = \{0, 1, 2\}$, as well as the topological pair $P = (P_1, P_2)$ with $P_1 = \{0, 1, 2\}$ and $P_2 = \emptyset$. Then one can easily see that P is an index pair for S_F in N_F with respect to F , as well as for S_G in N_G with respect to G . In addition, the definitions of F and G immediately imply $F \leq G$, which furnishes our claim. Thus, in view of Proposition 7.5 the Conley indices $C(S_F, F)$ and $C(S_G, G)$ are isomorphic. We leave it to the reader to verify that the only nontrivial homology group occurs in dimension zero, that it is one-dimensional, and that the index map is the identity. In other words, both isolated invariant sets have the Conley index of an attracting fixed point. We note that also (R_F, F) and (R_G, G) are related by an elementary continuation, but leave the verification of this and the index computation to the reader.

Yet, even more is true. Recall that we use the representation $X = \mathbb{Z}_{10}$ for our underlying topological space X . By using addition and subtraction modulo 10 we can then define the maps $F_i : X \multimap X$ via

$$(9) \quad \begin{aligned} F_i(a) &= F(a - i) + i & \subseteq X & \text{for even } i \in \mathbb{Z}_{10}, \\ F_i(a) &= G(a - i + 1) + i - 1 & \subseteq X & \text{for odd } i \in \mathbb{Z}_{10}, \end{aligned}$$

for every $a \in X$. These definitions give a fence $F_0 \leq F_1 \geq F_2 \leq \dots F_9 \geq F_0$ of lower semicontinuous multivalued maps with closed and acyclic values. By suitably adapting the argument from above, one can show that for odd i the map F_i has the isolated invariant set $S_{F_i} = \{i - 1, i, i + 1\}$. Furthermore, this set is related by an elementary continuation to both the isolated invariant set $S_{F_{i-1}} = \{i - 1\}$ for F_{i-1} , as well as to the isolated invariant set $S_{F_{i+1}} = \{i + 1\}$ for F_{i+1} . This in turn shows for example that $S_{F_0} = \{0\}$ and $S_{F_4} = \{4\}$ are related by continuation. This is illustrated in Figure 8, where we only depict the first six functions of the fence, and indicate the isolated invariant sets in orange.

8. FUTURE WORK AND OPEN PROBLEMS

In this paper, we have developed a notion of isolated invariant sets and Conley index for multivalued maps on finite topological spaces. Our theory requires these maps to be lower semicontinuous with closed and acyclic values. In addition, we

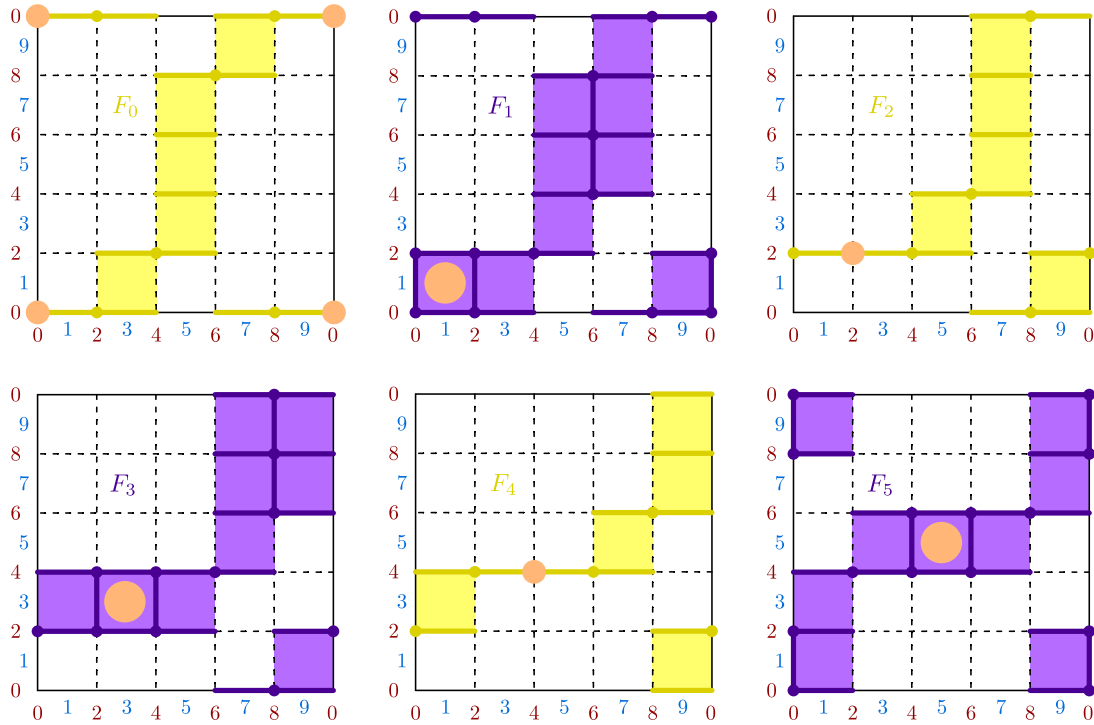


FIGURE 8. A sample fence $F_0 \leq F_1 \geq F_2 \leq \dots$ of lower semi-continuous multivalued maps $F_i : X \multimap X$ with closed and acyclic values, as defined in (9). The panels depict the first six functions of the fence. The associated isolated invariant sets S_{F_i} are indicated in orange, and they are related by continuation.

have established first properties of these objects, which mimic the corresponding results in the setting of classical dynamics. We would like to point out, however, that crucial assumptions concerning isolation had to be completely changed, due to poor separation in finite topological spaces. In addition, due to space constraints, we have omitted a number of properties of the Conley index, such as for example its additivity, and how it can be used to detect heteroclinic orbits.

While the results of this paper are very general and should be useful in a number of applied situations, we would like to close with a comment on one unresolved issue. To explain this in more detail, recall that classical dynamics can be broadly divided into continuous-time and discrete-time. As we saw earlier in this paper, on finite topological spaces the continuous-time analogue is trivial. Nevertheless, there is a dynamical theory which mimics the behavior of flows, and it is based on the concepts of combinatorial vector and multivector fields, see [17, 18, 24, 32]. In these approaches, the flow-like behavior is achieved by requiring solutions to move between adjacent elements of the space via their shared boundary. In contrast,

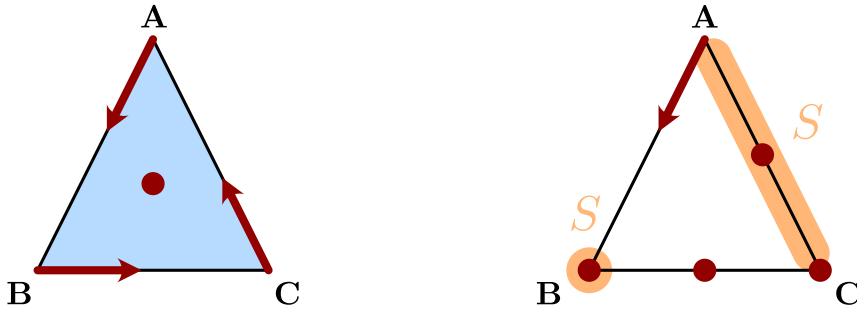


FIGURE 9. Two sample combinatorial vector fields in the sense of Forman. While the one depicted on the left can be represented via an admissible multivalued map $F : X \multimap X$ on the underlying finite topological space with the same overall dynamics, this is not possible for the vector field shown on the right. There exists no lower semicontinuous $G : Y \multimap Y$ with closed and acyclic values for which the set $S = \{B, AC\}$ is an isolated invariant set, and such that the map G has the same Morse graph as the indicated combinatorial vector field.

the results of the present paper allow for large jumps in the orbits via iteration of a multivalued map, i.e., our results mimic the discrete-time case.

It is natural to wonder what the relationship is between combinatorial vector and multivector fields, and the theory of this paper. For classical dynamics it has been shown in [27, 28] that every isolated invariant set for a continuous-time dynamical system is also an isolated invariant set for the discrete-time time-one-map. In this sense, continuous-time dynamical systems can also be studied via discrete-time results. Is the same true in the case of combinatorial vector fields? To illustrate this, Figure 9 shows two different combinatorial Forman vector fields. The one on the left is defined on a 2-simplex, while the one on the right is defined on a simplicial complex representing the boundary of a triangle. One can easily see that the dynamics of the left vector field can equivalently be described by a multivalued map $F : X \multimap X$, where X denotes the associated seven-point finite space. One just has to map every vertex to its opposite edge, every edge to everything along the boundary except itself, and the triangle to everything — and the resulting Morse graph induced by F is the same as the Morse graph associated with the depicted combinatorial vector field. However, this is not possible for the example on the right. If Y denotes the six-point finite space given by the boundary of the triangle, then one can show that there exists no lower semicontinuous multivalued map $G : Y \multimap Y$ with closed and acyclic values for which the set $S = \{B, AC\}$ (consisting of a vertex and the opposite edge) is an isolated invariant set, and such that the Morse graph of G equals the Morse graph of the indicated Forman vector field. This failure is due to our last two

requirements on G . It is therefore an interesting open problem as to whether our theory could be generalized to allow for a larger class of multivalued maps.

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