The complex of partial bases of a free group

Iván Sadofschi Costa

Abstract

We prove that the simplicial complex whose simplices are the nonempty partial bases of $\mathbb{F}_n$ is homotopy equivalent to a wedge of $(n-1)$-spheres. Moreover, we show that it is Cohen–Macaulay.

1. *Introduction*

The curve complex $\mathcal{C}(S_g)$ of an oriented surface S_g of genus g was introduced by Harvey as an analogue of Tits buildings for the mapping class group $\mathrm{Mod}(S_g)$. Harer proved that $\mathcal{C}(S_g)$ is homotopy equivalent to a wedge of $(g-1)$-spheres [**7**]. Masur and Minsky proved that $\mathcal{C}(S_g)$ is hyperbolic [**12**]. The curve complex became a fundamental object in the study of $\mathrm{Mod}(S_g)$. Since there is an analogy between $\mathrm{Aut}(\mathbb{F}_n)$ and $\mathrm{Mod}(S_g)$, it is natural to seek for an analogue of $\mathcal{C}(S_g)$ in this context. There are many candidates that share properties with the curve complex.

One of these analogues is the poset $\mathrm{FC}(\mathbb{F}_n)$ of proper free factors of $\mathbb{F}_n$. Hatcher and Vogtmann [**8**] proved that its order complex $\mathcal{K}(\mathrm{FC}(\mathbb{F}_n))$ is Cohen–Macaulay (in particular, that it is homotopy equivalent to a wedge of $(n-2)$-spheres). Bestvina and Feighn [**3**] proved that $\mathcal{K}(\mathrm{FC}(\mathbb{F}_n))$ is hyperbolic. Subsequently, different proofs of this fact appeared in [**10**] and [**9**].

Other natural analogues are defined in terms of partial bases. A *partial basis* of a free group $\mathbb{F}$ is a subset of a basis of $\mathbb{F}$. Day and Putman [**5**] defined the complex $\mathcal{B}(\mathbb{F}_n)$ whose simplices are sets $\{C_1,\dots,C_k\}$ of conjugacy classes of $\mathbb{F}_n$ such that there exists a partial basis $\{v_1,\dots,v_k\}$ with $C_i = [\![v_i]\!]$ for $1 \leq i \leq k$. They proved that $\mathcal{B}(\mathbb{F}_n)$ is 0-connected for $n \geq 2$ and 1-connected for $n \geq 3$ [**5**, Theorem A], that a certain quotient is $(n-2)$-connected [**5**, Theorem B] and they conjectured that $\mathcal{B}(\mathbb{F}_n)$ is $(n-2)$-connected [**5**, Conjecture 1.1]. As an application, they used $\mathcal{B}(\mathbb{F}_n)$ to prove that the Torelli subgroup is finitely generated.

In this paper we study the simplicial complex $\mathrm{PB}(\mathbb{F}_n)$ with simplices the nonempty partial bases of $\mathbb{F}_n$. Our main result is the following.

Theorem 6.2. *The complex* $\mathrm{PB}(\mathbb{F}_n)$ *is* $(n-1)$*-spherical. Moreover, it is Cohen–Macaulay.*

In Section 2 we review some basic definitions and results on posets.

In Section 3 we prove Theorem 3.1, which is a version of a result due to Quillen [**17**, Theorem 9.1]. Our version produces an explicit basis of the top homology group of X. The proof is based on Piterman's proof of Quillen's Theorem [**16**, Teorema 2.1.28], which uses Barmak–Minian's non-Hausdorff mapping cylinder argument [**2**].

In Section 4 we review the basic facts on partial bases and free factors and we state earlier results by Hatcher and Vogtmann [**8**] which relate different complexes and posets associated to a free group.

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In Section 5 we prove that the link $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ is connected if $n - |B_0| = 2$ and simply connected if $n - |B_0| = 3$.

In Section 6 we prove Theorem 6.2. The key idea is to compare the link $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ (which is $(n - |B_0| - 1)$-dimensional) with $\mathrm{FC}(\mathbb{F}_n)_{>\langle B_0\rangle}$ (which is $(n - |B_0| - 2)$-dimensional). In order do this, we have to consider the $(n - |B_0| - 2)$-skeleton of $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. Finally, using the basis given by Theorem 3.1, we can understand what happens when we pass from $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)})$ to $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. We proceed by induction on $n - |B_0|$.

Note. The author is grateful to Andrew Putman, who was the referee for this paper and pointed out that the original proof for Propositions 5.4 and 5.9 could be greatly simplified. To this end, he provided the statement of Theorems 5.3 and 5.7 and a proof of Theorem 5.7.

The original proof of Theorem 6.2 appeared in the author's thesis [**18**]. One of the main steps in that proof was to obtain a presentation for $\mathrm{SAut}(\mathbb{F}_n, B_0)$ analogous to Gersten's classical presentation of $\mathrm{SAut}(\mathbb{F}_n)$ [**6**]. This involved many computations using McCool's method [**13**, **14**] and the Reidemeister–Schreier method. Then to prove Propositions 5.4 and 5.9 we had to imitate the proof of [**5**, Theorem A] using the presentation for $\mathrm{SAut}(\mathbb{F}_n, B_0)$ instead of Gersten's presentation.

Another proof of Theorem 6.2 was obtained independently by Andrew Putman and Neil Fullarton about the same time as we finished the original version of this paper.

2. *Some preliminaries on posets*

If K is a simplicial complex, $\mathcal{X}(K)$ denotes the *face poset* of K, that is the poset of simplices of K ordered by inclusion. If X is a poset $\mathcal{K}(X)$ denotes the *order complex* of X, that is the simplicial complex with simplices the chains of X. Both the face poset and the order complex are functorial. The complex $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision K' of K. Throughout the paper we consider homology with integer coefficients and $\widetilde{C}_\bullet(K)$ is the augmented simplicial chain complex. Let $\lambda\colon \widetilde{C}_\bullet(K) \to \widetilde{C}_\bullet(K')$ be the subdivision operator $\alpha \mapsto \alpha'$. If X is a poset X^{op} denotes the poset X with the opposite order and we write $\widetilde{H}_\bullet(X)$ for the homology $\widetilde{H}_\bullet(\mathcal{K}(X))$. We thus have $\widetilde{H}_\bullet(X) = \widetilde{H}_\bullet(X^{\mathrm{op}})$. Recall that if X is a poset and $x \in X$ the *height* of x denoted $h(x)$ is the dimension of $\mathcal{K}(X_{\le x})$. If K is a simplicial complex we can identify $\mathcal{X}(\mathrm{lk}(\sigma, K)) = \mathcal{X}(K)_{>\sigma}$ by the map $\tau \mapsto \sigma \cup \tau$. If K_1, K_2 are simplicial complexes we have $\widetilde{C}_\bullet(K_1 * K_2) = \widetilde{C}_\bullet(K_1) * \widetilde{C}_\bullet(K_2)$ (here $*$ denotes the join of chain complexes, defined as the suspension of the tensor product). Recall that the *join* of two posets X_1, X_2 is the disjoint union of X_1 and X_2 keeping the given ordering within X_1 and X_2 and setting $x_1 < x_2$ for every $x_1 \in X_1$ and $x_2 \in X_2$ [**1**, Definition 2.7.1]. We have $\mathcal{K}(X_1 * X_2) = \mathcal{K}(X_1) * \mathcal{K}(X_2)$. If X is a poset and $x \in X$, then the *link* $\mathrm{lk}(x, X) = X_{<x} * X_{>x}$ is the subposet of X consisting of elements that can be compared with x. We have $\mathrm{lk}(x, \mathcal{K}(X)) = \mathcal{K}(\mathrm{lk}(x, X))$.

Definition 2.1. Let $f\colon X \to Y$ be an order preserving map. The *non-Hausdorff mapping cylinder* $M(f)$ is the poset given by the following order on the disjoint union of X and Y. We keep the given ordering within X and Y and for $x \in X$, $y \in Y$ we set $x < y$ in $M(f)$ if $f(x) \le y$ in Y.

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & M(f) \\ & Y \arrow[u, hook, "i"'] \end{tikzcd}$$

In the context of preorders (equivalently, Alexandrov spaces) this construction is an analogue of the usual mapping cylinder. If $j\colon X \to M(f)$, $i\colon Y \to M(f)$ are the inclusions, then $\mathcal{K}(i)$ is a homotopy equivalence. Since $j \leq if$ we also have $\mathcal{K}(j) \simeq \mathcal{K}(if)$. For more details see [**1**, 2.8].

DEFINITION 2.2. A simplicial complex K is said to be *n-spherical* if $\dim(K) = n$ and K is $(n-1)$-connected. We say that K is *homologically n-spherical* if $\dim(K) = n$ and $\widetilde{H}_i(K) = 0$ for every $i < n$. Recall that K is *Cohen–Macaulay* if K is n-spherical and the link $\operatorname{lk}(\sigma, K)$ is $(n - \dim(\sigma) - 1)$-spherical for every simplex $\sigma \in K$. A poset X is *(homologically) n-spherical* if $\mathcal{K}(X)$ is (homologically) n-spherical.

REMARK 2.3. Note that from this definition it follows that a homologically 0-spherical complex K is nonempty.

Recall that if $f\colon X \to Y$ is a map of posets, the *fiber of f under y* is the subposet $f/y = \{x \,:\, f(x) \leq y\} \subseteq X$.

DEFINITION 2.4. An order preserving map $f\colon X \to Y$ is *(homologically) n-spherical*, if $Y_{>y}$ is (homologically) $(n - h(y) - 1)$-spherical and f/y is (homologically) $h(y)$-spherical for all $y \in Y$.

PROPOSITION 2.5. *Let $f\colon X \to Y$ be homologically n-spherical. Then for every $x \in X$ we have $h(f(x)) \geq h(x)$.*

Proof. Let $y = f(x)$. Since $x \in f/y$ and f/y is homologically $h(y)$-spherical we have $h(x) \leq \dim(f/y) = h(y)$. □

PROPOSITION 2.6. *A homologically n-spherical map $f\colon X \to Y$ is surjective.*

Proof. Let $y \in Y$ and let $r = h(y)$. Since f/y is homologically r-spherical, $\dim(f/y) = r$. So there is $x \in f/y$ with $h(x) = r$. Let $\widetilde{y} = f(x)$. We obviously have $\widetilde{y} \leq y$. By Proposition 2.5 we have $h(\widetilde{y}) \geq h(x) = r$. Therefore we have $\widetilde{y} = y$. □

From the definition of spherical map we also have the following:

PROPOSITION 2.7. *If $f\colon X \to Y$ is homologically n-spherical then* $\dim(X) = \dim(Y) = n$.

3. *A variation on Quillen's result*

The first part of the following result is due to Quillen [**17**, Theorem 9.1]. To prove the second part we build on the proof of the first part given by Piterman [**16**, Teorema 2.1.28]. The idea of considering the non-Hausdorff mapping cylinder of $f\colon X \to Y$ and removing the points of Y from bottom to top is originally due to Barmak and Minian [**2**]. The word *homologically* in the statement of Theorem 3.1 is parenthesized because the result holds both with and without it. We prove the "homologically spherical" version of the result, the other version then follows from Quillen's original result [**17**, Theorem 9.1].

THEOREM 3.1. *Let $f\colon X \to Y$ be a (homologically) n-spherical map between posets such that Y is (homologically) n-spherical. Then X is (homologically) n-spherical, $f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ is an epimorphism and*

$$\widetilde{H}_n(X) \simeq \widetilde{H}_n(Y) \oplus \bigoplus_{y\in Y} \widetilde{H}_{h(y)}(f/y) \otimes \widetilde{H}_{n-h(y)-1}(Y_{>y}).$$

Moreover suppose that $X = \mathcal{X}(K)$ for a certain simplicial complex K and
(i) If $f(\sigma_1) \leq f(\sigma_2)$ then $\operatorname{lk}(\sigma_2, K) \subseteq \operatorname{lk}(\sigma_1, K)$.
(ii) If $f(\sigma_1) \leq f(\sigma_2)$ and $f(\tau_1) \leq f(\tau_2)$ then $f(\sigma_1 \cup \tau_1) \leq f(\sigma_2 \cup \tau_2)$, whenever $\sigma_1 \cup \tau_1, \sigma_2 \cup \tau_2 \in K$.
(iii) For every $y \in Y$ and every $\sigma \in f^{-1}(y)$, the map $f_*\colon \widetilde{H}_{n-h(y)-1}(X_{>\sigma}) \to \widetilde{H}_{n-h(y)-1}(Y_{>y})$ is an epimorphism.

Then we can produce a basis of $\widetilde{H}_n(K)$ as follows. Since f_* is an epimorphism, we can take $\{\gamma_i\}_{i\in I} \subseteq \widetilde{H}_n(K)$ such that $\{f_*(\gamma_i')\}_{i\in I}$ is a basis of $\widetilde{H}_n(Y)$. In addition, for every $y \in Y$ we choose $x \in f^{-1}(y)$ and we consider the subcomplexes $K_y = \{\sigma \,:\, f(\sigma) \leq y\}$ and $K^y = \operatorname{lk}(x, K)$. By (i), K^y does not depend on the choice of x. Also by (i), $K_y * K^y$ is a subcomplex of K. Let $\widetilde{f}\colon \mathcal{X}(K^y) \to Y_{>y}$ be defined by $\widetilde{f}(\tau) = f(x \cup \tau)$. By (ii), $\widetilde{f}$ does not depend on the choice of x. We take a basis $\{\alpha_i\}_{i\in I_y}$ of $\widetilde{H}_{h(y)}(K_y)$ and using (iii) we take $\{\beta_j\}_{j\in J_y} \subseteq \widetilde{H}_{n-h(y)-1}(K^y)$ such that $\{\widetilde{f}_*(\beta_j')\}_{j\in J_y}$ is a basis of $\widetilde{H}_{n-h(y)-1}(Y_{>y})$. Then

$$\{\gamma_i \,:\, i \in I\} \cup \{\alpha_i * \beta_j \,:\, y \in Y, i \in I_y, j \in J_y\}$$

is a basis of $\widetilde{H}_n(K)$.

Proof. Let $M = M(f)$ be the non-Hausdorff mapping cylinder of f and let $j\colon X \to M$, $i\colon Y \to M$ be the inclusions. We have $j_* = i_* f_*$. Since f is n-spherical we have $\dim(M) = n + 1$.

Let $Y_r = \{y \in Y \,:\, h(y) \geq r\}$. For each r we consider the subposet $M_r = X \cup Y_r$ of M. We have $M_{n+1} = X$ and $M_0 = M$. Let

$$L_r = \coprod_{h(y)=r} \operatorname{lk}(y, M_r) = \coprod_{h(y)=r} f/y * Y_{>y}.$$

For each r we consider the Mayer-Vietoris sequence for the open covering $\{U, V\}$ of $\mathcal{K}(M_{r-1})$ given by

$$\begin{aligned} U &= \mathcal{K}(M_{r-1}) - \{y \in Y \,:\, h(y) = r-1\} \\ V &= \bigcup_{h(y)=r-1} \overset{\circ}{\operatorname{st}}(y, \mathcal{K}(M_{r-1})) \end{aligned}$$

where $\overset{\circ}{\operatorname{st}}(v, K)$ denotes the open star of v in K. We have homotopy equivalences $U \simeq \mathcal{K}(M_r)$ and $U \cap V \simeq \mathcal{K}(L_r)$. Since f is a homologically n-spherical map $\operatorname{lk}(y, M_{r-1})$ is homologically n-spherical, so the homology of L_r is concentrated in degrees 0 and n. The tail of the sequence is

$0 \to \widetilde{H}_{n+1}(M_r) \to \widetilde{H}_{n+1}(M_{r-1})$ and since $\widetilde{H}_{n+1}(M_0) = \widetilde{H}_{n+1}(Y) = 0$ we have $\widetilde{H}_{n+1}(M_r) = 0$ for every r. We also have isomorphisms $\widetilde{H}_i(M_r) \to \widetilde{H}_i(M_{r-1})$ if $0 \le i \le n-1$ (since L_r may not be connected, we have to take some care when $i = 0, 1$). From this we conclude that X is homologically n-spherical and we also have short exact sequences

$$0 \to \widetilde{H}_n(L_n) \xrightarrow{i_{n+1}} \widetilde{H}_n(X) \xrightarrow{p_{n+1}} \widetilde{H}_n(M_n) \to 0$$
$$\cdots$$
$$0 \to \widetilde{H}_n(L_{r-1}) \xrightarrow{i_r} \widetilde{H}_n(M_r) \xrightarrow{p_r} \widetilde{H}_n(M_{r-1}) \to 0$$
$$\cdots$$
$$0 \to \widetilde{H}_n(L_0) \xrightarrow{i_1} \widetilde{H}_n(M_1) \xrightarrow{p_1} \widetilde{H}_n(M) \to 0.$$

Here the map i_r is the map induced by the map $L_{r-1} \to M_r$ given by the coproduct of the inclusions $\mathrm{lk}(y, M_{r-1}) \to M_r$ and the map p_r is induced by the inclusion $M_r \to M_{r-1}$. By induction on r, it follows that these sequences are split and that $\widetilde{H}_n(M_r)$ is free for every r. We have

$$\widetilde{H}_n(L_r) = \bigoplus_{h(y)=r} \widetilde{H}_r(f/y) \otimes \widetilde{H}_{n-r-1}(Y_{>y})$$

and therefore using the isomorphism $i_*\colon \widetilde{H}_n(Y) \to \widetilde{H}_n(M)$ we obtain

$$\widetilde{H}_n(X) = \widetilde{H}_n(Y) \bigoplus_{y \in Y} \widetilde{H}_{h(y)}(f/y) \otimes \widetilde{H}_{n-h(y)-1}(Y_{>y}).$$

Now $\widetilde{H}_n(j) = p_1 \cdots p_n$ is an epimorphism so $f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ is also an epimorphism. We will need the following claim which is proved at the end of the proof.

Claim. Let $y \in Y$, $r = h(y)$. Then for every $\alpha \in Z_r(K_y)$, $\beta \in Z_{n-r-1}(K^y)$ we have $[(\alpha * \beta)'] = [\alpha' * \widetilde{f}_*(\beta')]$ in $\widetilde{H}_n(M_{r+1})$.

Let $j_r\colon X \to M_r$ be the inclusion. We have $j_{r*} = p_{r+1} \circ \ldots \circ p_{n+1}$. Now by induction on r we prove that for $0 \le r \le n+1$

$$\{j_{r*}(\gamma_i') \;:\; i \in I\} \cup \{j_{r*}((\alpha_i * \beta_j)') \;:\; y \in Y, i \in I_y, j \in J_y, h(y) < r\}$$

is a basis of $\widetilde{H}_n(M_r)$. Since $j_0 = j$ and $j_* = i_* f_*$ it holds when $r = 0$. Now, assuming it holds for r, we prove it also holds for $r+1$. By the split exact sequence obtained above, it suffices to check that

$$\{j_{r+1}((\alpha_i * \beta_j)') \;:\; i \in I_y, j \in J_y\}$$

is a basis of $\widetilde{H}_n(\mathrm{lk}(y, M_r))$ for every $y \in Y$ of height r. Now in $\widetilde{H}_n(M_{r+1})$ we have

$$j_{r+1}((\alpha_i * \beta_j)') = (\alpha_i * \beta_j)' = \alpha_i' * \widetilde{f}_*(\beta_j')$$

and the induction is complete, for $\{\alpha_i' * \widetilde{f}_*(\beta_j')\}_{i \in I_y, j \in J_y}$ is a basis of $\widetilde{H}_n(\mathrm{lk}(y, M_r))$. We have $j_{n+1} = 1_X$ and taking $r = n+1$ we get the desired basis of $\widetilde{H}_n(K)$.

Proof (claim). We consider chain maps $\phi_1, \phi_2\colon \widetilde{C}_\bullet(K_y * K^y) \to \widetilde{C}_\bullet(\mathcal{K}(M_{r+1}))$ defined by

$$\phi_1\colon \widetilde{C}_\bullet(K_y * K^y) \hookrightarrow \widetilde{C}_\bullet(K) \xrightarrow{\lambda} \widetilde{C}_\bullet(\mathcal{K}(X)) \hookrightarrow \widetilde{C}_\bullet(\mathcal{K}(M_{r+1}))$$

and

$$\phi_2\colon \widetilde{C}_\bullet(K_y * K^y) = \widetilde{C}_\bullet(K_y) * \widetilde{C}_\bullet(K^y) \xrightarrow{\lambda * \lambda} \widetilde{C}_\bullet(\mathcal{K}(f/y)) * \widetilde{C}_\bullet(\mathcal{K}(\mathcal{X}(K^y))) \xrightarrow{1 * \widetilde{f}_*}$$
$$\widetilde{C}_\bullet(\mathcal{K}(f/y)) * \widetilde{C}_\bullet(\mathcal{K}(Y_{>y})) = \widetilde{C}_\bullet(\mathcal{K}(f/y * Y_{>y})) \hookrightarrow \widetilde{C}_\bullet(\mathcal{K}(M_{r+1})).$$

Note that $\phi_1(\alpha * \beta) = (\alpha * \beta)'$ and $\phi_2(\alpha * \beta) = \alpha' * \widetilde{f}_*(\beta')$. Using the Acyclic Carrier Theorem [**15**, Theorem 13.3] we will prove that ϕ_1 and ϕ_2 are chain homotopic. We define an acyclic carrier $\Phi\colon K_y * K^y \to \mathcal{K}(M_{r+1})$. If $\sigma \cup \tau$ is a simplex in $K_y * K^y$, with $\sigma \in K_y$ and $\tau \in K^y$, we define

$$\Phi(\sigma \cup \tau) = \begin{cases} \mathcal{K}\left(M_{r+1 \leq \widetilde{f}(\tau)}\right) & \text{if } \tau \neq \varnothing. \\ \mathcal{K}\left(M_{r+1 \leq \sigma}\right) & \text{if } \tau = \varnothing. \end{cases}$$

If $\sigma_1 \cup \tau_1 \subseteq \sigma_2 \cup \tau_2$ are simplices of $K_y * K^y$ where $\sigma_i \in K_y$ and $\tau_i \in K^y$ are possibly empty, we have $\sigma_1 \subseteq \sigma_2$ and $\tau_1 \subseteq \tau_2$. In M we have $\sigma_1 \leq \sigma_2 \leq y \leq \widetilde{f}(\tau_1) \leq \widetilde{f}(\tau_2)$ so in any case $\Phi(\sigma_1 \cup \tau_1) \subseteq \Phi(\sigma_2 \cup \tau_2)$. So Φ is a carrier. It is obviously acyclic.

Now we prove that ϕ_1 and ϕ_2 are carried by Φ. To show that ϕ_1 is carried by Φ we need to show that $\phi_1(\sigma \cup \tau) = (\sigma \cup \tau)'$ is supported on $\Phi(\sigma \cup \tau)$. If τ is empty it is clear. If τ is nonempty, we consider $x \in f^{-1}(y)$. In M, by (ii) we have $\sigma \cup \tau \leq f(\sigma \cup \tau) \leq f(x \cup \tau) = \widetilde{f}(\tau)$. Therefore $(\sigma \cup \tau)'$ is supported on $\Phi(\sigma \cup \tau) = \mathcal{K}\left(M_{r+1 \leq \widetilde{f}(\tau)}\right)$. It is easy to see that ϕ_2 is also carried by Φ.

Finally by the Acyclic Carrier Theorem we have

$$[(\alpha * \beta)'] = [\phi_1(\alpha * \beta)] = [\phi_2(\alpha * \beta)] = [\alpha' * \widetilde{f}_*(\beta')]$$

and we are done. □

□

Remark 3.2. We can consider $\varphi\colon X \to M_{r+1}$ given by

$$\varphi(x) = \begin{cases} x & \text{if } h(x) < r+1 \\ f(x) & \text{if } h(x) \geq r+1 \end{cases}.$$

Then $j_{r+1} \leq \varphi$. Therefore $j_{r+1*} \simeq \mathcal{K}(\varphi)$ and $j_* = \varphi_*$. In the previous proof we actually have $\varphi_*((\alpha * \beta)') = \alpha' * \widetilde{f}_*(\beta')$ in $Z_n(M_{r+1})$.

4. *Partial bases and free factors*

Recall that a subgroup H of a group G is a *free factor* if there is a subgroup $K \leq G$ such that the natural map $H * K \to G$ is an isomorphism. A *partial basis* of a free group $\mathbb{F}_n$ is a subset of a basis of $\mathbb{F}_n$. If H is a free factor of $\mathbb{F}_n$ and B is a basis of H then B is a partial basis of $\mathbb{F}_n$. If B is a partial basis of $\mathbb{F}_n$ then $H = \langle B \rangle$ is a free factor of $\mathbb{F}_n$.

Proposition 4.1 ([**11**, p. 117]). *Suppose H is a free factor of $\mathbb{F}_n$ and $K \leq H$. Then K is a free factor of H if and only if K is a free factor of $\mathbb{F}_n$.*

The *free factor poset* $\mathrm{FC}(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$ is the poset of nontrivial proper free factors of $\mathbb{F}_n$ ordered by inclusion. This poset was studied by Hatcher and Vogtmann [**8**]. There is an order preserving map

$$\begin{aligned} g\colon \mathcal{X}\left(\mathrm{PB}(\mathbb{F}_n)^{(n-2)}\right) &\to \mathrm{FC}(\mathbb{F}_n) \\ \sigma &\mapsto \langle \sigma \rangle \end{aligned}$$

and if B is a partial basis we have the restriction $g\colon \mathcal{X}\left(\mathrm{PB}(\mathbb{F}_n)^{(n-2)}\right)_{>B} \to \mathrm{FC}(\mathbb{F}_n)_{>\langle B\rangle}$. Instead of working with this map, we will work with the map $\widetilde{g}\colon \mathcal{X}\left(\mathrm{lk}\left(B, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}\right)\right) \to \mathrm{FC}(\mathbb{F}_n)_{>\langle B\rangle}$ given by $\sigma \mapsto \langle B \cup \sigma\rangle$ which can be identified with g.

Theorem 4.2 (Hatcher–Vogtmann, [**8**, §4]). *If $H \leq \mathbb{F}_n$ is a free factor, $\mathrm{FC}(\mathbb{F}_n)_{>H}$ is $(n - \mathrm{rk}\,(H) - 2)$-spherical.*

We will also need to consider the following simplicial complex Y with vertices the free factors of $\mathbb{F}_n$ that have rank $n-1$. A set of free factors $\{H_1, \dots, H_k\}$ is a simplex of Y if there is a basis $\{w_1, \dots, w_n\}$ of $\mathbb{F}_n$ such that for $1 \leq i \leq k$ we have $H_i = \langle w_1, \dots, \widehat{w_i}, \dots, w_n\rangle$. If $H \leq \mathbb{F}_n$ is a free factor, we can consider the full subcomplex Y_H of Y spanned by the vertices which are free factors containing H. There is another equivalent definition for Y and Y_H in terms of sphere systems, see [**8**, Remark after Corollary 3.4].

Theorem 4.3 (Hatcher–Vogtmann,[**8**, Theorem 2.4]). *Let H be a free factor of $\mathbb{F}_n$. Then Y_H is $(n - \mathrm{rk}\,(H) - 1)$-spherical.*

There is a spherical map $f\colon \mathcal{X}(Y_H^{(n-\mathrm{rk}(H)-2)}) \to (\mathrm{FC}(\mathbb{F}_n)_{>H})^{\mathrm{op}}$ that maps $\{H_1, \dots, H_k\}$ to $H_1 \cap \dots \cap H_k$. The map f was used in Hatcher and Vogtmann's proof of Theorem 4.2. The following lemma will be required in Sections 5 and 6.

Lemma 4.4. *Let $B = \{v_1, \dots, v_l\}$ be a partial basis of $\mathbb{F}_n$ and let $H = \langle B\rangle$. Let $\{H_{l+1}, \dots, H_n\}$ be a simplex of Y_H. Then B can be extended to a basis $\{v_1, \dots, v_l, w_{l+1}, \dots, w_n\}$ of $\mathbb{F}_n$ such that*

$$H_i = \langle v_1, \dots, v_l, w_{l+1}, \dots, \widehat{w_i}, \dots, w_n\rangle$$

for $l+1 \leq i \leq n$.

Proof. By definition of Y_H there is a basis $\{w_1, \dots, w_n\}$ of $\mathbb{F}_n$ such that

$$H_i = \langle w_1, \dots, \widehat{w_i}, \dots, w_n\rangle$$

for $l+1 \leq i \leq n$ and we have $H \leq \bigcap_{i=l+1}^n H_i = \langle w_1, \dots, w_l\rangle$. Then by Proposition 4.1 H is a free factor of $\langle w_1, \dots w_l\rangle$ and since the rank of both groups is l they must be equal. Thus $\{v_1, \dots, v_l, w_{l+1}, \dots, w_n\}$ is a basis of $\mathbb{F}_n$ with the desired property. □

5. *Connectivity and simple connectivity*

Definition 5.1. Let K be a CW complex. By a 0-loop in K we mean a pair of vertices of K which we think of as a map $S^0 \to K$. By a 1-loop we mean a closed edge path in K.

Definition 5.2. Let K be a CW complex. A set of 0-loops in K satisfies the π_0-*spanning property* if the space obtained from K by attaching 1-cells using these maps is 0-connected. A set of 1-loops in K satisfies the π_1-*spanning property* if the space obtained from K by attaching 2-cells using these maps is 1-connected.

The following result is immediate and can be seen as the case $n = 0$ of Theorem 3.1.

THEOREM 5.3. *Let $f\colon X \to Y$ be a 0-spherical map. If we choose:*

- *A set $\{\gamma_i\}_{i\in I}$ of 0-loops in X such that $\{f_*(\gamma_i)\}_{i\in I}$ has the π_0-spanning property in Y.*
- *For each $y \in Y$ a set $\{\alpha_i\}_{i\in I_y}$ of 0-loops with the π_0-spanning property in $|f^{-1}(y)|$.*

Then the set

$$\{\gamma_i \ : \ i \in I\} \cup \{\alpha_i \ : \ y \in Y, i \in I_y\}$$

has the π_0-spanning property in X.

Now using Theorem 5.3 we prove that in $\mathrm{PB}(\mathbb{F}_n)$ the links of $(n-3)$-simplices are 1-spherical.

PROPOSITION 5.4. *Let B_0 be a partial basis of $\mathbb{F}_n$ with $|B_0| = n-2$. Then $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ is connected.*

Proof. Let $H_0 = \langle B_0\rangle$. We will apply Theorem 5.3 to the map

$$\widetilde{g}\colon \mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}) \to \mathrm{FC}(\mathbb{F}_n)_{>H_0}.$$

We need to choose the γ_i and the α_i.

By Theorem 4.3 Y_{H_0} is connected. Thus the boundaries of the 1-simplices give a set of 0-loops in $Y_{H_0}^{(0)}$ with the π_0-spanning property. Since the map $f\colon \mathcal{X}(Y_{H_0}^{(0)}) \to (\mathrm{FC}(\mathbb{F}_n)_{>H_0})^{\mathrm{op}}$ is surjective we see that this set of 0-loops has the π_0-spanning property in $(\mathrm{FC}(\mathbb{F}_n)_{>H_0})^{\mathrm{op}}$. Now by Lemma 4.4, for each 1-simplex $\{H, H'\}$ in Y_{H_0} we can consider a 1-simplex $\{v, w\}$ in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ such that $H = \langle B_0, v\rangle$ and $H' = \langle B_0, w\rangle$. Then by applying $\widetilde{g}_*$ to the set of 0-loops $\{\gamma_i\}_{i\in I}$ given by the $\{v, w\}$ we obtain a set of 0-loops with the π_0-spanning property.

If $H \in \mathrm{FC}(\mathbb{F}_n)_{>H_0}$, we can take $\{\alpha_i\}_{i\in I_H}$ to be the set of all 0-loops in $\widetilde{g}/H$. Consider w such that $\mathbb{F}_n = \langle H, w\rangle$. If $v, v' \in \widetilde{g}/H$, then $v - w - v'$ is an edge path in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. Thus these 0-loops also are null-homotopic.

By Theorem 5.3 we have a set of 0-loops in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)})$ with the π_0-spanning property which are null-homotopic in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ and so we are done. □

If $\alpha = \{x, x'\}$ and $\beta = \{y, y'\}$ are 0-loops in X and Y respectively, the *join* $\alpha * \beta$ is the 1-loop in $X * Y$ given by $x - y - x' - y' - x$.

LEMMA 5.5. *Let X and Y be simplicial complexes of dimension 0. Suppose $\{\alpha_i\}_{i\in I}$ and $\{\beta_j\}_{j\in J}$ are sets of 0-loops in X and Y with the π_0-spanning property. Then the set of loops $\{\alpha_i * \beta_j\}_{(i,j)\in I\times J}$ has the π_1-spanning property in $X * Y$.*

Proof. Choose $x_0 \in X$, $y_0 \in Y$. Then $\{\{x_0, y\} \ : \ y \in Y\} \cup \{\{x, y_0\} \ : \ x \in X\}$ is a spanning tree for $X * Y$. Thus $\pi_1(X * Y, x_0)$ is free with basis the 1-loops $x_0 - y_0 - x - y - x_0$ with $x \neq x_0$, $y \neq y_0$. Then it is enough to show that these loops are null-homotopic in the space Z obtained from $X * Y$ by attaching 2-cells using the 1-loops $\alpha_i * \beta_j$ for $(i, j) \in I \times J$. Now since $\{\alpha_i\}_{i\in I}$ and $\{\beta_j\}_{j\in J}$ have the π_0-spanning property we can consider $x_1, \ldots, x_k = x$ and $y_1, \ldots, y_l = y$ such that for $s = 0, \ldots, k-1$ there is $i_s \in I$ such that $\{x_s, x_{s+1}\} = \alpha_{i_s}$ and for every $t = 0, \ldots, l-1$ there is $j_t \in J$ such that $\{y_t, y_{t+1}\} = \beta_{j_t}$. Now if $0 \leq t \leq l-1$, since the 1-loops $\alpha_{i_0} * \beta_{j_t}, \ldots, \alpha_{i_{k-1}} * \beta_{j_t}$ are null-homotopic in Z, the loop $\gamma_t = (x_0 - y_t - x_k - y_{t+1} - x_0)$ is null-homotopic in Z. Finally since $\gamma_0, \ldots, \gamma_{t-1}$ are null-homotopic in Z we have that $x_0 - y_0 - x_l - y_k - x_0$ is null-homotopic in Z, concluding the proof. □

In the following standard lemma $C(A)$ denotes the cone over A.

LEMMA 5.6. *Let X be a connected CW complex and let $\{A_i\}_{i\in I}$ be a family of connected subcomplexes of X. For each $i \in I$ take a set $\{\gamma_j\}_{j\in J_i}$ of 1-loops in A_i with the π_1-spanning property in A_i. Let $\widetilde{X}$ be the space obtained from X by attaching 2-cells using the maps γ_j with $i \in I$ and $j \in J_i$. Then the inclusion $X \to X \cup_{i\in I} C(A_i)$ extends to a map*

$$\widetilde{X} \to X \cup_{i\in I} C(A_i)$$

inducing an isomorphism on the fundamental group.

Proof. Let $\widetilde{A}_i = A_i \cup_{j\in J_i} e^2_j$. Then $\widetilde{X}$ is the pushout of $\coprod_{i\in I} \widetilde{A}_i \leftarrow \coprod_{i\in I} A_i \to X$. For every $i \in I$ let B_i be a contractible space obtained from $\widetilde{A}_i$ by attaching cells of dimension greater than 2. Let Y be the pushout of $\coprod_{i\in I} B_i \leftarrow \coprod_{i\in I} A_i \to X$. Then the inclusion $\widetilde{X} \hookrightarrow Y$ induces an isomorphism on π_1 since Y is obtained from $\widetilde{X}$ by attaching cells of dimension greater than 2. Now for each $i \in I$ we can extend the inclusion $A_i \hookrightarrow C(A_i)$ to a map $B_i \to C(A_i)$. Using the gluing theorem [**4**, 7.5.7] we obtain a homotopy equivalence $Y \to X \cup_{i\in I} C(A_i)$ which is the identity on X. □

THEOREM 5.7. *Let $f : X \to Y$ be a 1-spherical map. Assume Y is 1-spherical. Then X is 1-spherical and there is an epimorphism $f_* : \pi_1(X) \to \pi_1(Y)$.*

Moreover, if we choose:

- *A set $\{\gamma_i\}_{i\in I}$ of 1-loops in $\mathcal{K}(X)$ such that $\{f_*(\gamma_i)\}_{i\in I}$ has the π_1-spanning property in $\mathcal{K}(Y)$.*
- *For each $y \in Y$ a set $\{\alpha_i\}_{i\in I_y}$ of $h(y)$-loops in $\mathcal{K}(f/y)$ with the $\pi_{h(y)}$-spanning property.*
- *For each $y \in Y$ with $h(y) = 0$, a set $\{\overline{\beta}_j\}_{j\in J_y}$ of 0-loops in $\mathcal{K}(Y_{>y})$ with the π_0-spanning property.*
- *For each $y \in Y$ with $h(y) = 0$, each $i \in I_y$ and each $j \in J_y$ a 1-loop $\eta_{i,j}$ in $\mathcal{K}(X)$ as follows. If $\alpha_i = \{x, x'\}$ and $\overline{\beta}_j = \{y', y''\}$ then $\eta_{i,j}$ is the concatenation of an edge path from x to x' in $\mathcal{K}(f/y')$ and an edge path from x' to x in $\mathcal{K}(f/y'')$.*

Then the set

$$\{\gamma_i \;:\; i \in I\} \cup \{\alpha_i \;:\; y \in Y \text{ with } h(y) = 1 \text{ and } i \in I_y\}$$
$$\cup \{\eta_{i,j} \;:\; y \in Y \text{ with } h(y) = 0 \text{ and } (i, j) \in I_y \times J_y\}$$

has the π_1-spanning property in $\mathcal{K}(X)$.

Proof. We have that X is connected by Quillen's result in its original formulation (i.e. the first part of Theorem 3.1). Let $\overline{\gamma} = (y_1 - y'_1 - y_2 - y'_2 - \ldots - y_n - y'_n - y_{n+1})$ be an edge path in $\mathcal{K}(Y)$ with $h(y_i) = 0$ and $h(y'_i) = 1$. Since f/y_i is 0-spherical, we can take $x_i \in f/y_i$ for each i. Since f/y'_i is 1-spherical, we can take an edge path ξ_i from x_i to x_{i+1} in f/y'_i for each i. Then if γ is the concatenation of $\xi_1, \xi_2, \ldots, \xi_n$ we have that $f_*(\gamma)$ is path homotopic to $\overline{\gamma}$. Thus $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective.

Now we prove the second part of the result. The proof is similar to the proof of Theorem 3.1. Let $M = M(f)$ and consider the subposets M_r as before. We have $M_2 = X$ and $M_0 = M \simeq Y$. As before, we have

$$\mathcal{K}(M_1) = \mathcal{K}(X) \;\cup \bigcup_{h(y)=1} C(\mathcal{K}(f/y))$$

Then by Lemma 5.6,the map

$$\mathcal{K}(X) \;\cup \bigcup_{\substack{h(y)=1\\ i\in I_y}} e^2_i \to \mathcal{K}(M_1)$$

induces an isomorphism on π_1. Now

$$\mathcal{K}(M) = \mathcal{K}(M_1) \ \cup \bigcup_{h(y)=0} C(\mathcal{K}(Y_{>y} * f/y)).$$

Now consider $y \in Y$ with $h(y) = 0$, $i \in I_y$, $j \in J_y$. If $\alpha_i = \{x, x'\}$ and $\overline{\beta}_j = \{y', y''\}$, then in M_1 the loop $\eta_{i,j}$ is homotopic to the loop $\alpha_i * \overline{\beta}_j$. By Lemma 5.5, $\{\alpha_i * \overline{\beta}_j\}_{j \in J_y, k \in K_y}$ has the π_1-spanning property in $\mathcal{K}(Y_{>y} * f/y)$. Then by Lemma 5.6

$$\mathcal{K}(X) \ \cup \bigcup_{\substack{h(y)=1 \\ i \in I_y}} e_i^2 \ \cup \bigcup_{\substack{h(y)=0 \\ (i,j) \in I_y \times J_y}} e_{i,j}^2 \to \mathcal{K}(M)$$

induces an isomorphism on π_1. Now in M the loop γ_i is homotopic to $f_*(\gamma_i)$. Since $\pi_1(Y) = \pi_1(M)$, the set $\{f_*(\gamma_i)\}_{i \in I}$ has the π_1-spanning property in M. Thus

$$\mathcal{K}(X) \ \cup \bigcup_{\substack{h(y)=1 \\ i \in I_y}} e_i^2 \ \cup \bigcup_{\substack{h(y)=0 \\ (i,j) \in I_y \times J_y}} e_{i,j}^2 \ \cup \bigcup_{i \in I} e_i^2$$

is simply connected. □

Proposition 5.8. *Let B_0 be a partial basis of $\mathbb{F}_n$ with $|B_0| = n-3$. Consider the map $\widetilde{g}\colon \mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}) \to \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0 \rangle}$. Consider the set of 1-loops $\{\gamma_i : i \in I\}$ given by the barycentric subdivisions of the boundaries of the 2-simplices of $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. Then the set of 1-loops $\{\widetilde{g}_*(\gamma_i) : i \in I\}$ has the π_1-spanning property in $\mathcal{K}(\mathrm{FC}(\mathbb{F}_n)_{>\langle B_0 \rangle})$.*

Proof. Again, consider $Y_{\langle B_0 \rangle}$ which is simply-connected by Theorem 4.3. The boundaries of the 2-simplices of $Y_{\langle B_0 \rangle}$ give a set of 1-loops with the π_1-spanning property in $Y^{(1)}_{\langle B_0 \rangle}$. Thus by the first part of Theorem 5.7, if we apply f_* to these 1-loops we obtain a set of 1-loops in $\mathcal{K}(\mathrm{FC}(\mathbb{F}_n)_{>\langle B_0 \rangle})$ with the π_1-spanning property.

Now using Lemma 4.4, for each 2-simplex $\{H_{n-2}, H_{n-1}, H_n\}$ in $Y_{\langle B_0 \rangle}$ we can take a basis $\{v_1, \dots, v_n\}$ such that $H_i = \langle v_1, \dots, \widehat{v_i}, \dots, v_n \rangle$ and $B_0 = \{v_1, \dots, v_{n-3}\}$. Then $\{v_{n-2}, v_{n-1}, v_n\}$ is a 2-simplex in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. Moreover the 1-loop obtained by applying g_* to the barycentric subdivision of the boundary of $\{v_{n-2}, v_{n-1}, v_n\}$, is the loop obtained by applying f_* to the barycentric subdivision of the boundary of $\{H_{n-2}, H_{n-1}, H_n\}$. This completes the proof. □

Proposition 5.9. *Let B_0 be a partial basis of $\mathbb{F}_n$ with $|B_0| = n-3$. Then $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)$ is simply connected.*

Proof. By Proposition 5.4 the map $\widetilde{g}\colon \mathcal{X}(\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)})) \to \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0 \rangle}$ is 1-spherical. Thus by the first part of Theorem 5.7 the link $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ is connected. It remains to prove that it is simply connected. To this end, we need to choose the three families of loops. The set $\{\gamma_i\}_{i \in I}$ is the one given by Proposition 5.8.

If $H \in \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0 \rangle}$ has height 1, we consider a set of 1-loops in $\mathrm{lk}(B_0, \mathrm{PB}(H))$ with the π_1-spanning property. The set $\{\alpha_i\}_{i \in I_H}$ is then given by the barycentric subdivision of these 1-loops. Now since H is a free factor of rank $n-1$, there is an element $w \in \mathbb{F}_n$ such that $\mathbb{F}_n = \langle w \rangle * H$. Then by considering the cone $w * \mathrm{lk}(B_0, \mathrm{PB}(H))$, which is a subcomplex of $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$, we see that α_i is null-homotopic for every $i \in I_H$.

For each $H \in \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0 \rangle}$ of height 0 we choose the sets $\{\alpha_i\}_{i \in I_H}$ and $\{\overline{\beta}_j\}_{j \in J_H}$ arbitrarily. We explain how to choose the loops $\eta_{i,j}$ for $(i,j) \in I_H \times J_H$. Suppose $\alpha_i = \{x, x'\}$ and $\overline{\beta}_j = \{H', H''\}$. By Proposition 4.1, we can take $z', z'' \in \mathbb{F}_n$ so that $H' = \langle H, z' \rangle$ and $H'' = \langle H, z'' \rangle$. Then $\eta_{i,j}$ is the 1-loop given by $x - z' - x' - z'' - x$. To prove that $\eta_{i,j}$ is null-homotopic we note that $z', z'' \in \mathrm{lk}(B_0 \cup \{x\}, \mathrm{PB}(\mathbb{F}_n)) = \mathrm{lk}(B_0 \cup \{x'\}, \mathrm{PB}(\mathbb{F}_n))$ which is connected by Proposition 5.4.

Therefore by Theorem 5.7 we have a set of loops in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}) = \mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))^{(1)}$ with the π_1-spanning property which are, as we have seen, null-homotopic in $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. Thus $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ is simply connected. □

6. $\mathrm{PB}(\mathbb{F}_n)$ *is Cohen–Macaulay*

We first prove the following technical lemma.

LEMMA 6.1. *Let B be a partial basis of $\mathbb{F}_n$, $|B| = l$. Let $\overline{\gamma} \in \widetilde{H}_{n-l-2}(\mathrm{FC}(\mathbb{F}_n)_{>\langle B\rangle})$. There exists $\gamma \in B_{n-l-2}(\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n)))$ such that $\widetilde{g}_*(\gamma') = \overline{\gamma}$.*

Proof. We define a map $\phi\colon C_{n-l-1}(Y_{\langle B\rangle}) \to C_{n-l-1}(\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n)))$ as follows. For each $(n-l-1)$-simplex $\sigma = \{H_{l+1}, \dots, H_n\}$ of $Y_{\langle B\rangle}$ we use Lemma 4.4 to obtain a basis $B \cup \{w_{l+1}, \dots, w_n\}$. Therefore $\widetilde{\sigma} = \{w_{l+1}, \dots, w_n\}$ is an $(n-l-1)$-simplex of $\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n))$. Then we define the map ϕ on σ by $\phi(\sigma) = \widetilde{\sigma}$.

Now since $f\colon \mathcal{X}(Y_{\langle B\rangle}^{(n-l-2)}) \to (\mathrm{FC}(\mathbb{F}_n)_{>\langle B\rangle})^{\mathrm{op}}$ is $(n-l-2)$-spherical [**8**, §4], by Theorem 3.1 we have an epimorphism $f_*\colon \widetilde{H}_{n-l-2}(\mathcal{X}(Y_{\langle B\rangle}^{(n-l-2)})) \to \widetilde{H}_{n-l-2}(\mathrm{FC}(\mathbb{F}_n)_{>\langle B\rangle})$ and since $Y_{\langle B\rangle}$ is $(n-l-2)$-connected, there is $c \in C_{n-l-1}(Y_{\langle B\rangle})$ such that $f_*(d(c)') = \overline{\gamma}$. We define $\gamma = d\phi(c)$. We immediately have $\gamma \in B_{n-l-2}(\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n)))$. It is easy to verify that $\widetilde{g}_*(d\phi(\sigma)') = f_*(d\sigma')$ and from this it follows that $\widetilde{g}_*(\gamma') = \widetilde{g}_*(d\phi(c)') = f_*(dc') = \overline{\gamma}$. □

THEOREM 6.2. *The complex $\mathrm{PB}(\mathbb{F}_n)$ is $(n-1)$-spherical. Moreover, it is Cohen–Macaulay.*

Proof. We need to prove that $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ is $(n - |B_0| - 1)$-spherical for any partial basis B_0 of $\mathbb{F}_n$. We proceed by induction on $k = n - |B_0|$. For $k = 1$ the link is 0-dimensional and obviously nonempty. For $k = 2$ and $k = 3$ it follows from Propositions 5.4 and 5.9 respectively. Now if $k \geq 4$ we want to apply Theorem 3.1 to the map

$$\widetilde{g}\colon \mathcal{X}\left(\mathrm{lk}\left(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}\right)\right) \to \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0\rangle}$$

By Theorem 4.2, $\mathrm{FC}(\mathbb{F}_n)_{>\langle B_0\rangle}$ is $(n - |B_0| - 2)$-spherical. In addition $\widetilde{g}$ is $(n - |B_0| - 2)$-spherical, since $\mathrm{FC}(\mathbb{F}_n)_{>H}$ is $(n - \mathrm{rk}\,(H) - 2)$-spherical if $H \in \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0\rangle}$ and by the induction hypothesis $\widetilde{g}/H = \mathcal{X}(\mathrm{lk}(B_0, \mathrm{PB}(H)))$ is $(\mathrm{rk}\,(H) - |B_0| - 1)$-spherical. Then by Theorem 3.1, $\mathcal{X}(\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}))$ is $(n - |B_0| - 2)$-spherical.

Now we check the hypotheses (i), (ii) and (iii) of Theorem 3.1. If $\widetilde{g}(B_1) \subseteq \widetilde{g}(B_2)$ it is easy to see that $\mathrm{lk}(B_2, \mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)})) \subseteq \mathrm{lk}(B_1, \mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}))$ so (i) holds. Obviously (ii) holds. And by the induction hypothesis (iii) holds. Thus, the second part of Theorem 3.1 gives a basis of $\widetilde{H}_{n-|B_0|-2}(\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)^{(n-2)}))$. By Lemma 6.1 we can choose the γ_i to be borders. We need to prove that the remaining elements of this basis are trivial in $\widetilde{H}_{n-|B_0|-2}(\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)))$. We only have to show that for all $H \in \mathrm{FC}(\mathbb{F}_n)_{>\langle B_0\rangle}$, $i \in I_H$, $j \in J_H$

$$\alpha_i * \beta_j \in B_{n-|B_0|-2}(\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n)))$$

By Proposition 4.1, we can take a basis B of H with $B_0 \subset B$. Then $\mathrm{lk}(B_0, \mathrm{PB}(H)) * \mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n))$ is a subcomplex of $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$. By the induction hypothesis we have $\widetilde{H}_{n-|B|-2}(\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n))) = 0$. So there is $\omega \in C_{n-|B|-1}(\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n)))$ such that $d(\omega) = (-1)^{|\alpha_i|}\,\beta_j$. Therefore

$$d(\alpha_i * \omega) = d(\alpha_i) * \omega + (-1)^{|\alpha_i|}\,\alpha_i * d(\omega) = \alpha_i * \beta_j.$$

Therefore $\mathrm{lk}(B_0, \mathrm{PB}(\mathbb{F}_n))$ is $(n - |B_0| - 1)$-spherical and we are done. $\square$

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Iván Sadofschi Costa
CONICET-Universidad de Buenos Aires. Instituto de Investigaciones Matemáticas Luis A. Santaló (IMAS). Buenos Aires, Argentina.

isadofschi@dm.uba.ar