ON A GENERALIZATION OF THE SEATING COUPLES PROBLEM

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Abstract. We prove a conjecture of Adamaszek generalizing the seating couples problem to the case of $2n$ seats. Concretely, we prove that given a positive integer $n$ and $d_1, \ldots, d_n \in (\mathbb{Z}/2n)\times$ we can partition $\mathbb{Z}/2n$ into $n$ pairs with differences $d_1, \ldots, d_n$.

1. Introduction

Preissmann and Mischler [6] proved the following result, confirming a conjecture of R. Bacher.

Theorem 1.1. Let $p = 2n + 1$ be an odd prime. Suppose we are given $n$ elements $d_1, \ldots, d_n \in (\mathbb{Z}/p)\times$. Then there exists a partition of $\mathbb{Z}/p - \{0\}$ into pairs with differences $d_1, \ldots, d_n$.

We gave a simpler proof of this theorem using the Combinatorial Nullstellensatz [4]. Karasev and Petrov, independently, gave a proof of Theorem 1.1 along the same lines and provided further generalizations [3]. In this work, they also conjectured two generalizations of Theorem 1.1, replacing $p$ by an arbitrary integer $N$. The conjecture in the case that $N$ is even is originally due to Adamaszek.

Conjecture 1.2 ([3, Conjecture 1]). Let $N = 2n + 1$ be a positive integer. Suppose we are given $n$ elements $d_1, \ldots, d_n \in (\mathbb{Z}/N)\times$. Then there exists a partition of $\mathbb{Z}/N - \{0\}$ into pairs with differences $d_1, \ldots, d_n$.

We will prove the conjecture when $N$ is even:

Theorem 2.5 ([3, Conjecture 2]). Let $N = 2n$ be a positive integer. Suppose we are given $n$ elements $d_1, \ldots, d_n \in (\mathbb{Z}/N)\times$. Then there exists a partition of $\mathbb{Z}/N$ into pairs with differences $d_1, \ldots, d_n$.

While finishing this paper we found out that, in his master's thesis [5], T.R. Mezei suggests a possible way to solve the conjecture that is similar to ours. Furthermore, he shows that Theorem 2.5 holds whenever $N = 2p$ for $p$ a prime number.

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2. The even case

We recall the following version of the Cauchy-Davenport theorem.
Theorem 2.1 ([1 Theorem 1]). If $A$ and $B$ are nonempty subsets of $\mathbb{Z}/N$ where $0 \in B$, and $\gcd(b, N) = 1$ for all $b \in B \setminus \{0\}$, then

$$|A + B| \geq \min\{N, |A| + |B| - 1\}.$$ 

Suppose that we have a partition as in Theorem 2.5. Since the $d_i$ are odd numbers, each pair contains exactly one even number. Therefore, if Theorem 2.5 holds there must exist signs $s_i$ such that

$$s_1d_1 + \ldots + s_nd_n = 1 - 2 + 3 - \ldots + (2n - 1) - 2n = n \mod N.$$ 

Theorem 2.2. Let $N = 2n$ and let $d_1, \ldots, d_n \in (\mathbb{Z}/N)^\times$. Then there exists $s_1, \ldots, s_n \in \{1, -1\}$ such that

$$s_1d_1 + \ldots + s_nd_n = n \mod 2n.$$ 

Proof. It is enough to prove that there exists $I \subset \{1, \ldots, n\}$ such that

$$\sum_{i \in I} 2d_i = d_1 + \ldots + d_n + n \mod 2n.$$ 

Since $d_i$ is odd for every $i$, $d_1 + \ldots + d_n + n$ is even and therefore our task is equivalent to finding $I$ such that

$$\sum_{i \in I} d_i = \frac{d_1 + \ldots + d_n + n}{2} \mod n.$$ 

Let $A_i = \{d_i, 0\}$. Applying Theorem 2.1 inductively, we see that

$$\#(A_1 + \ldots + A_n) \geq \min\left\{n, \sum \#A_i - (n - 1)\right\} = n,$$

concluding the proof. \hfill \Box

Remark 2.3. Theorem 2.1 was stated in full strength for the benefit of the reader. However, in the previous proof we only needed to use this result in the case $|B| = 2$, which follows from the fact that $A + b = A$ and $\gcd(b, N) = 1$ imply that $A = \mathbb{Z}/N$.

The last ingredient is the following theorem due to Hall.

Theorem 2.4 ([2]). Let $A$ be an abelian group of order $n$ and $a_1, \ldots, a_n$ be a numbering of the elements of $A$. Let $d_1, \ldots, d_n \in A$ be elements such that $d_1 + \ldots + d_n = 0$. Then there are permutations $\sigma, \tau \in S_n$ such that

$$a_i - a_{\sigma(i)} = d_{\tau(i)}.$$ 

Theorem 2.5. Let $N = 2n$ be a positive integer. Suppose we are given $n$ elements $d_1, \ldots, d_n \in (\mathbb{Z}/N)^\times$. Then there exists a partition of $\mathbb{Z}/N$ into pairs with differences $d_1, \ldots, d_n$.

Proof. First, using Theorem 2.2 we may assume that $d_1 + \ldots + d_n = n \mod 2n$. Now it is enough to find a numbering $a_1, \ldots, a_n$ of the odd numbers in $\mathbb{Z}/N$ and $\sigma \in S_n$ such that $2i - a_i = d_{\sigma(i)} \mod 2n$ for every $i \in \{1, \ldots, n\}$, for then the partition in pairs $\{2a_1\}, \{4a_2\}, \ldots, \{2na_n\}$ works.

Equivalently, we need to find a numbering $b_1, \ldots, b_n$ of the even numbers in $\mathbb{Z}/N$ such that $2i - b_i = d_{\sigma(i)} + 1 \mod N$ for some $\sigma \in S_n$. Now since $d_i + 1$ is even for all $i$, this is the same as finding a permutation $c_1, \ldots, c_n$ of $\{1, \ldots, n\}$ such that

$$i - c_i = \frac{d_{\sigma(i)} + 1}{2} \mod n,$$
for some $\sigma \in S_n$. If we verify that
\[
\frac{d_1 + 1}{2} + \ldots + \frac{d_n + 1}{2} = 0 \bmod n
\]
this will follow from Theorem 2.4. But this holds, since $d_1 + \ldots + d_n = n \bmod 2n$ and therefore $(d_1 + 1) + \ldots + (d_n + 1) = 0 \bmod 2n$, proving that
\[
\frac{d_1 + 1}{2} + \ldots + \frac{d_n + 1}{2} = 0 \bmod n.
\]
\[\square\]

References


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