Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Akram Aldroubi, Carlos Cabrelli
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Editors

New Trends in Applied Harmonic Analysis

Sparse Representations, Compressed Sensing, and Multifractal Analysis

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems and of the metaplectic group for a meaningful interaction of signal decomposition methods.

The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA. We* intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in applicable topics such as the following, where harmonic analysis plays a substantial role:

Biomathematics, bioengineering, and biomedical signal processing;
Communications and RADAR;
Compressive sensing (sampling) and sparse representations;
Data science, data mining, and dimension reduction;
Fast algorithms;
Frame theory and noise reduction;
Image processing and super-resolution;
Machine learning;
Phaseless reconstruction;
Quantum informatics;
Remote sensing;
Sampling theory;
Spectral estimation;
Time-frequency and Time-scale analysis—Gabor theory and Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park, MD, USA John J. Benedetto

Foreword

The CIMPA13 Conference which took place in August 5–16, 2013, in Mar de Plata, Argentina, was entitled **New Trends in Applied Harmonic Analysis Sparse Representations, Compressed Sensing and Multifractal Analysis**. The event took place in a friendly atmosphere, encouraging interaction between speakers and participants, among them PhD students, postdocs, and senior scientists. Unfortunately not all the main speakers have been able to provide a written version of their presentation, but in many cases one may find slides of more formal talks through the Internet. General information about the conference can be found at

`http://www.nuhag.eu/cimpa13`

The topics of the articles which appear in this volume reflect the diversity of recent developments in harmonic analysis, both at the level of pure mathematics and applications. Some contributions concern interesting mathematical questions arising from a systematic investigation of structures which have not been sufficiently well explored so far, and others – such as sparsity with respect to non-orthogonal systems – are part of a current trend, related to compressed sensing.

To be more precise, let us take a look at the individual contributions: The first three chapters describe problems related to multifractal analysis (Kathryn E. Hare, Stephane Seuret, and Yanick Heurteaux).

We then find two chapters thematizing the sparsity of wavelet coefficients. In the first contribution (by Vladimir Temlyakov), Lebesgue-type inequalities for greedy approximations are discussed, demonstrating that many of the well-known expansions have the following nice property: Given the set of, say, wavelet coefficients of a given function in some Besov space (because these spaces can be characterized by weighted summability conditions with respect to a given wavelet system), it is a good strategy (not only in the Hilbert spaces setting) to just take more and more of the "large coefficients" in order to approximate the function, in fact with an optimal rate.

In the second chapter in this direction, written by Eugenio Hernandez and María de Natividade, we learn some *results on nonlinear approximation for wavelet bases in weighted function spaces*. Here Bernstein- and Jackson-type theorems for weighted L^p-spaces are provided, showing that wavelet expansions are doing a good job for the approximation of functions in this setting.

The chapter provided by Pete Casazza and Janet C. Tremain discusses *the consequences of the Marcus/Spielman/Srivasta solution to the Kadison-Singer problem* in the context of frame theory with some first glimpse on the consequences within harmonic analysis.

The chapter “Model Sets and New Versions of Shannon’s Sampling Theorem” by Basarab Matei presents some interesting insight on universal sampling sets, the so-called model sets and their relations to quasicrystals. While the classical Shannon theorem describes how one can recover a band-limited signal, given the *spectral support* Ω (the support of $\hat{f}$), with a formula which obviously depends on the choice of this set, the new approach discusses situations where the same sampling set can be used (with a more complicated recovery algorithm) for a large variety of sets Ω, as long as their measure is not too big.

The section written by Xianfeng Hu, Yang Wang, and Qiang Wu treats a somewhat unusual and therefore very interesting topic: *Stylometry and Mathematical Study of Authorship*.

The final contribution, entitled “Thoughts on Numerical and Conceptual Harmonic Analysis,” provided by the author of this introduction gives a glimpse on a problem within the community of harmonic analysts which should be given a bit more attention: the interaction between principles of abstract (or as he proposes conceptual harmonic analysis) and those who are involved in numerical resp. computational harmonic analysis. While the first group is searching for general structures, the second one is looking for efficient algorithms and their implementation, often using FFT-based algorithms. The aspect lost in this separation of duties is the connection between the two approaches, the question, which function spaces are suitable to describe the errors made by moving from the continuous, to the discrete, and then of course to the finite setting. The article is just providing a few thoughts in this direction and suggests to pay more attention to it, not just in the spirit of function spaces or pure functional analysis but more in the sense of constructive approximation theory, with quantitative error bounds, estimates for the required problem size if one needs a guaranteed estimate for the size of the error.

Thus in some sense the article describes the ideas and goals behind the material presented by the author during the conference in a more concrete but less reflected format. Important parts of those presentations are available in the form of PDF files from www.nuhag.eu.

Overall it is clear from this volume that harmonic analysis at large is and will provide a wide variety of interesting mathematical problems and that research in this direction will continue to be fruitful and rewarding for those interested in mathematical analysis in general, be it abstract or more application oriented.

Vienna, Austria Hans Feichtinger
October 2015

Preface

This book evolved from the written notes that were distributed to the students who participated in the CIMPA school, *New Trends in Applied Harmonic Analysis: Sparse Representations, Compressed Sensing and Multifractal Analysis*, which took place in Mar del Plata (Argentina) in August 2013.

This event was motivated by the recent interactions which developed between harmonic analysis and signal and image processing during the last 10 years. During that time, several technological deadlocks were solved through the resolution of deep theoretical problems in harmonic analysis. The purpose of this school was to focus on two particularly active areas which are representative of such advances: multifractal analysis and compressed sensing. The courses were taught by leaders in these areas and covered both theoretical aspects and applications. Most of the attendance was composed of PhD students and postdocs from diverse backgrounds (mathematics, signal and image processing, etc.), and the corresponding chapters of this book reflect the pedagogical care of the lecturers, in particular in the careful treatment of all needed prerequisites, and the illustration of the developments of each topic by several examples. Another original feature of this book is that some subjects overlap, with views taken from different perspectives, thus offering an in-depth picture of these scientific areas.

Let us be more specific. Multifractal analysis offers new tools of classification for signals and images derived from their scaling invariance properties. The part of the book concerning this subject include the contribution of K. Hare, "Multifractal Analysis of Cantor-like Measures," which deals with basics of fractal analysis and then focuses on the key example of Cantor-like measures. The contribution of Y. Heurteaux "An introduction to Mandelbrot cascades" goes one step further in modeling complexity and deals with the multifractal measures supplied by multiplicative cascades; a careful treatment of these examples is motivated both by the historical role played by these measures as models for the dissipation of energy in turbulent fluids and by the importance that they have recently acquired in other areas of mathematics (fragmentation, coalescence, harmonic measure associated with fractal sets, Schramm-Loewner evolution, etc.). Finally, the contribution of Stéphane Seuret "Multifractal analysis and Wavelets" deals with the extensions that these ideas have known in the setting of functions. The main tool here is wavelet analysis, a tool which is now prevalent in applied analysis and reappears in several other chapters of this book. Here its role is to yield a characterization of both pointwise and global regularity of functions. This property explains the success of wavelets in applied multifractal analysis, since this subject can be seen as unfolding the relationships between pointwise and global regularity and then deriving practical classification tools from these regularity characteristics.

Recently, many powerful techniques have been developed emphasizing the role of sparsity in signal and image processing. These new methods have had a substantial impact in areas like sampling, data compression and representation, atomic decompositions, wavelets, frames, and high-dimensional data analysis. In particular compressed sensing represents a new paradigm in signal and image processing, allowing to reconstruct compressible data from the knowledge of an underdetermined system, through an ℓ^1 minimization. The mathematics behind these methods is rich and sophisticated and presents new challenges. The chapters by Temlyakov "Lebesgue-type Inequalities for Greedy Approximation" and Hernández et. al "Results on Nonlinear Approximation for Wavelet Bases in Weighted Function Spaces" are excellent examples of the advances in this area.

On another note, just before the school took place, the *Kadison-Singer conjecture* was solved, and since this had deep impact on harmonic analysis – because of the implications with respect to the decomposition of frames into a finite number of Riesz bases *Feichtinger conjecture* – Pete Casazza gave a really nice lecture about the diverse attempts in the solution and agreed to write a chapter about all the implications.

Note that the contribution of Y. Heurteaux was not part of the courses taught at the CIMPA school of August 2013, but grew from the notes of another course taught at a fractal conference that took place in Porquerolles (France) in September 2013.

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- MinCyT, Ministerio de Ciencia y Tecnología, ARGENTINA
- IMU, International Mathematical Union

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Chapter 1
Multifractal Analysis of Cantor-Like Measures

Kathryn E. Hare

Abstract In this course we will study generalized Cantor sets and measures. We will see that they share many properties in common with self-similar sets and measures, although new geometric ideas are often needed in the proofs to replace the combinatorial structure of self-similar sets/measures. In particular, under a suitable separation condition the multifractal spectrum of generalized Cantor measures (the set of local dimensions) can be shown to be a closed interval, with one specific local dimension being attained at almost every point of the Cantor set.

Surprisingly, the property that the multifractal spectrum is a closed interval need not be true for convolutions of (even self-similar) Cantor measures. This seems to be a consequence of 'overlap' in their construction and was established first for certain examples of self-similar Cantor measures and subsequently for generalized Cantor measures. We will see that it is typically the case that the multifractal spectrum of a sufficiently large number of convolutions of fairly arbitrary, continuous measures admits an isolated point. This argument was motivated by the geometric ideas used in proving a special case of this property for generalized Cantor measures.

1.1 Introduction

Often in analysis one is interested in subsets of $\mathbb{R}$ of Lebesgue measure zero and the singular measures[1] concentrated on these sets. Many of the problems that arise have to do with quantifying the size of the set or the singularity of the measure; for such problems, fractal dimensions can be very helpful.

[1] By a measure, we mean a finite, positive, regular, compactly supported, Borel measure on $\mathbb{R}$.

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The classical middle-third Cantor set and its associated uniform measure is an important example of such a set and measure. The Cantor set and measure are often introduced in real analysis courses to illustrate unusual ideas or pathological behaviour. In this course, we will discuss generalizations of the classical Cantor set and measure, and investigate fractal concepts that help to quantify their singularity, such as local dimension and multifractal spectrum. These generalizations have interesting and unusual properties.

Generalized Cantor sets and measures are typically not self-similar and thus need not have the same symmetry or uniformity as the classical Cantor set/measure. Consequently, the concentration of the measure can vary at different points in its support, meaning general Cantor measures typically take on a range of different local dimensions. These different values are known as the multifractal spectrum. The study of the multifractal spectrum and the 'size' of the sets on which a given local dimension is attained is known as multifractal analysis.

For self-similar measures arising from an IFS which satisfies the open set condition, it is well known that the multifractal spectrum is a closed interval and formulas have been established for the Hausdorff dimension of the sets on which a given local dimension occurs. We will modify this argument to show that a similar result can be obtained for generalized Cantor measures, under reasonably weak assumptions. Another interesting fact we will establish is that the 'average' value of the local dimensions is attained at almost every point. These results can be found in Section 1.3.

Convolutions of the classical Cantor measure are again self-similar measures. However, they are not necessarily generated by an IFS that satisfies the open set condition so the general multifractal theory does not apply. In fact, the theory can fail in a striking way: the multifractal spectrum of the 3-fold convolution of the classical Cantor measure contains an isolated point. Here we will see that convolutions of quite general, continuous, probability measures typically admit isolated points in their multifractal spectrum, provided the number of convolutions is sufficiently large. In particular, this is the case for many generalized Cantor measures. These ideas are the content of Section 1.4.

Most of the proofs given in this note can be found in the literature, as detailed in the final section. There are many other important research papers on related topics; we have only mentioned those most relevant for the material discussed in the course.

1.2 Notation and Basic Facts

1.2.1 The Classical Cantor Set and Measure

The *classical middle-third Cantor set* C is a fascinating set which is often used in analysis to construct interesting examples. It is compact, totally disconnected, perfect (meaning, every point is an accumulation point), uncountable and of Lebesgue measure zero. By the *classical Cantor measure* we mean the singular, probability measure on $\mathbb{R}$ that is uniformly distributed on C. This measure, μ, can be defined in several equivalent ways:

1. As the self-similar measure that arises from the iterated function system (IFS) with contractions $F_i(x) = x/3 + 2i/3$, $i = 0,1$ and probabilities $1/2, 1/2$. This means the measure is invariant in the sense that
$$\mu(E) = \frac{1}{2}\left(\mu \circ F_0^{-1}(E) + \mu \circ F_1^{-1}(E)\right) \text{ for all Borel sets } E.$$
The classical Cantor set C is the self-similar set associated with this IFS.
2. As the Borel measure supported on C that assigns mass 2^{-k} to the Cantor intervals that arise at step k in the construction of the Cantor set.
3. As the weak limit of the discrete probability measures $\mu_K = 2^{-K}\sum_{j=1}^{2^K}\delta_{x_j}$, where $x_1, \ldots, x_{2^K}$ are the left end points of the 2^K Cantor intervals that are constructed at step K in the standard Cantor set construction. By a weak limit, we mean that for all continuous functions f on $[0,1]$ it is the case that $\int_0^1 f d\mu = \lim_K \int_0^1 f d\mu_K$.
4. As the probability measure whose cumulative distribution function is the Cantor ternary function.

From these different (but equivalent) descriptions of the Cantor measure one can easily establish many properties of the Cantor set/measure. Definition (2), for example, is useful in calculating the Hausdorff dimension of the set. From definition (3) it can be seen that the Fourier transform of μ is given by $\widehat{\mu}(y) = \prod_{k=1}^{\infty}(1 + e^{-4\pi i 3^{-k} y})/2$ for all y. Since the Cantor ternary function is a continuous function, it follows immediately from definition (4) that the Cantor measure is a continuous (or non-atomic) measure, meaning the measure of any singleton is 0.

The classical Cantor set and measure has been generalized in many ways. One obvious generalization is to consider the self-similar set arising from the IFS with contractions $F_i(x) = rx + i(1-r)$, $i = 0,1$ where $0 < r < 1/2$. This is the Cantor set with ratio of dissection r (rather than $1/3$rd, as in the classical case), meaning that at each step in the standard Cantor set construction one keeps the two outer closed intervals whose length is r times that of the parent interval. We will denote this Cantor set as $C(r)$, so that with this notation the classical Cantor set is $C(1/3)$. We can again define the associated *uniform Cantor measure* that assigns mass 2^{-k} to the Cantor intervals at step k, which in this case are of length r^k. This is the self-similar measure generated by the IFS given above, with probabilities $1/2, 1/2$.

Alternatively, rather than the uniform Cantor measure, we could consider the self-similar measure generated by the same iterated function systems again, but with probabilities p and $1-p$, where $0 \leq p \leq 1$. We call this the *p-Cantor measure* on $C(r)$. If $p = 0$ or 1, the p-Cantor measure is the point mass measure at 0 or 1, respectively. In all other cases, it is a continuous, singular, probability measure.

1.2.2 Cantor Sets and Measures with Varying Ratios of Dissection

In fractal geometry one is often interested in studying self-similar sets and measures arising from quite general iterated function systems. The IFS structure makes it possible to compute many important quantities and deduce various properties of the sets and measures. At the same time, the structure limits the kinds of examples that arise. If we relax this structure, we can create many other intriguing examples. One such variation is to allow the ratios of dissection in the construction of the Cantor set to vary at each step. We could also allow the probabilities to vary at different steps.

1.2.2.1 Cantor Sets with Varying Ratios of Dissection

Let $0 < r_j < 1/2$. We denote by $C(r_j)$[2] the Cantor set with varying ratios of dissection, r_j at step j, given by the following iterative Cantor-like construction: Let $C_0 = [0,1]$. Remove from C_0 the open middle interval of length $1-2r_1$, leaving two closed intervals of lengths r_1. Call these intervals the *Cantor intervals* of step one and their union C_1. At step j in the construction assume we have inductively constructed C_j as a union of 2^j closed intervals of length $r_1 \cdots r_j$, the Cantor intervals of step j. Remove the open middle interval of length $(1-2r_{j+1})r_1 \cdots r_j$ from each of the step j intervals and let C_{j+1} be the union of the remaining 2^{j+1} closed intervals of length $r_1 \cdots r_{j+1}$. Finally, define the Cantor set $C(r_j)$ by

$$C(r_j) = \bigcap_{j=1}^{\infty} C_j.$$

As with the classical Cantor set, $C(r_j)$ is compact, perfect, totally disconnected and uncountable. Its Lebesgue measure is $\liminf_{n\to\infty} 2^{-n} r_1 \cdots r_n$ and hence is zero if, for instance, the r_j are bounded away from $1/2$.

1.2.2.2 Labelling Cantor Intervals and the Elements of the Cantor Set

The Cantor intervals from this construction can be labelled by finite words with letters from $\{0,1\}$. The Cantor intervals of step one will be denoted I_0 (left interval) and I_1 (right interval). In general, if the Cantor interval of step n is labelled by the word w of length n, then its two descendants are I_{w0} and I_{w1}. Each $x \in C(r_j)$ belongs to a unique Cantor interval of step n for each n and these intervals are descendants of one another. Thus x corresponds to an infinite word w with the property that if $w|n$ denotes the truncation of w to length n, then $I_{w|n}$ is the step n Cantor interval to which x belongs. When we write $x = (w_j)$ we mean this correspondence.

[2] More properly, we should write $C(\{r_j\})$, but we prefer $C(r_j)$ for simplicity. This should not cause any confusion with the notation $C(r)$ for the Cantor set with fixed ratio of dissection r.

1.2.2.3 Uniform and p-Cantor Measures

Given $0 \le p \le 1$, by the *p-Cantor measure* associated with $C(r_j)$, we mean the probability measure μ with the property that

$$\mu(I_{w0}) = \mu(I_w)p \text{ and } \mu(I_{w1}) = \mu(I_w)(1-p).$$

Thus if $w = (w_1, \ldots, w_n)$ with $w_i \in \{0,1\}$, then $\mu(I_{w_1 \cdots w_n}) = p^{n_0}(1-p)^{n-n_0}$ where $n_0 = card\{i : w_i = 0\}$. As in the case for Cantor sets with fixed ratio of dissection, the p-Cantor measure μ is a singular measure whose support is the Cantor set $C(r_j)$. It is continuous provided $p \neq 0, 1$. If $p = 1/2$, we call μ the *uniform Cantor measure* on $C(r_j)$.

More generally, given a sequence of weights $\{p_j\}$, $0 \le p_j \le 1$, we could define a Cantor measure by the rule $\mu(I_{w_1 \ldots w_n}) = p_{w_1 1} p_{w_2 2} \cdots p_{w_n n}$ where $p_{0j} = p_j$ and $p_{1j} = 1 - p_j$.

One could consider still more general Cantor sets and measures by removing from $[0,1]$, k_1 equally spaced, open intervals of length g_1 at step one, so that C_1 is the union of $k_1 + 1$ closed intervals of length r_1 where $(k_1+1)r_1 + k_1 g_1 = 1$. Then inductively remove from each Cantor interval of step j, k_j equally spaced open intervals of length g_j so that C_j is the union of $\prod_{i=1}^{j}(k_j+1)$ closed intervals of length $r_1 \cdots r_j$ where $(k_j+1)r_j + k_j g_j = 1$. We can also define a general Cantor measure by putting weights p_{ij} on the $i = 1, \ldots, k_j + 1$ descendants at step j. In this note, we will focus on p-Cantor measures on $C(r_j)$, but much of what is said here is true for these very general Cantor sets and measures, at least under suitable assumptions. The technical details will be left for the reader.

1.2.3 Hausdorff Dimension

Let $\delta > 0$. By a δ-cover of a non-empty Borel subset $E \subseteq \mathbb{R}$ we mean a countable collection of sets $\{U_i\}$ of diameter at most δ, whose union contains E. We write $|U_i|$ to denote the diameter of the set U_i. Given $s \ge 0$, we define

$$H^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}$$

and put

$$H^s(E) = \sup_{\delta>0} H^s_\delta(E) = \lim_{\delta \to 0^+} H^s_\delta(E).$$

$H^s(\cdot)$ is a measure known as the *s-dimensional Hausdorff measure*. $H^s(E)$ is a decreasing function of s and can be positive and finite for at most one choice of s. The *Hausdorff dimension* of E, denoted $\dim_H E$, is defined to be the unique index s such that $H^t(E) = 0$ if $t > s$ and $H^t(E) = \infty$ for $t < s$. Thus

$$\dim_H F = \inf\{s : H^s(F) = 0\}$$
$$= \sup\{s : H^s(F) = \infty\}.$$

A useful fact is the Mass distribution principle: If there are a measure μ on E and real numbers $c, \delta > 0$ such that $\mu(U) \leq c|U|^s$ for all Borel sets U with diameter at most δ, then $H^s(E) \geq \mu(E)/c$ and $\dim_H E \geq s$.

We leave it as an exercise to verify that the Hausdorff dimension of $C = C(r_j)$ is given by the formula

$$\dim_H C = \liminf_{n\to\infty} \frac{\log 2}{\frac{1}{n}|\log r_1 \cdots r_n|}.$$

Exercise 1.1. Establish the formula given for the Hausdorff dimension of $C(r_j)$.

Exercise 1.2. Show that for every $s \leq 1$ there is a Cantor set with Hausdorff dimension equal to s.

Exercise 1.3. Construct a Cantor-like set, $C(r_j)$, with Hausdorff dimension one and Lebesgue measure zero.

1.3 Multifractal Analysis of p-Cantor Measures

1.3.1 Local Dimension

In many problems one is interested in quantifying the singularity of a measure, i.e., to specify, in some sense, how concentrated the measure is. One way to quantify this is through the *Hausdorff dimension of the measure* μ. This is defined as

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}.$$

This quantity provides global information on the singularity of the measure μ. For measures that are not uniformly distributed it is also of interest to quantify their local singularity. The local dimension is useful for this.

Definition 1.1. By the local dimension at x of a probability measure μ on $\mathbb{R}$ we mean the quantity

$$dim_{loc}\mu(x) = \lim_{r\to 0^+} \frac{\log(\mu(B(x,r)))}{\log r}$$

where $B(x,r)$ is the ball centred at x with radius r, provided this limit exists.

The upper and lower dimensions, denoted $\overline{dim}_{loc}\mu(x)$ and $\underline{dim}_{loc}\mu(x)$, are obtained by replacing the limit in the definition above with lim sup and lim inf, respectively.

The local dimension at x describes the power law behaviour of $\mu(B(x,r))$ for small r. Notice that if $x \notin \operatorname{supp}\mu$, then $dim_{loc}\mu(x) = \infty$, while if μ is Lebesgue measure on $[0,1]$, $dim_{loc}\mu(x) = 1$ at all $x \in [0,1]$.

One can prove that

$$\dim_H \mu = \sup\{s : \underline{dim}_{loc}\mu(x) \geq s \text{ for } \mu \text{ a.e. } x\}.$$

Moreover, the following is true.

Proposition 1.1. *Suppose μ is a probability measure, $F \subseteq \mathbb{R}$ is a Borel set and $0 < c < \infty$.*

(a) $H^s(F) \geq \mu(F)/c$ if

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r))}{r^s} \leq c \text{ for all } x \in F.$$

(b) $H^s(F) \leq 10^s \mu(\mathbb{R})/c$ if

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r))}{r^s} \geq c \text{ for all } x \in F.$$

Proof. (a) Fix $\varepsilon > 0$ and for each n let

$$F_n = \{x \in F : \mu(B(x,r)) \leq (c+\varepsilon)r^s \text{ for all } r \leq 1/n\}.$$

The sets F_n are increasing and the assumption of (a) guarantees that their union is all of F.

Temporarily fix n and let $\{U_i\}$ be a $1/2n$-cover of F and hence also of F_n. Each set U_i has diameter less than $1/n$ and thus $\mu(B(x,|U_i|)) \leq (c+\varepsilon)\,|U_i|^s$ for all $x \in F_n$. Notice that if $x \in U_i \cap F_n$, then $B(x,|U_i|) \supseteq U_i$ and $\mu(U_i) \leq (c+\varepsilon)\,|U_i|^s$. Thus

$$\mu(F_n) \leq \sum_{i:U_i\cap F_n \neq \text{empty}} \mu(U_i) \leq (c+\varepsilon)\sum |U_i|^s.$$

This is true for all $1/2n$-covers of F and consequently $\mu(F_n) \leq (c+\varepsilon)H^s_{1/2n}(F)$. But as $n \to \infty$, $\mu(F_n) \to \mu(F)$ and $H^s_{1/2n}(F) \to H^s(F)$. Since $\varepsilon > 0$ was arbitrary, $\mu(F) \leq cH^s(F)$.

(b) Fix $\varepsilon, \delta > 0$ and consider the collection of all balls, $B(x,r)$, with $x \in F$, $0 < r < \delta$ and $\mu(B(x,r)) \geq (c-\varepsilon)r^s$. By assumption, every $x \in F$ belongs to such a ball for arbitrarily small r. By the Vitali covering lemma there are countably many disjoint balls from this collection, $\{B_i\}$, such that $\mu(F \setminus \bigcup_i B_i) = 0$ and every ball in the collection is contained in the union of the sets $\widetilde{B}_i$, where $\widetilde{B}_i$ is a ball concentric with B_i and having five times the radius. Thus $F \subseteq \bigcup_i \widetilde{B}_i$ and $\left|\widetilde{B}_i\right|^s \leq 10^s \mu(B_i)/(c-\varepsilon)$. As $\left|\widetilde{B}_i\right| \leq 10\delta$ and the sets B_i are disjoint,

$$H^s_{10\delta}(F) \leq \sum_i \left|\widetilde{B}_i\right|^s \leq \frac{10^s}{c-\varepsilon}\sum_i \mu(B_i)$$
$$= \frac{10^s}{c-\varepsilon}\mu(\bigcup_i B_i) = \frac{10^s}{c-\varepsilon}\mu(F).$$

□

Corollary 1.1. *If there is a probability measure μ, concentrated on E, such that $\dim_{loc}\mu(x) = s$ for all $x \in E$, then $\dim_H E = s$.*

Proof. One can deduce from the previous proposition that for any $\varepsilon > 0$, $H^{s-\varepsilon}(E) > 0$ and $H^{s+\varepsilon}(E) < \infty$, from whence the result follows. □

We remark that there is a partial converse to this.

Proposition 1.2. *If $\dim_H E > s$, then there exists a probability measure μ, concentrated on E, such that $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$. Similarly, if $\dim_H E < s$, then there exists a probability measure μ, concentrated on $\overline{E}$, such that $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$.*

The proof of this is more sophisticated and can be found in the literature; see section 1.5.

It is an easy calculation to check that if μ is the uniform Cantor measure on the Cantor set $C(r)$, then

$$dim_{loc}\mu(x) = \frac{\log 2}{|\log r|} = \dim_H C(r) \text{ at all } x \in C(r).$$

In contrast, for measures that are not uniform the local dimension can vary at different points in the support of the measure. This is the case with the p-Cantor measures, for example, when $p \neq 1/2$. Indeed, suppose $C = C(r_j)$ and μ is the p-Cantor measure on C. To avoid technicalities we will also assume $\lim_n \frac{1}{n}\log(r_1 \cdots r_n) = \log r_0$. If $r = r_1 \cdots r_n$, then

$$\frac{\log(\mu(B(0,r)))}{\log r} = \frac{\log(\mu(I_{0\cdots 0}))}{\log r} = \frac{n\log p}{\log r_1 \cdots r_n} \to \frac{\log p}{\log r_0},$$

while

$$\frac{\log(\mu(B(1,r)))}{\log r} = \frac{\log(\mu(I_{1\ldots 1}))}{\log r} = \frac{n\log(1-p)}{\log r_1 \cdots r_n} \to \frac{\log(1-p)}{\log r_0}.$$

Thus

$$dim_{loc}\mu(0) = \frac{\log p}{\log r_0} \text{ and}$$
$$dim_{loc}\mu(1) = \frac{\log(1-p)}{\log r_0}.$$

In the next subsection, we will see that under a suitable separation assumption, these are the extreme values of the set of local dimensions and all numbers in between arise as local dimensions.

1.3.2 Multifractal Spectrum

Given $\alpha \geq 0$, we will denote

$$E_\alpha(\mu) = E_\alpha = \{x : dim_{loc}\mu(x) = \alpha\}.$$

We will call the set of all α such that E_α is non-empty the *multifractal spectrum* of μ. For measures that are not uniform it is of interest to determine the multifractal spectrum and the 'size' of the sets E_α, the so-called, *multifractal analysis*.

The multifractal analysis is well understood for self-similar measures generated by an IFS which satisfies the open set condition. In this section, we will establish similar results for p-Cantor measures supported on Cantor sets $C(r_j)$, under a suitable separation condition that plays the role of the open set condition, namely,

$$\text{Assumption: } \sup r_j < 1/2.$$

1.3.2.1 Local Dimensions Are Constant Almost Everywhere

First, we will show that the local dimension is constant at almost all points of the support of μ.

Theorem 1.1. *Suppose μ is a p-Cantor measure on the Cantor set $C = C(r_j)$ that satisfies* $\sup r_j < 1/2$, *and assume* $\lim \frac{1}{n}\log(r_1 \cdots r_n) = \log r_0$. *Then for μ a.e. $x \in C$,*

$$\dim_{loc}\mu(x) = \frac{p\log p + (1-p)\log(1-p)}{\log r_0}.$$

Remark 1.1. Assuming $\lim \frac{1}{n}\log(r_1 \cdots r_n)$ exists is a convenience. Similar results can be proved with the local dimension of μ at x replaced by the upper or lower local dimensions and with $\lim \frac{1}{n}\log(r_1 \cdots r_n)$ replaced by $\limsup \frac{1}{n}\log(r_1 \cdots r_n)$ (or $\liminf$).

The proof has two parts, a geometric and a probabilistic part. We begin with a geometric lemma which will have other applications. Its significance is to show that under the assumption $\sup r_j < 1/2$ we may replace balls by Cantor intervals in the definition of local dimension.

Notation 1.3.1 *If $x \in C$, by $I^{(k)}(x)$ we mean the unique Cantor interval of step k that contains x.*

Of course, $I^{(k)}(x) = I_{w_1,\dots,w_k}$ where x is associated with the infinite word whose first k letters are $w_1, \dots, w_k$.

Lemma 1.1. *Assume* $\sup r_j < 1/2$*,* μ *is a p-Cantor measure on* $C(r_j)$ *and* $x \in C(r_j)$*. Then*

$$\dim_{loc} \mu(x) = \lim_{k\to\infty} \frac{\log\left(\mu(I^{(k)}(x))\right)}{\log\left|I^{(k)}(x)\right|}.$$

Proof. Fix $x \in C$. Given $r > 0$, choose the minimum integer k so that $B(x,r)$ contains the Cantor interval of step k that contains x. As $I^{(k)}(x) \subseteq B(x,r)$, we must have

$$r_1 \cdots r_k = \left|I^{(k)}(x)\right| \le 2r.$$

On the other hand, as $I^{(k-1)}(x) \subsetneq B(x,r)$,

$$r_1 \cdots r_{k-1} = \left|I^{(k-1)}(x)\right| \ge r.$$

Assume $x = (w_j)$. Then $I^{(k)}(x) = I_{w_1 \dots w_k}$ and if t_k is the number of indices i such that $w_i = 0$ for $i = 1, \dots, k$, then putting $p_j = p$ if $j = 0$ and $p_j = 1 - p$ if $j = 1$ we have

$$\mu(I^{(k)}(x)) = p_{w_1} \cdots p_{w_k} = p^{t_k}(1-p)^{k-t_k}.$$

Since $B(x,r)$ does not contain $I^{(k-1)}(x)$, it must be the case that $B(x,r) \cap C$ is contained in the union of at most two Cantor intervals of step $k-1$. If it is actually the case that $B(x,r) \cap C \subseteq I^{(k-1)}(x)$, then $\mu(B(x,r)) \le \mu(I^{(k-1)}(x))$ and similar arguments to those used below, but easier, will complete the proof.

So assume $B(x,r) \cap C \subseteq I^{(k-1)}(x) \bigcup I^*$ and that the gap between these two step $k-1$ intervals was removed at step L in the construction, where $L \le k-1$. This means both $I^{(k-1)}(x)$ and I^* are descendants of a (common) step $L-1$ interval I. Furthermore, the step L gap is contained in $B(x,r)$ and thus

$$r_1 \cdots r_{L-1}(1 - 2r_L) \le r \le r_1 \cdots r_{k-1}.$$

By assumption there exists $\varepsilon > 0$ such that $r_j \le 1/2 - \varepsilon$ for all j. Consequently,

$$r_1 \cdots r_{L-1} 2\varepsilon \le r \le r_1 \cdots r_{L-1}(1/2)^{k-L}.$$

Hence there must be some integer m (depending only on ε) such that $k - L \le m$, in other words, $I^{(k-1)}(x)$ and I^* are both descendants of the Cantor interval $I = I_{w_1 \cdots w_{k-m}}$, of step $k-m$ and $B(x,r) \bigcap C \subseteq I$. Thus

$$p_{w_1} \cdots p_{w_k} \le \mu(B(x,r)) \le p_{w_1} \cdots p_{w_{k-m}}$$

and

$$\frac{\log p_{w_1} \cdots p_{w_{k-m}}}{\log r_1 \cdots r_k/2} \le \frac{\log(\mu(B(x,r)))}{\log r} \le \frac{\log p_{w_1} \cdots p_{w_k}}{\log r_1 \cdots r_{k-1}}.$$

Since m is bounded, we obtain the same limiting behaviour on both the left-and right-hand side as $r \to 0$, (or $k \to \infty$) and therefore

$$\dim_{loc} \mu(x) = \lim_{k \to \infty} \frac{\log p_{w_1} \cdots p_{w_k}}{\log r_1 \cdots r_k} = \lim_{k \to \infty} \frac{\log \left(\mu(I^{(k)}(x)) \right)}{\log \left| I^{(k)}(x) \right|}. \tag{1.1}$$

□

Remark 1.2. It follows easily from (1.1) that the local dimensions at 0 and 1 are the extreme values.

Proof (of Theorem). Define independent and identically distributed random variables on C by

$$X_k(x) = \begin{cases} 1 & \text{if } w_k = 0 \\ 0 & \text{if } w_k = 1 \end{cases} \text{ where } x = (w_k).$$

As the expected value of X_k is p, the Strong law of large numbers states that if $t_k(x)$ is the number of $0's$ occurring in the first k digits of x, then

$$\frac{t_k(x)}{k} = \frac{1}{k} \sum_{j=1}^{k} X_j(x) \to p \ \mu \text{ a.s.}$$

Thus, for μ almost all x,

$$\begin{aligned} \frac{\log \left(\mu(I^{(k)}(x)) \right)}{\log \left| I^{(k)}(x) \right|} &= \frac{\log p^{t_k} (1-p)^{k-t_k}}{\log r_1 \cdots r_k} \\ &= \frac{t_k \log p + (k - t_k) \log(1-p)}{\log r_1 \cdots r_k} \\ &\to \frac{p \log p + (1-p) \log(1-p)}{\log r_0}. \end{aligned}$$

□

1.3.2.2 Multifractal Formalism for p-Cantor Measures

An important feature of self-similar measures arising from an IFS that satisfies the open set condition is that the multifractal spectrum is a closed interval and the Hausdorff dimension of the sets E_α can be computed. Here we will see that the same property holds for many p-Cantor measures supported on Cantor sets with varying ratios of dissection. In place of the open set condition, we will assume that $\sup r_n < 1/2$. We will also continue to assume that $\frac{1}{n} \log (r_1 \cdots r_n) \to \log r_0$ so that we can work with limits, but related results can again be obtained using $\limsup$ or $\liminf$.

Theorem 1.2. *Suppose μ is the p-Cantor measure supported on the Cantor set $C = C(r_j)$ which satisfies $\sup r_n < 1/2$ and $\lim_n \frac{1}{n} \log (r_1 \cdots r_n) = \log r_0$. Without loss of* *generality, assume* $p \geq 1-p$. *Then the set* $E_\alpha = \{x \in C : \dim_{loc}\mu(x) = \alpha\}$ *is non-empty if and only if*

$$\alpha \in \left[\frac{\log p}{\log r_0}, \frac{\log(1-p)}{\log r_0}\right]$$

and $\dim_H E_\alpha = f(\alpha)$ *where*

$$f(\alpha) = \inf_{q \in \mathbb{R}} \left(q\alpha - \frac{\log(p^q + (1-p)^q)}{\log r_0} \right).$$

The proof we sketch below is similar to that known for self-similar sets satisfying the strong separation condition. Indeed, p-Cantor measures on Cantor sets with fixed ratio of dissection are examples of self-similar measures satisfying this separation property.

Proof (Sketch). The fact that E_α is non-empty only for the specified α is clear from (1.1).

For each $q \in \mathbb{R}$ we define the set function, ν_q, on C by

$$\nu_q(I_w) = \left(p_{w_1} \cdots p_{w_k}\right)^q \; (p^q + (1-p)^q)^{-k} \text{ if } w = w_1, \ldots, w_k.$$

One can check that ν_q is a probability measure concentrated on C and

$$\log \nu_q(I_w) = q \log \mu(I_w) - k \log(p^q + (1-p)^q)$$

Applying (a variant of) Lemma 1.1 to both μ and ν_q shows that

$$\begin{aligned} \dim_{loc} \nu_q(x) &= \lim_{k \to \infty} \frac{\log \nu_q(I^{(k)}(x))}{\log \left|I^{(k)}(x)\right|} \\ &= q \dim_{loc} \mu(x) - \frac{\log(p^q + (1-p)^q)}{\log r_0}. \end{aligned}$$

Thus, if $x \in E_\alpha$,

$$\dim_{loc} \nu_q(x) = q\alpha - \frac{\log(p^q + (1-p)^q)}{\log r_0}.$$

It is a routine calculus exercise to check that $f(\alpha)$ is achieved with the choice of $q = q(\alpha)$ satisfying

$$\alpha = \frac{p^q \log p + (1-p)^q \log(1-p)}{(p^q + (1-p)^q) \log r_0},$$

Thus $\dim_{loc} \nu_{q(\alpha)}(x) = f(\alpha)$ for all $x \in E_\alpha$.

If we can establish that $\nu_{q(\alpha)}$ is actually concentrated on E_α, then it will follow from Cor. 1.1 that $\dim_H E_\alpha = f(\alpha)$. To see this, fix $\varepsilon > 0$ and let $\delta > 0$ be small. Note that

$$\nu_{q(\alpha)}\left\{x : \mu(I^{(k)}(x)) \geq \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}\right\}$$

$$\begin{aligned}
&\leq \int \left(\mu(I^{(k)}(x))\right)^{\delta} \left|I^{(k)}(x)\right|^{-(\alpha-\varepsilon)\delta} d\nu(x) \\
&= \sum_{|w|=k} \mu(I_w)^{\delta}(r_1 \cdots r_k)^{-(\alpha-\varepsilon)\delta} \nu(I_w) \\
&= \sum_{|w|=k} (p_{w_1} \cdots p_{w_k})^{\delta+q} \left(\prod_{j=1}^{k} r_j^{-(\alpha-\varepsilon)\delta}\right) (p^q + (1-p)^q)^{-k} \\
&= \prod_{j=1}^{k} \left(\left(p^{q+\delta} + (1-p)^{q+\delta}\right) r_j^{-(\alpha-\varepsilon)\delta} (p^q + (1-p)^q)^{-1}\right) \equiv \Phi_1^{(k)}(\alpha).
\end{aligned}$$

Similarly,

$$\nu_{q(\alpha)}\left\{x : \mu(I^{(k)}(x)) \leq \left|I^{(k)}(x)\right|^{\alpha+\varepsilon}\right\}$$

$$\leq \prod_{j=1}^{k} \left(\left(p^{q+\delta} + (1-p)^{q+\delta}\right) r_j^{(\alpha+\varepsilon)\delta} (p^q + (1-p)^q)^{-1}\right) \equiv \Phi_2^{(k)}(\alpha).$$

Using Taylor series (in the variable δ), one can verify that for sufficiently large k (say, $k \geq k_1$) and suitable positive constants C_1, C_2,

$$\Phi_j^{(k)}(\alpha) \leq \exp(-k\delta C_j \varepsilon) \text{ for } j = 1, 2.$$

Thus

$$\sum_{k \geq k_1} \nu_{q(\alpha)}\{x : \mu(I^{(k)}(x)) \geq \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}\} \leq \sum_{k \geq k_1} \exp(-k\delta C_1 \varepsilon) < \infty$$

and similarly for $\sum \nu_{q(\alpha)}\{x : \mu(I^{(k)}(x)) \leq \left|I^{(k)}(x)\right|^{\alpha+\varepsilon}\}$. By the Borel Cantelli lemma, the probability that $\mu(I^{(k)}(x)) \geq \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}$ occurs infinitely often is zero and similarly for $\mu(I^{(k)}(x)) \leq \left|I^{(k)}(x)\right|^{\alpha+\varepsilon}$. Thus for $\nu_{q(\alpha)}$ a.e. x and large enough k,

$$\left|I^{(k)}(x)\right|^{\alpha+\varepsilon} \leq \mu(I^{(k)}(x)) \leq \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}.$$

Hence, for large enough k,

$$\alpha + \varepsilon \geq \frac{\log(\mu(I^{(k)}(x)))}{\log\left|I^{(k)}(x)\right|} \geq \alpha - \varepsilon$$

for $\nu_{q(\alpha)}$ a.e. x. As $\varepsilon > 0$ was arbitrary, it follows that $\dim_{loc} \mu(x) = \alpha$ for $\nu_{q(\alpha)}$ a.e. x, in other words, $\nu_{q(\alpha)}$ is concentrated on E_α as we desired to show.

We will leave it to the reader to check that $f(\alpha) \neq 0$ for $\alpha \in (\log p/\log r_0, \log(1-p)/\log r_0)$. As the endpoints of this interval are the local dimensions at 0 and 1, respectively, it follows that E_α is non-empty if and only if α belongs to the closure of the interval above. □

Exercise 1.4. Determine for which α the set E_α has maximal Hausdorff dimension and find that dimension.

1.4 Isolated Points in the Multifractal Spectrum

1.4.1 Isolated Points in the Spectrum of Convolutions of Cantor Measures

An important operation in many branches of analysis is convolution. Convolution is a binary operation on the space of measures on $\mathbb{R}$ defined in the following way.

Definition 1.2. If μ, ν are measures on $\mathbb{R}$, then their convolution, $\mu * \nu$, is defined as the measure with the property that for any Borel set $E \subseteq \mathbb{R}$,

$$\mu * \nu(E) = \int \mu(E - x) d\nu(x).$$

One can verify that the support of $\mu * \nu$ is contained in the sum of the supports of μ and ν.

Given a measure μ, we will write μ^m for the m-fold convolution of μ. When μ is the uniform Cantor measure on $C(r)$, then μ^m is a self-similar measure generated by the IFS with contractions $F_i(x) = rx + (1-r)i$ for $i = 0, 1, \ldots, m$ and weights $p_i = 2^{-m}\binom{m}{i}$. The support of the invariant measure is the m-fold sum of $C(r)$. For example, if μ is the classical Cantor measure and $m \geq 2$, then μ^m is supported on $[0, m]$. In this case, the IFS satisfies the open set condition if and only if $m \leq 2$.

Exercise 1.5. Determine the multifractal spectrum of $\mu * \mu$ for the classical Cantor measure μ.

In striking contrast to the case of self-similar measures associated with IFS satisfying the open set condition, the multifractal spectrum of μ^3 is known to consist of the union of a closed interval and an isolated point:

$$\{\alpha : E_\alpha(\mu^3) \neq \phi\} = \left[\frac{\log 8/3}{\log 3}, \frac{\log 8/\sqrt{b}}{\log 3}\right] \bigcup \left\{\frac{\log 8}{\log 3}\right\},$$

where $b = \left(7 + \sqrt{13}\right)/2$. It can be checked that $\log 8/\sqrt{b}/\log 3 \sim 1.1335$ and $\log 8/\log 3 \sim 1.89278$. It is also known that

$$\frac{\log 8/3}{\log 3} = \dim_{loc} \mu^3(x) \text{ for } x = (w_i) \text{ where } w_i \in \{1,2\}$$
$$\frac{\log 8/\sqrt{b}}{\log 3} = \dim_{loc} \mu^3(x) \text{ for } x = (w_i) \text{ where } w_{2i} = 0, w_{2i+1} = 1$$
$$\frac{\log 8}{\log 3} = \dim_{loc} \mu^3(0) = \dim_{loc} \mu^3(3) \text{ (and at no other } x).$$

The proof of these facts make strong use of the elegant combinatorial structure of the Cantor set and its 3-fold sum.

Similar results have been obtained for m-fold convolutions of the uniform Cantor measures on the Cantor sets $C(1/d)$ when $d \in \mathbb{N}$ and, more generally, for self-similar measures generated by an IFS consisting of contractions $F_i(x) = x/d + (d-1)i/d$ for $i = 0, 1, \ldots, m$ and probabilities $p_i > 0$, where $p_0, p_m \le p_i$ for all $i \ne 0, m$ and $d \ge 3$ is an integer. The algebraic and combinatorial structure of these self-similar measures can again be used to show that if $m \ge d$, then the multifractal spectrum is the union of a closed interval and one (or two) isolated points, the local dimensions at $0, m$. The significance of $m \ge d$ is that the support of μ^m is $[0, m]$.

A similar result holds, as well, for convolutions of p-Cantor measures μ supported on the Cantor sets $C(r_j)$. Provided $\inf r_j > 0$, the spectrum of μ^m has also been shown to have an isolated point for sufficiently large m, either $\dim_{loc} \mu^m(0)$ or $\dim_{loc} \mu^m(m)$, depending on whether p or $1-p$ is larger. Again, a key idea in the proof of this result is that the Cantor sets, $C(r_j)$, have the property that the M-fold sum of $C(r_j)$ (the support of μ^M) is the interval $[0, M]$ if $M + 1 \ge \sup 1/r_j$.

1.4.2 Isolated Points in the Spectrum of Convolutions of General Measures

It turns out a much more general result is true for convolutions of probability measures: If μ is any continuous, probability measure supported on $[0,1]$ and there is some integer M with the M-fold sum of the support of μ equal to $[0, M]$, then under rather mild assumptions, it is guaranteed that there will be an isolated point in the spectrum of μ^m for sufficiently large m.

Theorem 1.3. *Suppose μ is a continuous, probability measure supported on $[0,1]$ with $0, 1 \in supp\mu$ and assume $(M)supp\mu = [0, M]$ for some integer M. Assume, also, that*

1. $\overline{\dim}_{loc}\mu(0) > 0$ *and*
2. $\sup\{\overline{\dim}_{loc}\mu(x) : x \in supp\mu\} < \infty$.

Then there is an integer n_0 such that for all $n \ge n_0$, $\overline{\dim}_{loc}\mu^n(0)$ is isolated in the set of local dimensions of μ^n.

A similar statement holds with upper local dimensions replaced by lower local dimensions. We begin with two preliminary lemmas.

Lemma 1.2. *Suppose μ,ν are measures with supp$\nu=[0,n]$ and $0,1\in$supp$\mu\subseteq[0,1]$.*

(i) If $\overline{\dim}_{loc}\nu(x)\leq\lambda<\infty$ for all $x\in[0,n]$, then $\overline{\dim}_{loc}\nu*\mu(z)\leq\lambda$ for all $z\in(0,n+1)$.
(ii) If, in addition, μ is a continuous measure, then the same conclusion holds under the weaker assumption that $\overline{\dim}_{loc}\nu(x)\leq\lambda$ for all $x\in(0,n)$.

Proof. (i) Fix $z\in(0,n+1)$ and let $I=[0,1]\cap[z-n,z]$. Notice that at least one of 0 or 1 belongs to the relative interior of I. As $0,1$ both belong to suppμ it follows that $\mu(I)=\eta>0$.

Fix $\delta>0$. If $x\in I$, then $z-x\in[0,n]=$suppν, hence $\overline{\dim}_{loc}\nu(x)\leq\lambda$. This means that for every $x\in I$, there exists $r_x>0$ such that if $r<r_x$ then

$$\frac{\log(\nu(B(z-x,r)))}{\log r}\leq\lambda+\varepsilon.$$

Let $A_n=\{x\in I:r_x\geq 1/n\}$. As $\bigcup A_n=I$, by continuity of measure there is some n such that $\mu(A_n)>\eta/2$. For all $x\in A_n$,

$$\nu(B(z-x,r))\geq r^{\lambda+\varepsilon}\text{ for all }r\leq 1/n,$$

thus

$$\begin{aligned}\nu*\mu(B(z,r))&=\int\nu(B(z-x,r))d\mu(x)\geq\int_{A_n}r^{\lambda+\varepsilon}d\mu(x)\\&\geq r^{\lambda+\varepsilon}\mu(A_n)\geq r^{\lambda+\varepsilon}\eta/2.\end{aligned}$$

Hence

$$\frac{\log(\nu*\mu(B(z,r)))}{\log r}\leq\frac{\log\eta/2}{\log r}+\lambda+\varepsilon\text{ for all }r\leq 1/n.$$

Letting $r\to 0$, it follows that $\overline{\dim}_{loc}\nu*\mu(z)\leq\lambda+\varepsilon$ and since that holds for all $\varepsilon>0$, we conclude that $\overline{\dim}_{loc}\nu*\mu(z)\leq\lambda$.

(ii) Under the weaker assumption of (ii), it is still true that $\overline{\dim}_{loc}\nu(x)\leq\lambda$ for all $x\neq z,z-n$. But $\mu\{z\}=\mu\{z-n\}=0$, hence the sets $I\backslash\{z,z-n\}$ and I have the same positive μ-measure. Let $A_n=\{x\in I\backslash\{z,z-n\}:r_x\geq 1/n\}$ and choose n such that $\mu(A_n)>\mu(I)/2$. We conclude the proof as in the first part. □

Lemma 1.3. *Assume μ is supported on $[0,1]$.*

(i) Then $\overline{\dim}\mu^n(0)=n\overline{\dim}\mu(0)$.
(ii) If $x_j\in$suppμ and $x=\sum_{j=1}^n x_j$, then $\overline{\dim}\mu^n(x)\leq\sum_{j=1}^n\overline{\dim}\mu^n(x_j)$.

Proof. (i) This follows easily from the fact that

$$(\mu(B(0,r/n)))^n\leq\mu^n(B(0,r))\leq(\mu(B(0,r)))^n.$$

(ii) is similar. □

Proof (of Theorem). By assumption, if $x \in [0,M]$, then there are real numbers $x_j \in \text{supp}\mu$ such that $\sum_{j=1}^{M} x_j = x$. As $\overline{\dim}\mu(z) \leq \lambda$ for all $z \in \text{supp}\mu$, the previous lemma (ii) implies $\overline{\dim}\mu^M(x) \leq M\lambda$ for all $x \in [0,M] = \text{supp}\mu^M$.

Now apply Lemma 1.2 (either (i) or (ii)) with $\nu = \mu^M$ and $M = n$ to deduce that $\overline{\dim}\mu^{M+1}(x) \leq M\lambda$ for all $x \in (0, M+1)$.

Since $\text{supp}\mu^{M+T} = [0, M+T]$, we can repeatedly apply this argument (but with part (ii) of the lemma as we have only the weaker hypothesis satisfied) to deduce that $\overline{\dim}\mu^n(x) \leq M\lambda$ for all $x \in (0,n)$ and any $n \geq M$.

As Lemma 1.3 (i) implies $\overline{\dim}\mu^n(0) \to \infty$ as $n \to \infty$, $\overline{\dim}\mu^n(0)$ will be isolated in the spectrum for large enough n. □

Exercise 1.6. For what n can you be sure μ^n has an isolated point in its spectrum when μ is the uniform Cantor measure on the Cantor set $C(r)$?

1.5 Credits

The size of Cantor sets and their sums was explored in [3]. There the formula is given for the Hausdorff dimension of $C(r_k)$ and it is proven that if $\inf r_k > 0$, then some n-fold sum of $C(r_k)$ is the interval $[0,n]$.

An excellent exposition on local dimensions, including the proofs of Prop. 1.1 and 1.2, the probabilistic ideas in the proof of Theorem 1.1, and the multifractal analysis for self-similar measures arising from IFS satisfying the strong separation condition, can be in Falconer's books, [5] and [6], (particularly, chapters 17 and 10, 11, respectively). This is based in part upon the earlier work of Cawley and Mauldin [4], Mandelbrot [14], Riedi [17] and others. We refer the reader to the bibliographies given in [5] and [6] for further papers. In particular, Olsen in [15] developed a strong mathematical foundation for multifractal analysis.

Motivated in part by [16], the multifractal analysis of p-Cantor measures on $C(r_k)$ is investigated in [11]. There one can find the proofs of Theorem 1.2 and the geometric result, Lemma 1.1.

Hu and Lau in [12] established the existence of an isolated point in the spectrum of the 3-fold convolution of the uniform Cantor measure on $C(1/3)$. This fact was extended to various self-similar measures with overlap, generated by IFS with contraction factors $1/d$, $d \in \mathbb{N}$, in [8, 18] and [2]. In the latter paper, formulas are given for the spectrum in the case of convolutions of uniform Cantor measures on Cantor sets $C(1/d)$. A proof of the existence of an isolated point in the spectrum of convolutions of very general Cantor measures is given in [10]. Theorem 1.3 is proven in [1]. Pathological examples are constructed in [2] and [19].

Hu and Lau have extensively investigated the multifractal analysis of self-similar measures with overlap in a series of papers, including [7–9] and [13].

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Chapter 2
Multifractal Analysis and Wavelets

Stéphane Seuret

Abstract In this course, we give the basics of the part of multifractal theory that intersects wavelet theory. We start by characterizing the pointwise Hölder exponents by some decay rates of wavelet coefficients. Then, we give some examples of wavelet series having a multifractal behavior, and we explain how to build wavelet series with prescribed pointwise Hölder exponents. Next we develop the problematics of multifractal formalism, going from the intuitive formula by Frisch and Parisi to explicit and exploitable formulas. We prove that "multifractals are everywhere," in the sense that typical functions in Besov spaces or typical measures are multifractal in the sense of Baire categories. We finish by some well-known examples of multifractal wavelet series, random and deterministic, focusing on the influence of certain adaptive threshold procedures to the multifractal properties of signals.

2.1 Introduction

In the context of functional analysis, multifractal analysis is concerned with the local regularity and the scaling behavior of functions: it is an attempt to describe the geometric and statistic distribution of the singularities of a function. One major motivation for going inside multifractal theory is that multifractal studies have direct connections with many mathematical fields (harmonic and functional analysis, probability theory and stochastic processes, dynamical systems and ergodic theory, geometric measure theory, and even number theory), and simultaneously they

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have many natural application fields (physics, biology, and physiology, amongst many other upcoming examples) based on the developments of new numerical procedures in signal and image processing.

In this course, we develop the basic facts about multifractal analysis of functions, the tools being mainly geometric measure theory and wavelets.

Let us start by recalling how the local regularity of a locally bounded function is quantified.

Definition 2.1. Let $f \in L^\infty_{loc}(\mathbb{R}^d)$, and $x_0 \in \mathbb{R}^d$. Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$.

The function f is said to belong to $C^\alpha(x_0)$ if there exist two positive constants $C > 0, M > 0$, a polynomial P with degree less than $\lfloor \alpha \rfloor$ (the integer part of α), such that when $|x - x_0| \leq M$,

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{2.1}$$

Then, the *pointwise Hölder exponent of f at x* is

$$h_f(x_0) = \sup\{\alpha \geq 0 : \ f \in C^\alpha(x_0)\}, \tag{2.2}$$

If $h_f(x_0) = h$, the point x_0 is called a singularity of order h for f.

Observe that when the exponent $h_f(x_0)$ is strictly less than 1, it takes a much simpler form:

$$h_f(x_0) = \liminf_{x \to x_0} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|},$$

where by convention $\log 0 = -\infty$.

Exercise 2.1. Prove that the polynomial P in the definition of $C^s(x)$ is unique.

Exercise 2.2. Prove that if $s < s'$, $C^{s'}(x) \subset C^s(x)$.

Exercise 2.3. Let $f \in C^s(x)$, and call F a primitive of f. Prove that $F \in C^{s+1}(x)$. Build an example where $h_f(x) = s$ and $h_F(x) = s + 2$.

Exercise 2.4. Let $f \in C^s(x)$, with $s > 1$. Does one always have $f' \in C^{s-1}(x)$?

This exponent $h_f(x)$ encapsulates significant information about the local behavior of f around x: the less its value is, the more irregular the graph of the function f locally looks like.

As can be seen on real data signals (see Figure 2.1), or as can be computed on "pure" mathematical functions, the pointwise Hölder exponent $h_f(x)$ can be very erratic when viewed as a function of x, even for functions very easy to define. The most popular example of function whose pointwise Hölder exponent $h_f(x)$ depends highly on x (in a non-continuous manner) is certainly the "non-differentiable Riemann function," i.e. the lacunary Fourier series

$$R(x) = \sum_{n \geq 1} \frac{\sin(n^2 \pi x)}{n^2}.$$

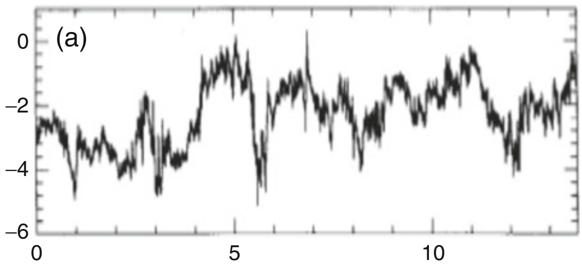

Fig. 2.1 1D-signal of the velocity of a turbulent fluid [18].

It took almost 140 years to complete the multifractal analysis of R, i.e. to compute the pointwise exponent of R at every x and to fully describe the geometric distribution of these singularities!! The graph is plotted on Figure 2.2. In particular, R is differentiable at the rationals p/q, where p and q are both odd.

Even if one is able to compute the pointwise Hölder exponent of a function f at every point x, the knowledge of all these exponents does not necessarily give a concrete idea of what the graph of the function looks like, or of which the most significant singularities (or the most frequent ones) are. In order to describe the diversity of the local behaviors of f, one focuses on the iso-Hölder sets associated with the pointwise Hölder exponents.

Definition 2.2. For every $h \in \mathbb{R}^+ \cup \{+\infty\}$, the iso-Hölder set $E_f(h)$ is the set

$$E_f(h) = \{x \in \mathbb{R}^d : h_f(x) = h\}$$

of all singularities of pointwise Hölder exponent for f equal to h.

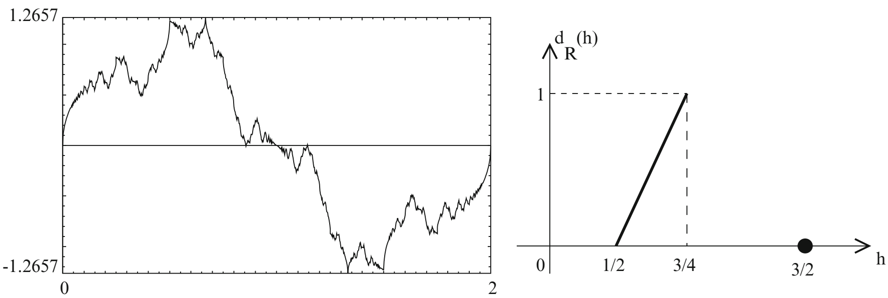

Fig. 2.2 "Non-differentiable" Riemann function, and its multifractal spectrum.

A single iso-Hölder set $E_f(h)$ may be concentrated around one region of $\mathbb{R}^d$, or spread all over the space. One thus needs a way to compare the sizes of the sets $E_f(h)$. It turns out that the right notion to distinguish them is the Hausdorff dimension, the main reason being the following: if one keeps in mind that the models we are interested in are built using procedures involving either random construction or dynamical systems, then our intuition (based on the law of large numbers or the Birkhoff ergodic theorem, depending on the context) makes us expect that there is a single value h_s such that Lebesgue-almost every $x \in \mathbb{R}^d$ has a pointwise Hölder exponent h_s for f (the same value h_s for Lebesgue every x!). So the Lebesgue measure is not the appropriate tool to measure the size of the iso-Hölder sets, since one $E_f(h)$ will have full Lebesgue measure, and all the other ones will have measure 0. It is natural idea to compare their "fractal" dimension. Actually, "fractal" dimension does not exist, it is either box (also called Minkowski) dimension, Hausdorff dimension or less frequently packing dimension. It appears that for many natural functions or sample paths of stochastic processes, the sets $E_f(h)$ are fractal (whatever this means!) and often dense in the support of the corresponding function. Unfortunately, the box dimension gives full dimension (i.e., dimension d) to any dense set, so it does not distinguish them. This is one of the heuristic reasons that explains the choice of the Hausdorff dimension (there are other explanations based on theoretical results, as will be explained below), and leads to the definition of the main object of study of this course.

Definition 2.3. The multifractal spectrum (also called the spectrum of singularities) of a locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ is the mapping $d_f : \mathbb{R}^+ \cup \{+\infty\} \to [0,d] \cup \{-\infty\}$ defined by

$$h \longmapsto d_f(h) := \dim_{\mathscr{H}} E_f(h),$$

where $\dim_{\mathscr{H}}$ stands for the Hausdorff dimension, and where by convention $\dim_{\mathscr{H}} \emptyset = -\infty$.

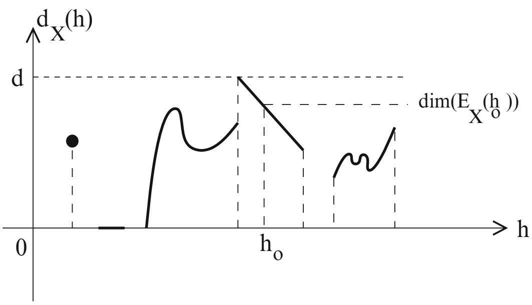

Fig. 2.3 Example of multifractal spectrum

The multifractal spectrum of a function contains information regarding the geometric distribution of the singularities of f. See Figure 2.3 for an example of multifractal spectrum. Of course, at first sight this quantity may seem difficult to compute, even more to estimate on real signals or images. Indeed, before accessing to the value of $d_f(h)$, many limits are needed, and the successive approximations would lead to results that are certainly meaningless. This is where the notion of multifractal formalism arises. We do not develop it here, just indicating the main idea of it: for "homogeneous" functions (i.e., functions which at least statistically have the same scaling behavior in any region of $\mathbb{R}^d$), following some heuristics from turbulence and thermodynamic formalism, it is reasonable to expect that the multifractal spectrum should be a concave function and should satisfy an equality having the following form:

$$d_f(h) = \inf_{q\in\mathbb{R}} (qh - \zeta_f(q) + d), \tag{2.3}$$

where $\zeta_f(q)$ is some global quantity computed from f, called the scaling function associated with f. In this situation, the multifractal spectrum is thus obtained as the Legendre transform of the scaling function, hence leading to the concave shape for d_f that we mentioned before. For continuous functions, a possible definition for $\zeta_f(q)$ is

$$\zeta_f(q) := \sup\left\{ s > 0 : f \in B^{s/q,\infty}_{q,loc}(\mathbb{R}^d) \right\}.$$

Of course the precise value of the scaling function $\zeta_f(q)$ may depend on the context, nevertheless, in all cases, when formula (2.3) (or an analog of it) holds true for a function f and an exponent h, one says that **the multifractal formalism holds for** f **at** h. See Section 2.4 for all details.

The intuition that the multifractal formalism holds for nice models is supported by numerous representative examples: many stochastic processes (Lévy processes, wavelet series, etc.) obey the multifractal formalism, as well as "typical" functions in many function spaces (the set of monotonic functions, Hölder and Besov spaces for instance). Moreover, even if the multifractal formalism does not hold, the Legendre spectrum

$$\zeta_f^*(h) := \inf_{q\in\mathbb{R}} (qh - \zeta_f(q) + d)$$

is meaningful since it encapsulates information about the histograms of oscillations or of wavelet coefficients associated with f. The key point is that the scaling function $\zeta_f(q)$ as we defined it just above, and thus its Legendre transform $\zeta_f^*(h)$, is accessible by numerical methods (using log-log diagrams for instance), while d_f is not. Hence, the Legendre spectrum $\zeta_f^*(h)$ is a quantity that can be estimated on every signal or image, and its form can be interpreted in terms of presence/relevance/density of the singularities of the object under consideration. The reader is referred to the course of P. Abry and S. Jaffard in the same volume to learn about efficient numerical procedures to estimate various scaling functions (see also [1, 2]).

Wavelets constitute a natural tool to study the multifractal nature of a function. For, there are two main reasons: the first one is that the pointwise Hölder exponent can be characterized by size estimates on the wavelet coefficients (see Theorem 2.4 below). The second one is that many function spaces (Hölder and Besov spaces for instance) can also be characterized by decay rates of the wavelet coefficients. Also, the fact that a wavelet basis is self-similar by construction (all the functions

$\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ are obtained through a translation and dilation of a same initial function ψ) is a priori an advantage to study "fractal"-like properties, but this could be discussed since it is self-similar with very specific ratio (powers of 2) while one aims at studying any irregular function. Anyway, wavelets are very important tools in this course, and some prior knowledge about their construction is advised, although we will only use their basic properties (vanishing moments, space and frequency localization).

The course is organized as follows. Section 2.2 contains the necessary material for the rest of the course: wavelet coefficients, Hausdorff dimension and some geometric measure theory, local dimensions of measures. In Section 2.3, we prove the characterization of the pointwise Hölder exponent by size estimates of the wavelet coefficients, or by size estimates of the wavelet leaders. We also explain how to build a function with prescribed local regularity, and give some examples of multifractal wavelet series. In Section 2.4, we develop the intuitive notion of multifractal formalism, and then give some theoretical results on multifractals; for instance, we explain how to obtain a priori upper bounds for multifractal spectra for Besov function and measures. In Section 2.5 it is proved that typical functions or measures (in the sense of Baire's category) in suitable function spaces are multifractal. There, we use methods described in Section 2.2 to effectively compute the Hausdorff dimensions of the iso-Hölder sets of some functions. Finally Section 2.6 contains examples of multifractal functions built as wavelet series (the proofs are essentially written as long exercises).

2.2 Recalls on Wavelets and Geometric Measure Theory

2.2.1 Wavelets

We recall very briefly the basics of multiresolution wavelet analysis (for details see, for instance, [15, 33]). For an arbitrary integer $N \geq 1$ one can construct compactly supported functions $\psi^0 \in C^N(\mathbb{R})$ (called the scaling function) and $\psi^1 \in C^N(\mathbb{R})$ (called the mother wavelet), with ψ^1 having at least $N+1$ vanishing moments (i.e., $\int_{\mathbb{R}} x^n \psi^1(x)dx = 0$ for $n \in \{0,\dots,N\}$), and such that the set of functions

$$\psi^1_{j,k} : x \mapsto \psi^1(2^j x - k)$$

for $j \in \mathbb{Z}, k \in \mathbb{Z}$ form an orthogonal basis of $L^2(\mathbb{R})$ (note that we choose the L^∞ normalization, not L^2). In this case, the wavelet is said to be N-regular.

Let us introduce the notations

$$0^d := (0,0,\cdots,0), \qquad 1^d := (1,1,\cdots,1), \qquad L^d = \{0,1\}^d \backslash 0^d.$$

An orthogonal basis of $L^2(\mathbb{R}^d)$ is then obtained by tensorization. For every $\lambda = (j,\mathbf{k},\mathbf{l}) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d$, let us define the tensorized wavelet

$$\Psi_\lambda(x) = \Psi^{\mathbf{l}}(2^j x - \mathbf{k}) := \prod_{i=1}^{d} \psi^{l_i}_{j,k_i}(x_i),$$

with obvious notations: $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ and $\mathbf{l} = (l_1, l_2, \cdots, l_d)$.

Any function $f \in L^2(\mathbb{R}^d)$ can be written (the equality being true in $L^2(\mathbb{R}^d)$)

$$f(x) = \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\, j\in\mathbb{Z},\, \mathbf{k}\in\mathbb{Z}^d,\, \mathbf{l}\in L^d} d_\lambda \Psi_\lambda(x), \tag{2.4}$$

where

$$d_\lambda = d_\lambda(f) := 2^{jd} \int_{R^d} f(x) \Psi_\lambda(x)\, dx. \tag{2.5}$$

It is implicit in (2.5) that the wavelet coefficients depend on f. Observe that in the wavelet decomposition (2.4), no wavelet Ψ_λ such that $\mathbf{l} = 0^d$ (where $\lambda = (j, \mathbf{k}, \mathbf{l})$) appears.

Assumption: We always assume that the wavelet has a number of vanishing moments larger than the index of regularity that we are looking at. Typically, if we focus on singularities on order h, we assume that ψ^1 has at least $\lfloor h \rfloor + 1$ vanishing moments.

The reason for this assumption is that wavelets with enough vanishing moments can be used to characterize Hölder functions.

Theorem 2.1. *For $s \in \mathbb{R}_+ \backslash \mathbb{N}$, a function f belongs to $C^s(\mathbb{R}^d)$ if and only if there exists a constant $C > 0$ such that*

$$\forall \lambda \in \mathbb{Z} \times \mathbb{Z}^d \times \{0,1\}^d \qquad |d_\lambda(f)| \leq C 2^{-js}. \tag{2.6}$$

We do not prove Theorem 2.1 here, it is a good exercise, the proof of which can be achieved by adapting the proof of Theorem 2.4 given later.

Wavelets can also be used to characterize functions in a Besov space, see Section 2.4.3.

2.2.2 Localization of the Problem

We are interested in the local behavior of functions, hence when focusing on $x_0 \in \mathbb{R}^d$, what happens far from x_0 should not interfere with our questions. Moreover, again because we concentrate on local phenomena, the low frequency terms have no importance in our analysis. This is why we focus only on functions supported by $[0,1]^d$, and when we deal with wavelets, we assume that the function we deal with have a wavelet decomposition like

$$f = \sum_{\substack{\lambda=(j,\mathbf{k},\mathbf{l}):\\ j\geq 0,\, \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d}} d_\lambda \Psi_\lambda(x). \tag{2.7}$$

2.2.3 Hausdorff and Box Dimension

Two notions of dimensions of sets in $\mathbb{R}^d$ will be used below: the Hausdorff dimension and the upper box dimension. We recall them quickly.

Let X be a bounded set in $\mathbb{R}^d$. For every $\varepsilon > 0$, denote by $N_\varepsilon(X)$ the minimal number of balls of diameter ε needed to fully cover the set X. The lower box dimension of X, denoted by $\underline{\dim}_B(X)$, is then the real number $\in [0,d]$ defined as

$$\underline{\dim}_B(X) = \liminf_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(X)}{-\log \varepsilon}. \tag{2.8}$$

Similarly, the upper box dimension of X is

$$\overline{\dim}_B(X) = \limsup_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(X)}{-\log \varepsilon}. \tag{2.9}$$

Exercise 2.5. Prove that $N_\varepsilon(X)$ is well defined, and that $\overline{\dim}_B(X) \leq d$.

Exercise 2.6. Build a set X such that $\underline{\dim}_B(X) < \overline{\dim}_B(X)$

When $\underline{\dim}_B(X) = \overline{\dim}_B(X)$, one denotes by $\dim_B(X)$ their common value.

Exercise 2.7. Prove that:

1. the box dimension of an open set is d.
2. the box dimension of the triadic Cantor set is $\log 2/\log 3$.
3. the box dimension of a set X dense in $[0,1]^d$ is d.

We also recall the definition of the Hausdorff dimension.

Definition 2.4. Let $s \geq 0$. The s-dimensional Hausdorff measure of a set X, $\mathscr{H}^s(X)$, is defined as

$$\mathscr{H}^s(X) = \lim_{r \searrow 0} \mathscr{H}^s_r(X), \quad \text{with } \mathscr{H}^s_r(X) = \inf\Big\{ \sum_i |X_i|^s \Big\},$$

the infimum being taken over all the countable families of sets X_i such that $|X_i| \leq r$ and $X \subset \bigcup_i X_i$. Then, the Hausdorff dimension of X, $\dim_{\mathscr{H}} X$, is defined as

$$\dim_{\mathscr{H}} X = \inf\{s \geq 0 : \mathscr{H}^s(X) = 0\} = \sup\{s \geq 0 : \mathscr{H}^s(X) = +\infty\}.$$

It is a good exercise to prove that for any bounded set $E \subset \mathbb{R}^d$, we have

$$0 \leq \dim_{\mathscr{H}}(E) \leq \underline{\dim}_B(E) \leq d.$$

In order to find an upper bound for the Hausdorff dimension of X, the most common method is the following: First guess what the dimension should be, call δ this expected value. Then, fix $s > \delta$, and find a covering $(X_i)_{i \in \mathbb{N}}$ of X such that

$$\mathscr{H}^s\left(\bigcup_{i\in\mathbb{N}} X_i\right) < +\infty.$$

This implies, by the definition of $\dim_{\mathscr{H}} X$, that $s \geq \dim_{\mathscr{H}} X$. This being true for any $s > \delta$, one deduces that $\delta \geq \dim_{\mathscr{H}} X$, hence the upper bound.

Obtaining a lower bound is in most cases much more difficult. Let us mention the mass distribution principle, on which most methods are based.

Theorem 2.2. *Let $X \subset \mathbb{R}^d$ be a Borel set, and assume that there exists a positive finite measure μ supported by X satisfying the following scaling property: there exists a positive real number $s > 0$ and a constant $C > 0$ such that*

$$\textit{for any } x \in X, \textit{ for any } 0 < r < 1, \;\; \mu(B(x,r)) \leq Cr^s.$$

Then $\mathscr{H}^s(X) > \frac{\mu(X)}{C}$, and thus $\dim_{\mathscr{H}} X \geq s$.

Exercise 2.8. Prove that the Hausdorff dimension of the triadic Cantor set is $\log 2/\log 3$ *(Hint: apply Theorem 2.2 with the uniform measure on the Cantor set).*

Another theorem that often allows one to compute Hausdorff dimension of iso-Hölder sets in multifractal analysis is the following theorem by Beresnevich and Velani [10]:

Theorem 2.3. *Let $(x_n)_{n\geq 1}$ be a sequence of points in $[0,1]^d$, and let $(l_n)_{n\geq 1}$ be a positive non-increasing sequence of radii. If*

$$\mathscr{L}^d\left(\limsup_{n\to+\infty} B(x_n,l_n)\right) = \mathscr{L}^d\left([0,1]^d\right),$$

($\mathscr{L}^d$ is the d-dimensional Lebesgue measure), then for every $\xi > 1$, one has

$$\mathscr{H}^{d/\xi}\left(\limsup_{n\to+\infty} B(x_n,(l_n)^\xi)\right) = +\infty.$$

Exercise 2.9. Let ξ_x be the approximation rate of an irrational number $x \in [0,1]$ by the dyadic numbers, defined by

$$\xi_x = \sup\{\xi \geq 0 : |x - k2^{-j}| \leq 2^{-j\xi} \text{ for i.m. pairs } (j,k),\ j \geq 1, k \text{ odd}\}.$$

1. Prove that for every irrational number $x \in [0,1]$, $\xi_x \geq 1$.
2. Let $S_\xi = \{x : |x - k2^{-j}| \leq 2^{-j\xi} \text{ for i.m. pairs } (j,k),\ j \geq 1, k \text{ odd}\}$ and $\widetilde{S}_\xi = \{x : \xi_x = \xi\}$. Prove that

$$\widetilde{S}_\xi = \left(\bigcap_{\xi'<\xi} S_{\xi'}\right) \setminus \left(\bigcup_{\xi'>\xi} S_{\xi'}\right).$$

3. Prove that for $\xi \geq 1$, $\dim_{\mathscr{H}}(S_\xi) \leq 1/\xi$ and $\dim_{\mathscr{H}}(\widetilde{S}_\xi) \leq 1/\xi$.

4. Prove that for $\xi > 1$, $\mathscr{H}^{1/\xi}(S_\xi) = \mathscr{H}^{1/\xi}(\widetilde{S}_\xi) = +\infty$.
5. Deduce the value of the Hausdorff dimension of S_ξ and $\widetilde{S}_\xi$, for every $\xi \geq 1$.

Exercise 2.10. Let ξ_x be the Diophantine approximation rate of an irrational number $x \in [0,1]$ by the rationals, defined by

$$\xi_x = \sup\{\xi \geq 0 : |x - p/q| \leq q^{-2\xi} \text{ for i.m. } q \geq 1 \text{ and } p \text{ with } p \wedge q = 1\}.$$

Here, the notation $p \wedge q$ stands for the largest common divisor of the two integers p and q.

1. Prove Dirichlet's theorem: for every irrational number $x \in [0,1]$, $\xi_x \geq 1$. (Hint: Use the pigeon-hole principle).
2. Let

$$S_\xi = \{x : |x - p/q| \leq q^{-2\xi} \text{ for infinitely many } q \geq 1 \text{ and } p \text{ with } p \wedge q = 1\}$$

and $\widetilde{S}_\xi = \{x : \xi_x = \xi\}$. Prove that

$$\widetilde{S}_\xi = \left(\bigcap_{\xi' < \xi} S_{\xi'} \right) \setminus \left(\bigcup_{\xi' > \xi} S_{\xi'} \right).$$

3. Prove that for $\xi \geq 1$, $\dim_{\mathscr{H}}(S_\xi) \leq 1/\xi$ and $\dim_{\mathscr{H}}(\widetilde{S}_\xi) \leq 1/\xi$.
4. Prove that for $\xi > 1$, $\mathscr{H}^{1/\xi}(S_\xi) = \mathscr{H}^{1/\xi}(\widetilde{S}_\xi) = +\infty$.
5. Deduce the value of the Hausdorff dimension of S_ξ and $\widetilde{S}_\xi$, for every $\xi \geq 1$.

Other methods will be probably used hereafter, we will mention them along the proofs.

2.2.4 Local Dimensions of Measures

Recall that the support of a Borel positive measure, denoted by $\mathrm{Supp}(\mu)$, is the smallest closed set $E \subset \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus E) = 0$.

Definition 2.5. Let μ be a positive measure supported on $\mathbb{R}^d$ at $x_0 \in \mathrm{Supp}(\mu)$. The (lower) local dimension $h_\mu(x_0)$ (also called local Hölder exponent) is

$$h_\mu(x_0) = \liminf_{r \to 0^+} \frac{\log \mu(B(x_0, r))}{\log r}, \tag{2.10}$$

where $B(x_0, r)$ denotes the open ball with center x_0 and radius r. When $x_0 \notin \mathrm{Supp}(\mu)$, by convention we set $h_\mu(x_0) = +\infty$.

Of course, in $\mathbb{R}$, there is a correspondence between the local dimension of a measure and the pointwise Hölder exponent of its primitive $F(x) = \int_0^x d\mu$.

Exercise 2.11. Prove that if $h_\mu(x_0) \notin \mathbb{N}$, then $h_F(x_0) = h_\mu(x_0)$. Is the converse true? (Hint: consider the Lebesgue measure).

Iso-Hölder sets and multifractal spectrum are quantities that can be defined for measures using the same ideas: One sets

$$E_\mu(h) = \{x \in \mathbb{R}^d : h_\mu(x) = h\}$$

and

$$d_\mu : h \longmapsto d_\mu(h) := \dim_{\mathscr{H}} E_\mu(h).$$

Contrary to what happens for measures, there is a strong constraint valid for all measures: for every $h \geq 0$, for every positive Borel measure on $\mathbb{R}^d$, one has (see next sections)

$$d_\mu(h) \leq \min(h, d).$$

This is one major difference between measures and functions from the multifractal standpoint (essentially due to the fact that measures have bounded variations).

2.2.5 Legendre Transform

The Legendre transform appears in many places in analysis, we recall the properties that are needed in the following.

Definition 2.6. Let $L : \mathbb{R} \to \mathbb{R}$ be a concave increasing function. The Legendre transform of L is the mapping $L^* : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$h \mapsto L^*(h) := \inf_{q \in \mathbb{R}} \big(qh - L(q)\big).$$

The assumption that L is increasing is not necessary for the definition of the Legendre transform, but it will be the case in our context in the following. In our cases, the function L satisfies $L(0) < 0$, and in this case one shall keep in mind the following properties:

- The support of L^* is included in the smallest interval containing $L'(\mathbb{R})$, the extreme points may or may not belong to the support, depending on L.
- L^* is concave on its support.
- If $h = L'(q)$ (i.e., L is differentiable at q), then $L^*(h) = qh - L(q)$.
- When $L'(0^+)$ exists, the Legendre transform L^* reaches its maximum at $h = L'(0^+)$, and $L^*(L'(0^+)) = -L(0)$.
- L^* is increasing on the interval when $h \leq L'(0^+)$, and is decreasing when $h \geq L'(0^+)$ (again, the extreme points may not belong to the support).

We draw the attention of the reader that L is not necessarily continuously differentiable, and that difficulties may appear to find the precise range of real numbers h such that $L^*(h) \geq 0$. These problems occur in many contexts, too numerous to list in detail now.

Exercise 2.12. Prove each of the preceding items.

2.3 Pointwise Hölder Exponent

2.3.1 Characterization by Decay Rate of Wavelet Coefficients

Recall the Definition 2.2 of the pointwise Hölder exponent of a locally bounded function f at a point $x_0 \in \mathbb{R}^d$. The definition of $h_f(x)$ involves some function spaces $C^s(x)$, which can be (almost) characterized by the decay rate of the coefficients located around x_0, as stated by the next theorem of Jaffard [22].

Theorem 2.4. *Let $s \in \mathbb{R}^+ \setminus \mathbb{N}$, and let $f \in L^2(\mathbb{R}^d)$.*

Assume that f belongs to $C^s(x_0)$. Then, there exist two constants $M > 0$ and $C > 0$ such that for every $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$, $|x_0 - \mathbf{k}2^{-j}| \leq M$, and for every $\mathbf{l} \in L^d$, one has

$$|d_\lambda| \leq C(2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^s. \tag{2.11}$$

Conversely, if the wavelet coefficients of a function $f \in \bigcup_{\varepsilon>0} C^\varepsilon([0,1]^d)$ satisfies (2.11), then $f \in C^s_{\log}(x_0)$.

Recall that $f \in C^s_{\log}(x_0)$ when locally around x_0, there exists a polynomial P of degree less than $\lfloor s \rfloor$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s |\log|x - x_0||. \tag{2.12}$$

As usual, the symbol $A \lesssim B$ means that inequality $A \leq CB$ holds for some constant C independent of the parameters involved in the formula.

Proof. We start by the direct implication. Assume that $f \in C^s(x_0)$, and let us call P the unique polynomial such that if $|x - x_0| \leq M$ (for some constant M), (2.1) holds true.

Fix $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$ and

$$|x_0 - \mathbf{k}2^{-j}| \leq \widetilde{M} := M/2. \tag{2.13}$$

One has

$$d_\lambda = 2^{dj} \int_{\mathbb{R}^d} f(x) \Psi_\lambda(x) dx = 2^{dj} \int_{\mathbb{R}^d} (f(x) - P(x - x_0)) \Psi_\lambda(x) dx,$$

where we used the vanishing moments up to the order $\lfloor s \rfloor$ to introduce the polynomial in the integral. Then,

$$|d_\lambda| \leq 2^{dj} \int_{|x-x_0|\leq M} |f(x) - P(x-x_0)||\Psi_\lambda(x)|dx$$
$$+2^{dj} \int_{|x-x_0|\geq M} |f(x)||\Psi_\lambda(x)|dx$$
$$+2^{dj} \int_{|x-x_0|\geq M} |P(x-x_0)||\Psi_\lambda(x)|dx.$$

Let us call I_M, J_M, and K_M the last three terms. Using (2.1), the first term is bounded above by

$$I_M \lesssim 2^{dj} \int_{|x-x_0|\leq M} |x-x_0|^s |\Psi^{\mathbf{l}}(2^j x - \mathbf{k})|dx$$
$$\lesssim \int_{|u|\leq M2^j} |2^{-j}(u+\mathbf{k}) - x_0|^s |\Psi^{\mathbf{l}}(u)|du.$$

Since each Ψ_λ is continuous and compactly supported, say, with support included in $[-M', M']^d$, it is uniformly bounded (independently of λ, due to the choice of the L^∞-normalization for the wavelet family) and one gets

$$I_M \lesssim \int_{[-M',M']^d} |2^{-j}(u+\mathbf{k}) - x_0|^s du$$
$$\lesssim \int_{[-M',M']^d} |2^{-j}u|^s + |x_0 - \mathbf{k}2^{-j}|^s du$$
$$\lesssim (2^{-js} + |x_0 - \mathbf{k}2^{-j}|^s) \lesssim (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^s,$$

where we used the double-sided inequality $(x+y)^s \leq 2^s(x^s+y^s) \leq 2^{s+1}(x+y)^s$.

Let us now treat the second term. By the Cauchy-Schwarz inequality and using that $f \in L^2(\mathbb{R}^d)$, one obtains

$$J_M \lesssim 2^{dj} \left(\int_{|x-x_0|\geq M} |f(x)|^2 dx \right)^{1/2} \left(\int_{|x-x_0|\geq M} |\Psi_\lambda(x)|^2 dx \right)^{1/2}$$
$$\lesssim 2^{dj} \left(\int_{|x-x_0|\geq M} |\Psi^{\mathbf{l}}(2^j x - \mathbf{k})|^2 dx \right)^{1/2}$$
$$\lesssim 2^{dj/2} \left(\int_{|2^{-j}(u+\mathbf{k})-x_0|\geq M} |\Psi^{\mathbf{l}}(u)|^2 du \right)^{1/2}. \tag{2.14}$$

Observe that our choice (2.13) for λ imposes that

$$\{u : |2^{-j}(u+\mathbf{k}) - x_0| \geq M\} \subset \{u : |u| \geq \widetilde{M}2^j\}. \tag{2.15}$$

The wavelets ψ^0 and ψ^1 being compactly supported, (2.15) tells us that the integral in (2.14) is 0 for j large enough.

The third term is treated almost similarly. The polynomial P being of degree at most $\lfloor s \rfloor$, one can write

$$\begin{aligned}
K_M &\lesssim 2^{dj} \left(\int_{|x-x_0|\geq M} \left(\frac{|P(x-x_0)|}{1+|x-x_0|^{s+d+2}} \right)^2 dx \right)^{1/2} \\
&\quad \times \left(\int_{|x-x_0|\geq M} (1+|x-x_0|^{s+d+2})^2 |\Psi_\lambda(x)|^2 dx \right)^{1/2} \\
&\lesssim 2^{dj} \left\| \frac{|P(.)|}{1+|.|^{s+d+2}} \right\|_{L^2(\mathbb{R}^d)} \\
&\quad \times \left(\int_{|x-x_0|\geq M} (1+|x-x_0|^{s+d+2})^2 |\Psi^{\mathbf{l}}(2^j x - \mathbf{k})|^2 dx \right)^{1/2} \\
&\lesssim 2^{dj/2} \left(\int_{|u|\geq \widetilde{M} 2^j} (1+ \ |2^{-j}(u+\mathbf{k}) - x_0|^{s+d+2})^2 |\Psi^{\mathbf{l}}(u)|^2 du \right)^{1/2}.
\end{aligned}$$

where (2.15) has been used. Again, the last integral is zero when j becomes large. Hence the first assertion.

Exercise 2.13. Prove that the same holds when the wavelets are not compactly supported (Hint: use their rapid decay at infinity).

Let us move to the converse implication, which is more delicate to handle with.

Assume that (2.11) holds for every $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$, $|x_0 - \mathbf{k}2^{-j}| \leq M$, and $\mathbf{l} \in L^d$.

We start from the decomposition (2.7). Since each Ψ_λ is at least $C^{\lfloor s \rfloor + 1}(\mathbb{R}^d)$, this is also true for every function f_j defined as the sum over each fixed generation j of the wavelet coefficients of f, i.e.

$$f_j(x) = \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d, \mathbf{l}\in L^d} d_\lambda \Psi_\lambda(x),$$

The partial derivatives of f_j are: for every $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| := \alpha_1 + \cdots + \alpha_d \leq \lfloor s \rfloor + 1$, one has

$$\partial^\alpha f_j(x) = \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d, \mathbf{l}\in L^d} d_\lambda \, \partial^\alpha \Psi_\lambda(x),$$

Each partial derivative of ψ^0 and ψ^1 is compactly supported, hence they satisfy the inequalities for all $v \in \mathbb{R}^d$

$$|\partial^\alpha \Psi^{\mathbf{l}}(v)| \leq \frac{C}{1+|v|^{2s+2d+4}} \tag{2.16}$$

where the constant C is uniform in α ranging in the set of indices such that $|\alpha| \leq \lfloor s \rfloor + 1$. Since $\Psi_\lambda(x) = \Psi^{\mathbf{l}}(2^j x - \mathbf{k})$, the last upper bound yields

$$|\partial^\alpha \Psi_\lambda(x)| \leq \frac{C 2^{j|\alpha|}}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}}.$$

From this we deduce that

$$|\partial^\alpha f_j(x)| \leq \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d, \mathbf{l}\in L^d} |d_\lambda| \frac{C 2^{j|\alpha|}}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}},$$

Observe now that, up to a modification of the constant C, (2.11) is also true for $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$, $|x_0 - \mathbf{k}2^{-j}| \leq 1$ (not only for $|x_0 - \mathbf{k}2^{-j}| \leq M$), since the sequence of the wavelet coefficients of f are necessarily bounded above by $\|f\|_{L^2}$, thus when for all λ such that $|x_0 - \mathbf{k}2^{-j}| \geq M$, one has $|d_\lambda| \leq \|f\|_{L^2} \lesssim |x_0 - \mathbf{k}2^{-j}|^s$ (with uniform constants). This yields the upper bound

$$\begin{aligned} |\partial^\alpha f_j(x)| &\lesssim \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d, \mathbf{l}\in L^d} \frac{2^{j|\alpha|}(2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^s}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}} \\ &\lesssim \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d, \mathbf{l}\in L^d} \frac{2^{j|\alpha|}(2^{-js} + |x - x_0|^s + |x - \mathbf{k}2^{-j}|^s)}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}}. \end{aligned}$$

The first two terms in the last sum are independent of $\mathbf{k}$, and thus the corresponding sums are bounded above by $2^{j|\alpha|}(2^{-js} + |x - x_0|^s)$. It is easy to see that the last one satisfies

$$\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d, \mathbf{l}\in L^d} \frac{2^{j|\alpha|}|x - \mathbf{k}2^{-j}|^s}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}} \lesssim 2^{-j(s+d+2)}.$$

We finally get the inequality

$$|\partial^\alpha f_j(x)| \lesssim 2^{j|\alpha|}(2^{-js} + |x - x_0|^s).$$

Since the wavelets are compactly supported, it is easily seen that f_j has at least same uniform Hölder regularity as ψ^0 and ψ^1, i.e. f_j belongs at least to $C^{\lfloor s \rfloor + 1}(\mathbb{R}^d)$. Using the Taylor polynomial P_j of order $\lfloor s \rfloor$ of each f_j, one has

$$\begin{aligned} |f_j(x) - P_j(x - x_0)| &\leq |x - x_0|^{\lfloor s \rfloor + 1} \sup_{|\alpha| = \lfloor s \rfloor + 1} |\partial^\alpha f_j(x)| \\ &\lesssim |x - x_0|^{\lfloor s \rfloor + 1} 2^{j(\lfloor s \rfloor + 1)}(2^{-js} + |x - x_0|^s). \qquad (2.17) \end{aligned}$$

It is time now to construct the polynomial associated with f. Obviously one has the decomposition $f(x) = \sum_{j\geq 0} f_j(x)$, hence it is natural to consider the polynomial $P = \sum_{j\geq 0} P_j$ as potential candidate. From the above estimates one easily sees that this polynomial is well defined.

We fix some x close to x_0. Let us call j_0 the unique integer such that

$$2^{-j_0} \leq |x - x_0| < 2^{-j_0+1}.$$

Recall that f is supposed to belong to $C^\eta(\mathbb{R}^d)$, for some $\eta > 0$. We then introduce the integer

$$j_1 = \left\lfloor j_0 \frac{s}{\eta} \right\rfloor > j_0,$$

where we can assume that $j_1 > j_0$ since η can be taken as small as we want. It remains us to bound above the difference $|f(x) - P(x - x_0)|$ by the desired quantity, i.e. $|x - x_0|^s |\log|x - x_0||$. Let us split this quantity into four terms, depending on the generations of the associated wavelet decomposition. More precisely,

$$\begin{aligned} f(x) - P(x - x_0) = & \sum_{j=0}^{j_0} (f_j(x) - P_j(x - x_0)) + \sum_{j=j_0+1}^{j_1} f_j(x) \\ & + \sum_{j \geq j_1+1} f_j(x) - \sum_{j=j_0+1}^{+\infty} P_j(x - x_0). \end{aligned}$$

We call S_1, S_2, S_3, and S_4 the four sums above.

First, we have by (2.17)

$$\begin{aligned} |S_1| & \lesssim \sum_{j=0}^{j_0} |f_j(x) - P_j(x - x_0)| \\ & \lesssim \sum_{j=0}^{j_0} |x - x_0|^{\lfloor s \rfloor + 1} 2^{j(\lfloor s \rfloor + 1)} (2^{-js} + |x - x_0|^s) \\ & \lesssim |x - x_0|^{\lfloor s \rfloor + 1} \sum_{j=0}^{j_0} \left(2^{j(\lfloor s \rfloor + 1 - s)} + 2^{j(\lfloor s \rfloor + 1)} |x - x_0|^s \right) \\ & \lesssim |x - x_0|^{\lfloor s \rfloor + 1} (2^{j_0(\lfloor s \rfloor + 1 - s)} + 2^{j_0(\lfloor s \rfloor + 1)} |x - x_0|^s) \\ & \lesssim |x - x_0|^s, \end{aligned}$$

where the last "miracle" follows from (2.18). Then, by (2.17) with $\alpha = 0^d$,

$$\begin{aligned} |S_2| & \lesssim \sum_{j=j_0+1}^{j_1} |f_j(x)| \lesssim \sum_{j=j_0+1}^{j_1} (2^{-js} + |x - x_0|^s) \\ & \lesssim (j_1 - j_0)(2^{-j_0 s} + |x - x_0|^s) \lesssim |x - x_0|^s |\log|x - x_0||, \end{aligned}$$

since $j_1 - j_0 \sim j_0(1 - s/\eta) \sim \log|x - x_0||$ by (2.13). Further, recalling that the function $f \in C^\eta(\mathbb{R}^d)$, the wavelet coefficients of f_j satisfy $|d_\lambda| \lesssim 2^{-j\eta}$, one sees that $\|f_j\|_\infty \lesssim 2^{-j\eta}$ (here the assumption that the wavelets are compactly supported makes the computations easier). Hence,

$$|S_3| \lesssim \sum_{j=j_1+1}^{+\infty} |f_j(x)| \lesssim \sum_{j=j_1+1}^{+\infty} 2^{-j\eta} \lesssim 2^{-j_1\eta} \lesssim 2^{-j_0 s} \lesssim |x-x_0|^s.$$

Finally, each polynomial P_j has the form

$$P_j(x-x_0) = \sum_{n=0}^{\lfloor s\rfloor} \sum_{|\alpha|=n} c_\alpha \partial^\alpha f_j(x_0)(x-x_0)^n,$$

for some universal coefficients c_α. Hence it can be bounded above as follows using (2.17)

$$\begin{aligned}|P_j(x-x_0)| &\lesssim \sum_{n=0}^{\lfloor s\rfloor} \sum_{|\alpha|=n} c_\alpha |\partial^\alpha f_j(x_0)||x-x_0|^n \\ &\lesssim \sum_{n=0}^{\lfloor s\rfloor} 2^{jn}|x-x_0|^n(2^{-js}+|x-x_0|^s) \\ &\lesssim 2^{j\lfloor s\rfloor}|x-x_0|^{\lfloor s\rfloor}(2^{-js}+|x-x_0|^s),\end{aligned}$$

where we used that $j \geq j_0$, implying $2^j|x-x_0| > 1$. Finally,

$$|S_4| \lesssim \sum_{j=j_0+1}^{+\infty} |P_j(x)| \lesssim \sum_{j=j_0+1}^{+\infty} 2^{j\lfloor s\rfloor}|x-x_0|^{\lfloor s\rfloor}(2^{-js}+|x-x_0|^s) \lesssim |x-x_0|^s.$$

This concludes the proof.

Let us end this section with an important remark: Theorem 2.4 tells us that to find the value of $h_f(x_0)$, it is not enough to look at the wavelet coefficients that lie inside the "cone of influence" around x_0, i.e. the λ such that $|\mathbf{k}2^{-j} - x_0| \leq M2^{-j}$. The cone of influence contains the wavelet coefficients whose value is influenced by the value of f at x, and one may believe that they are the only ones that play a role in the value of the pointwise Hölder exponent of f at x. In fact, when the largest coefficients are located within the cone of influence of x_0, x_0 is a cusp.

But it may happen that coefficients located very far from the cone of influence are the most important ones, in the sense that the inequality (2.11) is saturated for the coefficients. Actually, when (2.11) is saturated for wavelet coefficients d_λ satisfying $|x-x_0| \sim 2^{-j\rho}$ for some $\rho < 1$, one can prove that x_0 is an oscillating singularity with singularity exponent $1/\rho - 1$ (see the next sections for more details). So it is definitely not enough to concentrate on the cone of influence, especially when building local regularity algorithms. This is also one main motivation for introducing wavelet leaders.

Exercise 2.14. Construct a wavelet series in $\mathbb{R}$ such that all its wavelet coefficients are either 0 or equal to $2^{-j\alpha}$ and such that $h_f(0) = 2\alpha$, $h_f(x) = \alpha$ if $x \neq 0$.

Is it possible to have $h_f(x) = 2\alpha$ if $x \in \mathbb{Q}$, $h_f(x) = \alpha$ if $x \notin \mathbb{R}\setminus\mathbb{Q}$? What if one inverses the values α and 2α?

Exercise 2.15. Prove that, under the same assumption, it is enough for the reconstruction part to assume that (2.11) holds only for those wavelet coefficients d_λ such that the corresponding $\lambda = (j, \mathbf{k}, \mathbf{l})$ satisfies $|x_0 - \mathbf{k}2^{-j}| \leq 2^{-j/\log j}$.

Exercise 2.16. Consider the continuous wavelet transform defined for $a > 0$ and $b \in \mathbb{R}$ and for an L^2 function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$W_f(b,a) = \frac{1}{a^{d/2}} \int_{\mathbb{R}^d} f(t)\psi\left(\frac{t-b}{a}\right) dt.$$

Prove an analog of Theorem 2.4 for this wavelet transform.

Exercise 2.17. Consider the Riemann series

$$R(x) = \sum_{n\geq 1} \frac{\sin(\pi n^2 x)}{n^2}.$$

and the wavelet $\psi(x) = \frac{1}{(x+i)^2}$.

1. Compute the continuous wavelet transform of R, and relate it to the Jacobi Theta function $\Theta(z) = \sum_{n\in Z} e^{i\pi n^2 z}$. (Hint: use the residue formula.)
2. Prove that R is at least $C^{1/2}(x)$ at every x (difficult).
3. To learn more about R, see in chronological order [13, 19, 20, 25, 34].

Exercise 2.18. Let $0 < H < 1$, and consider the Weierstrass function

$$W_H(x) = \sum_{n\geq 1} 2^{-nH} \sin(2^n x).$$

1. Prove that $W_H \in C^H(\mathbb{R})$ (Hint: directly prove that $|W_H(x) - W_H(y)| \leq C|x-y|^H$.)
2. Using a suitable wavelet ψ (for instance, assuming that its Fourier transform $\widehat{\psi}$ has support in $[1/2, 2]$), prove that for every $x \in \mathbb{R}$, $h_{W_H}(x) = H$. Hence, W_H is monofractal.

2.3.2 Characterization by Decay Rate of Wavelet Leaders

Wavelet leaders are a theoretical tool introduced by S. Jaffard in [28] essentially for numerical reasons. The main idea comes from the fact that in multifractal analysis (see next section for the details), it is natural to consider sums of wavelet coefficients like

$$\sum_{\lambda:|\lambda|=j} |d_\lambda|^q$$

for a varying parameter $q \in \mathbb{R}$. In particular, as we will explain, the behavior of the sum when $j \to +\infty$ for $q < 0$ is related to the decreasing part of the multifractal spectrum of functions. It is thus natural to try to estimate the values of such sums.

Unfortunately numerical experiments show that this quantity is extremely unstable due to the presence of small wavelet coefficients, which, when they are taken to a negative power, can be extremely large. Wavelet leaders have been thought to stabilize these sums, and they are in fact related to multifractal analysis of capacities [30].

Definition 2.7. For every $\lambda = (j, \mathbf{k}, \mathbf{l})$, one defines the dyadic cube I_λ associated with λ by

$$I_\lambda = [\mathbf{k}2^{-j}, (\mathbf{k}+1)2^{-j}) \\ := [k_1 2^{-j}, k_1 2^{-j} + 2^{-j}) \times \cdots \times [k_d 2^{-j}, k_d 2^{-j} + 2^{-j}),$$

where $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$.

Let f be a function of the form (2.7). For every $\lambda = (j, \mathbf{k}, \mathbf{l})$ with $j \geq 0$ such that $\mathbf{k}2^{-j} \in [0,1]^d$, one defines the wavelet leader D_λ by

$$D_\lambda = \sup \left\{ |d_{\lambda'}| : I_{\lambda'} \subset \bigcup_{i \in \{-1,0,1\}^d} I_\lambda + i2^{-j} \right\}.$$

In other words, the wavelet leader D_λ is in fact the maximal value (in absolute value) amongst all the wavelet coefficients $d_{\lambda'}$ such that the corresponding cube $I_{\lambda'}$ lies inside I_λ or inside one of its $3^d - 1$ immediate neighbors.

Exercise 2.19. Prove that for every $f \in L^2$ each wavelet leader is a maximum (not only a supremum).

It is immediate that if $I_{\lambda'} \subset I_\lambda$, $D_{\lambda'} \leq D_\lambda$. Hence, instead of having wavelet coefficients that may be sparse, we end up with leader coefficients that enjoy a nice decreasing property (the set of wavelet leaders forms a *capacity* as a function of the dyadic cubes, i.e. a decreasing set function on the dyadic wavelet tree). Multifractal analysis of capacities has been studied in [5, 30] for instance.

Definition 2.8. For every $x_0 \in [0,1]^d$ and $j \geq 0$, let us denote by $\lambda_j(x_0)$ the unique cube (up to the value of $\mathbf{l} \in L^d$) such that $x_0 \in \lambda$ with $|\lambda| = j$, and we set

$$D_j(x_0) = D_{\lambda_j(x_0)} \quad \text{and} \quad I_j(x_0) = I_{\lambda_j(x_0)}.$$

We also set $\lambda_j(x_0) = (j, \mathbf{k}_j(x_0), \mathbf{l})$ (the index $\mathbf{l}$ has no importance here, only the location matters).

The main theorem relating wavelet leaders and pointwise regularity is the following.

Theorem 2.5. *Let f be locally bounded of the form (2.7). Then*

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log D_j(x_0)}{\log 2^{-j}},$$

where $\log 0 = -\infty$ *by convention.*

Proof. The proof is rather quick and is based on Theorem 2.4. Let $h := h_f(x_0)$.

Let $\varepsilon > 0$. Inequality (2.11) implies that for large j, all wavelet coefficients around x_0 satisfy

$$|d_\lambda| \le C(2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h-\varepsilon}.$$

Let $J \ge 0$ and $\lambda = (J, \mathbf{K}, \mathbf{L})$ be such that $|\mathbf{K}2^{-J} - x_0| \le M$ (the constant M being the one such that (2.11) holds).

Let $\lambda' = (j, \mathbf{k}, \mathbf{l})$ be such that $I_{\lambda'} \subset I_\lambda + i2^{-J}$, for some $i \in \{-1,0,1\}^d$. Obviously one has $|\mathbf{k}2^{-j} - x_0| \le 2 \cdot 2^{-J}$, thus

$$|d_{\lambda'}| \lesssim (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h-\varepsilon}| \lesssim 2^{-J(h-\varepsilon)}.$$

We deduce that $|D_\lambda| \le 2^{-J(h-\varepsilon)}$, and thus that

$$\liminf_{j\to+\infty} \frac{\log D_j(x_0)}{\log 2^{-j}} \ge h - \varepsilon.$$

Letting ε go to zero gives one inequality in (2.18).

Moving to the converse inequality, we know that (2.11) must be saturated for some coefficients. Let $\varepsilon > 0$, and consider one coefficient $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that

$$|d_\lambda| \ge (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h+\varepsilon}.$$

There are infinitely many such coefficients.

Let J be the unique integer such that $2^{-J-1} \le |x_0 - \mathbf{k}2^{-j}| < 2^{-J}$. Then, by construction, $I_\lambda \subset I_J(x_0) + i2^{-J}$ for some $i \in \{-1,0,1\}^d$. This yields that

$$D_J(x_0) \ge |d_\lambda| \ge (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h+\varepsilon} \ge 2^{-J(h+2\varepsilon)}.$$

Taking liming of both sides, we get

$$\liminf_{j\to+\infty} \frac{\log D_j(x_0)}{\log 2^{-j}} \le h + \varepsilon,$$

and the result follows when ε tends to zero.

Exercise 2.20. Prove that there must exist (an infinite number of) λ such that $D_\lambda = |d_\lambda|$.

2.3.3 Prescription of Hölder Exponents

As said in the introduction, the exponent mapping $x \mapsto h_f(x)$ of a locally bounded function may not be regular, and one may wonder what form this mapping can take. It is also natural, for practical purposes, to try to build functions with prescribed Hölder regularity. This problem is completely solved (see [23]).

Exercise 2.21. Let g be a strictly positive and continuous function. Build a function f such that its pointwise Hölder exponents are exactly $h_f(x) = g(x)$ at every x (Hint: modify the Weierstrass function W_H introduced in Exercise 2.18).

Proposition 2.1. *Let $f \in C^\eta(\mathbb{R}^d)$ for some $\eta > 0$. Then the mapping $x \mapsto h_f(x)$ is the liminf of a sequence of continuous functions.*

Conversely, if $(g_n)_{n\geq 1}$ is a sequence of continuous functions satisfying $g_n \geq \eta$, then there exists a function $f \in C^\eta(\mathbb{R}^d)$ such that

$$h_f(x) = \liminf_{n\to+\infty} g_n(x).$$

Proof. Let us start by remarking that any function f has the same pointwise Hölder exponents everywhere as the sum $f+g$ where g is the wavelet series whose wavelet coefficients are all equal to $\pm 2^{-j^2}$ for $j \geq 0$ (there is no need to precise how the signs are chosen). Hence, up to a modification that does not affect the pointwise Hölder coefficients, one may assume that the wavelet coefficients of f satisfy $|d_\lambda| \geq 2^{-j^2}$ for every $j \geq 2$.

Then, (2.11) implies that

$$h_f(x) = \liminf_{j\to+\infty, |\lambda|=j} \frac{\log|d_\lambda|}{\log(2^{-j} + |x - \mathbf{k}2^{-j}|)}.$$

Let us denote by g_λ the map $x \mapsto \frac{\log|d_\lambda|}{\log(2^{-j}+|x-\mathbf{k}2^{-j}|)}$. It is obviously continuous with respect to x. Hence, $h_f(x)$ is indeed the liminf of a sequence of continuous functions.

Conversely, consider a sequence $(g_n)_{n\geq 1}$ of continuous functions greater than $\eta > 0$. We work only on the cube $[0,1]^d$, the extension to $\mathbb{R}^d$ is immediate by concatenation. We build iteratively a wavelet series by the following method.

Let us first start by remarking that we can assume that each function g_n is C^1. Otherwise we replace g_n by any C^1 function $\widetilde{g}_n$ such that $\|g_n - \widetilde{g}_n\|_\infty \leq 1/2^n$. Then it is obvious that

$$g(x) = \liminf_{n\to+\infty} g_n(x) = \liminf_{n\to+\infty} \widetilde{g}_n(x).$$

We first construct a sequence J_n as follows.

Fix $J_0 = 1$, and assume that J_n is found. To find J_{n+1}, consider g_{n+1}. By uniform continuity, there exists $\widetilde{J}_{n+1}$ such that $|x-y| \leq 2^{-\widetilde{J}_{n+1}}$ implies $|g_{n+1}(x) - g_{n+1}(y)| \leq 2^{-(n+1)}$. We also assume that

$$2^{-J_n \inf\{g_n(x): x\in[0,1]^d\}} \geq 2^{-\widetilde{J}_{n+1}(\sup\{g_{n+1}(x): x\in[0,1]^d\})}.$$

Finally, we choose J_{n+1} as the integer

$$J_{n+1} = \max(J_n + n, \widetilde{J}_{n+1}, \sup\{|\nabla g_{n+1}(x)| : x \in [0,1]\}).$$

Then, we prescribe the wavelet coefficients d_λ for all $\lambda = (j, \mathbf{k}, \mathbf{l})$ as follows: if $J_n < j < J_{n+1}$, then $d_\lambda = 0$, and if $j = J_n$ for some $n \geq 1$, one sets

$$d_\lambda = 2^{-j g_n(\mathbf{k}2^{-j})}.$$

Fix $x_0 \in [0,1]^d$, and recall the Definition 2.8 of $\lambda_j(x_0)$ and $\mathbf{k}_j(x_0)$. It is a trivial matter to check that our construction implies that for every J_n, for the wavelet leader $D_{J_n}(x_0)$,

$$2^{-J_n/2^n} \leq \frac{D_{J_n}(x_0)}{2^{-J_n g_n(\mathbf{k}2^{-J_n})}} \leq 2^{J_n/2^n}.$$

By (2.18), one has $h_f(x_0) \leq \liminf_{n\to+\infty} g_n(\mathbf{k}_{J_n}(x_0)2^{-J_n})$. But our construction implies that for every n, $|g_n(\mathbf{k}_{J_n}(x_0)2^{-J_n}) - g_n(x_0)| \leq 1/2^n$, hence

$$\liminf_{n\to+\infty} g_n(\mathbf{k}_{J_n}(x_0)2^{-J_n}) = \liminf_{n\to+\infty} g_n(x_0),$$

from which we deduce that $h_f(x_0) \leq g(x_0)$.

Conversely, by Exercise 2.15, it is enough to consider those wavelet coefficients around x_0 that satisfy $|x_0 - \mathbf{k}2^{-J_n}| \leq 2^{-J_n/\log J_n}$. One sees that for such a coefficient $\lambda = (J_n, \mathbf{k}, \mathbf{l})$,

$$|d_\lambda| = 2^{-J_n g_n(\mathbf{k}2^{-J_n})} \leq 2^{-J_n\left(g_n(x_0) - \sup\{|\nabla g_n(x)|: x\in[0,1]\}\cdot|x_0 - \mathbf{k}2^{-J_n}|\right)}.$$

But our construction implies that

$$0 \leq \sup\{|\nabla g_{n+1}(x)| : x \in [0,1]\} \cdot |x_0 - \mathbf{k}2^{-J_n}| \leq J_n 2^{-J_n/\log J_n},$$

which tends to zero when n tends to infinity. This yields

$$|d_\lambda| \leq 2 \cdot 2^{-J_n g_n(x_0)}$$

for n large. In particular, $h_f(x_0) \geq \liminf_{n\to+\infty} g_n(x_0) = g(x_0)$. This gives the converse inequality.

Exercise 2.22. Extend last Proposition to functions that are only continuous, or only with bounded variations.

We finish this section by drawing the attention of the reader that what we have achieved for functions is not known for measures: one does not know what the possible forms of the local dimension map of a measure are like. The situation is much more complicated, as proved by next exercise [11].

Exercise 2.23. Consider the local dimension mapping $x \mapsto h_\mu(x)$ associated with a probability measure μ on $\mathbb{R}^d$. Prove that if it is continuous on an open set Ω, then it is constant and equal to d on Ω.

2.3.4 Other Exponents

The pointwise Hölder exponent does not fully describe the local behavior of a continuous function. For instance, it does not reflect the local oscillatory behavior: the functions $f_1(x) = |x|^{1/4}$ and $f_2(x) = |x|^{1/4}\sin(|x|^{-1})$ have the same exponent $1/4$ at 0, but they exhibit obviously a different behavior.

There are many other local regularity exponents that allow one to distinguish functions with the same pointwise Hölder exponent. Let us mention two of them.

Definition 2.9. The local Hölder exponent of f at x_0 is defined as

$$h^l_f(x_0) = \lim_{\varepsilon\to 0} \sup\{\alpha \geq 0 : f \in C^\alpha(B(x_0,\varepsilon))\},$$

where $B(x_0,\varepsilon)$ stands for the ball (using any norm) centered at x_0 of radius ε.

Exercise 2.24. Prove that the formula (2.18) makes sense, and that the value does not depend on the choice of the norm.

The local Hölder exponent is always lower than the pointwise Hölder exponent (Exercise: prove it!). This other exponent is often used when studying local regularity of stochastic processes, for which it is often difficult to obtain results that are valid almost surely for all points (while it is often easy to get an exact value for every point almost surely). For instance, for a multifractional Brownian motion (see [9, 29] for definitions), one can compute almost surely the value of every $h^l_f(x)$, and sometimes the value of $h_f(x)$ is not known (only for every x almost surely, not almost surely for every x: nevertheless under some conditions the pointwise Hölder exponent is known everywhere almost surely).

Another exponent encapsulates the oscillatory behavior of a function.

Definition 2.10. For every $\varepsilon > 0$, let f^ε be a fractional primitive of f of order ε. Then the oscillating exponent of f at x_0 is defined as

$$\beta_f(x_0) = \lim_{\varepsilon\to 0} \left(\frac{\partial h_{f^\varepsilon}(x_0)}{\partial \varepsilon}\right)_{|\varepsilon=0} - 1.$$

Recall that a fractional primitive f^ε of order ε of, say, a L^2 function can be defined via the formula

$$f^\varepsilon(x) = (-\Delta)^{\varepsilon/2}(f)(x),$$

or via its Fourier transform by

$$\widehat{f^\varepsilon}(\xi) = \frac{\hat{f}(\xi)}{(1+|\xi|^2)^{\varepsilon/2}}.$$

Exercise 2.25. Prove that for every $\varepsilon > 0$, $h_{f^\varepsilon}(x_0) \geq h_f(x_0) + \varepsilon$. Deduce that the formula (2.18) makes sense.

As an example, it is quite easy to see that for the functions f_1 and f_2 introduced at the beginning of this section, $\beta_{f_1}(0) = 0$ while $\beta_{f_2}(0) = 1$. The term $\sin(|x|^{-1})$ is responsible for the value $\beta_{f_2}(0) = 1$, and one can prove that the oscillatory content is indeed contained in $\beta_f(x)$.

When $\beta_f(x) > 0$, x is called an *oscillating singularity* of f. It is also often referred to as a *chirp*, on the opposite to the case where $\beta_f(x) = 0$, where the singularity is called a *cusp*.

Detecting oscillatory singularities is an important issue in signal processing, one knows that many phenomena occur only on such points (for instance, dissipation of energy in turbulent fluids may be due to this kind of singularities [17]).

2.3.5 An Example

Let μ be a positive Borel probability measure on $[0,1]^d$. Let us construct the wavelet series F_μ by prescribing its wavelet coefficients as follows: for every λ, we set

$$d_\lambda = \mu(I_\lambda).$$

Assume that the measure is uniformly regular, in the sense that there exist a constant $C > 0$ and an exponent $h_{\min} > 0$ such that for every ball with center x and radius $0 < r < 1$, one has

$$\mu(B(x,r)) \leq C r^{h_{\min}}.$$

These assumptions can be weakened, as the reader can easily check.

Proposition 2.2. *Under the assumption above, the wavelet series F_μ converges, $F_\mu \in C^{h_{\min}}(\mathbb{R}^d)$, and for every $x \in [0,1]^d$, one has*

$$h_{F_\mu}(x) = h_\mu(x).$$

In particular, $d_\mu \equiv d_{F_\mu}$.

Proof. We let the proof as an exercise. The idea is essentially to prove that there is a universal constant $C > 1$ such that for every $x_0 \in \mathrm{Supp}(\mu)$, for every $j \geq 0$, one has

$$C^{-1} \leq \frac{D_j(x_0)}{\mu(B(x_0, 2^{-j}))} \leq C,$$

where $D_j(x_0)$ is the wavelet leader of F_μ.

Exercise 2.26. Let (ξ_λ) be a family of i.i.d random variables with common law the normal Gaussian law. Consider the (random) wavelet series $\widetilde{F_\mu}$ whose wavelet coefficients are

$$d_\lambda = \mu(I_\lambda)\xi_\lambda.$$

This is a random modification of (2.18) and of F_μ.

1. Prove that, almost surely, for every x, $h_{F_\mu}(x) = h_{\widetilde{F_\mu}}(x)$, and thus $d_{F_\mu} \equiv d_{\widetilde{F_\mu}}$.
2. Can one weaken the i.i.d. assumption on the random coefficients? (the answer is yes, but to what extent…)

Exercise 2.27. What happens if (2.18) is replaced by

$$d_\lambda = 2^{-j\alpha}\mu(I_\lambda)^\beta$$

for some $\alpha, \beta > 0$?

2.4 Multifractal Formalism

2.4.1 The Intuition of U. Frisch and G. Parisi

Multifractal analysis and formalism for functions were introduced by physicists in order to interpret some experimental observations related to Kolmogorov's theory of fully developed turbulence. Since the 1940s, Kolmogorov emphasized the role in fluid mechanics played by the scaling function associated with the fluid velocity, defined as follows. Let $v(x)$ be the velocity at time t and position x of a turbulent fluid contained in a bounded domain Ω. For every $q \in \mathbb{R}$, one studies the q-th moment of v defined by

$$S(q,l) = \int_\Omega |v(x+l) - v(x)|^q dx.$$

In his K41 model, Kolmogorov models the small fluctuations of the velocity by a fractional Brownian motion with Hurst exponent $H = 1/3$, for which one can prove $S(q,l)$ enjoys a nice scaling behavior of the form, for every $q \geq 0$,

$$S(q,l) \sim |l|^{qH} \quad \text{when } |l| \text{ tends to 0.}$$

But very quickly, some experiments showed that in reality

$$S(q,l) \sim |l|^{\zeta(q)} \quad \text{when } |l| \text{ tends to 0,} \tag{2.18}$$

where the mapping $q \longmapsto \zeta(q)$, called the *scaling function* of the velocity, is a strictly concave, increasing function. This has been eventually confirmed by experiments that took place at the ONERA in Modane by Y. Gagne [18] (see Figure 2.1 for the one-dimensional trace of the 3D velocity of a turbulent fluid).

Uriel Frisch and Georgio Parisi had this insightful idea that the non-linearity of the scaling function shall be a consequence of the multifractality of the velocity, i.e. the fact that there are different pointwise Hölder exponents occurring at different places, the corresponding iso-Hölder sets $E_v(h) = \{x \in \Omega : h_v(h) = h\}$ having non-zero Hausdorff dimension, whose value depends on h.

Their heuristics was the following: Assume that there are many exponents h such that their associated iso-Hölder set $E_v(h)$ is non-empty, with Hausdorff dimension $\dim E_v(h) = d_v(h) > 0$. Intuitively, around every point $x \in \mathbb{R}^3$ with $h_v(x) = h$, one has

$$|v(x+l) - v(x)| \sim |l|^h.$$

Since $\dim E_v(h) = d_v(h) > 0$, there are approximately $|l|^{-d_v(h)}$ cubes of size length $|l|$ (hence, of volume $|l|^3$) that contain points x whose exponent is h. Hence,

$$S(q,l) = \int_\Omega |v(x+l) - v(x)|^q dx \sim \int_h |l|^{qh} |l|^{-d_v(h)} |l|^3 dh \sim \int_h |l|^{qh - d_v(h) + 3} dh.$$

When $l \to 0$, the most important contribution in the integral comes from the smallest possible value for the exponent $qh - d_v(h) + 3$. Combining this with (2.18), one deduces that

$$\zeta(q) = \inf_h \, (qh - d_v(h) + 3).$$

This expression is a Legendre transform, which explains a priori the concavity of the scaling function ζ. Moreover, by inverse Legendre transform, one gets

$$d_v(h) = \inf_{q \in \mathbb{R}} \, (qh - \zeta(q) + 3), \tag{2.19}$$

which suggests us that the multifractal spectrum of v should also have a concave shape.

The remarkable, and surprising, point is that despite the successive approximations made along this proof, the formula (2.19) (or resembling formulas) holds true for many mathematical objects, from self-similar functions and measures to generic functions. In fact, as soon as the function enjoys some nice scaling properties, this kind of formula is expected to hold.

Definition 2.11. When a formula like (2.19) holds true, one says that *the multifractal formalism is true for the function f at the exponent h.*

The definition is intentionally imprecise, since the right formulation for the scaling function and for the range of parameters q ($q \in \mathbb{R}$, $q \in \mathbb{R}^+$, q belongs to some interval, ...) may depend on the context (the support of the function, the function space, ...).

It is important at this point to emphasize once again that this multifractal formalism is the main reason for the use of multifractals in applications. Indeed, as said in the introduction, it is useless to try to estimate directly the multifractal spectrum of a signal or an image, too many limits are involved. Nevertheless, when the object under consideration enjoys some specific scaling properties (deterministic or statistical self-similarity, independence or stationarity of increments, ...), it is natural to look for a multifractal formalism-like formula involving a scaling function $\zeta(q)$, which is hopefully easy to estimate numerically.

2.4.2 A Rigorous Formulation of the Multifractal Formalism

It is possible to give an effective meaning to the multifractal formalism in many contexts. The easiest one is obtained through a scaling function associated with the wavelet leaders.

Definition 2.12. For every $q \in \mathbb{R}$, one considers the leader scaling function of the function f defined by

$$L_f(q) = \liminf_{j \to +\infty} \frac{1}{-j} \log_2 \left(\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\mathbf{k}2^{-j}\in[0,1]^d} |D_\lambda|^q \right).$$

From this value one deduces an upper bound a priori for the multifractal spectrum of f.

Theorem 2.6. *For every function $f \in C^\eta(\mathbb{R}^d)$ for some $\eta > 0$, one has*

$$d_f(h) \leq L_f^*(h) := \inf_{q \in \mathbb{R}} (qh - L_f(q)).$$

L_f^* is called the Leader Legendre Spectrum of f.

Actually we will prove a much stronger result:

- for every $h \leq L^*(L'(0^+))$, i.e. in the increasing part of the Leader Legendre spectrum,

$$\dim_{\mathscr{H}} \{x : h_f(x) \leq h\} \leq L_f^*(h),$$

- on the decreasing part $h \geq L^*(L'(0^+))$, one has

$$\dim_{\mathscr{H}} \{x : \overline{h_f}(x) \geq h\} \leq L_f^*(h),$$

where $\overline{h_f}(x)$ is the limsup exponent defined by

$$\overline{h_f}(x) = \limsup_{j \to +\infty} \frac{\log D_j(x_0)}{\log 2^{-j}}.$$

From (2.20) and (2.20) one easily deduces Theorem 2.6, since in the increasing part

$$d_f(h) = \dim_{\mathscr{H}} E_f(h) = \dim_{\mathscr{H}} \{x \in [0,1] : h_f(x) = h\} \leq \dim_{\mathscr{H}} \{x : h_f(x) \leq h\},$$

and in the decreasing part, one has

$$d_f(h) = \dim_{\mathscr{H}} E_f(h) = \dim_{\mathscr{H}} \{x \in [0,1] : h_f(x) = h\} \leq \dim_{\mathscr{H}} \{x : \overline{h_f}(x) \geq h\}.$$

Proof. This is actually a standard proof coming from large deviations theory, which is only based on formula (2.18)

$$h_f(x_0) = \liminf_{j\to+\infty} \frac{\log D_j(x_0)}{\log 2^{-j}},$$

and on a counting argument.

From the recalls on the Legendre transform in Section 2.2.5, one knows that L_f^* reaches its maximum at $L_f^*(L'(0^+))$. It is obvious that this maximum is equal to $L_f(0) = d$.

Let $h \le L_f^*(L'(0^+))$. We are going to prove (2.20). In that case, the maximal value of $L^*(h)$ is reached for a positive value of q.

Let $\varepsilon > 0$ be small.

Since $f \in C^\eta(\mathbb{R}^d)$, it is enough to consider $h \ge \eta$, and the set $\widetilde{E}_f(h) = \{x \in [0,1]^d : h_f(x) \le h\}$. For every x in this set, by (2.18), there exists an infinite number of generations j such that

$$2^{-j(h+\varepsilon)} \le D_j(x_0).$$

Let us denote by $N_j(h,\varepsilon)$ the number of wavelet leaders D_λ of generation j such that (2.20) holds for D_λ (instead of $D_j(x_0)$). From Definition 2.12 of L_f, there exists a generation J_ε such that for every $j \ge J_\varepsilon$,

$$\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\mathbf{k}2^{-j}\in[0,1]^d} |D_\lambda|^q \le 2^{-j(L_f(q)-\varepsilon)}.$$

One deduces that when $q > 0$,

$$2^{-j(L_f(q)-\varepsilon)} \ge \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\mathbf{k}2^{-j}\in[0,1]^d} |D_\lambda|^q \ge N_j(h,\varepsilon) 2^{-qj(h+\varepsilon)}.$$

In particular,

$$N_j(h,\varepsilon) \le 2^{j(qh-L_f(q)+\varepsilon(1+q))}.$$

From (2.20), the set $\widetilde{E}_f(h)$ is included in

$$\widetilde{E}_f(h) \subset \bigcap_{J\ge1} \bigcup_{j\ge J} \bigcup_{\lambda=(j,\mathbf{k},\mathbf{l}):D_\lambda\ge 2^{-j(h+\varepsilon)}} B(\mathbf{k}2^{-j}, 2.2^{-j}).$$

Hence a covering of $\widetilde{E}_f(h)$ by sets of diameter less than $\delta > 0$ is given by the union

$$\bigcup_{j\ge J} \bigcup_{\lambda=(j,\mathbf{k},\mathbf{l}):D_\lambda\ge 2^{-j(h+\varepsilon)}} B(\mathbf{k}2^{-j}, 2.2^{-j}),$$

where J is such that $4.2^{-J} \le \delta$. Let $s > qh - L_f(q) + \varepsilon(1+q)$. We use this covering to bound from above the $\mathscr{H}_\eta^s$-Hausdorff pre-measure of $\widetilde{E}_f(h)$ as follows:

$$\mathscr{H}^s_\eta(\widetilde{E}_f(h)) \le \sum_{j\ge J} \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):D_\lambda \ge 2^{-j(h+\varepsilon)}} |B(\mathbf{k}2^{-j}, 2.2^{-j})|^s$$
$$\lesssim \sum_{j\ge J} N_j(h,\varepsilon)2^{-js} \le \sum_{j\ge J} 2^{j(qh-L_f(q)+\varepsilon(1+q)-s)},$$

which is finite by our choice of s. Hence, $\dim_{\mathscr{H}} \widetilde{E}_f(h) \le s$, and letting s tend to $qh - L_f(q) + \varepsilon(1+q)$ and then ε to zero, one deduces that

$$\dim_{\mathscr{H}} \widetilde{E}_f(h) \le qh - L_f(q).$$

This holds true for every $q > 0$, hence

$$\dim_{\mathscr{H}} \widetilde{E}_f(h) \le \inf_{q\ge 0} qh - L_f(q).$$

Finally, as said above, the positive q's are the only ones that matter in the range $h \le L^*_f(L'(0^+))$, hence (2.20).

Inequality (2.20) is obtained similarly, by inverting liminf and limsup and replacing $h_f(x)$ by the limsup exponent (2.20) $\overline{h_f}(x)$.

Exercise 2.28. Prove (2.20).

Theorem 2.6 yields an (adaptive) upper bound for the multifractal spectrum of every function f. This is of course important for the applications, since the Legendre transform of the Leader scaling function can be estimated numerically, at least if the data set is large enough (i.e., there are many generations j available).

2.4.3 Upper Bounds for the Multifractal Spectrum of Functions in Classical Function Spaces

In the previous section we found an upper bound for the multifractal spectrum, but this upper bound is not related directly to "classical" function spaces. In other words, the value of the Leader scaling function of f is not equivalent to the fact that f belongs to a Sobolev or a Besov space. Stéphane Jaffard introduced new function spaces, that he named "Oscillation spaces," which are naturally associated with the leader scaling function; we refer the reader to [28] for further details.

We explain now how to obtain a priori upper bounds for the multifractal spectrum of a function f that belongs to a Hölder or a Besov space, or when $f \in \mathscr{M}([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ monotonic}\}$.

1. For the Hölder space $C^s(\mathbb{R}^d)$: it is quite straightforward.

Exercise 2.29. Prove that for every $f \in C^s(\mathbb{R}^d)$, $d_f(h) = -\infty$ if $h < s$, and $d_f(h) \le d$ if $h \ge s$.

Exercise 2.30. Construct a function $f \in C^s(\mathbb{R}^d)$ for which $d_f(h) = d \cdot \mathbb{1}_{\{s\}}(h)$.

Exercise 2.31. Construct a function $f \in C^s(\mathbb{R}^d)$ for which $d_f(h) = d \cdot \mathbb{1}_{\{h \geq s\} \cap \mathbb{Q}}(h)$.

Exercise 2.32. Construct a function $f \in C^s(\mathbb{R}^d)$ for which $d_f(h) = d \cdot \mathbb{1}_{\{h \geq s\}}(h)$.

2. For a Besov space: It is more tricky. Let $0 < s < \infty$, $0 < p, q \leq \infty$. Assume that the wavelets ψ^0 and ψ^1 are at least $[s+1]$-regular. The $B^s_{p,q}([0,1]^d)$ Besov norm (quasi-norm when $p < 1$ or $q < 1$) of a distribution f on $[0,1]^d$ (with wavelet coefficients d_λ) is

$$|||B^s_{p,q} f = \left(\sum_{j \geq 1} \left(2^{(sp-d)j} \sum_{|\lambda|=j} |d_\lambda|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

with the obvious modifications when $p = \infty$ or $q = \infty$. The Besov space $B^s_{p,q}([0,1]^d)$ is the set of functions with finite norm. It is a complete metrizable space, normed when p and $q \geq 1$, separable when both are finite.

The following standard embeddings are easy to deduce from (2.20): For any $0 < s < \infty$, $0 < p \leq \infty$, $0 < q < q' \leq \infty$, $\varepsilon > 0$,

$$B^s_{p,q}([0,1]^d) \hookrightarrow B^s_{p,q'}([0,1]^d) \hookrightarrow B^{s-\varepsilon}_{p,q}([0,1]^d) \tag{2.20}$$

We prove the result of Jaffard [24]: belonging to a Besov space yields an upper bound on the multifractal spectrum.

Theorem 2.7. *Let $0 < p < \infty$ and $d/p < s < \infty$. For every $f \in B^s_{p,\infty}([0,1]^d)$ and every $h \geq s - d/p$,*

$$d_f(h) \leq \min\big(d, d + (h-s)p\big),$$

and $E_f(h) = \emptyset$ if $h < s - d/p$.

Remark 2.1. The results have been stated for Besov spaces with $q = \infty$ but it is clear from classical Besov embeddings (2.20) that they hold identically for any $q > 0$.

Theorem 2.7 is not only optimal, the upper bound is actually an *almost sure* equality in $B^s_{p,q}([0,1]^d)$ in the sense of genericity or prevalence, as explained next in Section 2.5.

Proof. The proof follows the same lines as the one of Theorem 2.6. We indicate the main steps, and let the reader complete the missing parts as exercises. Let $f \in B^s_{p,q}([0,1]^d)$. Hence $|||B^s_{p,q} f < +\infty$.

1. The Sobolev embedding $B^s_{p,q}([0,1]^d) \hookrightarrow C^{s-d/p}([0,1]^d)$ implies that $E_f(h) = \emptyset$ for all $h < s - d/p$.

2. The inequality (2.21) is trivial when $h \geq s$, hence we fix $h \in [s-d/p, s)$. Then, for every $h' \leq h$, one has

$$N_j(h') = \#\{\lambda : |\lambda| = j \text{ and } |d_\lambda| \geq 2^{-jh'}\} \leq C2^{j(ph'-ps+d)},$$

this inequality following from the fact that $|||B^s_{p,q} f < +\infty$.

3. Let $\lambda = (j, \mathbf{k}, \mathbf{l})$ and D_λ be a wavelet leader such that $D_\lambda \geq 2^{-jh}$. This means that there exists $\lambda' = (j', \mathbf{k}', \mathbf{l}')$ such that $j' \geq j$, $I_{\lambda'} \subset \bigcup_{i \in \{-1,0,1\}^d} I_\lambda + i2^{-j}$ and $|d_{\lambda'}| \geq 2^{-jh}$. For every $j' \geq j$, the number of wavelet coefficients satisfying $|d_{\lambda'}| \geq 2^{-jh} = 2^{-j' \frac{j}{j'} h}$ is less than

$$N_{j'}\left(\frac{j}{j'} h\right) \leq 2^{j'p\left(\frac{j}{j'}h - s + d/p\right)} = 2^{jph - j'ps + j'd}.$$

Hence,

$$\begin{aligned} \#\{\lambda = (j, \mathbf{k}, \mathbf{l}) : D_\lambda \geq 2^{-jh}\} &\lesssim \sum_{j'=j}^{+\infty} 2^{jph - j'ps + j'd} \\ &\lesssim 2^{j(ph - ps + d)}. \end{aligned}$$

4. The last argument implies that the leader scaling function L_f associated with f satisfies
$$L_f(p) \geq ps - d.$$
at the specific value p associated with the Besov space $B^s_{p,q}([0,1]^d)$ we have chosen. Indeed, one has

$$\sum_\lambda |D_\lambda|^p \geq \sum_{\lambda : D_\lambda \geq 2^{-jh}} |D_\lambda|^p \geq 2^{j(ph-ps+d)} 2^{-jph} = 2^{j(-ps+d)}.$$

Finally, apply Theorem 2.6 to get

$$d_f(h) \leq ph - ps + d.$$

3. For monotonic functions: There are many constraints on the multifractal spectrum of a monotonic function $f \in \mathscr{M}([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ monotonic}\}$. The main reason is the fact that a monotonic function has (obviously) bounded variations, and that f is the integral of a positive measure. We will work with measures rather than functions, but the two are equivalent.

The first constraint on the multifractal spectrum is due to the famous Lebesgue theorem on Lebesgue density, which implies that for every positive and finite Borel measure μ, for Lebesgue-almost every $x \in \mathbb{R}^d$,

$$\lim_{\varepsilon \to 0} \frac{\mu(B(x, \varepsilon))}{\varepsilon^d} \in [0, +\infty).$$

This obviously implies that for Lebesgue-almost every $x \in \mathbb{R}^d$, $h_\mu(x) \geq d$.

Theorem 2.8. *For every probability measure μ supported on $[0,1]^d$,*

$$d_\mu(h) \leq \min(h,d).$$

Proof. Fix a measure μ, and an exponent $0 < h < d$. As in the preceding proofs, we find an upper bound for $\widetilde{E}_\mu(h) = \{x : h_\mu(x) \leq h\}$.

Fix $\varepsilon > 0$, and $\eta > 0$.

Let $x \in \widetilde{E}_\mu(h)$. By definition of the local dimension of a measure, there exists $0 < r_x < \eta$ such that $\mu(B(x,r)) \geq (r_x)^{h+\varepsilon}$.

The set of balls $(B(x,r_x))_{x\in\widetilde{E}_\mu(h)}$ forms a covering of $\widetilde{E}_\mu(h)$ by balls centered at points belonging to it. By the Besicovich's covering lemma, there exists $Q(d)$ disjoint families $F^1, \ldots, F^{Q(d)}$, each of them being formed by pairwise disjoint balls $F^j = (B(x_i^j, r_i^j))_{i\in\mathbb{N}}$, such that

$$\widetilde{E}_\mu(h) \subset \bigcup_{j=1}^{Q(d)} \bigcup_{i\in\mathbb{N}} B(x_i^j, r_i^j).$$

Let us estimate by above the $\mathscr{H}_\eta^{h+\varepsilon}$ Hausdorff pre-measure of $\widetilde{E}_\mu(h)$ using this covering. One gets

$$\begin{aligned}\mathscr{H}_\eta^{h+\varepsilon}(\widetilde{E}_\mu(h)) &\leq \sum_{j=1}^{Q(d)} \sum_{i\in\mathbb{N}} |B(x_i^j, r_i^j)|^{h+\varepsilon} \leq \sum_{j=1}^{Q(d)} \sum_{i\in\mathbb{N}} \mu(B(x_i^j, r_i^j)) \\ &\leq \sum_{j=1}^{Q(d)} \mu([0,1]^d) = Q(d),\end{aligned}$$

the last inequality following from the fact the balls constituting one family F^j are pairwise disjoint. Hence $\dim_{\mathscr{H}} \widetilde{E}_\mu(h) \leq h + \varepsilon$, for every $\varepsilon > 0$.

Exercise 2.33. Let $\mathscr{G}_j$ be the partition of $[0,1)^d$ into dyadic boxes that we denote I_λ where $\lambda = (j, \mathbf{k})$ (in this section the index $\mathbf{l}$ (due the use of wavelets) does not exist).

The L^q-spectrum of a measure $\mu \in \mathscr{M}([0,1]^d)$, which is the analog of the scaling function associated with functions, is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j\to\infty} -\frac{1}{j} \log_2 s_j(q) \quad \text{where} \quad s_j(q) = \sum_{|\lambda|=j,\ \mu(I_\lambda)\neq 0} \mu(I_\lambda)^q.$$

Prove that for every probability measure on $[0,1]^d$,

$$d_\mu(h) \leq (\tau_\mu)^*(h) := \inf_{q\in\mathbb{R}} (qh - \tau_\mu(q)).$$

Hint: use the same ideas as the one developed in Theorems 2.6 and 2.8.

Exercise 2.34. Consider the measure

$$\nu = \sum_{j\geq 1} \frac{1}{j^2} \sum_{k \text{ odd}} 2^{-j} \delta_{k2^{-j}},$$

where δ_x is the Dirac mass at x. Let

$$\xi_x = \sup\{\xi \geq 0 : |x - k2^{-j}| \leq 2^{-j\xi} \text{ for infinitely many } j \geq 1 \text{ and odd } k\}.$$

ξ_x is called the approximation rate of x by the dyadic numbers.

1. Prove that $\xi_x \in [1, +\infty]$ for every $x \in [0,1]$.
2. For every $\xi \in [1, +\infty]$, construct a real number $x \in [0,1]$ such that $\xi_x = \xi$ (Hint: use a dyadic decomposition).
3. Prove that for every x, $h_\nu(x) \leq 1/\xi_x$.
4. Using that the dyadic numbers are well distributed in $[0,1]$, prove that one has $h_\nu(x) = 1/\xi_x$.
5. Conclude that the support of the multifractal spectrum of ν is exactly the interval $[0,1]$.
6. (Difficult) Prove that for every $h \in [0,1]$, $d_\nu(h) = h$.
   Hint: One can:
   - either prove it directly by computing $\dim_{\mathscr{H}} E_\nu(h) = \dim_{\mathscr{H}}\{x : \xi_x = 1/h\}$ and use the mass distribution principle (Theorem 2.2),
   - or use the theorem by Beresnevich and Velani [10], recalled before (Theorem 2.3 and Exercise 2.9). The method consists in applying Theorem 2.3 to the family $\Big((\mathbf{k}2^{-j}, 2^{-j})\Big)_{j\geq 1, \mathbf{k}2^{-j}\in[0,1]^d}$ of dyadic balls in $[0,1]^d$, to prove that

$$\dim_{\mathscr{H}}\{x : \xi_x = 1/h\} = 1/(1/h) = h.$$

Exercise 2.35. Build a measure μ supported on $[0,1]$ such that for Lebesgue-almost every $x \in [0,1]$, $h_\mu(x) \geq 2$. (Hint: build a devil's staircase.)

2.4.4 Another Multifractal Spectrum: The Large Deviations Spectrum

The *large deviations* spectrum d_f^{ld} of a function is related on the asymptotic histogram of wavelet coefficients, see [4, 6, 31] for a complete study of this spectrum and an application to heart beat rates analysis. It is also relatively easy to estimate, in practical cases.

Definition 2.13. Let f of the form (2.7), and let $\varepsilon > 0$. For every λ such that $|\lambda| = j$ and $\mathbf{k}2^{-j} \in [0,1]^d$, let $h_\lambda = -j^{-1}\log_2|d_\lambda|$ (we set $h_\lambda = +\infty$ if $d_\lambda = 0$). We set

$$N_j^\varepsilon(h) = \#\{\lambda : |h_\lambda - h| \leq \varepsilon\}. \tag{2.21}$$

and $d_f^{ld}(\varepsilon, h) = \limsup_{j\to+\infty} j^{-1}\log_2 N_j^\varepsilon(h)$.

The *large deviations spectrum* $d_f^{ld}(h)$ is defined as the mapping $d_f^{ld}(h) = \lim_{\varepsilon\to 0} d_f^{ld}(\varepsilon, h)$.

Exercise 2.36. Prove that the definition makes sense, and that the mapping $h \mapsto d_f^{ld}(h)$ is lower semi-continuous.

The large deviations spectrum clearly depends on the choice of the wavelet ψ. While one always has $d_f^{ld}(h) \le (\eta_f - d)^*(h)$ (the Legendre transform of the wavelet scaling function), there is no general relationship between d_f^{ld} and d_f. The examples we later consider illustrate this statement.

2.5 Generic Results for the Multifractality of Functions

In the previous section, we obtained upper bounds for the multifractal spectrum of many functions, based on the function spaces to which these functions belong. It is a natural question to ask whether these bounds are optimal. In the cases developed before, they are indeed. Even more, one can show that "almost every function" in these spaces realizes the upper bound. This can be interpreted by the fact that the worst regularity is the most common one, since the iso-Hölder sets for typical functions have the greatest possible dimension, as we will see.

Let us start by recalling how one can talk about "almost every" element in infinite dimensional spaces.

Definition 2.14. A property $\mathscr{P}$ is said to be *generic* in a complete metric space E when it holds on a residual set, i.e. a set with a complement of first Baire category. A set is of first Baire category if it is the union of countably many nowhere dense sets. As it is often the case, it is enough to build a residual set which is a countable intersection of dense open sets in E.

Genericity is essentially a topological notion, and this is the one that we are going to use in this course.

Exercise 2.37. Prove that a generic set in $\mathbb{R}$ must be uncountably dense.

Exercise 2.38. Find a generic set in $\mathbb{R}$ of Lebesgue measure 0.

There is another notion for describing the "size" of a set. Prevalence theory is used to supersede the Lebesgue measure in any topological vector space E. This notion was proposed by Christensen [14] and later by Hunt [21]. The space E is endowed with its Borel σ-algebra $\mathscr{B}(E)$.

Definition 2.15. A Borel set $A \subset E$ is said to be *shy* if there exists a positive Borel measure μ, supported on some compact subset K of E, such that

$$\text{for every } x \in E, \;\; \mu(A + x) = 0.$$

A set that is included in a shy Borel set is also called shy.
The complement of a shy set A in E is called prevalent.

Prevalent sets are stable under translation, dilation, union, and countable intersection. Moreover, when E has finite dimension, being prevalent in E is equivalent to having full Lebesgue measure. This justifies that a prevalent set A is referred to as a "large" set in E and extends reasonably the notion of full Lebesgue measure to infinite dimensional spaces.

Exercise 2.39. Prove the above claims about prevalent sets.

In what follows, we essentially deal with multifractal properties of generic functions (i.e., of all functions in a generic set of some function space), but most of the time, these results also hold true for prevalent functions (i.e., for all functions in a prevalent set). The reader can have a look at the numerous results on the subject for further details.

2.5.1 Hölder Spaces

Theorem 2.9. *There exists a dense open set (hence, generic) $\mathscr{R} \subset C^s([0,1]^d)$ such that for every $f \in \mathscr{R}$ and every $x \in [0,1]^d$, $h_f(x) = s$.*

In particular, generic functions in $C^s([0,1]^d)$ are monofractal, i.e. $E_f(h) = \emptyset$ if $h \neq s$.

Proof. Let us recall that for any $f \in C^s([0,1]^d)$, there exists a constant $C > 0$ such that

$$f = \sum_{\lambda : |\lambda| \geq 1} d_\lambda \Psi_\lambda(x) \quad \text{ with } \ |d_\lambda| \leq C 2^{-js}$$

and $\|f\|_{C^s} = \inf\{C > 0 : (2.22) \text{is satisfied for all } \lambda\}$ is a Banach norm on $C^s(\mathbb{R}^d)$.

For each integer $N \geq 1$, let us introduce the sets:

$$\begin{aligned} \mathscr{E}_N &= \left\{ f \in C^s([0,1]^d) \ : \ \forall \lambda,\ 2^{js+N} d_\lambda \in \mathbb{Z}^* \right\} \\ \mathscr{F}_N &= \left\{ g \in C^s([0,1]^d) \ : \ \exists f \in \mathscr{E}_N,\ \|f-g\|_{C^s([0,1]^d)} < 2^{-N-2} \right\}. \end{aligned} \tag{2.22}$$

Lemma 2.1. *For every $N \geq 1$, all functions in $\mathscr{F}_N$ are monofractal with exponent s.*

Proof. This follows from the fact that, given $f \in \mathscr{E}_N$, all the wavelet coefficients of f satisfy

$$2^{-N-js} \leq |d_\lambda| \leq \|f\|_{C^s} 2^{-js}.$$

Thus for any function $g \in \mathscr{F}_N$ with coefficients g_λ and its associated $f \in \mathscr{E}_N$:

$$2^{-N-js} - 2^{-N-2-js} \leq |g_\lambda| \leq \|f\|_{C^s} 2^{-js} + 2^{-N-2-js}$$

i.e.

$$2^{-N-1-js} \leq |g_\lambda| \leq \left(\|f\|_{C^s} + 2^{-N-2}\right) 2^{-js}.$$

In particular, $g \in C^s(x)$ for any $x \in [0,1]^d$ and there is no $x_0 \in [0,1]^d$ and $s' > s$ such that $g \in C^{s'}(x_0)$. Indeed, (2.11) with $s' > s$ is not compatible when j tends to infinity with the left hand-side of the above inequality.

We prove now that the set

$$\mathscr{R} = \bigcup_{N \geq 1} \mathscr{F}_N$$

is a dense open set in $C^s([0,1]^d)$ containing only monofractal functions with exponent s.

The preceding lemma ensures that $\mathscr{R}$ is composed of monofractal functions. According to (2.22), $\mathscr{F}_N$ is an open set and thus, so is $\mathscr{R}$. Let us check the density. Fix $f \in C^s([0,1]^d)$ with wavelet coefficients d_λ.

Let $\eta > 0$, choose $N \geq 1$ so that $2^{-N} < \eta$. We use the "non-zero integer part" function

$$E^*(x) = \begin{cases} -1 & \text{if } -2 < x < 0, \\ 1 & \text{if } 0 \leq x < 2, \\ \lfloor x \rfloor & \text{if } |x| \geq 2. \end{cases}$$

Obviously $E^* : \mathbb{R} \to \mathbb{Z}^*$ and $|x - E^*(x)| \leq 1$. Let us finally define a function $g \in \mathscr{F}_N$ by its wavelet coefficients g_λ:

$$g_\lambda = 2^{-js-N} E^*\left(2^{js+N} d_\lambda\right).$$

By construction,

$$2^{js}\,|d_\lambda - g_\lambda| = 2^{-N} |2^{js+N} d_\lambda - E^*\left(2^{js+N} d_\lambda\right)| \leq 2^{-N} < \eta$$

thus $\|f - g\|_{C^s} < \eta$. This proves the density of $\mathscr{R}$ in $C^s([0,1]^d)$.

2.5.2 Besov Spaces

One starts by constructing a measure whose multifractal spectrum is the "worst possible" in a given Besov space $B^s_{p,q}([0,1]^d)$.

Lemma 2.2. *Let $\beta = 1/p + 1/q$. Consider the measure ν built in Exercise 2.34, and the wavelet series F_ν whose wavelet coefficients f_λ are given by*

$$f_\lambda = \frac{1}{j^\beta} 2^{-j(s-d/p)} \nu(I_\lambda)^p.$$

Then, $F_\nu \in B^s_{p,q}([0,1]^d)$ *and its multifractal spectrum is*

$$\text{for every } h \in [s - d/p, s], \;\; d_{F_\nu}(h) = p(h-s) + d,$$

and $E_{F_\nu}(h) = \emptyset$ *if* $h > s$*.*

Exercise 2.40. Prove Lemma 2.2 by combining Exercises 2.27 and 2.34.
Prove that for all λ such that $|\lambda| = j$, $|f_\lambda| \geq \frac{1}{j^\beta} 2^{-js}$.

We now prove the multifractal nature of generic functions in $B^s_{p,q}([0,1]^d)$.

Theorem 2.10. *In* $B^s_{p,q}([0,1]^d)$*, generic functions* f *are multifractal with the "as worse as possible" multifractal spectrum, i.e.*

$$\text{for every } h \in [s - d/p, s], \;\; d_f(h) = p(h-s) + d,$$

and $E_f(h) = \emptyset$ *if* $h > s$*.*

Proof. The strategy to build a residual set with the desired multifractal properties is the following. Consider a dense sequence of functions $(\widetilde{f_n})_{n\geq 1}$ in the separable space $B^s_{p,q}([0,1]^d)$ (each $\widetilde{f_n}$ having $(\widetilde{d^n_\lambda})$ as wavelet coefficients) and replace it by the sequence $(f_n)_{n\geq 1}$ whose wavelet coefficients (d^n_λ) are defined as follows:

$$d^n_\lambda = \begin{cases} \widetilde{d^n_\lambda} & \text{if } |\lambda| < n, \\ F_\lambda & \text{if } |\lambda| \geq n. \end{cases}$$

In other words, one replaces the wavelet coefficients of $\widetilde{f_n}$ by those of F for large $|\lambda|$.

It is easy to see that each f_n has the same multifractal behavior as F, since only the wavelet coefficients of large generation (corresponding to high frequencies) are important for the local behavior, and that the sequence is still dense in $B^s_{p,q}([0,1]^d)$.

Exercise 2.41. Prove that (f_n) is indeed dense in $B^s_{p,q}([0,1]^d)$.

Definition 2.16. Let $\beta = 1/p + 1/q$, and $r_n = n^{-\beta} 2^{-nd/p}/2$. One defines the set $\widetilde{\mathscr{R}}$

$$\widetilde{\mathscr{R}} = \bigcap_{N\geq 1} \bigcup_{n\geq N} \mathscr{B}(f_n, r_n)$$

where $\mathscr{B}(g,r) = \{f \in B^s_{p,q}([0,1)^d) \,:\, \|f-g\|_{B^s_{p,q}([0,1]^d)} < r\}$.

The set $\widetilde{\mathscr{R}}$ is an intersection of dense open set, hence a residual set in $B^s_{p,q}([0,1]^d)$. The choice for the radius r_n is small enough to ensure that any function f in $\mathscr{B}(f_n, r_n)$ has its wavelet coefficients at generation n close to those of f_n (and thus to those of F).

Lemma 2.3. *If $f \in \mathscr{B}(f_n, r_n)$ has wavelet coefficients d_λ, then $|d_\lambda - d_\lambda^n| \leq |d_\lambda^n|/2$.*

Proof. By definition, one has $d_\lambda^n = F_\lambda$, $\forall \lambda$ such that $|\lambda| = n$. Hence, by definition of the Besov norm and the inclusion $\ell^q \subset \ell^\infty$:

$$\left(2^{pn(s-d/p)} \sum_{\lambda:\, |\lambda|=n} |d_\lambda - F_\lambda|^p \right)^{1/p} < r_n.$$

In particular, for any λ such that $|\lambda| = n$,

$$|d_\lambda - F_\lambda| \leq r_n 2^{-n(s-d/p)} = 2^{-ns} n^{-\beta}/2.$$

By Exercise 2.40, if $|\lambda| = j$, $|F_\lambda| \geq 2^{-js}/j^\beta$. Combining both inequalities ensures the result.

Let us now prove Theorem 2.10.

Let $f \in \widetilde{\mathscr{R}}$. There exists a strictly increasing sequence $(n_m)_{m\geq 1}$ of integers such that $f \in \mathscr{B}(g_{n_m}, r_{n_m})$.

Lemma 2.3 provides a precise estimate of the wavelet coefficients of f, namely for any $m \geq 1$: if $|\lambda| = n_m$,

$$\frac{1}{2} F_\lambda \leq |d_\lambda| \leq \frac{3}{2} F_\lambda.$$

The (almost) same proof as the one used for Exercise 2.34, Exercise 2.27, and Lemma 2.2 ensures that for any $x \in [0,1]^d$:

$$s - d/p \leq h_f(x) \leq s - d/p + d/(p\widetilde{\xi}_x) \leq s,$$

where $\widetilde{\xi}_x$ is the approximation rate by the family $(n_m)_{m\geq 1}$, defined by

$$\widetilde{\xi}_x = \sup\{\xi \geq 0 : |x - k2^{-n_m}| \leq 2^{-n_m \xi} \text{ for infinitely many } m \geq 1 \text{ and odd } k\}.$$

The definition is almost the same as in Exercise 2.34, except that only a subsequence of the integers is used in the dyadic approximation.

Given $h \in [s - Q/p, s]$ and the unique ξ^h such that $h = s - d/p + d/(p\xi^h)$, one introduces the set

$$\mathscr{E} = \{x :\ \widetilde{\xi}_x = \xi^h\} \setminus \bigcup_{i=1}^{+\infty} \{x \in [0,1]^d : h_f(x) \leq h - 1/i\}.$$

By Theorem 2.7 and the remarks thereafter, one knows that $\dim_{\mathscr{H}}\{x \in [0,1]^d : h_f(x) \leq h'\} \leq p(h' - s - d/p)$ for any $h' < h$. In particular, for every $i \geq 1$, one has:

$$\begin{aligned}\dim_{\mathscr{H}} \{x \in [0,1]^d : h_f(x) \leq h - 1/i\} &\leq p\,(h - 1/i - s - d/p) \\ &< p\,(h - s - d/p) \\ &= d/\xi^h.\end{aligned}$$

But according to Theorem 2.3 that can be applied to the subsequence of dyadic numbers

$$(\mathbf{k}2^{-n_m}, 2^{-n_m})_{m\geq 1},$$

one has $\mathscr{H}^{Q/\xi^h i}(\{x:\ \widetilde{\xi}_x = \xi^h\}) = +\infty$, thus $\mathscr{H}^{Q/\xi^h}(\mathscr{E}) = +\infty$ and

$$\dim_H \mathscr{E} \geq d/\xi^h.$$

Next, one observes that $\mathscr{E} \subset E_f(h)$, since every $x \in \{x:\ \widetilde{\xi}_x = \xi^h\}$ satisfies $h_f(x) \leq s - d/p + 1/(p\xi^h) = h$ and, by definition, $\mathscr{E}$ does not contain those elements x which have a pointwise Hölder exponent strictly smaller than h. One finally infers that:

$$\dim_{\mathscr{H}} E_f(h) \geq \dim_{\mathscr{H}} \mathscr{E} \geq d/\xi^h = p\,(h - s - d/p).$$

The converse inequality is provided by Theorem 2.7 because $f \in B^s_{p,q}([0,1]^d)$.

2.5.3 Measures (or Monotonic Functions)

We do not give the complete proof of the main result of this section, we refer the reader to [11] for the details. Nevertheless we explain the context and we let the reader observe that the main ideas to prove the results are comparable to the ones used in last sections. Theorem 2.11 is an important result because it does not require anything specific on the measures (no self-similarity or scaling behavior), it simply says that typical measures are multifractal.

We recall the notion of L^q-spectrum for a probability measure μ supported on $[0,1]^d$. We denote I_λ, where $\lambda = (j, \mathbf{k})$ and $\mathbf{k} \in \{0, 1, \cdots, 2^j - 1\}^d$, the dyadic boxes of generation j included in $[0,1]^d$ (in this section the index $\mathbf{l}$ (due to wavelets) does not exist).

The L^q-spectrum of a measure $\mu \in \mathscr{M}([0,1]^d)$, which is the analog of the scaling function associated with functions, is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j\to\infty} -\frac{1}{j}\log_2 s_j(q) \quad \text{where} \quad s_j(q) = \sum_{|\lambda|=j,\ \mu(I_\lambda)\neq 0} \mu(I_\lambda)^q.$$

To be able to talk about "generic" or "typical" measures in the sense of Baire categories, we need to define the topology on the set of probability measures on $[0,1]^d$. We endow it with the weak topology induced by the following metric: if $\mathrm{Lip}([0,1]^d)$ stands for the set of Lipschitz functions on $[0,1]^d$ with Lipschitz constant ≤ 1, and if μ and ν belong to $\mathscr{M}([0,1]^d)$, we set

$$d(\mu,\nu) = \sup\left\{ \left| \int f\,d\mu - \int f\,d\nu \right| : f \in \mathrm{Lip}([0,1]^d) \right\}. \tag{2.23}$$

Theorem 2.11. *There is a dense G_δ set $\mathscr{R}$ included in $\mathscr{M}([0,1]^d)$ such that for every measure $\mu \in \mathscr{R}$, we have*

$$\forall\, h \in [0,d] \quad d_\mu(h) = h, \tag{2.24}$$

and $E_\mu(h) = \emptyset$ if $h > d$.

In particular, for every $q \in [0,1]$, $\tau_\mu(q) = d(1-q)$, and μ satisfies the multifractal formalism at every $h \in [0,d]$, i.e. $d_\mu(h) = \tau_\mu^*(h)$.

Remark 2.2. Theorem 2.11 has been extended by F. Bayart in [8] to measures supported on every compact set $K \subset \mathbb{R}^d$.

2.5.4 Traces, Slices, Projections...

Given a multifractal function $f : \mathbb{R}^d \to \mathbb{R}$, it is very natural to ask whether its traces, i.e. its restrictions to subspaces or submanifolds of $\mathbb{R}^d$ are still multifractal. This is actually a fundamental question since, for instance, multifractality has been proved for 1D-traces of the 3D-velocity of turbulent fluids, not for the 3D-velocity itself. Only few is known, we give one theorem proved in [3] and [32].

Theorem 2.12. *Let $1 \le d' < d$ be two integers. For every $a \in \mathbb{R}^{d-d'}$, let $\mathscr{H}_a$ be the affine space*

$$\mathscr{H}_a = \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_{d'+1} = a_1, x_{d'+2} = a_2, \dots, x_d = a_{d-d'}\}.$$

Consider two positive real numbers s and p such that $s - d/p > 0$.

For every $f \in B^s_{p,q}(\mathbb{R}^d)$, for Lebesgue-almost every $a \in \mathbb{R}^{d-d'}$, the trace of f over $\mathscr{H}_a$, denoted by f_a, belongs to $\bigcap_{s'<s} B^{s'}_{p,\infty}(\mathbb{R}^{d'})$. If $q < p$, one has $f_a \in B^s_{p,qp/(p-q)}([0,1]^{d'})$.

Moreover, for typical functions in $B^s_{p,q}(\mathbb{R}^d)$, for Lebesgue-almost every $a \in \mathbb{R}^{d-d'}$, the trace of f over $\mathscr{H}_a$ has the following multifractal properties:

- *the exponents of f_a all belong to the interval $[s - d'/p, s]$,*
- *for every $h \in [s - d'/p, s]$, the multifractal spectrum of f_a is*

$$d_{f_a}(h) = p(h - (s - d'/p)).$$

What is surprising in this theorem is that the typical traces of f on affine subspaces possess a regularity which is better than the one guaranteed by the standard trace theorems (one knows that traces of functions belonging to $B^s_{p,q}(\mathbb{R}^d)$ all belong to $B^{s-(d-d'd)/p}_{p,q}(\mathbb{R}^{d'})$, but we prove that their regularity is actually better than expected). In addition, we compute their exact multifractal spectrum which **is not the worst possible regularity**, as it is the case in the preceding section. A lot is still to be done on multifractal properties of traces of functions, and also of slices and projections of multifractal measures (see [3, 12, 16, 26]).

2.6 Some Examples of Multifractal Wavelet Series

2.6.1 Hierarchical Wavelet Series

We come back to the example of Section 2.3.5, which was originally developed in [7]. Let μ be a positive Borel probability measure on $[0,1]^d$, and let F_μ be the wavelet series whose wavelet coefficients are

$$d_\lambda = 2^{-j\alpha}\mu(I_\lambda)^\beta.$$

Assuming that the measure is uniformly regular, i.e. there exists a constant $C>0$ and an exponent $h_{\min}>0$ such that for every ball with center x and radius $0<r<1$, one has

$$\mu(B(x,r)) \geq Cr^{h_{\min}}.$$

Then by an adaptation of Proposition 2.2, $f\in C^{h_{\min}}(\mathbb{R}^d)$, and for every $x\in[0,1]^d$, one has

$$h_F(x) = \alpha + \beta\cdot h_\mu(x).$$

In particular, $d_F(h) = d_\mu\left(\frac{h-\alpha}{\beta}\right)$.

Also, by Exercise 2.26, the multifractal spectrum is quite stable if one perturbs the wavelet coefficients by multiplying them by random variables which are "not too bad."

Such wavelet series are nice models, since they are relatively easy to simulate numerically, and there are some parameters that allow one to fit the multifractal parameters of real data, by choosing relevant values of α, β, and the measure μ. In addition, the hierarchical structure of the wavelet coefficients (due to the measure, if $I_{\lambda'}\subset I_\lambda$, then $d_{\lambda'}\leq d_\lambda$) imply that large coefficients must be located around the same position through scales, which correspond to real situations (for instance, contours of images are irregular and large coefficients are located around them). Unfortunately, it has many unrealistic characteristics: there is no sparsity, and almost no small coefficients.

2.6.2 Lacunary Wavelet Series

A model somehow orthogonal to the previous one was introduced by S. Jaffard in [27]. In his model, all coefficients are independent, but not with common law. The model is as follows (we focus on the one-dimensional model, the extension to $\mathbb{R}^d$ is immediate): Let $\alpha > 0$, $0 < \eta < 1$ be two parameters. Let $(g_\lambda)_{\lambda:|\lambda|\geq 1}$ be a sequence of independent random variables in a probability space $(\Omega, \mathscr{B}, \mathbb{P})$, whose law are Bernoulli laws with parameter $2^{-|\lambda|\eta}$, i.e.

$$\text{for every } \lambda \text{ such that } |\lambda| = j \geq 1, \quad g_\lambda = \begin{cases} 1 & \text{with probability } 2^{-j\eta} \\ 0 & \text{with probability } 1 - 2^{-j\eta}. \end{cases}$$

Then consider the (random) wavelet series $R_{\alpha,\eta}$ whose wavelet coefficients d_λ are

$$\text{for every } \lambda = (j,k,l) \text{ such that } |\lambda| = j \text{ and } k2^{-j} \in [0,1], \;\; d_\lambda = 2^{-j\alpha} g_\lambda.$$

Theorem 2.13. *There exists an event of probability one in* $(\Omega, \mathscr{B}, \mathbb{P})$ *such that*

$$\textit{for every } h \in [\alpha, \alpha/\eta], \qquad d_{R_{\alpha,\eta}}(h) = \frac{\eta}{\alpha} h,$$

and $E_{R_{\alpha,\eta}}(h) = \emptyset$ *otherwise.*

The proof is written as a succession of Lemmas and exercises that are individually accessible (hopefully…).

Proof. 1. Observe that for all λ, $|d_\lambda| \leq 2^{-|\lambda|\alpha}$, hence $R_{\alpha,\eta} \in C^\alpha(\mathbb{R})$.

2. The upper bound for the exponents is obtained thanks to the following uniform lower bound for the wavelet leaders.

   Lemma 2.4. *For every* $\varepsilon > 0$*, almost surely, there exists* $J \geq 1$ *such that for all* $j \geq J$*, for all* λ *such that* $|\lambda| = j$ *and* $k2^{-j} \in [0,1]$*, the wavelet leader* D_λ *satisfies*

   $$D_\lambda \geq 2^{-j(\alpha/\eta+\varepsilon)}.$$

   Prove Lemma 2.4 using the Borel-Cantelli Lemma.

3. By (2.25), one infers that almost surely, for every $x \in [0,1]$,

   $$h_{R_{\alpha,\eta}}(x) \leq \alpha/\eta + \varepsilon.$$

   Since (2.25) holds true for every $\varepsilon > 0$, one gets that almost surely, $h_{R_{\alpha,\eta}}(x) \leq \alpha/\eta$ for every $x \in [0,1]$ (observe that it is stronger than "for every $x \in [0,1]$, almost surely, we have…").
   Hence we get the correct range $[\alpha, \alpha/\eta]$ for the possible exponents for $R_{\alpha,\eta}$, and it remains us to compute the Hausdorff dimension of the iso-Hölder sets.

4. There is a relationship between the value of the pointwise Hölder exponent h of $R_{\alpha,\eta}$ at a point x and the approximation rate of x by some random family of intervals, which cover the interval $[0,1]$.

Lemma 2.5. *Let us denote by $(\lambda_n = (j_n, k_n, l_n))_{n\geq 1}$ the sequence of cubes for which $g_{\lambda_n} = 1$, re-ordered so that $j_n \leq j_{n+1}$ for every $n \geq 1$. With probability one, there exists a positive non-increasing sequence $(\varepsilon_n)_{n\geq 1}$, converging to zero, such that*

$$[0,1] \subset \limsup_{n\to+\infty} B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)}).$$

Prove Lemma 2.5 using the Borel-Cantelli Lemma (this is a sort of refinement of Lemma 2.4).

5. Following Lemma 2.5 and also exercises 2.9 and 2.10, let us introduce the approximation rate of a real number $x \in [0,1]$ by the random family $(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)})_{n\geq 1}$ as

$$\xi_x = \sup\{\xi \geq 1 : x \in \limsup_{n\to+\infty} B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)\xi})\},$$

the associated (random) sets

$$S_\xi = \{x \in [0,1] : x \in \limsup_{n\to+\infty} B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)\xi})\}$$

and finally

$$\widetilde{S}_\xi = \{x \in [0,1] : \xi_x = \xi\}.$$

Using the same techniques as in Exercise 2.9 and 2.10, prove that almost surely, for every $\xi \geq 1$, $\dim_{\mathscr{H}} S_\xi = \dim_{\mathscr{H}} \widetilde{S}_\xi = 1/\xi$.

6. We now find a first inequality between the approximation rate and the pointwise Hölder exponent.

Lemma 2.6. *If $\xi \in [1, 1/\eta]$ and $x \in S_\xi$, then $h_{R_{\alpha,\eta}}(x) \leq \alpha/(\eta\xi)$.*

Prove Lemma 2.6. (Hint: when $x \in B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)\xi})$, find a lower bound for the wavelet leader $D_{j_n(1-\varepsilon_n)\xi}(x)$.)

7. Prove that if $x \notin S_\xi$, then for every $\varepsilon > 0$, for all j large enough, $D_j(x) \geq 2^{-j(\alpha/(\eta\xi)+\varepsilon)}$.
8. Deduce that if $h = \alpha/(\eta\xi)$ with $\xi \in [1, 1/\eta]$, then $E_{R_{\alpha,\eta}}(h) = \widetilde{S}_\xi$.
9. Conclude.
10. Compute the almost-sure large deviations spectrum of $R_{\alpha,\eta}$. Does this spectrum satisfy a multifractal formalism?

A generalization of these random wavelet series is developed in [4].

2.6.3 Thresholded Wavelet Series

Thresholding is an important method in signal and image processing. As is well known, it provides efficient methods in compression (the JPEG 2000 algorithm uses such techniques for instance).

In this section, we give some connections between multifractal properties and adaptive threshold methods, which are essentially based on results published in [35].

Theorem 2.14. *Let f be a function satisfying (2.7), and assume that $f \in C^{\varepsilon}([0,1])$ for some $\varepsilon > 0$. Assume that there exists an exponent $h > 0$ such that $d_f^{ld}(h) < d_f(h)$. Then there exists a set $E \subset E_f(h)$ of dimension $d_f(h)$ of oscillating singularities for f.*

Exercise 2.42. Demonstrate Theorem 2.14 by proving that for $\varepsilon < d_f(h) - d_f^{ld}(h)$, there is not enough wavelet coefficients $\lambda = (j, \mathbf{k}, \mathbf{l})$ satisfying $d_\lambda \sim 2^{-|\lambda|h}$ to create a set of Hausdorff dimension $d_f(h)$ of points such that $|x - \mathbf{k}2^{-j}| \leq 2^{-j(1-\varepsilon)}$.

Essentially, the last Theorem asserts that if the multifractal and the large deviations spectra do not coincide, **there are oscillating singularities!** It is thus a simple theoretical way to detect chirp-like behaviors. Unfortunately it is not applicable, since it requires the knowledge of the multifractal spectrum.

This theorem is completed by the next one. Let us introduce an adaptive threshold to keep, at each generation, the greatest wavelet coefficients. The same can be achieved by keeping only the smallest wavelet coefficients.

Definition 2.17. Let f be a function satisfying (2.7). Let $\gamma > 0$. The function series f^γ, defined by

$$f^\gamma = \sum_{j \geq 1} \sum_{\lambda : |\lambda| = j} d_\lambda \cdot \mathbb{1}_{|d_\lambda| \geq 2^{-j\gamma}} \Psi_\lambda$$

is said to be obtained from f after *an adaptive threshold of order γ*.

We learn from Theorem 2.14 that for any function f which is sufficiently smooth, $d_f^{ld}(h) < d_f(h)$ for some exponent $h > 0$ ensures the existence of oscillating singularities for f. For such a function f, if $d_f(h) > 0$ for some $h > 0$, a threshold of order $\gamma < h$ imposes $d_{f^\gamma}^{ld}(h) = 0$. But since a threshold increases (local and global) regularity, every point $x \in E_f(h)$ has a pointwise Hölder exponent for f^γ at x which is greater than h. These points are good candidates to be oscillating singularities for f^γ.

Theorem 2.15. *Let f be a function satisfying (2.7), $\gamma > 0$, and assume that $f \in C^{\varepsilon}([0,1])$ for some $\varepsilon > 0$. Assume that $E_f^{(}h) \neq \emptyset$. Let f^γ be the function obtained after an adaptive threshold of f of order $\gamma < h$. Then for every $x \in E_f(h)$, either $h_{f^\gamma}(x) = +\infty$, or x is an oscillating singularity for f^γ.*

Exercise 2.43. Prove Theorem 2.15. One can first observe that necessarily $h_{f^\gamma}(x) \geq h_f(x)$, and then estimate what is the loss of value for the wavelet leader $D_j(x)$ in terms of $h-\gamma$.

This theorem can be interpreted as a sort of Gibbs phenomenon for the adaptive threshold we proposed. It is also a very convenient method to create functions with homogenous non-concave multifractal spectra, as stated by the following (last) theorem, proved in [35], in which a concrete case is treated.

Theorem 2.16. *Let μ be the binomial measure on the interval $[0,1]$ with parameter $0<p<1/2$, whose multifractal spectrum ranges in $[-\log_2(1-p), -\log_2(p)]$, and consider the wavelet series g whose wavelet coefficients are built according to the hierarchical model of Section 2.6.1, i.e. $d_\lambda = \mu(I_\lambda)$.*

Let $\omega_\gamma : [\gamma, -\log_2(p)] \to (0,+\infty)$ be the increasing function

$$u \to \gamma \frac{u+1+\log_2 p}{\gamma + \log_2 p}.$$

Let $h^\gamma_{\max} = \omega_\gamma^{-1}(-\log_2(p))$. The multifractal spectrum of g^γ ranges in $[-\log_2(p), h^\gamma_{\max}]$, and equals

$$d_{g^\gamma}(h) = \begin{cases} d_g(h) & \text{if } h \in [-\log_2(1-p), \gamma], \\ d_g(\omega_\gamma^{-1}(h)) & \text{if } \quad h \in (\gamma, h^\gamma_{\max}]. \end{cases}$$

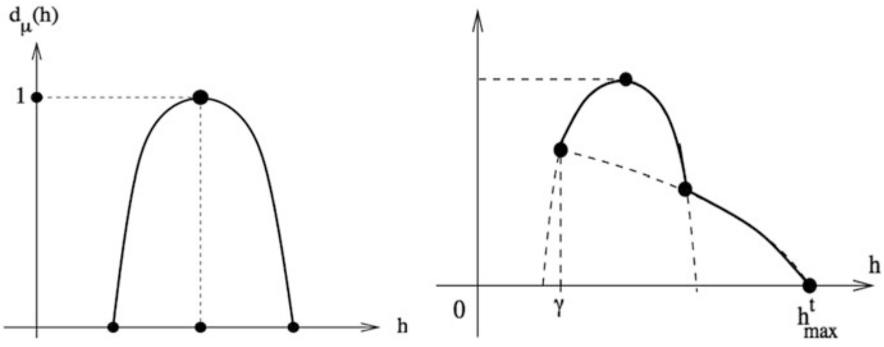

Fig. 2.4 Multifractal spectrum of g (left) and of g^γ (right) when $-\log_2(1-p) < \gamma < -\log_2(p)$.

See Fig. 2.4 for the multifractal spectra of g and g^γ.

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Chapter 3
An Introduction to Mandelbrot Cascades

Yanick Heurteaux

Abstract In this course, we propose an elementary and self-contained introduction to canonical Mandelbrot random cascades. The multiplicative construction is explained and the necessary and sufficient condition of non-degeneracy is proved. Then, we discuss the problem of the existence of moments and the link with non-degeneracy. We also calculate the almost sure dimension of the measures. Finally, we give an outline on multifractal analysis of Mandelbrot cascades. This course was delivered in September 2013 during a meeting of the "Multifractal Analysis GDR" (GDR n^{o} 3475 of the french CNRS).

3.1 Introduction

At the beginning of the seventies, Mandelbrot proposed a model of random measures based on an elementary multiplicative construction. This model, known as canonical Mandelbrot cascades, was introduced to simulate the energy dissipation in intermittent turbulence [19]. It was probably inspired by previous heuristics described by Richardson in [22]. In two notes [20, 21] published in '74, Mandelbrot described the fractal nature of the sets in which the energy is concentrated and proved or conjectured the main properties of this model. Two years later, in the fundamental paper [15], Kahane and Peyrière proposed a complete proof of the results announced by Mandelbrot. In particular, the questions of non-degeneracy, existence of moments, and dimension of the measures were rigorously solved.

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Mandelbrot also observed that in a multiplicative cascade, the energy is distributed along a large deviations principle: this was the beginning of the multifractal analysis.

Multifractal analysis has been developed a lot in the 1980s. Frisch and Parisi observed that in the context of the fully developed turbulence, the pointwise Hölder exponent of the dissipation of energy varies widely from point to point. They proposed in [12] a heuristic argument, showing that the Hausdorff dimension of the level sets of a measure or a function can be obtained as the Legendre transform of a free energy function (which will be called in this text the structure function). This principle is known as *multifractal formalism*. Such a formalism was then rigorously proved by Brown Michon and Peyrière for the so-called quasi-Bernoulli measures [10]. In particular, they highlighted the link between the multifractal formalism and the existence of auxiliary measures.

The problem of the multifractal analysis of Mandelbrot cascades appeared as a natural question at the end of the 1980s. Holley and Waymire were the first to obtain results in this direction. Under restrictive hypotheses, they proved in [14] that for any value of the Hölder exponent, the multifractal formalism is almost surely satisfied. The expected stronger result which says that, almost surely, for any value of the Hölder exponent, the multifractal formalism is satisfied was finally proved by Barral at the end of the 20$^{\text{th}}$ century [2].

Let us finish this overview by saying that there exist now many generalizations of the Mandelbrot cascades (see, for example, [8] for the description of the principal ones).

In the following pages, we want to relate the beginning of the story of canonical Mandelbrot cascades. As a preliminary, we explain the well known determinist case of binomial cascades. It allows us to describe the multiplicative principle, to introduce the most important notations and definitions, and to show the way to calculate the dimension and to perform the multifractal analysis. Then, we introduce the canonical random Mandelbrot cascades (Theorem 3.1), solve the problem of non-degeneracy (Theorem 3.2) and its link with the existence of moments for the total mass of the cascade (Theorem 3.3). In Section 3.5, we prove that the Mandelbrot cascades are almost surely unidimensional and give the value of the dimension (Theorem 3.4). Finally, in a last section, we deal with the problem of multifractal analysis, and prove that for any value of the parameter β the Hausdorff dimension of the level set of points with Hölder exponent β is almost surely given by the multifractal formalism (Theorem 3.8). To obtain such a result, we use auxiliary cascades and we need to describe the simultaneous behavior of two cascades (Theorem 3.6) and to prove the existence of negative moments for the total mass (Proposition 3.5).

Part of this text is inspired by the founding article [15] by Kahane and Peyrière.

3.2 Binomial Cascades

In order to understand the multiplicative construction principle, we begin with a very simple and classical example, known as Bernoulli product, which can be regarded as an introduction to the following.

Let $\mathscr{F}_n$ be the family of dyadic intervals of the n^{th} generation on $[0,1)$, $0 < p < 1$ and define the measure m as follows. If $\varepsilon_1 \ldots \varepsilon_n$ are integers in $\{0,1\}$, and if

$$I_{\varepsilon_1\cdots\varepsilon_n} = \left[\sum_{i=1}^{n} \frac{\varepsilon_i}{2^i}, \sum_{i=1}^{n} \frac{\varepsilon_i}{2^i} + \frac{1}{2^n}\right) \in \mathscr{F}_n,$$

then

$$m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{S_n}(1-p)^{n-S_n}, \quad \text{where} \quad S_n = \varepsilon_1 + \cdots + \varepsilon_n \,. \tag{3.1}$$

The measure m is constructed using a multiplicative principle: if $I = I_{\varepsilon_1\cdots\varepsilon_n} \in \mathscr{F}_n$ and in $I' = I_{\varepsilon_1\cdots\varepsilon_n 0}$ and $I'' = I_{\varepsilon_1\cdots\varepsilon_n 1}$ are the two children of I in $\mathscr{F}_{n+1}$, then

$$m(I') = pm(I) \quad \text{and} \quad m(I'') = (1-p)m(I).$$

If $x \in [0,1)$, we can find $\varepsilon 1, \ldots, \varepsilon n, \cdots \in \{0,1\}$ uniquely determined and such that for any $n \geq 1$, $x \in I_{\varepsilon 1\ldots\varepsilon n}$. We also denote $I_{\varepsilon 1\ldots\varepsilon n} = I_n(x)$ and we observe that

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{S_n}{n}\log_2 p + \left(1 - \frac{S_n}{n}\right)\log_2(1-p)\right)$$

where $|I|$ is the length of the interval I. By the strong law of large numbers applied to the sequence (εn), we can then conclude that

$$\lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = h(p) \quad dm-\text{almost surely}$$

where $h(p) = -(p\log_2 p + (1-p)\log_2(1-p))$.

Using Billingsley's theorem (see, for example, [11]), it is then easy to conclude that

$$\dim_*(m) = \dim^*(m) = h(p)$$

where $\dim_*(m)$ and $\dim^*(m)$ are the lower and the upper dimension defined by

$$\begin{cases} \dim_*(m) = \inf(\dim(E)\ ;\ m(E) > 0) \\ \\ \dim^*(m) = \inf(\dim(E)\ ;\ m([0,1] \setminus E) = 0). \end{cases} \tag{3.2}$$

It means that the measure m is supported by a set of Hausdorff dimension $h(p)$ and that every set of dimension less that $h(p)$ is negligible. We say that the measure m is unidimensional with dimension $h(p)$.

If $\mathrm{Dim}(E)$ is the packing dimension of a set E and if

$$\begin{cases} \mathrm{Dim}_*(m) = \inf(\mathrm{Dim}(E)\ ;\ m(E) > 0) \\ \\ \mathrm{Dim}^*(m) = \inf(\mathrm{Dim}(E)\ ;\ m([0,1] \setminus E) = 0), \end{cases} \tag{3.3}$$

we can also conclude that

$$\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h(p).$$

3.2.1 Multifractal Analysis of Binomial Cascades

Binomial cascades are also known to be multifractal measures and it is easy to compute their multifractal spectrum. Let

$$E_\beta = \left\{ x\ ;\ \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = \beta \right\}.$$

The multifractal spectrum of the measure m is the function $\beta \mapsto \dim(E_\beta)$.

Recall that

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{S_n}{n} \log_2 p + \left(1 - \frac{S_n}{n}\right) \log_2(1-p) \right).$$

Without lost of generality, we can suppose that $1/2 \le p < 1$, so that $-\log_2 p \le -\log_2(1-p)$.

Suppose that $\beta \in [-\log_2 p, -\log_2(1-p)]$. We can find $\theta \in [0,1]$ such that

$$\beta = -(\theta \log_2 p + (1-\theta)\log_2(1-p)).$$

It follows that $E_\beta = \left\{ \frac{S_n}{n} \to \theta \right\}$ and we can conclude that

$$\dim(E_\beta) = -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) = h(\theta) := F(\beta) \tag{3.4}$$

where $F(\beta) = h\left(\frac{\beta + \log_2(1-p)}{\log_2(1-p) - \log_2 p} \right)$.

Suppose now that $\beta \notin [-\log_2 p, -\log_2(1-p)]$ and observe that

$$-\log_2(p) \le \frac{\log m(I_n(x))}{\log |I_n(x)|} \le -\log_2(1-p).$$

It follows that $E_\beta = \emptyset$ and $\dim(E_\beta) = -\infty$.

We can finally give the graph of the multifractal spectrum of the measure m (Fig. 3.1).

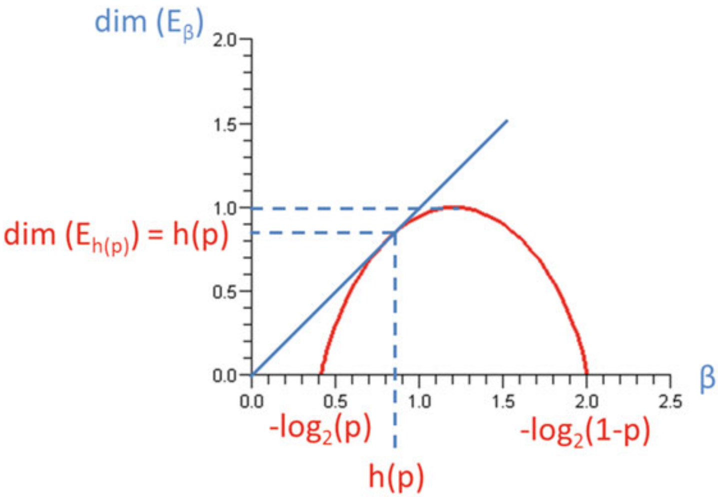

Fig. 3.1 The graph of the multifractal spectrum of the measure m. The dimension of the measure m is the abscissa of the point where the curve intersects the first bisector.

3.2.2 Binomial Cascades Satisfy the Multifractal Formalism

We can also rewrite formula (3.4) in the following way. If m_θ be the binomial cascade with parameter θ, the measure m_θ is supported by E_β and we have

$$\dim(E_\beta) = \dim(m_\theta) = h(\theta).$$

Moreover, if $q \in \mathbb{R}$ is such that

$$\theta = \frac{p^q}{p^q + (1-p)^q},$$

and if $I \in \mathscr{F}_n$, we have

$$\begin{aligned} m_\theta(I) &= \theta^{S_n}(1-\theta)^{n-S_n} \\ &= \frac{p^{qS_n}(1-p)^{q(n-S_n)}}{(p^q+(1-p)^q)^n} \\ &= m(I)^q |I|^{\tau(q)} \end{aligned}$$

where $\tau(q) = \log_2(p^q + (1-p)^q)$ is the structure function of the measure m at state q.

Finally, if we observe that $\beta = -(\theta \log_2 p + (1-\theta)\log_2(1-p)) = -\tau'(q)$, we can conclude that

$$\begin{aligned}\dim(E_\beta) &= -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) \\ &= -q\tau'(q) + \tau(q) \\ &= \tau^*(-\tau'(q)) \\ &= \tau^*(\beta)\end{aligned}$$

where $\tau^*(\beta) = \inf_t(t\beta + \tau(t))$ is the Legendre transform of τ.

We say that the measure m satisfies the multifractal formalism and that m_θ is a Gibbs measure at state q. Such a construction of an auxiliary cascade will be used in Section 3.7.

Remark 3.1. The new measure m_θ is obtained from m by changing the parameters $(p, 1-p)$ in $\left(\frac{p^q}{p^q+(1-p)^q}, \frac{(1-p)^q}{p^q+(1-p)^q}\right)$. The quantity $\frac{1}{p^q+(1-p)^q}$ is just the renormalization needed to ensure that the sum of the two parameters is equal to 1. A similar idea will be used to construct auxiliary Mandelbrot cascades (see the beginning of Section 3.7).

Remark 3.2. If m is a binomial cascade, we have

$$\sum_{I\in\mathscr{F}_{n+1}} m(I)^q = \sum_{I\in\mathscr{F}_n} p^q m(I)^q + (1-p)^q m(I)^q = (p^q + (1-p)^q) \sum_{I\in\mathscr{F}_n} m(I)^q.$$

Finally,

$$\log_2(p^q + (1-p)^q) = \limsup_{n\to+\infty} \frac{1}{n} \log_2\left(\sum_{I\in\mathscr{F}_n} m(I)^q\right)$$

which is the classical definition of the structure function τ (see Section 3.6).

3.2.3 Back to the Existence of Binomial Cascades

We want to finish this section with an elementary proposition which gives a rigorous proof of the existence of a measure m satisfying (3.1). Denote by λ the Lebesgue measure on $[0,1)$ and let

$$m_n = f_n d\lambda \quad \text{where} \quad f_n = 2^n \sum_{\varepsilon_1\cdots\varepsilon_n} p^{S_n}(1-p)^{n-S_n} 1\!\!1_{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

If $I = I_{\varepsilon 1\cdots\varepsilon j} \in \mathscr{F}_j$, we have

$$m_j(I) = p^{S_j}(1-p)^{j-S_j} = m_{j+1}(I) = \cdots = m_{j+k}(I) = \cdots$$

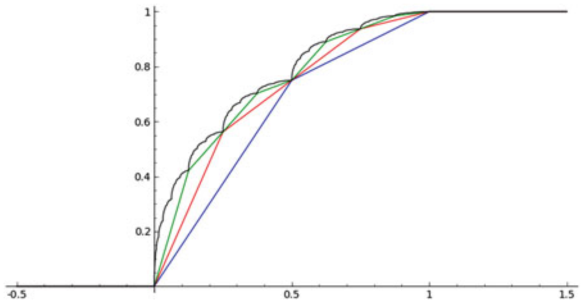

Fig. 3.2 The repartition function of the measures m_1, m_2, m_3, and m.

and the sequence $(m_n(I))_{n\geq 1}$ is convergent. We can then use the following elementary proposition (Fig. 3.2).

Proposition 3.1. *Let $(m_n)_{n\geq 1}$ be a sequence of finite Borel measures on $[0,1)$. Suppose that for any dyadic interval $I \in \bigcup_{j\geq 0}\mathscr{F}_j$, the sequence $(m_n(I))_{n\geq 1}$ is convergent. Then, the sequence $(m_n)_{n\geq 1}$ is weakly convergent to a finite Borel measure m.*

Remark 3.3. In Proposition 3.1, we can of course replace the family of dyadic intervals by the family of ℓ-adic intervals ($\ell \geq 2$). Proposition 3.1 will be used in Section 3.3 to prove the existence of Mandelbrot cascades.

Proof (Proof of Proposition 3.1). Observe that if f is a continuous function on $[0,1]$ and $\varepsilon > 0$, we can find a function φ which is a linear combination of functions $\mathbb{1}_I$ with $I \in \bigcup_{j\geq 0}\mathscr{F}_j$ and such that $\|f-\varphi\|_\infty \leq \varepsilon$. By the hypothesis, the sequence $\int \varphi(x)\,dm_n(x)$ is convergent and we have

$$\begin{aligned}\left|\int f\,dm_n - \int f\,dm_p\right| &\leq \left|\int \varphi\,dm_n - \int \varphi\,dm_p\right| + \|f-\varphi\|_\infty (m_n([0,1]) + m_p([0,1]))\\ &\leq \left|\int \varphi\,dm_n - \int \varphi\,dm_p\right| + C\varepsilon.\end{aligned}$$

It follows that the sequence $\int f(x)\,dm_n(x)$ is convergent. The conclusion is then a consequence of the Banach-Steinhaus theorem and of the Riesz representation theorem.

3.3 Canonical Mandelbrot Cascades: Construction and Non-degeneracy Conditions

3.3.1 Construction

In all the sequel, $\ell \geq 2$ is an integer and $\mathscr{F}_n$ is the set of ℓ-adic intervals of the n^{th} generation on $[0,1)$. We denote by $\mathscr{M}_n$ the set of words of length n written with the letters $0,\dots,\ell-1$ and $\mathscr{M}=\bigcup_n \mathscr{M}_n$. If $\varepsilon=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n$, let

$$I_{\varepsilon_1\cdots\varepsilon_n}=\left[\sum_{k=1}^n \frac{\varepsilon_k}{\ell^k},\sum_{k=1}^n \frac{\varepsilon_k}{\ell^k}+\frac{1}{\ell^n}\right)\in\mathscr{F}_n.$$

Let W be a non-negative random variable such that $E[W]=1$ and $(W_\varepsilon)_{\varepsilon\in\mathscr{M}}$ be a family of independent copies of W.

If λ is the Lebesgue on $[0,1]$, we can define the sequence of random measures by

$$m_n=f_n\lambda \quad \text{where} \quad f_n=\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}} W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\mathbb{1}_{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

The construction of the measure m_n uses a multiplicative principle and

$$m(I_{\varepsilon_1\cdots\varepsilon_n})=\ell^{-n}W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}.$$

We have the following existence theorem :

Theorem 3.1 (existence of m**).** *Almost surely, the sequence* $(m_n)_{n\geq 1}$ *is weakly convergent to a (random) measure* m*. The measure* m *is called the Mandelbrot cascade associated with the weight* W*.*

Remark 3.4. The condition $E[W]=1$ is a natural condition. Indeed if

$$Y_n:=m_n([0,1])=\int_0^1 f_n(t)\,d\lambda(t)=\ell^{-n}\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n},$$

then

$$\begin{aligned}E[Y_n]&=\ell^{-n}\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} E\left[W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_{n-1}}\right]E\left[W_{\varepsilon_1\cdots\varepsilon_n}\right]\\&=E[Y_{n-1}]\times E[W]\end{aligned}$$

and the condition $E[W]=1$ ensures that the expectation of the total mass does not go to 0 or to $+\infty$.

Proof. Let $\mathscr{A}_n$ be the σ-algebra generated by the W_ε, $\varepsilon\in\mathscr{M}_1\cup\cdots\cup\mathscr{M}_n$. Define $Y_n:=m_n([0,1])$. An easy calculation says

$$\begin{aligned}E[Y_{n+1}|\mathscr{A}_n]&=\ell^{-(n+1)}\sum_{\varepsilon_1\cdots\varepsilon_{n+1}\in\mathscr{M}_{n+1}} E[W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}W_{\varepsilon_1\cdots\varepsilon_{n+1}}|\mathscr{A}_n]\\&=\ell^{-(n+1)}\sum_{\varepsilon_1\cdots\varepsilon_{n+1}\in\mathscr{M}_{n+1}} W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}E[W_{\varepsilon_1\cdots\varepsilon_{n+1}}]\\&=Y_n\end{aligned}$$

and the sequence Y_n is a non-negative martingale. So it is almost surely convergent.

More generally, if $I = I_{\alpha_1 \cdots \alpha_k} \in \mathscr{F}_k$,

$$m_{k+n}(I) = \ell^{-(k+n)} \sum_{\varepsilon_{k+1} \cdots \varepsilon_{k+n} \in \mathscr{M}_n} W_{\alpha_1} \cdots W_{\alpha_1 \cdots \alpha_k} W_{\alpha_1 \cdots \alpha_k \varepsilon_{k+1}} \cdots W_{\alpha_1 \cdots \alpha_k \varepsilon_{k+1} \cdots \varepsilon_{k+n}}$$

and a similar calculation says that $m_{k+n}(I)$ is a non-negative martingale. Finally, for any $I \in \bigcup_{k \geq 0} \mathscr{F}_k$ the random quantity $m_n(I)$ is almost surely convergent (Fig. 3.3).

Observing that the set $\bigcup_{k \geq 0} \mathscr{F}_k$ is countable, we can also say that almost surely, for any $I \in \bigcup_{k \geq 0} \mathscr{F}_k$, $m_n(I)$ is convergent and the conclusion is a consequence of Proposition 3.1.

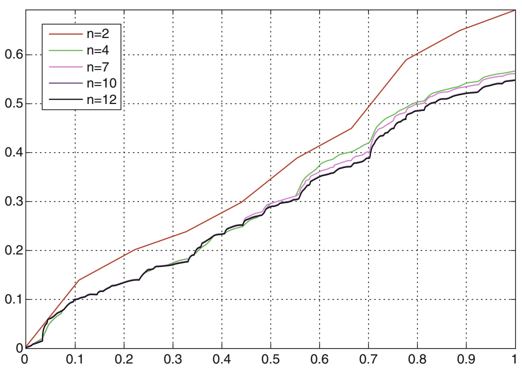

Fig. 3.3 The repartition function of the random measures m_2, m_4, m_7, m_{10}, and m_{12} (from [9]).The total mass is not equal to 1.

3.3.2 Examples

3.3.2.1 Birth and Death Processes

We suppose in this example that the random variable W only takes the value 0 and another positive value. Let $p = 1 - P[W = 0]$. To ensure that $E[W] = 1$ we need to take $P\left[W = \frac{1}{p}\right] = p$. When $m \neq 0$, its support is a random Cantor set.

3.3.2.2 Log-Normal Cascades

This is the case where W is a log-normal random variable, that is, $W = e^X$ where X follows a normal distribution with expectation m and variance σ^2. An easy calculation says that

$$
\begin{aligned}
E\left[e^X\right] &= \int e^x\, e^{-(x-m)^2/2\sigma^2}\,\frac{dx}{\sigma\sqrt{2\pi}}\\
&= \int e^{(m+\sigma u)}\, e^{-u^2/2}\,\frac{du}{\sqrt{2\pi}}\\
&= \int e^{-(u-\sigma)^2/2}\, e^{m+\sigma^2/2}\,\frac{du}{\sqrt{2\pi}}\\
&= e^{m+\sigma^2/2}.
\end{aligned}
$$

In order to have $E[W]=1$ we need to choose $m=-\sigma^2/2$. In other words,

$$W = e^{\sigma N-\sigma^2/2}$$

where N follows a standard normal distribution.

3.3.3 The Fundamental Equations

Define $Y_n = m_n([0,1])$ as above. Then,

$$
\begin{aligned}
Y_{n+1} &= \ell^{-(n+1)} \sum_{\varepsilon_1\cdots\varepsilon_{n+1}} W_{\varepsilon_1} W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_{n+1}}\\
&= \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j \left[\ell^{-n}\sum_{\varepsilon_2\cdots\varepsilon_{n+1}} W_{j\varepsilon_2}\cdots W_{j\cdots\varepsilon_{n+1}}\right]
\end{aligned}
\tag{3.5}
$$

and the sequence (Y_n) is a solution in law of the equation

$$Y_{n+1} = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_n(j). \tag{3.6}$$

where the $Y_n(0),\dots,Y_n(\ell-1)$ are independent copies of Y_n, and are independent to $W_0,\dots,W_{\ell-1}$.

Taking the limit in the equality (3.5), the total mass $Y_\infty = m([0,1]$ is also a solution in law of the equation

$$Y_\infty = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_\infty(j) \tag{3.7}$$

where $Y_\infty(0),\dots,Y_\infty(\ell-1)$ are independent copies of Y_∞, and are independent to $W_0,\dots,W_{\ell-1}$.

Equations (3.6) and (3.7) are called the fundamental equations and will be very useful in the following.

3.3.4 Non-degeneracy

As proved in Theorem 3.1, the sequence $Y_n = m_n([0,1])$ is a non-negative martingale and we only know in the general case that $E[Y_\infty] \leq 1$. In particular, the situation where $E[Y_\infty] = 0$ is possible and is called the degenerate case. The first natural problem related to the random measure m is then to find conditions that ensure that m is not almost surely equal to 0 (i.e., $E[Y_\infty] \neq 0$). An abstract answer is given by an equi-integrability property. We will see further a more concrete necessary and sufficient condition (Theorem 3.2) and more concrete sufficient conditions (Proposition 3.4 and Theorem 3.3).

Proposition 3.2. *Let m be a Mandelbrot cascade associated with a weight W. Denote as before $Y_n = m_n([0,1])$ and $Y_\infty = m([0,1])$. The following are equivalent*

1. $E[Y_\infty] = 1$
2. $E[Y_\infty] > 0$ *(i.e., $P[m([0,1]) \neq 0] > 0$)*
3. *The martingale (Y_n) is equi-integrable.*

In that case, we say that the Mandelbrot cascade m is non-degenerate.

Proof. Suppose that 2 is true. Considering $Z = \frac{Y_\infty}{E[Y_\infty]}$, it follows that the fundamental equation

$$Z = \frac{1}{\ell} \sum_{j=0}^{\ell-1} W_j Z(j)$$

has a solution satisfying $E[Z] = 1$.

Iterating the fundamental equation, we get

$$Z = \frac{1}{\ell^n} \sum_{\varepsilon_1 \cdots \varepsilon_n \in \mathscr{M}_n} W_{\varepsilon_1} \cdots W_{\varepsilon_1 \cdots \varepsilon_n} Z(\varepsilon_1 \cdots \varepsilon_n)$$

in which the $Z(\varepsilon 1 \cdots \varepsilon n)$ are independent copies of Z, independent to the W_ε. Let $\mathscr{A}_n$ be again the σ-algebra generated by the W_ε, $\varepsilon \in \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_n$. We get

$$E[Z|\mathscr{A}_n] = \frac{1}{\ell^n} \sum_{\varepsilon_1 \cdots \varepsilon_n \in \mathscr{M}_n} W_{\varepsilon_1} \cdots W_{\varepsilon_1 \cdots \varepsilon_n} E[Z(\varepsilon_1 \cdots \varepsilon_n)] = Y_n$$

and the martingale (Y_n) is equi-integrable.

The proof of $3 \Rightarrow 1$ is elementary. Indeed, if we suppose that the martingale (Y_n) is equi-integrable, it converges almost surely and in L^1 to its limit Y_∞. In particular, $E[Y_\infty] = \lim_{n\to\infty} E[Y_n] = 1$.

Remark 3.5. In fact, the proof of Proposition 3.2 says that the condition of non-degeneracy of the cascade m is equivalent to the existence of a non-negative solution Z satisfying $E[Z] = 1$ for the fundamental equation

$$Z = \frac{1}{\ell} \sum_{j=0}^{\ell-1} W_j Z(j). \tag{3.8}$$

Remark 3.6. Equation (3.8) may have non-integrable solutions. For example, if $\ell = 2$ and $W = 1$, equation (3.8) becomes

$$Z = \frac{1}{2}(Z(1) + Z(2)).$$

If $Z(1)$ et $Z(2)$ are two independent Cauchy variables (with density $\frac{dz}{\pi(1+z^2)}$), then Z is also a Cauchy variable.

In the non-degenerate case, we only know that $P[m \neq 0] > 0$ almost surely. A natural question is then to ask if $P[m \neq 0] = 1$ almost surely. The answer to this question is easy.

Proposition 3.3. *Suppose that the Mandelbrot cascade m is non-degenerate. Then,*

$$P[m \neq 0] = 1 \quad \textit{if and only if} \quad P[W = 0] = 0.$$

Proof. Suppose that (Y_n) is equi-integrable. Let us write again the fundamental equation

$$Y_\infty = \frac{1}{\ell} \sum_{j=0}^{\ell-1} W_j Y_\infty(j).$$

Then

$$\begin{aligned} P[Y_\infty = 0] &= P\left[W_0 Y_\infty(0) = 0 \text{ and } \cdots \text{ and } W_{\ell-1} Y_\infty(\ell-1) = 0\right] \\ &= P[WY_\infty = 0]^\ell \\ &= (1 - P[W \neq 0 \text{ and } Y_\infty \neq 0])^\ell \end{aligned}$$

If $r = P[W = 0]$, it follows that $P[Y_\infty = 0]$ is a fixed point of the function

$$f(x) = (r + (1-r)x)^\ell.$$

We know that $P[Y_\infty = 0] < 1$. The second fixed point of the function f is equal to 0 if and only if $r = 0$. The conclusion follows (Fig. 3.4).

In the L^2 case it is easy to obtain a condition on the second order moment which gives non-degeneracy.

Proposition 3.4. *Suppose that $E[W^2] < +\infty$. The following are equivalent*

1. $E[W^2] < \ell$
2. *The sequence (Y_n) is bounded in L^2*
3. $0 < E[Y_\infty^2] < +\infty$.

In particular, if 1 is true, the sequence (Y_n) is equi-integrable and the cascade m is non-degenerate.

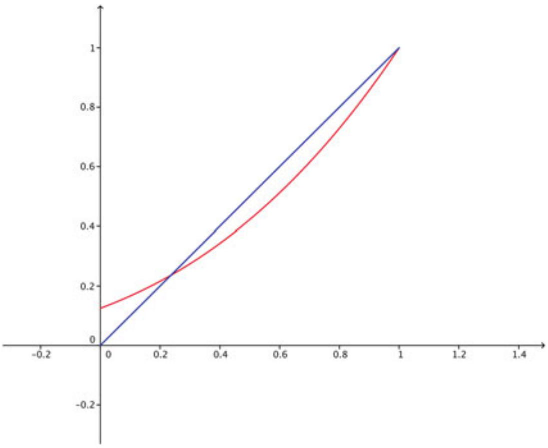

Fig. 3.4 The function f has two fixed points.

Proof. Let us write the fundamental equation

$$Y_{n+1} = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_n(j).$$

We get

$$\begin{aligned} E[Y_{n+1}^2] &= \frac{1}{\ell^2}\left(\sum_{j=0}^{\ell-1} E[(W_j^2 Y_n(j))^2] + \sum_{i\neq j} E[W_i Y_n(i) W_j Y_n(j)]\right) \\ &= \frac{1}{\ell}E[W^2]E[Y_n^2] + \frac{1}{\ell^2}\times \ell(\ell-1) \end{aligned}$$

It follows that the sequence $(E[Y_n^2])$ is bounded if and only if the common ratio $\frac{1}{\ell}E[W^2]$ is lower than 1. So 1 is equivalent to 2.

2. $\Rightarrow$ 3. Suppose that the sequence (Y_n) is bounded in L^2. We know that the martingale (Y_n) converges in L^2. In particular

$$E[Y_\infty^2] = \lim_{n\to+\infty} E[Y_n^2] < +\infty.$$

Moreover the sequence (Y_n^2) is a submartingale and the sequence $(E[Y_n^2])$ is non-decreasing. It follows that $E[Y_\infty^2] > 0$, which gives 3.

3. $\Rightarrow$ 1. Suppose that $0 < E[Y_\infty^2] < +\infty$. According to Proposition 3.2, the martingale (Y_n) is non-degenerate. In particular, $E[Y_\infty] = 1$. The fundamental equation says that

$$Y_\infty = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_\infty(j).$$

It follows that

$$E[Y_\infty^2] = \frac{1}{\ell^2}\left(\sum_{j=0}^{\ell-1} E[(W_j^2 Y_\infty(j))^2] + \sum_{i\neq j} E[W_i Y_\infty(i) W_j Y_\infty(j)]\right)$$
$$= \frac{1}{\ell}E[W^2]E[Y_\infty^2] + \frac{1}{\ell^2}\times \ell(\ell-1)$$

so that

$$\left(\ell - E[W^2]\right)E[Y_\infty^2] = \ell - 1.$$

In particular, $E[W^2] < \ell$.

A generalization of Proposition 3.4 in the case where the weight W admits an L^q moment is possible. This is the object of Section 3.4. Nevertheless, we can also give a characterization on the non-degeneracy of the cascade m. It is given in terms of the $L\log L$ moment of the weight W.

Theorem 3.2 (Kahane, 1976, [15]). *Let m be a Mandelbrot cascade associated with a weight W. The following are equivalent*

1. *The cascade m is non-degenerate*
2. *The martingale (Y_n) is equi-integrable*
3. *$E[W\log W] < \log \ell$.*

We begin with a geometric interpretation of the condition $E[W\log W] < \log\ell$. Let us introduce the structure function τ, which is defined by

$$\tau(q) = \log_\ell E\left[\sum_{j=0}^{\ell-1}\left[\frac{1}{\ell}W_j\right]^q\right] = \log_\ell\left(E[W^q]\right) - (q-1). \tag{3.9}$$

Such a formula makes sense when $0 \le q \le 1$ (and perhaps for other values of q) and we always use the convention $0^q = 0$. In particular,

$$\tau(0) = 1 + \log_\ell(P[W \neq 0])$$

which will be seen as the almost sure Hausdorff dimension of the closed support of the measure m.

The function τ is continuous and convex on $[0,1]$ and we will show that

$$\tau'(1^-) = E[W\log_\ell W] - 1 \le +\infty.$$

It follows that Condition 3 in Theorem 3.2 is equivalent to $\tau'(1^-) < 0$.

Set $\phi(q) = E[W^q]$. In order to prove that $\tau'(1^-) = E[W\log_\ell W] - 1$, we have to understand why we can write $\phi'(1^-) = E[W\log W]$, with a possible value equal to $+\infty$. Indeed, using the dominated convergence theorem, we have $\phi'(q) = E[W^q \log W]$ when $0 \le q < 1$. On one hand, the convexity of the function ϕ allows us to write

$$\lim_{q \to 1^-} \phi'(q) = \phi'(1^-) \le +\infty.$$

On the other hand,

$$\phi'(q) = E[W^q \log W] = E[W^q \log W \, \mathbb{1}_{\{W<1\}}] + E[W^q \log W \, \mathbb{1}_{\{W\ge 1\}}].$$

The non-negative quantity $E[W^q \log W \, \mathbb{1}_{\{W\ge 1\}}]$ increases to $E[W \log W \, \mathbb{1}_{\{W\ge 1\}}]$ and by the dominated convergence theorem, the quantity $E[W^q \log W \, \mathbb{1}_{\{W<1\}}]$ goes to $E[W \log W \, \mathbb{1}_{\{W<1\}}]$. The formula $\phi'(1^-) = E[W \log W]$ follows.

Proof (Proof of Theorem 3.2). According to Proposition 3.2, we just have to prove that Conditions 2 and 3 are equivalent.

Step 1. $\tau'(1^-) \le 0$ *is a necessary condition.*

Suppose that the sequence (Y_n) is equi-integrable. Then, the fundamental equation

$$Z = \frac{1}{\ell} \sum_{j=0}^{\ell-1} W_j Z(j)$$

has a non-negative solution with expectation equal to 1. If $0 < q \le 1$, the function $x \mapsto x^q$ is subadditive (that is, satisfies $(a+b)^q \le a^q + b^q$). We get

$$E\left[\ell^q Z^q\right] \le \sum_{j=0}^{\ell-1} E[W_j^q Z(j)^q] = \ell E[W^q] E[Z^q].$$

Observe that $E[Z^q] > 0$, so that

$$\ell^q \le \ell E[W^q].$$

Finally, $\tau(q) \ge 0$ if $q \le 1$ and $\tau'(1^-) \le 0$.

Step 2. *More precisely,* $\tau'(1^-) < 0$ *is a necessary condition.*

We have to improve the previous result. We need a lemma which gives a more precise estimate than the subadditivity of the function $x \mapsto x^q$.

Lemma 3.1. *If* $0 < q < 1$ *and if* $0 < y \le x$*, then* $(x+y)^q \le x^q + q y^q$.

Proof. Using homogeneity, we may assume that $y = 1$ and $x \ge 1$. The inequality $(x+1)^q - x^q \le q$ is then an easy consequence of the mean value theorem.

We also need the following elementary lemma on random variables.

Lemma 3.2. *Let* X *and* X' *be two non-negative i.i.d. random variables such that* $E[X] > 0$*. There exists* $\delta > 0$ *such that for any* $q \in [0,1]$, $E[X^q \mathbb{1}_{X' \ge X}] \ge \delta E[X^q]$.

Proof. We claim that for any $q \in [0,1]$, $E[X^q \mathbb{1}_{X' \ge X}] > 0$. Indeed, if $E[X^q \mathbb{1}_{X' \ge X}] = 0$ for some q, then X is almost surely equal to 0 on the set $\{X' \ge X\}$. By symmetry,

X' is almost surely equal to 0 on the set $\{X \geq X'\}$. Then $XX' = 0$ almost surely, which is in contradiction with $E[XX'] = E[X]E[X'] > 0$. Moreover, the functions $q \mapsto E[X^q \mathbb{1}_{X' \geq X}]$ and $q \mapsto E[X^q]$ are continuous on $[0,1]$ and the conclusion follows.

We can now prove that $\tau'(1^-) < 0$ is a necessary condition. Let

$$A = \{W_1 Z(1) \geq W_0 Z(0)\}.$$

Using subadditivity of $x \mapsto x^q$ and Lemma 3.1, we have :

$$\begin{cases} (\ell Z)^q \leq \sum_{j=0}^{\ell-1} (W_j Z(j))^q \\ (\ell Z)^q \leq q(W_0 Z(0))^q + \sum_{j=1}^{\ell-1} (W_j Z(j))^q \qquad \text{on } A. \end{cases}$$

Then,

$$\begin{aligned} E\left[(\ell Z)^q\right] &= E\left[(\ell Z)^q \mathbb{1}_A\right] + E\left[(\ell Z)^q \mathbb{1}_{A^c}\right] \\ &\leq qE\left[(W_0 Z(0))^q \mathbb{1}_A\right] + \sum_{j=1}^{\ell-1} E\left[(W_j Z(j))^q \mathbb{1}_A\right] + \sum_{j=0}^{\ell-1} E\left[(W_j Z(j))^q \mathbb{1}_{A^c}\right] \\ &= (q-1)E\left[(W_0 Z(0))^q \mathbb{1}_A\right] + \ell E[W^q]E[Z^q] \\ &\leq (q-1)\delta E[W^q]E[Z^q] + \ell E[W^q]E[Z^q]. \end{aligned}$$

We get

$$\ell^{1-q} E[W^q] \geq \frac{1}{1 + (q-1)\frac{\delta}{\ell}}$$

so that

$$\tau(q) \geq -\log_\ell \left(1 + (q-1)\frac{\delta}{\ell}\right).$$

Finally,

$$\tau'(1^-) \leq -\frac{\delta}{\ell \log \ell} < 0.$$

Step 3. $\tau'(1^-) < 0$ *is a sufficient condition.*

We suppose that $E[W \log W] < \log \ell$ (i.e., $\tau'(1^-) < 0$) and, according to Proposition 3.2, we want to prove that $E[Y_\infty] > 0$. Now, we need a precise lower bound of quantities such as $\left(\sum_{j=1}^{\ell} x_j\right)^q$. We will use the following lemma.

Lemma 3.3. *If* $x_1, \ldots, x_\ell \geq 0$*, and if* $0 < q \leq 1$*, then*

$$\left(\sum_{j=1}^{\ell} x_j\right)^q \geq \sum_{j=1}^{\ell} x_j^q - 2(1-q)\sum_{i<j}(x_i x_j)^{q/2}. \tag{3.10}$$

Suppose first that the lemma is true and let us write again the fundamental equation

$$\ell Y_n = \sum_{j=0}^{\ell-1} W_j Y_{n-1}(j).$$

Lemma 3.3 ensures that

$$(\ell Y_n)^q \geq \sum_{j=0}^{\ell-1}(W_j Y_{n-1}(j))^q - 2(1-q)\sum_{i<j}(W_i Y_{n-1}(i) W_j Y_{n-1}(j))^{q/2}.$$

Taking the expectation and using that Y_n^q is a supermartingale, we get

$$\begin{aligned}\ell^q E[Y_n^q] &\geq \ell E[W^q] E\left[Y_{n-1}^q\right] - \ell(\ell-1)(1-q)E[W^{q/2}]^2 \times E\left[Y_{n-1}^{q/2}\right]^2 \\ &\geq \ell E[W^q] E\left[Y_n^q\right] - \ell(\ell-1)(1-q)E[W^{q/2}]^2 \times E\left[Y_{n-1}^{q/2}\right]^2.\end{aligned}$$

Finally,

$$\begin{aligned}E\left[Y_n^q\right](\ell^{\tau(q)} - 1) &= E\left[Y_n^q\right](\ell^{1-q}E[W^q] - 1) \\ &\leq \ell^{1-q}(\ell-1)(1-q)E\left[Y_{n-1}^{q/2}\right]^2 \times E\left[W^{q/2}\right]^2 \\ &\leq \ell^{1-q}(\ell-1)(1-q)E\left[Y_{n-1}^{q/2}\right]^2 \times E\left[W^q\right].\end{aligned}$$

Dividing by $1-q$ and taking the limit when q goes to 1^-, we get

$$1 \times (-\tau'(1^-) \times \log \ell) \leq (\ell-1)E\left[Y_{n-1}^{1/2}\right]^2 \times 1$$

which gives that $E\left[Y_{n-1}^{1/2}\right] \geq C > 0$. Observing that the supermartingale $\left(Y_n^{1/2}\right)$ converges almost surely to $Y_\infty^{1/2}$ and is bounded in L^2, we conclude that $\left(Y_n^{1/2}\right)$ is equi-integrable and converges in L^1. In particular,

$$E[Y_\infty^{1/2}] = \lim_{n\to+\infty} E[Y_n^{1/2}] \geq C.$$

So $E[Y_\infty] > 0$ and the cascade m is non-degenerate.

Let us now finish this part with the proof of Lemma 3.3. Suppose first that $\ell = 2$. By homogeneity the inequality is equivalent to

$$\left(x+x^{-1}\right)^q \geq x^q + x^{-q} - 2(1-q)$$

for any $x > 0$. Let

$$\varphi(x) = x^q + x^{-q} - \left(x+x^{-1}\right)^q.$$

If $0 < x \leq 1$, we have

$$\begin{aligned} \varphi'(x) &= qx^{-(q+1)}\left[x^{2q} - 1 + \left(1-x^2\right)\left(1+x^2\right)^{q-1}\right] \\ &\geq qx^{-(q+1)}\left[x^{2q} - 1 + \left(1-x^2\right)\left(1+(q-1)x^2\right)\right] \\ &= qx^{-(q+1)}\left[x^{2q} + (q-2)x^2 - (q-1)x^4\right]. \end{aligned}$$

By studying the function $\psi(y) = y^q + (q-2)y - (q-1)y^2$, it is then easy to see that $\psi(y) \geq 0$ for any $y \in [0,1]$.

Finally, for any $x > 0$,

$$\varphi(x) = \varphi(x^{-1}) \leq \varphi(1) = 2 - 2^q \leq 2\ln 2(1-q) \leq 2(1-q),$$

and the proof is done in the case $\ell = 2$.

The general case is easily obtained by induction on ℓ, using once again that the function $x \mapsto x^{q/2}$ is subadditive if $0 < q < 1$.

Remark 3.7. In fact, the proof of Lemma 3.3 says that the constant $-2(1-q)$ in (3.10) can replaced by $-2\ln 2(1-q)$ which is the optimal one.

Example 3.1 (Birth and death processes). Suppose that the law of the random variable W is given by $dP_W = (1-p)\delta_0 + p\delta_{\frac{1}{p}}$. Then

$$E\left[W^q\right] = 0P[W=0] + \left(\frac{1}{p}\right)^q P\left[W = \frac{1}{p}\right] = p^{1-q}$$

and

$$\tau(q) = \log_\ell(E[W^q]) - (q-1) = (1-q) \times (1 + \log_\ell p).$$

The cascade is non-degenerate if and only if $p > 1/\ell$, that is, if and only if $P[W = 0] < 1 - \frac{1}{\ell}$. In that case, the box dimension of the closed support of the measure m is almost surely $d = \tau(0) = 1 + \log_\ell p$ on the set $\{m \neq 0\}$.

Example 3.2 (Log-normal cascades). Suppose that

$$W = e^{\sigma N - \sigma^2/2}$$

where N follows a standard normal distribution.

$$E[W^q] = \int e^{q(\sigma x - \sigma^2/2)} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$
$$= \int e^{-(x-q\sigma)^2/2} e^{q^2\sigma^2/2} e^{-q\sigma^2/2} \frac{dx}{\sqrt{2\pi}}$$
$$= e^{q^2\sigma^2/2} e^{-q\sigma^2/2}$$

and

$$\tau(q) = \log_\ell(E[W^q]) - (q-1) = \frac{\sigma^2}{2\ln\ell}(q^2 - q) - (q-1).$$

The cascade is non-degenerate if and only if $\sigma^2 < 2\log\ell$.

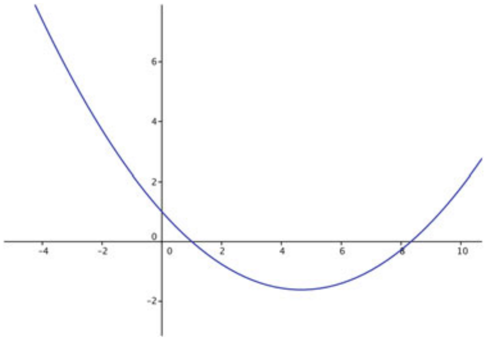

Fig. 3.5 The graph of the structure function τ for a non-degenerate log-normal cascade.

3.4 On the Existence of Moments for the Random Variable Y_∞

In Proposition 3.4, we obtained a necessary and sufficient condition for the martingale (Y_n) to be bounded in L^2. This condition can be generalized in the following way.

Theorem 3.3 (Kahane, 1976, [15]). *Let* $q > 1$. *Suppose that* $E[W^q] < +\infty$. *The following are equivalent*

1. $E[W^q] < \ell^{q-1}$ *(i.e.,* $\tau(q) < 0$*)*
2. *The sequence* (Y_n) *is bounded in* L^q
3. $0 < E[Y_\infty^q] < +\infty$.

In particular, if 1 is true, the sequence (Y_n) *is equi-integrable and the cascade m is non-degenerate.*

Remark 3.8. The condition $E[W^q] < \ell^{q-1}$ is equivalent to $\tau(q) < 0$. The graph of the function τ allows us to determine the set of values of $q > 1$ such that (Y_n) is bounded in L^q (see Figure 3.5 for the case of log-normal cascades).

Proof (Proof of Theorem 3.3).

2. $\Rightarrow$ 3. If (Y_n) is bounded in L^q, the martingale (Y_n) converges in L^q. In particular,

$$E[Y_\infty^q] = \lim_{n\to+\infty} E[Y_n^q] < +\infty.$$

Moreover the sequence (Y_n^q) is a submartingale and the sequence $(E[Y_n^q])$ is non-decreasing. It follows that $E[Y_\infty^q] > 0$, which gives 3.

3. $\Rightarrow$ 1. Suppose that $0 < E[Y_\infty^q] < +\infty$ and write the fundamental equation

$$(\ell Y_\infty)^q = \left(\sum_{j=0}^{\ell-1} W_j Y_\infty(j)\right)^q. \tag{3.11}$$

Recall that the function $x \mapsto x^q$ is superadditive. More precisely, $(a+b)^q \geq a^q + b^q$ with equality if and only if $ab = 0$.

Taking the expectation in (3.11) we get

$$E[\ell^q Y_\infty^q] \geq \sum_{j=0}^{\ell-1} E\left[(W_j Y_\infty(j))^q\right] = \ell E[W^q] E[Y_\infty^q]$$

and the equality case would imply that $\prod_{j=0}^{\ell-1}(W_j Y_\infty(j)) = 0$ almost surely, which is impossible by independence. In particular, $E[W^q] < \ell^{q-1}$.

1. $\Rightarrow$ 2. This is the difficult part of the theorem. Let us begin with the easier case $1 < q \leq 2$. Recall once again the fundamental equation

$$\ell Y_{n+1} = \sum_{j=0}^{\ell-1} W_j Y_n(j).$$

The function $x \mapsto x^{q/2}$ is subadditive so that

$$\begin{aligned}(\ell Y_{n+1})^q &\leq \left(\sum_{j=0}^{\ell-1} (W_j Y_n(j))^{q/2}\right)^2 \\ &= \sum_{j=0}^{\ell-1} (W_j Y_n(j))^q + \sum_{i\neq j} (W_i Y_n(i))^{q/2} (W_j Y_n(j))^{q/2}.\end{aligned}$$

Taking the expectation, and using that (Y_n^q) is a submartingale, we get

$$\begin{aligned}\ell^q E\left[Y_{n+1}^q\right] &\leq \ell E\left[Y_n^q\right] E\left[W^q\right] + \ell(\ell-1) E[W^{q/2}]^2 E\left[Y_n^{q/2}\right]^2 \\ &\leq \ell E\left[Y_n^q\right] E\left[W^q\right] + \ell(\ell-1) E[W]^q E[Y_n]^q \\ &= \ell E\left[Y_{n+1}^q\right] E\left[W^q\right] + \ell(\ell-1).\end{aligned}$$

Finally,

$$E[Y_{n+1}^q] \le \frac{\ell - 1}{\ell^{q-1} - E[W^q]}. \tag{3.12}$$

Suppose now that $k < q \le k+1$ where $k \ge 2$ is an integer and write

$$(\ell Y_{n+1})^q \le \left(\sum_{j=0}^{\ell-1} (W_j Y_n(j))^{q/(k+1)} \right)^{k+1} = \sum_{j=0}^{\ell-1} (W_j Y_n(j))^q + T$$

where the quantity T is a sum of $\ell^{k+1} - \ell$ terms of the form

$$(W_{j_1} Y_n(j_1))^{\alpha_1 q/(k+1)} \times \cdots \times (W_{j_p} Y_n(j_p))^{\alpha_p q/(k+1)}$$

with $p \ge 2$ and $\alpha_1 + \cdots + \alpha_p = k+1$. The expectation of such a term satisfies

$$\begin{aligned} & E\left[(W_{j_1} Y_n(j_1))^{\alpha_1 q/(k+1)} \times \cdots \times (W_{j_p} Y_n(j_p))^{\alpha_p q/(k+1)} \right] \\ \le\; & E\left[(W_{j_1} Y_n(j_1))^k \right]^{\alpha_1 q/k(k+1)} \times \cdots \times E\left[(W_{j_p} Y_n(j_p))^k \right]^{\alpha_p q/k(k+1)} \\ =\; & \left(E\left[W^k \right] E\left[Y_n^k \right] \right)^{q/k} \end{aligned}$$

so that

$$\ell^q E\left[Y_{n+1}^q \right] \le \ell E\left[Y_n^q \right] E\left[W^q \right] + (\ell^{k+1} - \ell) \left(E\left[W^k \right] E\left[Y_n^k \right] \right)^{q/k}.$$

Using that (Y_n^q) is a submartingale, we get

$$E\left[Y_{n+1}^q \right] (\ell^{q-1} - E[W^q]) \le (\ell^k - 1) \left(E[W^k] E\left[Y_n^k \right] \right)^{q/k} \tag{3.13}$$

which is the generalization of (3.12). It follows that (Y_n) is bounded in L^q as soon as (Y_n) is bounded in L^k.

Let us finally observe that the hypothesis $E[W^q] < \ell^{q-1}$ (i.e., $\tau(q) < 0$) implies that $E[W^t] < \ell^{t-1}$ (i.e., $\tau(t) < 0$) for any t such that $1 < t < q$. Replacing q by $j+1$ in (3.13), we also have

$$E\left[Y_{n+1}^{j+1} \right] (\ell^j - E[W^{j+1}]) \le (\ell^j - 1) \left(E[W^j] E\left[Y_n^j \right] \right)^{q/j}$$

for any integer j such that $2 \le j < k$. Step by step we get that (Y_n) is bounded in L^2, $L^3, \ldots, L^k, L^q$.

3.5 On the Dimension of Non-degenerate Cascades

The Mandelbrot cascade is almost surely a unidimensional measure as was proved by Peyrière in [15].

Theorem 3.4 (Peyrière, 1976, [15]). *Suppose that* $0 < E[Y_\infty \log Y_\infty] < +\infty$. *Then, almost surely,*

$$\lim_{n\to+\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = 1 - E[W \log_\ell W] \quad dm - \text{almost everywhere}.$$

Let us recall that it is possible that $m = 0$ with positive probability. So, the good way to rewrite Theorem 3.4 is

Corollary 3.1. *Suppose that* $0 < E[Y_\infty \log Y_\infty] < +\infty$. *Almost surely on* $\{m \neq 0\}$ *we have :*

1. *There exists a Borel set E such that*

$$\dim(E) = 1 - E[W \log_\ell W] \quad \text{and} \quad m([0,1] \setminus E) = 0$$

2. *If* $\dim(F) < 1 - E[W \log_\ell W]$, *then* $m(F) = 0$.

It follows that

$$\dim_*(m) = \dim^*(m) = 1 - E[W \log_\ell W]$$

where $\dim_*(m)$ *and* $\dim^*(m)$ *are, respectively, the lower and the upper dimension of the measure m as defined on* (3.2).
The measure m is unidimensional with dimension

$$\dim(m) = 1 - E[W \log_\ell W].$$

Remark 3.9. The condition $0 < E[Y_\infty \log Y_\infty] < +\infty$ is stronger than $E[W \log W] < \log \ell$ which ensures the non-degeneracy of the cascade m. Indeed, suppose that $0 < E[Y_\infty \log Y_\infty] < +\infty$ and observe that the function $t \mapsto t \log t$ is superadditive. More precisely, $(a+b)\log(a+b) \geq a \log a + b \log b$ with equality if and only if $ab = 0$. The fundamental equation implies that

$$\ell Y_\infty \log(\ell Y_\infty) \geq \sum_{j=0}^{\ell-1} (W_j Y_\infty(j)) \log(W_j Y_\infty(j)).$$

Taking the expectation,

$$E[\ell Y_\infty \log(\ell Y_\infty)] \geq \ell E[(W Y_\infty) \log(W Y_\infty)]$$

and the equality case would imply that $\prod_{j=0}^{\ell-1}(W_j Y_\infty(j)) = 0$ almost surely, which is impossible by independence.

Finally,

$$E[Y_\infty \log(\ell Y_\infty)] > E[W \log W]E[Y_\infty] + E[Y_\infty \log Y_\infty]E[W]$$

so that

$$E[Y_\infty]\log \ell > E[W \log W]E[Y_\infty].$$

It follows that $E[W \log W] < \log \ell$ and the cascade is non-degenerate.

Remark 3.10. Under the hypothesis of Theorem 3.4 and Corollary 3.1, we have the following relation:

$$\dim(m) = 1 - E[W \log_\ell W] = -\tau'(1^-).$$

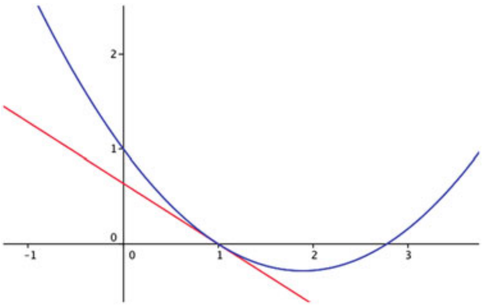

Fig. 3.6 The graph of the structure function τ and its tangente at point 1. The quantity $\dim(m)$ is the intercept of the tangente.

In particular, $0 < \dim(m) \leq 1$ almost surely on the event $\{m \neq 0\}$ (Fig. 3.6).

Example 3.3 (Birth and death processes). Suppose that $dP_W = (1-p)\delta_0 + p\delta_{\frac{1}{p}}$ with $p > 1/\ell$. Then,

$$\tau(q) = (1-q) \times (1 + \log_\ell p) \qquad \text{and} \qquad \dim(m) = 1 + \log_\ell p$$

almost surely on the event $\{m \neq 0\}$.

Let $\mathscr{M}_n^*$ be the set of words $\varepsilon \in \mathscr{M}_n$ such that $m(I_\varepsilon) > 0$. The closed support of m is nothing else but the Cantor set

$$\operatorname{supp}(m) = K = \bigcap_{n \geq 1} \bigcup_{\varepsilon \in \mathscr{M}_n^*} \overline{I_\varepsilon}.$$

Theorem 3.4 and Corollary 3.1 ensure that almost surely on the event $\{m \neq 0\}$, the Hausdorff dimension of K satisfies $\dim(K) \geq 1 + \log_\ell p$ which is also known as the box dimension of K. Finally,

$$\dim(K) = 1 + \log_\ell p$$

almost surely on the event $\{m \neq 0\} = \{K \neq \emptyset\}$.

Example 3.4 (Log-normal cascades). Suppose that N follows a standard normal distribution and $W = e^{\sigma N - \sigma^2/2}$ with $\sigma^2 < 2\log\ell$. We know that

$$\tau(q) = \frac{\sigma^2}{2\log\ell}(q^2 - q) + 1 - q$$

and we find

$$\dim(m) = 1 - \frac{\sigma^2}{2\log\ell} \quad \text{almost surely.}$$

Proof (Proof of Theorem 3.4). As observed before, under the hypothesis

$$0 < E[Y_\infty \log Y_\infty] < +\infty,$$

the cascade m is non-degenerate. In particular, $E[Y_\infty] = 1$.

We first need to precisely define the sentence "almost surely dm-almost everywhere." Let $\tilde{\Omega} = \Omega \times [0,1]$ endowed with the product σ-algebra. Define the measure Q by

$$Q[A] = E\left[\int \mathbb{1}_A \, dm\right].$$

Observe that the measure m depends on $\omega \in \Omega$ so that Q is not a product measure. Nevertheless,

$$Q[\tilde{\Omega}] = E\left[\int dm\right] = E[Y_\infty] = 1$$

so that Q is a probability measure. If a property is true on a set $A \subset \tilde{\Omega}$ satisfying $Q[A] = 1$, then, almost surely, the property is true dm-almost everywhere. The measure Q is now very often referred as the Peyrière measure.

Recall that the measure m is constructed as the weak limit of the sequence $m_n = f_n \lambda$ where

$$f_n = \sum_{\varepsilon_1 \cdots \varepsilon_n \in \mathscr{M}_n} W_{\varepsilon_1} W_{\varepsilon_1 \varepsilon_2} \cdots W_{\varepsilon_1 \cdots \varepsilon_n} \mathbb{1}_{I_{\varepsilon_1 \cdots \varepsilon_n}}.$$

The proof of Theorem 3.4 is an easy consequence of the two following lemmas.

Lemma 3.4. *Suppose that $E[W \log W] < \log\ell$. Then, almost surely,*

$$\lim_{n \to +\infty} \frac{\log f_n(x)}{n} = E[W \log W] \qquad \textit{for } dm-\textit{almost every } x.$$

Lemma 3.5. *Suppose that $E[Y_\infty \log Y_\infty] < +\infty$. Let $\mu_n = \frac{1}{f_n} m$. Then, almost surely,*

$$\lim_{n \to +\infty} \frac{\log \mu_n(I_n(x))}{n} = -\log\ell \qquad \textit{for } dm-\textit{almost every } x.$$

Suppose first that Lemma 3.4 and Lemma 3.5 are true and recall that $dm = f_n d\mu_n$. The density f_n is constant on any interval of the n^{th} generation, so that

$$m(I_n(x)) = \int_{I_n(x)} f_n(y) d\mu_n(y) = f_n(x)\mu_n(I_n(x)).$$

It follows that

$$\begin{aligned}\frac{\log(m(I_n(x))}{\log|I_n(x)|} &= \frac{\log f_n(x) + \log \mu_n(I_n(x))}{-n\log \ell} \\ &\to -E[W\log_\ell W] + 1\end{aligned}$$

almost surely dm-almost everywhere.

Proof (Proof of Lemma 3.4). Let us write

$$f_n = g_1 \times \cdots \times g_n \qquad \text{where} \qquad g_n = \sum_{\varepsilon 1 \cdots \varepsilon n \in \mathscr{M}_n} W_{\varepsilon 1 \cdots \varepsilon n} 1\!\!1_{I_{\varepsilon 1 \cdots \varepsilon n}}.$$

We get

$$\frac{\log f_n}{n} = \frac{1}{n}\sum_{k=1}^{n} \log g_k$$

and Lemma 3.4 will be a consequence of the strong law of large numbers in the space $\tilde{\Omega}$ associated with the probability Q.

Let us calculate the law of the random variable $g_n = \sum_{\varepsilon\in\mathscr{M}_n} W_\varepsilon 1\!\!1_{I_\varepsilon}$. If $\mathbb{E}$ is the expectation related to the probability Q and if ϕ is bounded and measurable,

$$\begin{aligned}\mathbb{E}[\phi(g_n)] &= E\left[\int \sum_{\varepsilon\in\mathscr{M}_n} \phi(W_\varepsilon) 1\!\!1_{I_\varepsilon} \, dm\right] \\ &= \sum_{\varepsilon\in\mathscr{M}_n} E\left[\phi(W_\varepsilon) m(I_\varepsilon)\right]\end{aligned}$$

Moreover, if $k \geq 0$, using the independence properties,

$$\begin{aligned}E\left[\phi(W_\varepsilon) m_{n+k}(I_\varepsilon)\right] &= \sum_{\alpha_1\cdots\alpha_k\in\mathscr{M}_k} E\left[\phi(W_\varepsilon)\ell^{-(n+k)} W_{\varepsilon 1}\cdots W_\varepsilon W_{\varepsilon\alpha_1}\cdots W_{\varepsilon\alpha_1\cdots\alpha_k}\right] \\ &= \ell^{-n} E\left[\phi(W)W\right].\end{aligned}$$

Taking the limit, we get

$$\mathbb{E}[\phi(g_n)] = E[\phi(W)W]. \tag{3.14}$$

Equation (3.14) remains true if ϕ is such that $E\left[|\phi(W)W|\right] < +\infty$. In particular, the random variables g_n have the same law and $\log(g_n)$ are integrable with respect to Q.

The independence of the sequence (g_n) is obtained in a similar way. If $\phi_1,\ldots,\phi_n$ are bounded and measurable, we can also write

$$\begin{aligned}\mathbb{E}\left[\phi_1(g_1)\cdots\phi_n(g_n)\right] &= \sum_{\varepsilon\in\mathscr{M}_n} E\left[\phi_1(W_{\varepsilon_1})\cdots\phi_n(W_{\varepsilon_1\cdots\varepsilon_n})m(I_\varepsilon)\right]\\ &= \cdots\\ &= E[\phi_1(W)W]\times\cdots\times E[\phi_n(W)W]\\ &= \mathbb{E}\left[\phi_1(g_1)\right]\times\cdots\times\mathbb{E}\left[\phi_n(g_n)\right]\end{aligned}$$

and the independence follows. Finally, the strong law of large numbers gives

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\log g_k = \mathbb{E}[\log g_1] = E[W\log W] \quad dQ-\text{almost surely},$$

which says that almost surely,

$$\lim_{n\to+\infty}\frac{f_n(x)}{n} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\log g_k(x) = E[W\log W] \quad \text{for } dm-\text{almost every } x.$$

Remark 3.11 (On the importance of the order of the quantifiers). Let $x\in[0,1]$ and $\varepsilon_1\ldots\varepsilon_n\ldots$ such that $x\in I_{\varepsilon_1\cdots\varepsilon_n}$ for any n. We have

$$\frac{\log(f_n(x))}{n} = \frac{1}{n}(\log W_{\varepsilon_1}+\cdots+\log W_{\varepsilon_1\cdots\varepsilon_n}).$$

Using the strong law of large numbers, we get:

$$\text{For any } x\in[0,1],\ \text{almost surely,} \quad \lim_{n\to+\infty}\frac{\log(f_n(x))}{n} = E[\log W]$$

which is different from the conclusion of Lemma 3.4 !

Proof (Proof of Lemma 3.5). Let us begin with a comment on the definition of the measure μ_n. The function f_n is constant on any interval of the n^{th} generation. Moreover, if f_n is equal to zero on some interval I of the n^{th} generation, then $m(I)=0$. Finally, $f_n\neq 0$ dm-almost surely and $\mu_n=\frac{1}{f_n}m$ is well defined. We can also write $m=f_n\mu_n$ and if $\varepsilon=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n$,

$$m(I_\varepsilon) = W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\,\mu_n(I_\varepsilon).$$

We claim that $\mu_n(I_\varepsilon)$ is independent to $W_{\varepsilon_1},\ldots,W_{\varepsilon_1\cdots\varepsilon_n}$ and has the same distribution as $\ell^{-n}Y_\infty$. Indeed,

$$m_{n+k} = f_n(g_{n+1}\cdots g_{n+k})\,d\lambda$$

and

$$\mu_n = \lim_{k\to+\infty}(g_{n+1}\cdots g_{n+k})\,d\lambda.$$

In particular,

$$\mu_n(I_\varepsilon) = \lim_{k\to+\infty}\int_{I_\varepsilon} g_{n+1}(x)\cdots g_{n+k}(x)\,d\lambda(x)$$

is clearly independent to $W_{\varepsilon_1}, \dots, W_{\varepsilon_1 \cdots \varepsilon_n}$ and an easy calculation gives that it has the same distribution as $\ell^{-n} Y_\infty$.

Using the previous remark, we get

$$\begin{aligned}
\mathbb{E}\left[(\ell^n \mu_n(I_n(x)))^{-1/2}\right] &= E\left[\ell^{-n/2} \int \mu_n(I_n(x))^{-1/2} dm(x)\right] \\
&= \ell^{-n/2} \sum_{\varepsilon \in \mathscr{M}_n} E\left[\mu_n(I_\varepsilon)^{-1/2} m(I_\varepsilon)\right] \\
&= \ell^{-n/2} \sum_{\varepsilon \in \mathscr{M}_n} E\left[W_{\varepsilon_1} \cdots W_{\varepsilon_1 \cdots \varepsilon_n}\right] E\left[\mu_n(I_\varepsilon)^{1/2}\right] \\
&= E\left[Y_\infty^{1/2}\right].
\end{aligned}$$

It follows that

$$\mathbb{E}\left[\sum_{n \geq 1} \frac{1}{n^2} (\ell^n \mu_n(I_n(x)))^{-1/2}\right] < +\infty.$$

In particular, dQ-almost surely, $\ell^n \mu_n(I_n(x)) \geq 1/n^4$ if n is large enough and we can conclude that almost surely,

$$\liminf_{n \to +\infty} \frac{\log(\ell^n \mu_n(I_n(x)))}{n} \geq 0 \quad dm - \text{almost everywhere.}$$

In other words, almost surely,

$$\liminf_{n \to +\infty} \left(\frac{\log(\mu_n(I_n(x)))}{n}\right) \geq -\log \ell \quad dm - \text{almost everywhere.}$$

We have now to prove that almost surely,

$$\limsup_{n \to +\infty} \left(\frac{\log(\mu_n(I_n(x)))}{n}\right) \leq -\log \ell \quad dm - \text{almost everywhere.}$$

Recall that $m(I_\varepsilon) = W_{\varepsilon_1} \cdots W_{\varepsilon_1 \cdots \varepsilon_n} \mu_n(I_\varepsilon)$ with independence properties. If $\alpha > 0$,

$$\begin{aligned}
Q[\ell^n \mu_n(I_n(x)) > \alpha^n] &= E\left[\int \mathbb{1}_{\{\ell^n \mu_n(I_n(x)) > \alpha^n\}}(x)\, dm(x)\right] \\
&= \sum_{\varepsilon \in \mathscr{M}_n} E\left[\int_{I_\varepsilon} \mathbb{1}_{\{\ell^n \mu_n(I_\varepsilon) > \alpha^n\}}(x)\, dm(x)\right] \\
&= \sum_{\varepsilon \in \mathscr{M}_n} E\left[m(I_\varepsilon) \mathbb{1}_{\{\ell^n \mu_n(I_\varepsilon) > \alpha^n\}}\right] \\
&= \sum_{\varepsilon \in \mathscr{M}_n} E[W_{\varepsilon_1} \cdots W_{\varepsilon_1 \cdots \varepsilon_n}] E\left[\mu_n(I_\varepsilon) \mathbb{1}_{\{\ell^n \mu_n(I_\varepsilon) > \alpha^n\}}\right] \\
&= E\left[Y_\infty \mathbb{1}_{\{Y_\infty > \alpha^n\}}\right].
\end{aligned}$$

In particular,

$$\begin{aligned}\sum_{n\geq 1} Q[\ell^n \mu_n(I_n(x)) > \alpha^n] &= E\left[\sum_{n\geq 1} Y_\infty 1\!\!1_{\{Y_\infty>\alpha^n\}}\right]\\ &\leq E\left[Y_\infty \log_\alpha^+(Y_\infty)\right]\\ &< +\infty.\end{aligned}$$

Using Borel Cantelli's lemma, we get

$$dQ - \text{almost surely}, \quad \ell^n \mu_n(I_n(x)) \leq \alpha^n \quad \text{if } n \text{ is large enough.}$$

In particular, almost surely,

$$\limsup_{n\to+\infty} \frac{\log(\ell^n \mu_n(I_n(x)))}{n} \leq \alpha \quad dm - \text{almost everywhere}$$

and the conclusion is a consequence of the arbitrary value of α.

Remark 3.12. In the eighties, Kahane proved that the condition $0 < E[Y_\infty \log Y_\infty] < +\infty$ is not necessary.

3.6 A Digression on Multifractal Analysis of Measures

In order to understand the approach developed in Section 3.7, let us recall some basic facts on multifractal analysis of measures. In this part, m is a deterministic measure on $[0,1]$ with finite total mass. As usual, we define the structure function as

$$\tau(q) = \limsup_{n\to+\infty} \frac{1}{n} \log_\ell \left(\sum_{I\in\mathscr{F}_n} m(I)^q\right)$$

and we want to briefly recall the way to improve the formula

$$\dim(E_\beta) = \tau^*(\beta)$$

where

$$E_\beta = \left\{x\ ;\ \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log|I_n(x)|} = \beta\right\}$$

and

$$\tau^*(\beta) = \inf_{q\in\mathbb{R}} (q\beta + \tau(q))$$

is the Legendre transform of τ.

The function τ is known to be a non-increasing convex function on $\mathbb{R}$ such that $\tau(1) = 0$. Moreover, the right and the left derivative $-\tau'(1^+)$ and $-\tau'(1^-)$ are related to the dimensions of the measure m which are defined in formula (3.2) and (3.3). In the general case, as we can see, for example, in [13], we have

Theorem 3.5.

$$-\tau'(1^+) \leq \dim_*(m) \leq \mathrm{Dim}^*(m) \leq -\tau'(1^-).$$

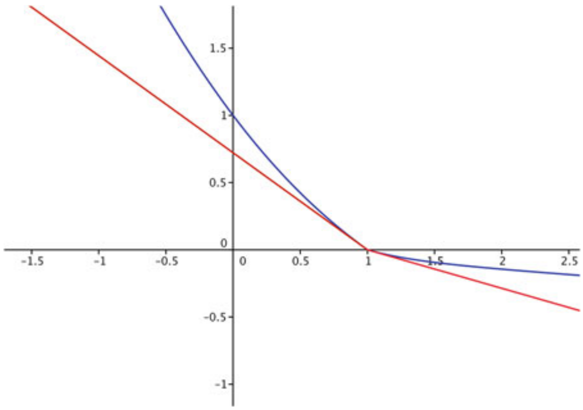

Fig. 3.7 A structure function τ such that $\tau'(1)$ does not exist.

We cannot ensure in general that $\tau'(1^+) = \tau'(1^-)$ (Fig. 3.7). Nevertheless, if $\tau'(1)$ exists, the measure m is unidimensional and the following are true.

Corollary 3.2. *Suppose that $\tau'(1)$ exists. Then*

1. *dm-almost surely,* $\displaystyle \lim_{n\to+\infty} \frac{\log(m(I_n(x)))}{\log|I_n(x)|} = -\tau'(1)$
2. $\dim\left(E_{-\tau'(1)}\right) = -\tau'(1)$
3. $\dim_*(m) = \dim^*(m) = \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = -\tau'(1)$.

The equality $\dim\left(E_{-\tau'(1)}\right) = -\tau'(1)$ can be rewritten in terms of the Legendre transform of the function τ. More precisely, if $\beta = -\tau'(1)$, then $\tau^*(\beta) = \beta$ and $\dim(E_\beta) = \tau^*(\beta)$. This is the first step in multifractal formalism (Fig. 3.8).

In order to obtain the formula $\dim(E_\beta) = \tau^*(\beta)$ for another value of β, the usual way is to write $\beta = -\tau'(q)$ and to construct an auxiliary measure m_q satisfying for any ℓ-adic interval

$$\frac{1}{C} m(I)^q |I|^{\tau(q)} \leq m_q(I) \leq C m(I)^q |I|^{\tau(q)}.$$

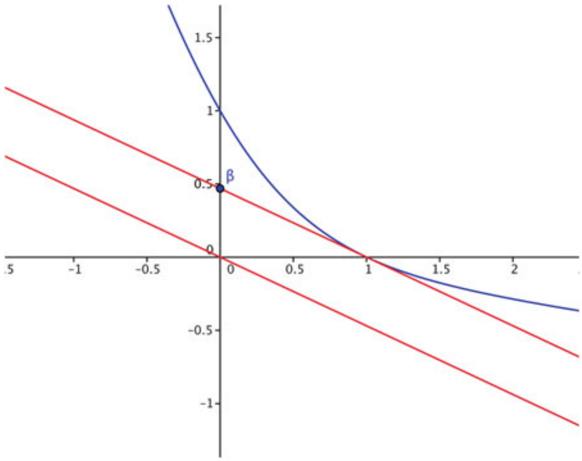

Fig. 3.8 If $\beta = -\tau'(1)$, then $\tau^*(\beta) = \beta$

This is the way used in [10]. If such a measure m_q exists, its structure function τ_{m_q} is such that

$$\tau_{m_q}(t) = \tau(qt) - t\tau(q).$$

In particular,

$$-\tau'_{m_q}(1) = -q\tau'(q) + \tau(q) = \tau^*(\beta).$$

If we observe that

$$\frac{\log(m_q(I_n(x)))}{\log|I_n(x)|} = q\frac{\log(m(I_n(x)))}{\log|I_n(x)|} + \tau(q) + o(1),$$

we can conclude that

$$\dim(E_\beta) = \dim(m_q) = -\tau'_{m_q}(1) = \tau^*(\beta).$$

3.7 Multifractal Analysis of Mandelbrot Cascades: An Outline

In this section, we make the following additional assumptions:

$$\begin{cases} P[W=0]=0 \\ \text{For any real } q, \quad E[W^q] < +\infty. \end{cases} \tag{3.15}$$

In particular, assumption (3.15) is satisfied when m is a log-normal cascade or when $\frac{1}{C} \leq W \leq C$ almost surely.

We can then list some easy consequences.

- The function $\tau(q) = \log_\ell(E[W^q]) - (q-1)$ is defined on $\mathbb{R}$, convex and of class C^∞
- There exists $r > 1$ such that $\tau(r) < 0 \quad$ (and so $E[Y_\infty^r] < +\infty$)
- The cascade is non-degenerate
- $P[m = 0] = P[Y_\infty = 0] = 0$.

In order to perform the multifractal analysis of the Mandelbrot cascades, we want to mimic the proof developed for the binomial cascades. It is then natural to introduce the auxiliary cascade m' associated with the weight $W' = \frac{W^q}{E[W^q]}$ (the renormalization ensures that $E[W'] = 1$). The structure function $\tau_{m'}$ of the cascade m' is

$$\begin{aligned}\tau_{m'}(t) &= \log_\ell(E[W'^t]) - (t-1)\\ &= \log_\ell\left(E\left[\frac{W^{qt}}{E[W^q]^t}\right]\right) - (t-1)\\ &= \log_\ell\left(E\left[W^{qt}\right]\right) - t\log_\ell(E[W^q]) - (t-1)\\ &= \tau(tq) - t\tau(q).\end{aligned}$$

In particular,

$$-\tau'_{m'}(1^-) = -q\tau'(q) + \tau(q) = \tau^*(-\tau'(q))$$

and the cascade m' is non-degenerate if and only if $\tau^*(-\tau'(q)) > 0$. This suggests to consider the interval

$$(q_{min}, q_{max}) = \{q \in \mathbb{R} \ ; \ \tau^*(-\tau'(q)) > 0\}.$$

Example 3.5 (The interval (q_{min}, q_{max}) in the case of log-normal cascades). If m is a log-normal cascade, the function τ is given by

$$\tau(q) = \log_\ell(E[W^q]) - (q-1) = \frac{\sigma^2}{2\ln\ell}(q^2 - q) - (q-1)$$

and the numbers q_{min} and q_{max} are the solutions of the equation

$$\tau(q) = q\tau'(q).$$

We find

$$q_{min} = -\frac{\sqrt{2\ln\ell}}{\sigma} \qquad \text{and} \qquad q_{max} = \frac{\sqrt{2\ln\ell}}{\sigma}.$$

See Fig. 3.9.

When $q \in (q_{min}, q_{max})$, we would like to compare the behavior of the cascades m and m'. In the following subsection, we give a general result which can be applied to the present situation.

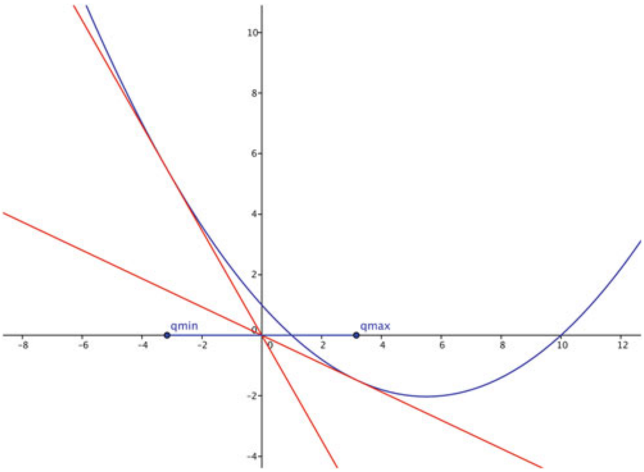

Fig. 3.9 The structure function τ and the interval (q_{min}, q_{max}) in the case of a log-normal cascade.

3.7.1 Simultaneous Behavior of Two Mandelbrot Cascades

Theorem 3.6. *Let (W, W') be a random vector (with coordinates not necessarily independent) such that the Mandelbrot cascades m and m' associated with the weight W et W' are non-degenerate. Suppose that :*

- *There exists $r > 1$ such that $E[Y_\infty^r] < +\infty$ and $E[Y_\infty'^r] < +\infty$*
- *There exists $\alpha > 0$ such that $E[Y_\infty^{-\alpha}] < +\infty$.*

Then, almost surely,

$$\lim_{n\to+\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = 1 - E[W' \log_\ell W] \quad dm' - \textit{almost everywhere.}$$

Proof. The ideas are quite similar to those developed in the proof of Theorem 3.4. The probability measure on the product space $\tilde{\Omega} = \Omega \times [0,1]$ is now

$$Q'(A) = E\left[\int 1\!\!1_A \, dm'\right]$$

and the related expectation is denoted by $\mathbb{E}'$. The notations are the same as in Theorem 3.4. In particular

$$dm = f_n \, d\mu_n, \qquad f_n = g_1 \times \cdots \times g_n, \qquad \frac{\log(m(I_n(x))}{\log |I_n(x)|} = \frac{\log f_n(x) + \log \mu_n(I_n(x))}{-n \log \ell}$$

and we have to prove :

1. $\frac{1}{n}\sum_{j=1}^n \log g_j(x)$ converges to $E[W' \log W] \quad dQ'$ almost surely
2. $\frac{1}{n}\log \mu_n(I_n(x))$ converges to $-\log \ell \quad dQ'$ almost surely.

Step 1: *behavior of* $\frac{1}{n}\sum_{j=1}^{n}\log g_j$.
In the same way as in Lemma 3.4, we have :

$$\mathbb{E}'[\phi(g_n)] = \sum_{\varepsilon\in\mathscr{M}n} E\left[\phi(W_\varepsilon)m'(I_\varepsilon)\right] = E[\phi(W)W']$$

when ϕ is a bounded measurable function and the g_n are identically distributed. On the other hand,

$$\begin{aligned}\mathbb{E}'\left[\phi_1(g_1)\cdots\phi_n(g_n)\right] &= E[\phi_1(W)W']\times\cdots\times E[\phi_n(W)W'] \\ &= \mathbb{E}'\left[\phi_1(g_1)\right]\times\cdots\times\mathbb{E}'\left[\phi_n(g_n)\right]\end{aligned}$$

which proves the independence of the random variables (g_n) with respect to Q'. Observing that $\mathbb{E}'[|\log g_n|] = E[W'|\log W|] < +\infty$, the strong law of large numbers says that

$$\frac{1}{n}\sum_{k=1}^{n}\log g_k \xrightarrow[n\to+\infty]{} E[W'\log W] \quad dQ' - \text{almost surely.}$$

Step 2 : *behavior of* $\frac{1}{n}\log\mu_n(I_n(x))$.
Let $\varepsilon=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n$ and recall that $m(I_\varepsilon)=W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\mu_n(I_\varepsilon)$. It is easy to see that $\mu_n(I_\varepsilon)$ is independent to $W_{\varepsilon_1},\dots,W_{\varepsilon_1\cdots\varepsilon_n}$ and has the same law as $\ell^{-n}Y_\infty$. If we write $m'(I_\varepsilon)=W'_{\varepsilon_1}\cdots W'_{\varepsilon_1\cdots\varepsilon_n}\mu'_n(I_\varepsilon)$, we can more precisely say that the vector $(m(I_\varepsilon),m'(I_\varepsilon))$ is identically distributed to $(\ell^{-n}Y_\infty,\ell^{-n}Y'_\infty)$ and independent to $W_{\varepsilon_1},\dots,W_{\varepsilon_1\dots\varepsilon_n},W'_{\varepsilon_1},\dots,W'_{\varepsilon_1\dots\varepsilon_n}$. It follows that

$$\begin{aligned}\mathbb{E}'\left[(\ell^n\mu_n(I_n(x)))^{-\eta}\right] &= E\left[\ell^{-n\eta}\int\mu_n(I_n(x))^{-\eta}dm'(x)\right] \\ &= \ell^{-n\eta}\sum_{\varepsilon\in\mathscr{M}_n}E\left[\mu_n(I_\varepsilon)^{-\eta}m'(I_\varepsilon)\right] \\ &= \ell^{-n\eta}\sum_{\varepsilon\in\mathscr{M}_n}E\left[W'_{\varepsilon_1}\cdots W'_{\varepsilon_1\cdots\varepsilon_n}\right]E\left[\mu_n(I_\varepsilon)^{-\eta}\mu'_n(I_\varepsilon)\right] \\ &= E[Y_\infty^{-\eta}Y'_\infty] \\ &\le E[Y_\infty^{-\eta r'}]^{1/r'}E\left[Y'^{r}_\infty\right]^{1/r}\end{aligned}$$

where r' is such that $\frac{1}{r}+\frac{1}{r'}=1$. If we choose η such that $\eta r'=\alpha$, we get

$$\mathbb{E}'\left[\sum_{n=1}^{+\infty}\frac{1}{n^2}\left(\ell^n\mu_n(I_n(x))\right)^{-\eta}\right] < +\infty$$

and we can conclude as in Lemma 3.4 that

$$\text{almost surely,}\quad \liminf_{n\to+\infty}\frac{\log\left(\ell^n\mu_n(I_n(x))\right)}{n}\ge 0 \quad dm' - \text{almost everywhere.}$$

In the same way,

$$\mathbb{E}'\left[(\ell^n \mu_n(I_n(x)))^\eta\right] = E[Y_\infty^\eta Y'_\infty] \\ \leq E[Y_\infty^{\eta r'}]^{1/r'} E\left[Y'^r_\infty\right]^{1/r}$$

which is finite and independent of n if we choose η such that $\eta r' = r$. We can then conclude that

$$\text{almost surely,} \quad \limsup_{n\to+\infty} \frac{\log(\ell^n \mu_n(I_n(x)))}{n} \leq 0 \quad dm' - \text{almost everywhere.}$$

3.7.2 Application to the Multifractal Analysis of Mandelbrot Cascades

If we apply Theorem 3.6 to the case where $W' = \frac{W^q}{E[W^q]}$, we obtain the following result on multifractal analysis of Mandelbrot cascades.

Theorem 3.7. *Let m be a Mandelbrot cascade associated with a weight W. Suppose that* (3.15) is satisfied and define q_{min} and q_{max} as above. Let $\beta = -\tau'(q)$ with $q \in (q_{min}, q_{max})$. Then

$$\dim(E_\beta) \geq \tau^*(\beta)$$

where

$$E_\beta = \left\{x\,;\ \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log|I_n(x)|} = \beta\right\}.$$

Proof. As suggested at the beginning of Section 3.7, let $W' = \frac{W^q}{E[W^q]}$. The condition $q \in (q_{min}, q_{max})$ ensures that the associated cascade m' is non-degenerate. More precisely, observing that $\tau'(1) < 0$ and $\tau'_q(1) = -\tau^*(-\tau'(q)) = -\tau^*(\beta) < 0$, we can find $r > 1$ such that $E[Y^r_\infty] < +\infty$ and $E[Y'^r_\infty] < +\infty$. Finally, all the hypotheses of Theorem 3.6 are satisfied. Observe that

$$1 - E[W' \log_\ell W] = 1 - E\left[\frac{W^q}{E[W^q]} \log_\ell W\right] = -\tau'(q) = \beta.$$

The conclusion of Theorem 3.6 says that almost surely, the set E_β is of full measure m'. It follows that

$$\dim(E_\beta) \geq \dim(m') = -\tau'_q(1) = -q\tau'(q) + \tau(q) = \tau^*(-\tau'(q)) = \tau^*(\beta).$$

Remark 3.13. We can observe some analogies between the construction proposed in Theorem 3.7 and similar ones developed in the context of the thermodynamic formalism (see, for example, [23]) or in the context of quasi-Bernoulli measures (see [10] or [13]).

3.7.3 To Go Further

It is natural to ask if the inequality proved in Theorem 3.7 is an equality. Indeed we know that the inequality

$$\dim(E_\beta) \le \tilde{\tau}^*(\beta) \quad \text{where} \quad \tilde{\tau}(q) = \limsup_{n\to+\infty} \frac{1}{n} \log_\ell \left(\sum_{I\in\mathscr{F}_n} m(I)^q \right)$$

is always true (see, for example, [10]).

Our goal is then to compare the convex functions τ and $\tilde{\tau}$. Such a comparison can be deduced from the existence of negative moments for the random variable Y_∞.

Proposition 3.5 (Existence of negative moments). *Suppose that* (3.15) is satisfied. Then, for any $\alpha > 0$, $E\left[Y_\infty^{-\alpha}\right] < +\infty$.

Proof. The argument is developed, for example, in [1] or [18]. Let

$$F(t) = E\left[e^{-tY_\infty}\right]$$

be the generating function of Y_∞. The fundamental equation $\ell Y_\infty = \sum_{j=0}^{\ell-1} W_j Y_\infty(j)$ gives the following duplication formula :

$$F(\ell t) = \left(\int_0^{+\infty} F(tw)\, dP_W(w) \right)^\ell . \tag{3.16}$$

We claim that it is sufficient to prove that for any $\alpha > 0$, $F(t) = O(t^{-\alpha})$ when $t \to +\infty$. Indeed, if it is the case,

$$P\left[Y_\infty \le t^{-1}\right] = P\left[e^{-tY_\infty} \ge e^{-1}\right] \le eF(t) = O(t^{-\alpha})$$

and we can conclude that

$$E\left[Y_\infty^{-\alpha'}\right] = \int_0^{+\infty} P\left[Y_\infty^{-\alpha'} \ge t\right] dt = \int_0^{+\infty} P\left[Y_\infty \le t^{-1/\alpha'}\right] dt < +\infty$$

for any $\alpha' < \alpha$.

Let us now observe that (3.16) gives for any $t > 0$ and any $u \in (0,1]$,

$$\begin{aligned} F(t) &\le \left(\int_0^{+\infty} F\left((t\ell^{-1})w\right) dP_W(w) \right)^2 \\ &\le (P[W \le \ell u] + F(tu))^2 \\ &\le 2P[W \le \ell u]^2 + 2F(tu)^2. \end{aligned}$$

Moreover, for any $\beta > 0$, assumption (3.15) ensures that

$$P[W \le \ell u] = P\left[W^{-\beta} \ge (\ell u)^{-\beta}\right] \le (\ell u)^\beta E\left[W^{-\beta}\right] = Cu^\beta .$$

Proposition 3.5 is then a consequence of the following elementary lemma.

Lemma 3.6. *Let $\beta > 0$ and $\psi : [0,+\infty) \to [0,+\infty)$ a continuous function such that*

$$\lim_{t\to+\infty} \psi(t) = 0.$$

Suppose that there exists $K > 0$ such that for any $t > 0$ and any $u \in (0,1]$,

$$\psi(t) \le Ku^{2\beta} + 2\psi(tu)^2. \tag{3.17}$$

Then, for any $\alpha < \beta$, $\psi(t) = O(t^{-\alpha})$ when $t \to +\infty$.

Proof. Let $\alpha < \beta$ and $t_0 > 1$ such that $4Kt_0^{2(\alpha-\beta)} + \frac{1}{2} \le 1$. Let $\lambda > 1$ such that

$$\text{for any } t \in \left[t_0, t_0^2\right], \quad \psi(\lambda t) \le \frac{1}{4t^\alpha}.$$

Define $\psi_\lambda(t) = \psi(\lambda t)$. Equation (3.17) remains true if we replace ψ by ψ_λ. Moreover, if $u = \frac{1}{t}$, we get

$$\psi_\lambda(t^2) \le Kt^{-2\beta} + 2\psi_\lambda(t)^2.$$

If $t \in \left[t_0, t_0^2\right]$, we obtain

$$\begin{aligned}\psi_\lambda(t^2) &\le Kt^{-2\beta} + 2\left(\frac{1}{4t^\alpha}\right)^2\\ &= \frac{1}{4t^{2\alpha}}\left[4Kt^{2(\alpha-\beta)} + \frac{1}{2}\right]\\ &\le \frac{1}{4(t^2)^\alpha}.\end{aligned}$$

Define the sequence (t_n) by $t_{n+1} = t_n^2$. Using the same argument, we obtain step by step

$$\text{for any } n \ge 0, \quad \text{for any } t \in [t_n, t_{n+1}], \qquad \psi_\lambda(t) \le \frac{1}{4t^\alpha}$$

and the conclusion follows.

Corollary 3.3. *Suppose that* (3.15) is satisfied. Then,

$$\text{almost surely,} \quad \text{for any } q \in \mathbb{R}, \qquad \tilde\tau(q) \le \tau(q).$$

Proof. Using the continuity of the convex functions $\tilde\tau$ and τ, it is sufficient to prove that for any $q \in \mathbb{R}$, almost surely, $\tilde\tau(q) \le \tau(q)$. Let

$$q_0 = \sup\{q > 1 \; ; \; \tau(q) < 0\}.$$

It is possible that $q_0 = +\infty$. Nevertheless, if $q_0 < +\infty$ and if $q \ge q_0$, we obviously have $\tilde\tau(q) \le 0 \le \tau(q)$.

We can now suppose that $q < q_0$ and we claim that

$$E\left[Y_\infty^q\right] < +\infty. \tag{3.18}$$

Indeed, the case $q < 0$ is due to Proposition 3.5, the case $0 \le q \le 1$ is obvious and the case $1 < q < q_0$ is due to Theorem 3.3.

Let $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathscr{M}_n$. As observed in Lemma 3.5, we have

$$m(I_\varepsilon) = W_{\varepsilon_1} \cdots W_{\varepsilon_1 \cdots \varepsilon_n} \mu_n(I_\varepsilon).$$

where $\mu_n(I_\varepsilon)$ is independent to $W_{\varepsilon_1}, \cdots, W_{\varepsilon_1 \cdots \varepsilon_n}$ and has the same distribution as $\ell^{-n} Y_\infty$. It follows that

$$\begin{aligned} E\left[\sum_{\varepsilon \in \mathscr{M}_n} m(I_\varepsilon)^q\right] &= \sum_{\varepsilon \in \mathscr{M}_n} E\left[W_{\varepsilon_1}^q \cdots W_{\varepsilon_1 \cdots \varepsilon_n}^q \mu_n(I_\varepsilon)^q\right] \\ &= \ell^n E\left[W^q\right]^n \ell^{-nq} E\left[Y_\infty^q\right] \\ &= \ell^{n\tau(q)} E\left[Y_\infty^q\right]. \end{aligned}$$

Let $t > \tau(q)$. In view of (3.18) we get

$$E\left[\sum_{n \ge 1} \ell^{-nt} \sum_{\varepsilon \in \mathscr{M}_n} m(I_\varepsilon)^q\right] = \sum_{n \ge 1} \ell^{-nt} \ell^{n\tau(q)} E\left[Y_\infty^q\right] < +\infty.$$

It follows that almost surely, $\sum_{\varepsilon \in \mathscr{M}_n} m(I_\varepsilon)^q \le \ell^{nt}$ if n is large enough and we can conclude that $\tilde{\tau}(q) \le t$ almost surely. This gives the conclusion.

We can now prove the following result.

Theorem 3.8. *Suppose that* (3.15) *is satisfied. Then, for any* $\beta \in (-\tau'(q_{max}), -\tau'(q_{min}))$,

$$\dim(E_\beta) = \tau^*(\beta) \quad \text{almost surely.}$$

Indeed Theorem 3.7 and Corollary 3.3 ensure that for any $\beta \in (-\tau'(q_{max}), -\tau'(q_{min}))$,

$$\tau^*(\beta) \le \dim(E_\beta) \le \tilde{\tau}^*(\beta) \le \tau^*(\beta)$$

which gives the conclusion of Theorem 3.8.

Remark 3.14. The proof of Theorem 3.8 shows that

$$\tau^*(\beta) = \tilde{\tau}^*(\beta) \quad \text{for any } \beta \in (-\tau'(q_{max}), -\tau'(q_{min})).$$

It follows that $\tau(q) = \tilde{\tau}(q)$ for any $q \in (q_{min}, q_{max})$. When q_{min} and q_{max} are finite, it is possible to prove that

$$\tilde{\tau}(q) = \tau'(q_{min})q \quad \text{if } q \le q_{min} \qquad \text{and} \qquad \tilde{\tau}(q) = \tau'(q_{max})q \quad \text{if } q \ge q_{max}$$

(see, for example, [8]).

Let us finish this text by recalling that Barral proved in [2] the much more difficult result:

$$\text{almost surely,} \quad \begin{cases} \text{for any } \beta \in (-\tau'(q_{max}), -\tau'(q_{min})), & \dim(E_\beta) = \tau^*(\beta) \\ \\ \text{for any } \beta \notin [-\tau'(q_{max}), -\tau'(q_{min})], & E_\beta = \emptyset. \end{cases}$$

This text about Mandelbrot cascades can be viewed as the beginning of the story. Since then, important generalizations have been proposed. In 1985 Kahane introduced the so-called T-martingales (see [16, 17]). In particular, log-infinitely divisible multifractal processes [4] and compound Poisson cascades [3] are models for which similar fine results on multifractal analysis have been proved (see [5–7]).

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Chapter 4
Lebesgue-Type Inequalities for Greedy Approximation

Vladimir Temlyakov

Abstract The estimation of a solution that satisfies a certain optimality criterion is the goal of many engineering applications. In many contemporary problems one would often like to obtain an approximation of the solution using a sparse linear combination of elements of a given system (dictionary). This paper is devoted to theoretical aspects of sparse approximation. The main motivation for the study of sparse approximation is that many real world signals can be well approximated by sparse ones. Sparse approximation automatically implies a need for nonlinear approximation, in particular, for greedy approximation. The paper is a survey on results in constructive sparse approximation. Two directions are discussed here: (1) Lebesgue-type inequalities for thresholding greedy algorithms with respect to bases, and (2) Lebesgue-type inequalities for Chebyshev Greedy Algorithms with respect to a special class of dictionaries. In particular, these algorithms provide constructive sparse approximation with respect to the trigonometric system. The technique used is based on fundamental results from the theory of greedy approximation. Results in the direction (2) are based on deep methods developed recently in compressed sensing. We present some of these results with detailed proofs.

4.1 Introduction

The estimation of a solution that satisfies a certain optimality criterion is the goal of many engineering applications. In many contemporary problems one would often like to obtain an approximation of the solution using a sparse linear combination of elements of a given system (dictionary). This paper is devoted to theoretical aspects of sparse approximation. The main motivation for the study of sparse approximation is that many real world signals can be well approximated by sparse ones. Sparse

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approximation automatically implies a need for nonlinear approximation, in particular, for greedy approximation. We give a brief description of a sparse approximation problem. In a general setting we are working in a Banach space X with a redundant system of elements $\mathscr{D}$ (dictionary $\mathscr{D}$). There is a solid justification of importance of a Banach space setting in numerical analysis in general and in sparse approximation in particular (see, for instance, [28], Preface, and [22]). An element (function, signal) $f \in X$ is said to be K-sparse with respect to $\mathscr{D}$ if it has a representation $f = \sum_{i=1}^{K} x_i g_i$, $g_i \in \mathscr{D}$, $i = 1,\dots,K$. The set of all K-sparse elements is denoted by $\Sigma_K(\mathscr{D})$. For a given element f we introduce the error of best K-term approximation

$$\sigma_K(f,\mathscr{D}) := \inf_{a \in \Sigma_K(\mathscr{D})} \|f - a\|.$$

We are interested in the following fundamental problem of sparse approximation.

Problem. How to design a practical algorithm that builds sparse approximations comparable to best K-term approximations?

Clearly, the most difficult part of the above problem is to identify the dictionary elements which are good for the K-term approximation. In the case of an orthonormal basis in a Hilbert space the recipe is evident: pick the elements with the largest in absolute value coefficients from the expansion. This idea leads to a greedy algorithm with respect to a basis in a Banach space that we call the Thresholding Greedy Algorithm (TGA). It turns out that this algorithm works excellent for bases alike the univariate Haar basis. However, it does not work well for the trigonometric system where interference between harmonics is essential. Recently, we discovered that another greedy algorithm—the Weak Chebyshev Greedy Algorithm (WCGA)—works well for both the trigonometric system and the Haar basis. Results for both algorithms the TGA and the WCGA are presented in this paper. We begin our discussion with the TGA.

Let a Banach space X, with a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$, be given. We assume that $\|\psi_k\| \ge C > 0$, $k = 1,2,\dots$, and consider the following theoretical greedy algorithm. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f,\Psi)\psi_k. \tag{4.1}$$

For an element $f \in X$ we say that a permutation ρ of the positive integers is decreasing if

$$|c_{k_1}(f,\Psi)| \ge |c_{k_2}(f,\Psi)| \ge \dots, \tag{4.2}$$

where $\rho(j) = k_j$, $j = 1,2,\dots$, and write $\rho \in D(f)$. If the inequalities are strict in (4.2), then $D(f)$ consists of only one permutation. We define the mth greedy approximant of f, with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$, by the formula

$$G_m(f) := G_m(f,\Psi) := G_m(f,\Psi,\rho) := \sum_{j=1}^{m} c_{k_j}(f,\Psi)\psi_{k_j}.$$

The above algorithm $G_m(\cdot,\Psi)$ is a simple algorithm which describes the theoretical scheme for m-term approximation of an element f. We call this algorithm the Thresholding Greedy Algorithm (TGA) or simply the Greedy Algorithm (GA). In order to understand the efficiency of this algorithm we compare its accuracy with the best-possible accuracy when an approximant is a linear combination of m terms from Ψ. We define the best m-term approximation with regard to Ψ as follows:

$$\sigma_m(f) := \sigma_m(f,\Psi)_X := \inf_{c_k,\Lambda} \|f - \sum_{k\in\Lambda} c_k\psi_k\|_X,$$

where the infimum is taken over coefficients c_k and sets of indices Λ with cardinality $|\Lambda| = m$. The best we can achieve with the algorithm G_m is

$$\|f - G_m(f,\Psi,\rho)\|_X = \sigma_m(f,\Psi)_X,$$

or the slightly weaker

$$\|f - G_m(f,\Psi,\rho)\|_X \le G\sigma_m(f,\Psi)_X, \tag{4.3}$$

for all elements $f \in X$, and with a constant $G = C(X,\Psi)$ independent of f and m. It is clear that, when $X = H$ is a Hilbert space and $\mathscr{B}$ is an orthonormal basis, we have

$$\|f - G_m(f,\mathscr{B},\rho)\|_H = \sigma_m(f,\mathscr{B})_H.$$

The following concept of a greedy basis has been introduced in [13].

Definition 4.1.1 *We call a basis Ψ a greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$\|f - G_m(f,\Psi,\rho)\|_X \le C\sigma_m(f,\Psi)_X$$

with a constant C independent of f and m.

Lebesgue [15] proved the following inequality: for any 2π-periodic continuous function f we have

$$\|f - S_n(f)\|_\infty \le (4 + \frac{4}{\pi^2}\ln n)E_n(f)_\infty, \tag{4.4}$$

where $S_n(f)$ is the nth partial sum of the Fourier series of f and $E_n(f)_\infty$ is the error of the best approximation of f by the trigonometric polynomials of order n in the uniform norm $\|\cdot\|_\infty$. The inequality (4.4) relates the error of a particular method (S_n) of approximation by the trigonometric polynomials of order n to the best-possible error $E_n(f)_\infty$ of approximation by the trigonometric polynomials of order n. By the Lebesgue-type inequality we mean an inequality that provides an upper estimate for the error of a particular method of approximation of f by elements of a special form, say, form $\mathscr{A}$, by the best-possible approximation of f by elements of the form $\mathscr{A}$. In the case of approximation with regard to bases (or minimal systems), the Lebesgue-type inequalities are known both in linear and in nonlinear settings (see surveys [14, 26] and [27]).

By the Definition 4.1.1 greedy bases are those for which we have ideal (up to a multiplicative constant) Lebesgue inequalities for greedy approximation. In Section 4.2 we obtain Lebesgue-type inequalities for greedy approximation with respect to the trigonometric system. In Section 4.3 we study Lebesgue-type inequalities for greedy approximation with respect to the Haar basis and prove that the Haar basis is a greedy basis for L_p, $1 < p < \infty$. In Sections 4.2 and 4.3 we obtain the Lebesgue-type inequalities for the TGA. Sections 4.4–4.6 are devoted to the Lebesgue-type inequalities for the WCGA.

In Section 4.2 we consider the case $X = L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, $\Psi = \mathscr{T} := \{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$ is the trigonometric system. We give a remark on approximation of one special function by trigonometric polynomials that shows an advantage of nonlinear approximation over linear approximation.

Let us denote for $f \in L_p(\mathbb{T})$

$$E_n(f,\mathscr{T})_p := \inf_{c_k,|k|\le n} \|f(x) - \sum_{|k|\le n} c_k e^{ikx}\|_p.$$

De la Vallée Poussin (1908) and S.N. Bernstein (1912) proved that

$$E_n(|\sin x|,\mathscr{T})_\infty \asymp n^{-1}.$$

R.S. Ismagilov [12] (1974) proved that

$$\sigma_n(|\sin x|,\mathscr{T})_\infty \le C_\varepsilon n^{-6/5+\varepsilon}$$

with arbitrary $\varepsilon > 0$. Later V.E. Maiorov [18] (1986) proved that

$$\sigma_n(|\sin x|,\mathscr{T})_\infty \asymp n^{-3/2}.$$

These results showed an advantage of nonlinear approximation over linear approximation for typical individual functions. Now, when we know that efficiency of the m-term best approximation is good the following important problem arises. Construct an algorithm which realizes good m-term approximation. It is clear from the definition of $\sigma_m(f,\mathscr{T}^d)_p$ that a good algorithm should be a nonlinear algorithm. In Section 4.2 we concentrate on efficiency of the Thresholding Greedy Algorithm. We prove the following theorem [23].

Theorem 4.1.1 *For each $f \in L_p(\mathbb{T}^d)$ we have*

$$\|f - G_m(f,\mathscr{T}^d)\|_p \le (1 + 3m^{h(p)})\sigma_m(f,\mathscr{T}^d)_p, \quad 1 \le p \le \infty,$$

where $h(p) := |1/2 - 1/p|$.

In Section 4.3 we discuss another important class of bases, wavelet type bases. We discuss in detail the simplest representative of wavelet bases, the Haar basis. Denote $\mathscr{H} := \{H_k\}_{k=1}^\infty$ the Haar basis on $[0,1)$ normalized in $L_2(0,1)$: $H_1 = 1$ on $[0,1)$ and for $k = 2^n + l$, $n = 0,1,\dots$, $l = 1,2,\dots,2^n$

$$H_k = \begin{cases} 2^{n/2}, & x \in [(2l-2)2^{-n-1}, (2l-1)2^{-n-1}) \\ -2^{n/2}, & x \in [(2l-1)2^{-n-1}, 2l2^{-n-1}) \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathscr{H}_p := \{H_{k,p}\}_{k=1}^{\infty}$ the Haar basis $\mathscr{H}$ renormalized in $L_p(0,1)$. We will use the following definition of the L_p-equivalence of bases. We say that $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is L_p-equivalent to $\Phi = \{\phi_k\}_{k=1}^{\infty}$ if for any finite set Λ and any coefficients c_k, $k \in \Lambda$, we have

$$C_1(p,\Psi,\Phi)\|\sum_{k\in\Lambda} c_k\phi_k\|_p \le \|\sum_{k\in\Lambda} c_k\psi_k\|_p \le C_2(p,\Psi,\Phi)\|\sum_{k\in\Lambda} c_k\phi_k\|_p \tag{4.5}$$

with two positive constants $C_1(p,\Psi,\Phi), C_2(p,\Psi,\Phi)$ which may depend on p, Ψ, and Φ. For sufficient conditions on Ψ to be L_p-equivalent to $\mathscr{H}$ see, for instance, [28], Section 1.10. We prove the following theorem in Section 3 (see [24]).

Theorem 4.1.2 *Let $1 < p < \infty$ and a basis Ψ be L_p-equivalent to the Haar basis $\mathscr{H}_p$. Then for any $f \in L_p(0,1)$ and any $\rho \in D(f)$ we have*

$$\|f - G_m(f,\Psi,\rho)\|_p \le C(p,\Psi)\sigma_m(f,\Psi)_p \tag{4.6}$$

with a constant $C(p,\Psi)$ independent of f, ρ, and m.

Theorem 4.1.2 shows that each basis Ψ which is L_p-equivalent to the univariate Haar basis $\mathscr{H}_p$ is a greedy basis for $L_p(0,1)$, $1 < p < \infty$. We note that in the case of Hilbert space each orthonormal basis is a greedy basis.

We now give a definition of democratic basis (see [13]) that is needed in a characterization theorem.

Definition 4.1.2 *We say that a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a democratic basis for X if there exists a constant $D := D(X,\Psi)$ such that for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$ we have*

$$\|\sum_{k\in P} \psi_k\| \le D\|\sum_{k\in Q} \psi_k\|.$$

We proved in [13] the following theorem.

Theorem 4.1.3 *A basis is greedy if and only if it is unconditional and democratic.*

In Section 4.4 we consider the general setting of greedy approximation in Banach spaces. We demonstrate in Sections 4.4–4.6 that the Weak Chebyshev Greedy Algorithm (WCGA) which we define momentarily is very good for construction of sparse approximations comparable with best m-term approximations.

Let X be a real Banach space with norm $\|\cdot\| := \|\cdot\|_X$. We say that a set of elements (functions) $\mathscr{D}$ from X is a dictionary if each $g \in \mathscr{D}$ has norm one ($\|g\| = 1$), and the closure of span $\mathscr{D}$ is X. For a nonzero element $g \in X$ we let F_g denote a norming (peak) functional for g:

$$\|F_g\|_{X^*} = 1, \qquad F_g(g) = \|g\|_X.$$

The existence of such a functional is guaranteed by the Hahn-Banach theorem.

Let $\tau := \{t_k\}_{k=1}^{\infty}$ be a given weakness sequence of nonnegative numbers $t_k \le 1$, $k = 1, \dots$. We define the Weak Chebyshev Greedy Algorithm (WCGA) (see [25]) as a generalization for Banach spaces of the Weak Orthogonal Matching Pursuit (WOMP). In a Hilbert space the WCGA coincides with the WOMP. The WOMP is very popular in signal processing, in particular, in compressed sensing. In approximation theory the WOMP is called the Weak Orthogonal Greedy Algorithm (WOGA). We study in detail the WCGA in this paper.

Weak Chebyshev Greedy Algorithm (WCGA). Let f_0 be given. Then for each $m \ge 1$ we have the following inductive definition.

(1) $\varphi_m := \varphi_m^{c,\tau} \in \mathscr{D}$ is any element satisfying

$$|F_{f_{m-1}}(\varphi_m)| \ge t_m \sup_{g \in \mathscr{D}} |F_{f_{m-1}}(g)|.$$

(2) Define

$$\Phi_m := \Phi_m^{\tau} := \operatorname{span}\{\varphi_j\}_{j=1}^m,$$

and define $G_m := G_m^{c,\tau}$ to be the best approximant to f_0 from Φ_m.

(3) Let

$$f_m := f_m^{c,\tau} := f_0 - G_m.$$

In this paper we only consider the case when $t_k = t \in (0,1]$, $k = 1, 2, \dots$.

The trigonometric system is a classical system that is known to be difficult to study. In Sections 4.4–4.6 we study among other problems the problem of nonlinear sparse approximation with respect to it. Let $\mathscr{RT}$ denote the real trigonometric system $1, \sin 2\pi x, \cos 2\pi x, \dots$ on $[0,1]$ and let $\mathscr{RT}_p$ to be its version normalized in $L_p(0,1)$. Denote $\mathscr{RT}_p^d := \mathscr{RT}_p \times \dots \times \mathscr{RT}_p$ the d-variate trigonometric system. We need to consider the real trigonometric system because the algorithm WCGA is well studied for the real Banach space. In order to illustrate performance of the WCGA we discuss in this section the abovementioned problem for the trigonometric system. We discuss performance of the Weak Chebyshev Greedy Algorithm (WCGA) with respect to the trigonometric system. We prove here the following Lebesgue-type inequality for the WCGA from [29] (see Example 2 in Section 6).

Theorem 4.1.4 *Let $\mathscr{D}$ be the normalized in L_p, $2 \le p < \infty$, real d-variate trigonometric system. Then for any $f_0 \in L_p$ the WCGA with weakness parameter t gives*

$$\|f_{C(t,p,d)m\ln(m+1)}\|_p \le C\sigma_m(f_0, \mathscr{D})_p. \tag{4.7}$$

The Open Problem 7.1 (p. 91) from [26] asks if (4.7) holds without an extra $\ln(m+1)$ factor. Theorem 4.1.4 is the first result on the Lebesgue-type inequalities for the WCGA with respect to the trigonometric system. It provides a progress in solving the abovementioned open problem, but the problem is still open.

Theorem 4.1.4 shows that the WCGA is very well designed for the trigonometric system. We show in Example 1 of Section 4.6 that an analog of (4.7) holds for uniformly bounded orthogonal systems. We note that it is known (see [28] and Theorem 4.1.2 above) that the TGA is very well designed for bases L_p-equivalent to the Haar basis, $1 < p < \infty$. We discuss performance of the WCGA in more detail in Sections 4.6 and 4.7.

The proof of Theorem 4.1.4 uses technique developed in [29] for proving the Lebesgue-type inequalities for redundant dictionaries with special properties. We present these results in Sections 4.4–4.6. These results are an extension of earlier results from [17]. In Section 4.6 we test the power of general results from Section 4.4 on specific dictionaries, namely, on bases. Section 4.6 provides a number of examples, including the trigonometric system, where the technique from Sections 4.4 and 4.5 can be successfully applied. In particular, results from Section 4.6 demonstrate that the general technique from Sections 4.4 and 4.5 provides almost optimal m-term approximation results for uniformly bounded orthogonal systems (see Example 1). Example 7 shows that an extra assumption that a uniformly bounded orthogonal system Ψ is a quasi-greedy basis allows us to improve inequality (4.7):

$$\|f_{C(t,p,\Psi)m\ln\ln(m+3)}\|_p \le C\sigma_m(f_0,\Psi)_p.$$

The paper is a survey based on papers [17, 23, 25], and [29].

4.2 The Trigonometric System

We prove Theorem 4.1.1 from Introduction in this section. We formulate it here for convenience.

Theorem 4.2.1 *For each $f \in L_p(\mathbb{T}^d)$ we have*

$$\|f - G_m(f,\mathscr{T}^d)\|_p \le (1+3m^{h(p)})\sigma_m(f,\mathscr{T}^d)_p, \quad 1 \le p \le \infty,$$

where $h(p) := |1/2 - 1/p|$.

Proof. We treat separately the two cases $1 \le p \le 2$ and $2 \le p \le \infty$. Before splitting into these two cases we prove one auxiliary statement for $1 \le p \le \infty$. Here we use the notation

$$\hat{f}(k) := (2\pi)^{-d}\int_{\mathbb{T}^d} f(x)e^{-i(k,x)}dx.$$

Lemma 4.2.1 *Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset with cardinality $|\Lambda| = m$. Then for the operator S_Λ defined on $L_1(\mathbb{T}^d)$ by*

$$S_\Lambda(f) := \sum_{k\in\Lambda} \hat{f}(k)e^{i(k,x)},$$

we have for all $1 \le p \le \infty$,

$$\|S_\Lambda(f)\|_p \le m^{h(p)}\|f\|_p. \tag{4.8}$$

Proof. For a given linear operator A denote by $\|A\|_{a\to b}$ the norm of this operator as an operator from $L_a(\mathbb{T}^d)$ to $L_b(\mathbb{T}^d)$. Then it is obvious that

$$\|S_\Lambda\|_{2\to 2} = 1. \tag{4.9}$$

Consider

$$\mathscr{D}_\Lambda(x) := \sum_{k\in\Lambda} e^{i(k,x)}, \tag{4.10}$$

then

$$S_\Lambda(f) = f * \mathscr{D}_\Lambda := (2\pi)^{-d}\int_{\mathbb{T}^d} f(x-y)\mathscr{D}_\Lambda(y)dy$$

and for $p=1$, or $p=\infty$ we have

$$\|S_\Lambda\|_{p\to p} \le \|\mathscr{D}_\Lambda\|_1 \le \|\mathscr{D}_\Lambda\|_2 = m^{1/2}. \tag{4.11}$$

The relations (4.9) and (4.11) and the Riesz-Thorin theorem (see [36]) imply (4.8).

We now return to the proof of Theorem 4.2.1.

Case 1: $2 \le p \le \infty$. Take any function $f \in L_p(\mathbb{T}^d)$. Let t_m be a trigonometric polynomial which realizes the best m-term approximation to f in $L_p(\mathbb{T}^d)$. For the existence of t_m see Theorem 1.7 from [28], p. 10. Denote by Λ the set of frequencies of t_m, i.e. $\Lambda := \{k : \hat{t}_m(k) \neq 0\}$. Then $|\Lambda| \le m$. Denote by Λ' the set of frequencies of $G_m(f) := G_m(f, \mathscr{T}^d)$. Then $|\Lambda'| = m$. Let us use the representation

$$f - G_m(f) = f - S_{\Lambda'}(f) = f - S_\Lambda(f) + S_\Lambda(f) - S_{\Lambda'}(f).$$

From this representation we derive

$$\|f - G_m(f)\|_p \le \|f - S_\Lambda(f)\|_p + \|S_\Lambda(f) - S_{\Lambda'}(f)\|_p. \tag{4.12}$$

We use Lemma 4.2.1 to estimate the first term in the right-hand side of (4.12)

$$\|f - S_\Lambda(f)\|_p = \|f - t_m - S_\Lambda(f - t_m)\|_p \le (1 + m^{h(p)})\sigma_m(f, \mathscr{T}^d)_p. \tag{4.13}$$

In estimating the second term in (4.12) we use the well-known inequality $\|f\|_2 \le \|f\|_p$ for $2 \le p \le \infty$ and the following lemma.

Lemma 4.2.2 *Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset with cardinality $|\Lambda| = n$. Then, for $2 \le p \le \infty$, we have*

$$\|S_\Lambda(f)\|_p \le n^{h(p)}\|f\|_2. \tag{4.14}$$

Proof. For $p = \infty$ we have

$$\|S_\Lambda(f)\|_\infty \le \sum_{k\in\Lambda} |\hat{f}(k)| \le n^{1/2}(\sum_{k\in\Lambda} |\hat{f}(k)|^2)^{1/2} \le n^{1/2}\|f\|_2. \tag{4.15}$$

For $2 < p < \infty$ we use (4.15) and the following well-known inequality

$$\|g\|_p \le \|g\|_2^{2/p}\|g\|_\infty^{1-2/p}.$$

We continue estimating $\|S_\Lambda(f) - S_{\Lambda'}(f)\|_p$. Using Lemma 4.2.2 we get

$$\|S_\Lambda(f) - S_{\Lambda'}(f)\|_p = \|S_{\Lambda\setminus\Lambda'}(f) - S_{\Lambda'\setminus\Lambda}(f)\|_p \leq$$

$$\|S_{\Lambda\setminus\Lambda'}(f)\|_p + \|S_{\Lambda'\setminus\Lambda}(f)\|_p \leq m^{h(p)}(\|S_{\Lambda\setminus\Lambda'}(f)\|_2 + \|S_{\Lambda'\setminus\Lambda}(f)\|_2). \tag{4.16}$$

The definition of Λ' and the relations $|\Lambda'| = m$, $|\Lambda| \leq m$ imply

$$\|S_{\Lambda\setminus\Lambda'}(f)\|_2 \leq \|S_{\Lambda'\setminus\Lambda}(f)\|_2. \tag{4.17}$$

Finally, we have

$$\|S_{\Lambda'\setminus\Lambda}(f)\|_2 \leq \|f - S_\Lambda(f)\|_2 \leq \|f - t_m\|_2 \leq \|f - t_m\|_p = \sigma_m(f, \mathscr{T}^d)_p. \tag{4.18}$$

Combining the relations (4.16)–(4.18) we get

$$\|S_\Lambda(f) - S_{\Lambda'}(f)\|_p \leq 2m^{h(p)}\sigma_m(f, \mathscr{T}^d)_p. \tag{4.19}$$

The relations (4.12), (4.13) and (4.19) result in

$$\|f - G_m(f)\|_p \leq (1 + 3m^{h(p)})\sigma_m(f, \mathscr{T}^d)_p.$$

This completes the proof of Theorem 4.2.1 in the case $2 \leq p \leq \infty$.

Case 2: $1 \leq p \leq 2$. We keep the notation of Case 1. We start again with the inequality (4.12). Next, the inequality (4.13) holds also for $1 \leq p \leq 2$ because it is based on Lemma 4.8 which covers the whole range $1 \leq p \leq \infty$ of the parameter p. Thus, it remains to estimate $\|S_\Lambda(f) - S_{\Lambda'}(f)\|_p$. Using the inequality $\|f\|_p \leq \|f\|_2$ we get

$$\|S_\Lambda(f) - S_{\Lambda'}(f)\|_p = \|S_{\Lambda\setminus\Lambda'}(f) - S_{\Lambda'\setminus\Lambda}(f)\|_p \leq$$

$$\|S_{\Lambda\setminus\Lambda'}(f)\|_p + \|S_{\Lambda'\setminus\Lambda}(f)\|_p \leq \|S_{\Lambda\setminus\Lambda'}(f)\|_2 + \|S_{\Lambda'\setminus\Lambda}(f)\|_2. \tag{4.20}$$

In order to estimate $\|S_{\Lambda'\setminus\Lambda}(f)\|_2$ we use the part of the Hausdorff-Young theorem (see [36]) which states that

$$\|(\hat{f}(k))_{k\in\mathbb{Z}^d}\|_{\ell_{p'}} \leq \|f\|_p, \quad 1 \leq p \leq 2, \quad p' := \frac{p}{p-1}.$$

We have

$$\|S_{\Lambda'\setminus\Lambda}(f)\|_2 = \|(\hat{f}(k))_{k\in\Lambda'\setminus\Lambda}\|_{\ell_2} \leq$$

$$|\Lambda' \setminus \Lambda|^{1/p-1/2}\|(\hat{f}(k))_{k\in\Lambda'\setminus\Lambda}\|_{\ell_{p'}} \leq m^{h(p)}\|(\hat{f}(k) - \hat{t}_m(k))_{k\in\mathbb{Z}^d}\|_{\ell_{p'}} \leq$$

$$m^{h(p)}\|f - t_m\|_p = m^{h(p)}\sigma_m(f, \mathscr{T}^d)_p. \tag{4.21}$$

Gathering (4.12), (4.13), (4.17), (4.20), and (4.21) we get

$$\|f - G_m(f)\|_p \leq (1 + 3m^{h(p)})\sigma_m(f, \mathscr{T}^d)_p,$$

which completes the proof of Theorem 4.2.1.

REMARK 4.22 *Lemma 4.2.1 implies for all* $1 \le p \le \infty$

$$\|G_m(f)\|_p \le m^{h(p)} \|f\|_p. \tag{4.23}$$

REMARK 4.24 *There is a positive absolute constant* C *such that for each* m *and* $1 \le p \le \infty$ *there exists a function* $f \ne 0$ *with the property*

$$\|G_m(f)\|_p \ge C m^{h(p)} \|f\|_p.$$

REMARK 4.25 *The trivial inequality* $\sigma_m(f, \mathscr{T}^d)_p \le \|f\|_p$ *and Remark 4.24 show that the factor* $m^{h(p)}$ *in Theorem 4.8 is sharp in the sense of order.*

REMARK 4.26 *Using Remark 4.24 it is easy to construct for each* $p \ne 2$ *a function* $f \in L_p(\mathbb{T})$ *such that the sequence* $\{\|G_m(f)\|_p\}_{m=1}^\infty$ *is not bounded.*

Remarks 4.24–4.26 show that the TGA does not work well for the trigonometric system. Here are two more results in this direction from [26].

Theorem 4.2.2 *There exists a continuous function* f *such that* $G_m(f, \mathscr{T})$ *does not converge to* f *in* L_p *for any* $p > 2$.

Theorem 4.2.3 *There exists a function* f *that belongs to any* L_p, $p < 2$, *such that* $G_m(f, \mathscr{T})$ *does not converge to* f *in measure.*

We now make some remarks about possible generalizations of Theorem 4.2.1. Reviewing the proof of Theorem 4.2.1 one verifies that all arguments hold true for any orthonormal system $\{\phi_j\}_{j=1}^\infty$ of uniformly bounded functions $\|\phi_j\|_\infty \le M, j = 1, 2, \ldots$. The only difference is that instead of the Hausdorff–Young theorem we shall use the F. Riesz theorem and the constants in Lemmas 4.2.1 and 4.2.2 will depend on M. Let us formulate the corresponding analog of Theorem 4.2.1. Let $\Phi := \{\phi_j\}_{j=1}^\infty$ be an orthonormal system in $L_2(\mathbb{T}^d)$ such that $\|\phi_j\|_\infty \le M, j = 1, 2, \ldots$.

Theorem 4.2.4 *For any orthonormal system* $\Phi = \{\phi_j\}_{j=1}^\infty$ *of uniformly bounded functions* $\|\phi_j\|_\infty \le M$ *there exists a constant* $C(M)$ *such that*

$$\|f - G_m(f, \Phi)\|_p \le C(M) m^{h(p)} \sigma_m(f, \Phi)_p, \quad 1 \le p \le \infty,$$

where $h(p) := |1/2 - 1/p|$.

4.3 The Wavelet Bases

In this section it will be convenient for us to index elements of bases by dyadic intervals: $\psi_1 =: \psi_{[0,1]}$ and

$$\psi_{2^n+l} =: \psi_I, \quad I = [(l-1)2^{-n}, l2^{-n}).$$

We note that there is another natural greedy-type algorithm based on ordering $\|c_k(f,\Psi)\psi_k\|$ instead of ordering absolute values of coefficients. In this case we do not need the restriction $\|\psi_k\| \geq C > 0$, $k = 1,2,\dots$. Let $\Lambda_m(f)$ be a set of indices such that

$$\min_{k\in\Lambda_m(f)} \|c_k(f,\Psi)\psi_k\| \geq \max_{k\notin\Lambda_m(f)} \|c_k(f,\Psi)\psi_k\|.$$

We define $G_m^X(f,\Psi)$ by the formula

$$G_m^X(f,\Psi) := S_{\Lambda_m(f)}(f,\Psi), \quad \text{where} \quad S_E(f) := S_E(f,\Psi) := \sum_{k\in E} c_k(f,\Psi)\psi_k.$$

It is clear that for a normalized basis ($\|\psi_k\| = 1$, $k = 1,2,\dots$) the above greedy algorithm coincides with the TGA. It is also clear that the above greedy algorithm $G_m^X(\cdot,\Psi)$ can be considered as a greedy algorithm $G_m(\cdot,\Psi')$, with $\Psi' := \{\psi_k/\|\psi_k\|\}_{k=1}^{\infty}$ being a normalized version of the Ψ. Thus, we will concentrate on studying the algorithm $G_m(\cdot,\Psi)$. In the above definition of $G_m(\cdot,\Psi)$ we impose an extra condition on a basis Ψ: $\inf_k \|\psi_k\| > 0$. This restriction allows us to define $G_m(f,\Psi)$ for all $f \in X$. We begin with proving Theorem 4.3.1 (see [24]) and note that Theorem 4.1.2 from the Introduction follows from Theorem 4.3.1 by a simple renormalization argument.

Theorem 4.3.1 *Let $1 < p < \infty$ and a basis $\Psi := \{\psi_I\}_I$ be L_p-equivalent to $\mathscr{H}$. Then for any $f \in L_p$ we have*

$$\|f - G_m^p(f,\Psi)\|_p \leq C(p,\Psi)\sigma_m(f,\Psi)_p.$$

Proof. Let us take a parameter $0 < t \leq 1$ and consider the following greedy type algorithm $G^{p,t}$ with regard to the Haar system. For the Haar basis $\mathscr{H}$ we define

$$c_I(f) := \langle f, H_I\rangle = \int_0^1 f(x)H_I(x)dx.$$

Denote $\Lambda_m(t)$ any set of m dyadic intervals such that

$$\min_{I\in\Lambda_m(t)} \|c_I(f)H_I\|_p \geq t \max_{J\notin\Lambda_m(t)} \|c_J(f)H_J\|_p, \tag{4.27}$$

and define

$$G_m^{p,t}(f) := G_m^{p,t}(f,\mathscr{H}) := \sum_{I\in\Lambda_m(t)} c_I(f)H_I. \tag{4.28}$$

For a given function $f \in L_p$ we define

$$g(f) := \sum_I c_I(f,\Psi)H_I. \tag{4.29}$$

It is clear that $g(f) \in L_p$ and

$$\sigma_m(g(f),\mathscr{H})_p \leq C_1(p)^{-1}\sigma_m(f,\Psi)_p, \tag{4.30}$$

here and later on we use brief notation $C_i(p) := C_i(p,\Psi,\mathscr{H})$, $i = 1,2$, for the constants from (4.5). Let

$$G^p_m(f,\Psi) = \sum_{I\in\Lambda_m} c_I(f,\Psi)\psi_I.$$

Next, for any two intervals $I \in \Lambda_m$, $J \notin \Lambda_m$ by the definition of Λ_m we have

$$\|c_I(f,\Psi)\psi_I\|_p \geq \|c_J(f,\Psi)\psi_J\|_p.$$

Using (4.5) we get from here

$$\|c_I(g(f))H_I\|_p = \|c_I(f,\Psi)H_I\|_p \geq C_2(p)^{-1}\|c_I(f,\Psi)\psi_I\|_p \geq$$

$$\geq C_2(p)^{-1}\|c_J(f,\Psi)\psi_J\|_p \geq C_1(p)C_2(p)^{-1}\|c_J(g(f))H_J\|_p. \tag{4.31}$$

This inequality implies that for any m we can find a set $\Lambda_m(t)$, where $t = C_1(p)$ $C_2(p)^{-1}$, such that $\Lambda_m(t) = \Lambda_m$ and, therefore,

$$\|f - G^p_m(f,\Psi)\|_p \leq C_2(p)\|g(f) - G^{p,t}_m(g(f))\|_p. \tag{4.32}$$

The relations (4.30) and (4.32) show that Theorem 4.3.1 follows from Theorem 4.3.2.

Theorem 4.3.2 *Let* $1 < p < \infty$ *and* $0 < t \leq 1$. *Then for any* $g \in L_p$ *we have*

$$\|g - G^{p,t}_m(g,\mathscr{H})\|_p \leq C(p,t)\sigma_m(g,\mathscr{H})_p.$$

Proof. The Littlewood-Paley Theorem for the Haar system gives for $1 < p < \infty$

$$C_3(p)\|(\sum_I |c_I(g)H_I|^2)^{1/2}\|_p \leq \|g\|_p \leq C_4(p)\|(\sum_I |c_I(g)H_I|^2)^{1/2}\|_p. \tag{4.33}$$

We formulate first two simple corollaries from (4.33):

$$\|g\|_p \leq C_5(p)(\sum_I \|c_I(g)H_I\|_p^p)^{1/p}, \qquad 1 < p \leq 2, \tag{4.34}$$

$$\|g\|_p \leq C_6(p)(\sum_I \|c_I(g)H_I\|_p^2)^{1/2}, \qquad 2 \leq p < \infty. \tag{4.35}$$

The dual inequalities to (4.34) and (4.35) are

$$\|g\|_p \geq C_7(p)(\sum_I \|c_I(g)H_I\|_p^2)^{1/2}, \qquad 1 < p \leq 2, \tag{4.36}$$

$$\|g\|_p \geq C_8(p)(\sum_I \|c_I(g)H_I\|_p^p)^{1/p}, \qquad 2 \leq p < \infty. \tag{4.37}$$

We proceed to the proof of Theorem 4.3.2. Let T_m be an m-term Haar polynomial of best m-term approximation to g in L_p (for existence see [1, 7] and also Theorems 1.8 and 1.9 from [28]):

$$T_m = \sum_{I \in \Lambda} a_I H_I, \quad |\Lambda| = m.$$

For any finite set Q of dyadic intervals we denote by S_Q the projector

$$S_Q(f) := \sum_{I \in Q} c_I(f) H_I.$$

From (4.33) we get

$$\|g - S_\Lambda(g)\|_p = \|g - T_m - S_\Lambda(g - T_m)\|_p \le \|Id - S_\Lambda\|_{p \to p} \sigma_m(g, \mathscr{H})_p \le$$

$$C_4(p) C_3(p)^{-1} \sigma_m(g, \mathscr{H})_p, \tag{4.38}$$

where Id denotes the identical operator. Further, we have

$$G_m^{p,t}(g) = S_{\Lambda_m(t)}(g),$$

and

$$\|g - G_m^{p,t}(g)\|_p \le \|g - S_\Lambda(g)\|_p + \|S_\Lambda(g) - S_{\Lambda_m(t)}(g)\|_p. \tag{4.39}$$

The first term in the right side of (4.39) has been estimated in (4.38). We now estimate the second term. We represent it in the form

$$S_\Lambda(g) - S_{\Lambda_m(t)}(g) = S_{\Lambda \setminus \Lambda_m(t)}(g) - S_{\Lambda_m(t) \setminus \Lambda}(g)$$

and remark that similarly to (4.38) we get

$$\|S_{\Lambda_m(t) \setminus \Lambda}(g)\|_p \le C_9(p) \sigma_m(g, \mathscr{H})_p. \tag{4.40}$$

The key point of the proof of Theorem 4.3.2 is the estimate

$$\|S_{\Lambda \setminus \Lambda_m(t)}(g)\|_p \le C(p,t) \|S_{\Lambda_m(t) \setminus \Lambda}(g)\|_p \tag{4.41}$$

which will be derived from the following two lemmas.

Lemma 4.3.1 *Consider*

$$f = \sum_{I \in Q} c_I H_I, \qquad |Q| = N.$$

Let $1 \le p < \infty$. *Assume*

$$\|c_I H_I\|_p \le 1, \qquad I \in Q. \tag{4.42}$$

Then

$$\|f\|_p \le C_{10}(p) N^{1/p}.$$

Lemma 4.3.2 *Consider*

$$f = \sum_{I \in Q} c_I H_I, \qquad |Q| = N.$$

Let $1 < p \le \infty$. *Assume*

$$\|c_I H_I\|_p \ge 1, \qquad I \in Q.$$

Then

$$\|f\|_p \ge C_{11}(p) N^{1/p}.$$

Proof. First we prove Lemma 4.3.1. We note that in the case $1 < p \le 2$ the statement of Lemma 4.3.1 follows from (4.34). We will give a proof of this lemma for all $1 \le p < \infty$. We have

$$\|c_I H_I\|_p = |c_I||I|^{1/p-1/2}.$$

The assumption (4.42) implies

$$|c_I| \le |I|^{1/2-1/p}.$$

Next, we have

$$\|f\|_p \le \| \sum_{I\in Q} |c_I H_I| \|_p \le \| \sum_{I \in Q} |I|^{-1/p} \chi_I(x) \|_p, \tag{4.43}$$

where $\chi_I(x)$ is a characteristic function of the interval I

$$\chi_I(x) = \begin{cases} 1, & x \in I \\ 0, & x \notin I. \end{cases}$$

In order to proceed further we need a simple lemma. Statements similar to Lemma 4.3.3 are often used in the theory of wavelets (see, for instance, [11]).

Lemma 4.3.3 *Let* $n_1 < n_2 < \cdots < n_s$ *be integers and let* $E_j \subset [0,1]$ *be measurable sets,* $j = 1,\dots,s$. *Then for any* $0 < q < \infty$ *we have*

$$\int_0^1 (\sum_{j=1}^s 2^{n_j/q} \chi_{E_j}(x))^q dx \le C_{12}(q) \sum_{j=1}^s 2^{n_j} |E_j|.$$

Proof. Denote

$$F(x) := \sum_{j=1}^s 2^{n_j/q} \chi_{E_j}(x)$$

and estimate it on the sets

$$E_l^- := E_l \setminus \cup_{k=l+1}^s E_k, \quad l = 1,\dots,s-1; \quad E_s^- := E_s.$$

We have for $x \in E_l^-$

$$F(x) \le \sum_{j=1}^l 2^{n_j/q} \le C(q) 2^{n_l/q}.$$

Therefore,

$$\int_0^1 F(x)^q dx \le C(q)^q \sum_{l=1}^{s} 2^{n_l}|E_l^-| \le C(q)^q \sum_{l=1}^{s} 2^{n_l}|E_l|,$$

which proves the lemma.

We return to the proof of Lemma 4.3.1. Denote by $n_1 < n_2 < \cdots < n_s$ all integers such that there is $I \in Q$ with $|I| = 2^{-n_j}$. Introduce the sets

$$E_j := \cup_{I\in Q; |I|=2^{-n_j}} I.$$

Then the number N of elements in Q can be written in the form

$$N = \sum_{j=1}^{s} |E_j| 2^{n_j}. \tag{4.44}$$

Using these notations the right-hand side of (4.43) can be rewritten as

$$Y := \left(\int_0^1 \left(\sum_{j=1}^{s} 2^{n_j/p}\chi_{E_j}(x)\right)^p dx\right)^{1/p}.$$

Applying Lemma 4.3.3 with $q = p$ we get

$$\|f\|_p \le Y \le C_{13}(p)\left(\sum_{j=1}^{s} |E_j| 2^{n_j}\right)^{1/p} = C_{13}(p)N^{1/p}.$$

On the last step we used (4.44). Lemma 4.3.1 is proved now.

Proof. We now prove Lemma 4.3.2. We derive Lemma 4.3.2 from Lemma 4.3.1. Define

$$u := \sum_{I\in Q} \bar{c}_I |c_I|^{-1} |I|^{1/p-1/2} H_I,$$

where the bar means complex conjugate number. Then for $p' = \frac{p}{p-1}$ we have

$$\|\bar{c}_I |c_I|^{-1} |I|^{1/p-1/2} H_I\|_{p'} = 1$$

and by Lemma 4.3.1

$$\|u\|_{p'} \le C_{10}(p) N^{1/p'}. \tag{4.45}$$

Consider $\langle f, u\rangle$. We have on one hand

$$\langle f, u\rangle = \sum_{I\in Q} |c_I||I|^{1/p-1/2} = \sum_{I\in Q} \|c_I H_I\|_p \ge N, \tag{4.46}$$

on the other hand

$$\langle f, u\rangle \le \|f\|_p \|u\|_{p'}. \tag{4.47}$$

Combining (4.45)–(4.47) we get the statement of Lemma 4.3.2.

We now complete the proof of Theorem 4.3.2. It remained to prove inequality (4.41). Denote

$$A := \max_{I\in\Lambda\setminus\Lambda_m(t)} \|c_I(g)H_I\|_p,$$

and

$$B := \min_{I\in\Lambda_m(t)\setminus\Lambda} \|c_I(g)H_I\|_p.$$

Then by the definition of $\Lambda_m(t)$ we have

$$B \geq tA. \tag{4.48}$$

Using Lemma 4.3.1 we get

$$\|S_{\Lambda\setminus\Lambda_m(t)}(g)\|_p \leq AC_{10}(p)|\Lambda\setminus\Lambda_m(t)|^{1/p} \leq t^{-1}BC_{10}(p)|\Lambda\setminus\Lambda_m(t)|^{1/p}. \tag{4.49}$$

Using Lemma 4.3.2 we get

$$\|S_{\Lambda_m(t)\setminus\Lambda}(g)\|_p \geq BC_{11}(p)|\Lambda_m(t)\setminus\Lambda|^{1/p}. \tag{4.50}$$

Taking into account that $|\Lambda_m(t)\setminus\Lambda| = |\Lambda\setminus\Lambda_m(t)|$ we get from (4.49) and (4.50) inequality (4.41).

The proof of Theorem 3.2 is complete.

4.4 Lebesgue-Type Inequalities: General Results

A very important advantage of the WCGA is its convergence and rate of convergence properties. The WCGA is well defined for all m. Moreover, it is known (see [25] and [28]) that the WCGA with $\tau = \{t\}$ converges for all f_0 in all uniformly smooth Banach spaces with respect to any dictionary. That is, when X is a real Banach space and the modulus of smoothness of X is defined as follows

$$\rho(u) := \frac{1}{2}\sup_{x,y;\|x\|=\|y\|=1} |\|x+uy\|+\|x-uy\|-2|, \tag{4.51}$$

then the uniformly smooth Banach space is the one with $\rho(u)/u \to 0$ when $u \to 0$.

We discuss here the Lebesgue-type inequalities for the WCGA with $\tau = \{t\}$, $t \in (0,1]$. This discussion is based on papers [17] and [29]. For notational convenience we consider here a countable dictionary $\mathscr{D} = \{g_i\}_{i=1}^{\infty}$. The following assumptions **A1** and **A2** were used in [17]. For a given f_0 let sparse element (signal)

$$f := f^{\varepsilon} = \sum_{i\in T} x_i g_i, \quad g_i \in \mathscr{D},$$

be such that $\|f_0 - f^{\varepsilon}\| \leq \varepsilon$ and $|T| = K$. For $A \subset T$ denote

$$f_A := f_A^{\varepsilon} := \sum_{i\in A} x_i g_i.$$

A1. We say that $f=\sum_{i\in T}x_ig_i$ satisfies the Nikol'skii-type $\ell_1 X$ inequality with parameter r if

$$\sum_{i\in A}|x_i| \le C_1|A|^r\|f_A\|, \quad A\subset T. \tag{4.52}$$

We say that a dictionary $\mathscr{D}$ has the Nikol'skii-type $\ell_1 X$ property with parameters K, r if any K-sparse element satisfies the Nikol'skii-type $\ell_1 X$ inequality with parameter r.

A2. We say that $f=\sum_{i\in T}x_ig_i$ has incoherence property with parameters D and U if for any $A\subset T$ and any Λ such that $A\cap\Lambda=\emptyset$, $|A|+|\Lambda|\le D$ we have for any $\{c_i\}$

$$\|f_A-\sum_{i\in\Lambda}c_ig_i\| \ge U^{-1}\|f_A\|. \tag{4.53}$$

We say that a dictionary $\mathscr{D}$ is (K,D)-unconditional with a constant U if for any $f=\sum_{i\in T}x_ig_i$ with $|T|\le K$ inequality (4.53) holds.

The term *unconditional* in **A2** is justified by the following remark. The above definition of (K,D)-unconditional dictionary is equivalent to the following definition. Let $\mathscr{D}$ be such that any subsystem of D distinct elements $e_1,\dots,e_D$ from $\mathscr{D}$ is linearly independent and for any $A\subset[1,D]$ with $|A|\le K$ and any coefficients $\{c_i\}$ we have

$$\|\sum_{i\in A}c_ie_i\| \le U\|\sum_{i=1}^{D}c_ie_i\|.$$

It is convenient for us to use the following assumption **A3** introduced in [29] which is a corollary of assumptions **A1** and **A2**.

A3. We say that $f=\sum_{i\in T}x_ig_i$ has ℓ_1 incoherence property with parameters D, V, and r if for any $A\subset T$ and any Λ such that $A\cap\Lambda=\emptyset$, $|A|+|\Lambda|\le D$ we have for any $\{c_i\}$

$$\sum_{i\in A}|x_i| \le V|A|^r\|f_A-\sum_{i\in\Lambda}c_ig_i\|. \tag{4.54}$$

A dictionary $\mathscr{D}$ has ℓ_1 incoherence property with parameters K, D, V, and r if for any $A\subset B$, $|A|\le K$, $|B|\le D$ we have for any $\{c_i\}_{i\in B}$

$$\sum_{i\in A}|c_i| \le V|A|^r\|\sum_{i\in B}c_ig_i\|.$$

It is clear that **A1** and **A2** imply **A3** with $V=C_1U$. Also, **A3** implies **A1** with $C_1=V$ and **A2** with $U=VK^r$. Obviously, we can restrict ourselves to $r\le 1$.

We now proceed to main results of [17] and [29] on the WCGA with respect to redundant dictionaries. The following Theorem 4.4.1 from [29] in the case $q=2$ was proved in [17].

Theorem 4.4.1 *Let X be a Banach space with $\rho(u)\le\gamma u^q$, $1<q\le 2$. Suppose K-sparse f^ε satisfies* **A1**, **A2** *and $\|f_0-f^\varepsilon\|\le\varepsilon$. Assume that $rq'\ge 1$. Then the WCGA with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,\gamma,C_1)U^{q'}\ln(U+1)K^{rq'}}\| \le C\varepsilon \quad \textit{for} \quad K+C(t,\gamma,C_1)U^{q'}\ln(U+1)K^{rq'}\le D$$

with an absolute constant C.

It was pointed out in [17] that Theorem 4.4.1 provides a corollary for Hilbert spaces that gives sufficient conditions somewhat weaker than the known RIP conditions on $\mathscr{D}$ for the Lebesgue-type inequality to hold. We formulate the corresponding definitions and results. Let $\mathscr{D}$ be the Riesz dictionary with depth D and parameter $\delta \in (0,1)$. This class of dictionaries is a generalization of the class of classical Riesz bases. We give a definition in a general Hilbert space (see [28], p. 306).

Definition 4.4.1 *A dictionary $\mathscr{D}$ is called the Riesz dictionary with depth D and parameter $\delta \in (0,1)$ if, for any D distinct elements $e_1,\dots,e_D$ of the dictionary and any coefficients $a=(a_1,\dots,a_D)$, we have*

$$(1-\delta)\|a\|_2^2 \le \|\sum_{i=1}^{D} a_i e_i\|^2 \le (1+\delta)\|a\|_2^2. \tag{4.55}$$

We denote the class of Riesz dictionaries with depth D and parameter $\delta \in (0,1)$ by $R(D,\delta)$.

The term Riesz dictionary with depth D and parameter $\delta \in (0,1)$ is another name for a dictionary satisfying the Restricted Isometry Property (RIP) with parameters D and δ. The following simple lemma holds.

Lemma 4.4.1 *Let $\mathscr{D} \in R(D,\delta)$ and let $e_j \in \mathscr{D}$, $j=1,\dots,s$. For $f=\sum_{i=1}^{s} a_i e_i$ and $A \subset \{1,\dots,s\}$ denote*

$$S_A(f) := \sum_{i\in A} a_i e_i.$$

If $s \le D$, then

$$\|S_A(f)\|^2 \le (1+\delta)(1-\delta)^{-1}\|f\|^2.$$

Lemma 4.4.1 implies that if $\mathscr{D} \in R(D,\delta)$ then it is (D,D)-unconditional with a constant $U=(1+\delta)^{1/2}(1-\delta)^{-1/2}$.

Theorem 4.4.2 *Let X be a Hilbert space. Suppose K-sparse f^ε satisfies* **A2** *and $\|f_0-f^\varepsilon\| \le \varepsilon$. Then the WOMP with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,U)K}\| \le C\varepsilon \quad \textit{for} \quad K+C(t,U)K \le D$$

with an absolute constant C.

Theorem 4.4.2 implies the following corollaries.

Corollary 4.4.1 *Let X be a Hilbert space. Suppose any K-sparse f satisfies* **A2**. *Then the WOMP with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,U)K}\| \le C\sigma_K(f_0,\mathscr{D}) \quad \textit{for} \quad K+C(t,U)K \le D$$

with an absolute constant C.

Corollary 4.4.2 *Let X be a Hilbert space. Suppose $\mathscr{D} \in R(D,\delta)$. Then the WOMP with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,\delta)K}\| \le C\sigma_K(f_0,\mathscr{D}) \quad \text{for} \quad K + C(t,\delta)K \le D$$

with an absolute constant C.

We emphasized in [17] that in Theorem 4.4.1 we impose our conditions on an individual function f^ε. It may happen that the dictionary does not have the Nikol'skii $\ell_1 X$ property and (K,D)-unconditionality but the given f_0 can be approximated by f^ε which does satisfy assumptions **A1** and **A2**. Even in the case of a Hilbert space the above results from [17] add something new to the study based on the RIP property of a dictionary. First of all, Theorem 4.4.2 shows that it is sufficient to impose assumption **A2** on f^ε in order to obtain exact recovery and the Lebesgue-type inequality results. Second, Corollary 4.4.1 shows that the condition **A2**, which is weaker than the RIP condition, is sufficient for exact recovery and the Lebesgue-type inequality results. Third, Corollary 4.4.2 shows that even if we impose our assumptions in terms of RIP we do not need to assume that $\delta < \delta_0$. In fact, the result works for all $\delta < 1$ with parameters depending on δ.

Theorem 4.4.1 follows from the combination of Theorems 4.4.3 and 4.4.4. In case $q = 2$ these theorems were proved in [17] and in general case $q \in (1,2]$ – in [29].

Theorem 4.4.3 *Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Suppose for a given f_0 we have $\|f_0 - f^\varepsilon\| \le \varepsilon$ with K-sparse $f := f^\varepsilon$ satisfying* **A3**. *Then for any $k \ge 0$ we have for $K + m \le D$*

$$\|f_m\| \le \|f_k\| \exp\left(-\frac{c_1(m-k)}{K^{rq'}}\right) + 2\varepsilon, \quad q' := \frac{q}{q-1},$$

where $c_1 := \frac{t^{q'}}{2(16\gamma)^{\frac{1}{q-1}} V^{q'}}$.

In all theorems that follow we assume $rq' \ge 1$.

Theorem 4.4.4 *Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Suppose K-sparse f^ε satisfies* **A1**, **A2** *and $\|f_0 - f^\varepsilon\| \le \varepsilon$. Then the WCGA with weakness parameter t applied to f_0 provides*

$$\|f_{C'U^{q'}\ln(U+1)K^{rq'}}\| \le CU\varepsilon \quad \text{for} \quad K + C'U^{q'}\ln(U+1)K^{rq'} \le D$$

with an absolute constant C and $C' = C_2(q)\gamma^{\frac{1}{q-1}} C_1^{q'} t^{-q'}$.

We formulate an immediate corollary of Theorem 4.4.4 with $\varepsilon = 0$.

Corollary 4.4.3 *Let X be a Banach space with $\rho(u) \le \gamma u^q$. Suppose K-sparse f satisfies* **A1**, **A2**. *Then the WCGA with weakness parameter t applied to f recovers it exactly after $C'U^{q'}\ln(U+1)K^{rq'}$ iterations under condition $K + C'U^{q'}\ln(U+1)$ $K^{rq'} \le D$.*

We formulate versions of Theorem 4.4.4 with assumptions **A1**, **A2** replaced by a single assumption **A3** and replaced by two assumptions **A2** and **A3**. The corresponding modifications in the proofs go as in the proof of Theorem 4.4.3.

Theorem 4.4.5 *Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Suppose K-sparse f^ε satisfies* **A3** *and $\|f_0 - f^\varepsilon\| \le \varepsilon$. Then the WCGA with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,\gamma,q)V^{q'}\ln(VK)K^{rq'}}\| \le CVK^r\varepsilon \quad \textit{for} \quad K + C(t,\gamma,q)V^{q'}\ln(VK)K^{rq'} \le D$$

with an absolute constant C and $C(t,\gamma,q) = C_2(q)\gamma^{\frac{1}{q-1}}t^{-q'}$.

Theorem 4.4.6 *Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Suppose K-sparse f^ε satisfies* **A2**, **A3** *and $\|f_0 - f^\varepsilon\| \le \varepsilon$. Then the WCGA with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,\gamma,q)V^{q'}\ln(U+1)K^{rq'}}\| \le CU\varepsilon \quad \textit{for} \quad K + C(t,\gamma,q)V^{q'}\ln(U+1)K^{rq'} \le D$$

with an absolute constant C and $C(t,\gamma,q) = C_2(q)\gamma^{\frac{1}{q-1}}t^{-q'}$.

Theorems 4.4.5 and 4.4.3 imply the following analog of Theorem 4.4.1.

Theorem 4.4.7 *Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Suppose K-sparse f^ε satisfies* **A3** *and $\|f_0 - f^\varepsilon\| \le \varepsilon$. Then the WCGA with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,\gamma,q)V^{q'}\ln(VK)K^{rq'}}\| \le C\varepsilon \quad \textit{for} \quad K + C(t,\gamma,q)V^{q'}\ln(VK)K^{rq'} \le D$$

with an absolute constant C and $C(t,\gamma,q) = C_2(q)\gamma^{\frac{1}{q-1}}t^{-q'}$.

The following edition of Theorems 4.4.1 and 4.4.7 is also useful in applications. It follows from Theorems 4.4.6 and 4.4.3.

Theorem 4.4.8 *Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Suppose K-sparse f^ε satisfies* **A2**, **A3** *and $\|f_0 - f^\varepsilon\| \le \varepsilon$. Then the WCGA with weakness parameter t applied to f_0 provides*

$$\|f_{C(t,\gamma,q)V^{q'}\ln(U+1)K^{rq'}}\| \le C\varepsilon \quad \textit{for} \quad K + C(t,\gamma,q)V^{q'}\ln(U+1)K^{rq'} \le D$$

with an absolute constant C and $C(t,\gamma,q) = C_2(q)\gamma^{\frac{1}{q-1}}t^{-q'}$.

4.5 Proofs

Proof of Theorem 4.4.3. We begin with a proof of Theorem 4.4.3.

Proof. Let

$$f := f^{\varepsilon} = \sum_{i\in T} x_i g_i, \quad |T| = K, \quad g_i \in \mathscr{D}.$$

Denote by T^m the set of indices of $g_j \in \mathscr{D}$ picked by the WCGA after m iterations, $\Gamma^m := T \setminus T^m$. Denote by $A_1(\mathscr{D})$ the closure in X of the convex hull of the symmetrized dictionary $\mathscr{D}^{\pm} := \{\pm g, g \in D\}$. We will bound $\|f_m\|$ from above. Assume $\|f_{m-1}\| \geq \varepsilon$. Let $m > k$. We bound from below

$$S_m := \sup_{\phi\in A_1(\mathscr{D})} |F_{f_{m-1}}(\phi)|.$$

Denote $A_m := \Gamma^{m-1}$. Then

$$S_m \geq F_{f_{m-1}}(f_{A_m}/\|f_{A_m}\|_1),$$

where $\|f_A\|_1 := \sum_{i\in A} |x_i|$. Next, by Lemma 6.9, p. 342, from [28] we obtain

$$F_{f_{m-1}}(f_{A_m}) = F_{f_{m-1}}(f^{\varepsilon}) \geq \|f_{m-1}\| - \varepsilon.$$

Thus

$$S_m \geq \|f_{A_m}\|_1^{-1}(\|f_{m-1}\| - \varepsilon). \tag{4.56}$$

From the definition of the modulus of smoothness we have for any λ

$$\|f_{m-1} - \lambda\varphi_m\| + \|f_{m-1} + \lambda\varphi_m\| \leq 2\|f_{m-1}\| \left(1 + \rho\left(\frac{\lambda}{\|f_{m-1}\|}\right)\right) \tag{4.57}$$

and by (1) from the definition of the WCGA and Lemma 6.10 from [28], p. 343, we get

$$|F_{f_{m-1}}(\varphi_m)| \geq t \sup_{g\in\mathscr{D}} |F_{f_{m-1}}(g)| =$$

$$t \sup_{\phi\in A_1(\mathscr{D})} |F_{f_{m-1}}(\phi)| = tS_m.$$

Then either $F_{f_{m-1}}(\varphi_m) \geq tS_m$ or $F_{f_{m-1}}(-\varphi_m) \geq tS_m$. Both cases are treated in the same way. We demonstrate the case $F_{f_{m-1}}(\varphi_m) \geq tS_m$. We have for $\lambda \geq 0$

$$\|f_{m-1} + \lambda\varphi_m\| \geq F_{f_{m-1}}(f_{m-1} + \lambda\varphi_m) \geq \|f_{m-1}\| + \lambda t S_m.$$

From here and from (4.57) we obtain

$$\|f_m\| \leq \|f_{m-1} - \lambda\varphi_m\| \leq \|f_{m-1}\| + \inf_{\lambda\geq 0}(-\lambda t S_m + 2\|f_{m-1}\|\rho(\lambda/\|f_{m-1}\|)).$$

We discuss here the case $\rho(u) \le \gamma u^q$. Using (4.56) we get

$$\|f_m\| \le \|f_{m-1}\| \left(1 - \frac{\lambda t}{\|f_{A_m}\|_1} + 2\gamma \frac{\lambda^q}{\|f_{m-1}\|^q}\right) + \frac{\varepsilon \lambda t}{\|f_{A_m}\|_1}.$$

Let λ_1 be a solution of

$$\frac{\lambda t}{2\|f_{A_m}\|_1} = 2\gamma \frac{\lambda^q}{\|f_{m-1}\|^q}, \quad \lambda_1 = \left(\frac{t\|f_{m-1}\|^q}{4\gamma\|f_{A_m}\|_1}\right)^{\frac{1}{q-1}}.$$

Our assumption (4.54) gives

$$\begin{aligned}\|f_{A_m}\|_1 &= \|(f^\varepsilon - G_{m-1})_{A_m}\|_1 \le VK^r\|f^\varepsilon - G_{m-1}\| \\ &\le VK^r(\|f_0 - G_{m-1}\| + \|f_0 - f^\varepsilon\|) \le VK^r(\|f_{m-1}\| + \varepsilon).\end{aligned}$$

Specify

$$\lambda = \left(\frac{t\|f_{A_m}\|_1^{q-1}}{16\gamma(VK^r)^q}\right)^{\frac{1}{q-1}}.$$

Then, using $\|f_{m-1}\| \ge \varepsilon$ we get

$$\left(\frac{\lambda}{\lambda_1}\right)^{q-1} = \frac{\|f_{A_m}\|_1^q}{4\|f_{m-1}\|^q(VK^r)^q} \le 1$$

and obtain

$$\|f_m\| \le \|f_{m-1}\| \left(1 - \frac{t^{q'}}{2(16\gamma)^{\frac{1}{q-1}}(VK^r)^{q'}}\right) + \frac{\varepsilon t^{q'}}{(16\gamma)^{\frac{1}{q-1}}(VK^r)^{q'}}.$$

Denote $c_1 := \frac{t^{q'}}{2(16\gamma)^{\frac{1}{q-1}}V^{q'}}$. Then

$$\|f_m\| \le \|f_k\| \exp\left(-\frac{c_1(m-k)}{K^{rq'}}\right) + 2\varepsilon.$$

Proof of Theorem 4.4.4. We proceed to a proof of Theorem 4.4.4. Modifications of this proof which are in a style of the above proof of Theorem 4.4.3 give Theorems 4.4.5 and 4.4.6.

Proof. We use the above notations T^m and $\Gamma^m := T \setminus T^m$. Let $k \ge 0$ be fixed. Suppose

$$2^{n-1} < |\Gamma^k| \le 2^n.$$

For $j = 1, 2, \dots, n, n+1$ consider the following pairs of sets A_j, B_j: $A_{n+1} = \Gamma^k$, $B_{n+1} = \emptyset$; for $j \le n$, $A_j := \Gamma^k \setminus B_j$ with $B_j \subset \Gamma^k$ is such that $|B_j| \ge |\Gamma^k| - 2^{j-1}$ and for any set $J \subset \Gamma^k$ with $|J| \ge |\Gamma^k| - 2^{j-1}$ we have

$$\|f_{B_j}\| \le \|f_J\|.$$

We note that this implies that if for some $Q \subset \Gamma^k$ we have

$$\|f_Q\| < \|f_{B_j}\| \quad \text{then} \quad |Q| < |\Gamma^k| - 2^{j-1}. \tag{4.58}$$

For a given $b > 1$, to be specified later, denote by L the index such that $(B_0 := \Gamma^k)$

$$\|f_{B_0}\| < b\|f_{B_1}\|,$$

$$\|f_{B_1}\| < b\|f_{B_2}\|,$$

$$\dots$$

$$\|f_{B_{L-2}}\| < b\|f_{B_{L-1}}\|,$$

$$\|f_{B_{L-1}}\| \ge b\|f_{B_L}\|.$$

Then

$$\|f_{B_j}\| \le b^{L-1-j}\|f_{B_{L-1}}\|, \quad j = 1,2,\dots,L. \tag{4.59}$$

We now proceed to a general step. Let $m > k$ and let $A, B \subset \Gamma^k$ be such that $A = \Gamma^k \setminus B$. As above we bound S_m from below. It is clear that $S_m \ge 0$. Denote $A_m := A \cap \Gamma^{m-1}$. Then

$$S_m \ge F_{f_{m-1}}(f_{A_m}/\|f_{A_m}\|_1).$$

Next,

$$F_{f_{m-1}}(f_{A_m}) = F_{f_{m-1}}(f_{A_m} + f_B - f_B).$$

Then $f_{A_m} + f_B = f^{\varepsilon} - f_{\Lambda}$ with $F_{f_{m-1}}(f_{\Lambda}) = 0$. Moreover, it is easy to see that $F_{f_{m-1}}(f^{\varepsilon}) \ge \|f_{m-1}\| - \varepsilon$. Therefore,

$$F_{f_{m-1}}(f_{A_m} + f_B - f_B) \ge \|f_{m-1}\| - \varepsilon - \|f_B\|.$$

Thus

$$S_m \ge \|f_{A_m}\|_1^{-1} \max(0, \|f_{m-1}\| - \varepsilon - \|f_B\|).$$

By (4.52) we get

$$\|f_{A_m}\|_1 \le C_1|A_m|^r\|f_{A_m}\| \le C_1|A|^r\|f_{A_m}\|.$$

Then

$$S_m \ge \frac{\|f_{m-1}\| - \|f_B\| - \varepsilon}{C_1|A|^r\|f_{A_m}\|}. \tag{4.60}$$

From the definition of the modulus of smoothness we have for any λ

$$\|f_{m-1} - \lambda\varphi_m\| + \|f_{m-1} + \lambda\varphi_m\| \le 2\|f_{m-1}\|(1 + \rho(\frac{\lambda}{\|f_{m-1}\|}))$$

and by (1) from the definition of the WCGA and Lemma 6.10 from [28], p. 343, we get

$$|F_{f_{m-1}}(\varphi_m)| \geq t \sup_{g\in\mathscr{D}} |F_{f_{m-1}}(g)| =$$

$$t \sup_{\phi\in A_1(\mathscr{D})} |F_{f_{m-1}}(\phi)|.$$

From here we obtain

$$\|f_m\| \leq \|f_{m-1}\| + \inf_{\lambda\geq 0}(-\lambda t S_m + 2\|f_{m-1}\|\rho(\lambda/\|f_{m-1}\|)).$$

We discuss here the case $\rho(u) \leq \gamma u^q$. Using (4.60) we get

$$\|f_m\| \leq \|f_{m-1}\| \left(1 - \frac{\lambda t}{C_1|A|^r\|f_{A_m}\|} + 2\gamma\frac{\lambda^q}{\|f_{m-1}\|^q}\right) + \frac{\lambda t(\|f_B\|+\varepsilon)}{C_1|A|^r\|f_{A_m}\|}.$$

Let λ_1 be a solution of

$$\frac{\lambda t}{2C_1|A|^r\|f_{A_m}\|} = 2\gamma\frac{\lambda^q}{\|f_{m-1}\|^q}, \quad \lambda_1 = \left(\frac{t\|f_{m-1}\|^q}{4\gamma C_1|A|^r\|f_{A_m}\|}\right)^{\frac{1}{q-1}}.$$

Our assumption (4.53) gives

$$\|f_{A_m}\| \leq U(\|f_{m-1}\|+\varepsilon).$$

Specify

$$\lambda = \left(\frac{t\|f_{A_m}\|^{q-1}}{16\gamma C_1|A|^r U^q}\right)^{\frac{1}{q-1}}.$$

Then $\lambda \leq \lambda_1$ and we obtain

$$\|f_m\| \leq \|f_{m-1}\| \left(1 - \frac{t^{q'}}{2(16\gamma)^{\frac{1}{q-1}}(C_1 U|A|^r)^{q'}}\right) + \frac{t^{q'}(\|f_B\|+\varepsilon)}{(16\gamma)^{\frac{1}{q-1}}(C_1|A|^r U)^{q'}}. \tag{4.61}$$

Denote $c_1 := \frac{t^{q'}}{2(16\gamma)^{\frac{1}{q-1}}(C_1 U)^{q'}}$. This implies for $m_2 > m_1 \geq k$

$$\|f_{m_2}\| \leq \|f_{m_1}\|(1 - c_1/|A|^{rq'})^{m_2-m_1} + \frac{2c_1(m_2-m_1)}{|A|^{rq'}}(\|f_B\|+\varepsilon). \tag{4.62}$$

Define $m_0 := k$ and, inductively,

$$m_j = m_{j-1} + [\beta|A_j|^{rq'}], \quad j = 1,\dots,n,$$

where $[x]$ denotes the integer part of x. The parameter β is any which satisfies the following inequalities

$$\beta \geq 1, \qquad e^{-c_1\beta/2} < 1/2, \qquad 16Ue^{-c_1\beta/2} < 1.$$

We note that the inequality $\beta \ge 1$ implies that

$$[\beta|A_j|^{rq'}] \ge \beta|A_j|^{rq'}/2.$$

At iterations from $m_{j-1}+1$ to m_j we use $A = A_j$ and obtain from (4.61) that

$$\|f_m\| \le \|f_{m-1}\|(1-u) + 2u(\|f_B\| + \varepsilon), \quad u := c_1|A|^{-rq'}.$$

Using $1-u \le e^{-u}$ and $\sum_{k=0}^{\infty}(1-u)^k = 1/u$ we derive from here

$$\|f_{m_j}\| \le \|f_{m_{j-1}}\|e^{-c_1\beta/2} + 2(\|f_{B_j}\| + \varepsilon).$$

We continue it up to $j = L$. Denote $\eta := e^{-c_1\beta/2}$. Then

$$\|f_{m_L}\| \le \|f_k\|\eta^L + 2\sum_{j=1}^{L}(\|f_{B_j}\| + \varepsilon)\eta^{L-j}.$$

We bound the $\|f_k\|$. It follows from the definition of f_k that $\|f_k\|$ is the error of best approximation of f_0 by the subspace Φ_k. Representing $f_0 = f + f_0 - f$ we see that $\|f_k\|$ is not greater than the error of best approximation of f by the subspace Φ_k plus $\|f_0 - f\|$. This implies $\|f_k\| \le \|f_{B_0}\| + \varepsilon$. Therefore we continue

$$\le (\|f_{B_0}\| + \varepsilon)\eta^L + 2\sum_{j=1}^{L}(\|f_{B_{L-1}}\|(\eta b)^{L-j}b^{-1} + \varepsilon\eta^{L-j})$$

$$\le b^{-1}\|f_{B_{L-1}}\|\left((\eta b)^L + 2\sum_{j=1}^{L}(\eta b)^{L-j}\right) + \frac{2\varepsilon}{1-\eta}.$$

Our choice of β guarantees $\eta < 1/2$. Choose $b = \frac{1}{2\eta}$. Then

$$\|f_{m_L}\| \le \|f_{B_{L-1}}\|8e^{-c_1\beta/2} + 4\varepsilon. \tag{4.63}$$

By (4.53) we get

$$\|f_{\Gamma^{m_L}}\| \le U(\|f_{m_L}\| + \varepsilon) \le U(\|f_{B_{L-1}}\|8e^{-c_1\beta/2} + 5\varepsilon).$$

We note that in the proof of Theorem 4.4.5 we use the above inequality with $U = VK^r \le VK$. If $\|f_{B_{L-1}}\| \le 10U\varepsilon$, then by (4.63)

$$\|f_{m_L}\| \le CU\varepsilon.$$

If $\|f_{B_{L-1}}\| \ge 10U\varepsilon$, then by our choice of β we have $16Ue^{-c_1\beta/2} < 1$ and

$$U(\|f_{B_{L-1}}\|8e^{-c_1\beta/2} + 5\varepsilon) < \|f_{B_{L-1}}\|.$$

Therefore

$$\|f_{\Gamma^{m_L}}\| < \|f_{B_{L-1}}\|.$$

This implies

$$|\Gamma^{m_L}| < |\Gamma^k| - 2^{L-2}.$$

We begin with f_0 and apply the above argument (with $k = 0$). As a result we either get the required inequality or we reduce the cardinality of support of f from $|T| = K$ to $|\Gamma^{m_{L_1}}| < |T| - 2^{L_1-2}$, $m_{L_1} \le \beta 2^{aL_1}$, $a := rq'$. We continue the process and build a sequence m_{L_j} such that $m_{L_j} \le \beta 2^{aL_j}$ and after m_{L_j} iterations we reduce the support by at least 2^{L_j-2}. We also note that $m_{L_j} \le \beta 2^a K^a$. We continue this process till the following inequality is satisfied for the first time

$$m_{L_1} + \cdots + m_{L_n} \ge 2^{2a}\beta K^a. \tag{4.64}$$

Then, clearly,

$$m_{L_1} + \cdots + m_{L_n} \le 2^{2a+1}\beta K^a.$$

Using the inequality

$$(a_1 + \cdots + a_n)^\theta \le a_1^\theta + \cdots + a_n^\theta, \quad a_j \ge 0, \quad \theta \in (0,1]$$

we derive from (4.64)

$$2^{L_1-2} + \cdots + 2^{L_n-2} \ge \left(2^{a(L_1-2)} + \cdots + 2^{a(L_n-2)}\right)^{\frac{1}{a}}$$

$$\ge 2^{-2}\left(2^{aL_1} + \cdots + 2^{aL_n}\right)^{\frac{1}{a}}$$

$$\ge 2^{-2}\left((\beta)^{-1}(m_{L_1} + \cdots + m_{L_n})\right)^{\frac{1}{a}} \ge K.$$

Thus, after not more than $N := 2^{2a+1}\beta K^a$ iterations we recover f exactly and then $\|f_N\| \le \|f_0 - f\| \le \varepsilon$.

4.6 Examples

In this section we discuss applications of Theorems from Section 4.4 for specific dictionaries $\mathscr{D}$. Mostly, $\mathscr{D}$ will be a basis Ψ for X. Because of that we use m instead of K in the notation of sparse approximation. In some of our examples we take $X = L_p$, $2 \le p < \infty$. Then it is known that $\rho(u) \le \gamma u^2$ with $\gamma = (p-1)/2$. In some other examples we take $X = L_p$, $1 < p \le 2$. Then it is known that $\rho(u) \le \gamma u^p$, with $\gamma = 1/p$.

Example 1. Let Ψ be a uniformly bounded orthogonal system normalized in $L_p(\Omega)$, $2 \le p < \infty$, Ω is a bounded domain. Then we have

$$C_1(\Omega,p)\|\psi_j\|_2 \le \|\psi_j\|_p \le C_2(\Omega,p)\|\psi_j\|_2, \quad j=1,2\dots.$$

Next, for $f=\sum_i c_i(f)\psi_i$

$$\sum_{i\in A}|c_i(f)| = \langle f, \sum_{i\in A}(\operatorname{sign} c_i(f))\psi_i\|\psi_i\|_2^{-2}\rangle$$

$$\le \|f\|_2\|\sum_{i\in A}(\operatorname{sign} c_i(f))\psi_i\|\psi_i\|_2^{-2}\|_2 \le C_3(\Omega,p)|A|^{1/2}\|f\|_p.$$

Therefore Ψ satisfies **A3** with $D=\infty$, $V=C_3(\Omega,p)$, $r=1/2$. Theorem 4.4.7 gives

$$\|f_{C(t,p,D)m\ln(m+1)}\|_p \le C\sigma_m(f_0,\Psi)_p. \tag{4.65}$$

Example 1q. Let Ψ be a uniformly bounded orthogonal system normalized in $L_p(\Omega)$, $1<p\le 2$, Ω is a bounded domain. Then we have

$$C_1(\Omega,p)\|\psi_j\|_2 \le \|\psi_j\|_p \le C_2(\Omega,p)\|\psi_j\|_2, \quad j=1,2\dots.$$

Next, for $f=\sum_i c_i(f)\psi_i$

$$\sum_{i\in A}|c_i(f)| = \langle f, \sum_{i\in A}(\operatorname{sign} c_i(f))\psi_i\|\psi_i\|_2^{-2}\rangle$$

$$\le \|f\|_p\|\sum_{i\in A}(\operatorname{sign} c_i(f))\psi_i\|\psi_i\|_2^{-2}\|_{p'} \le C_4(\Omega,p)|A|^{1-1/p'}\|f\|_p.$$

Therefore Ψ satisfies **A3** with $D=\infty$, $V=C_4(\Omega,p)$, $r=1-1/p'$. Theorem 4.4.7 gives

$$\|f_{C(t,p,D)m^{p'-1}\ln(m+1)}\|_p \le C\sigma_m(f_0,\Psi)_p. \tag{4.66}$$

Example 2. Let Ψ be the normalized in L_p, $2\le p<\infty$, real d-variate trigonometric system. Then Example 1 applies and gives for any $f_0\in L_p$

$$\|f_{C(t,p,d)m\ln(m+1)}\|_p \le C\sigma_m(f_0,\Psi)_p. \tag{4.67}$$

We note that (4.67) provides some progress in Open Problem 7.1 (p. 91) from [26].

Example 2q. Let Ψ be the normalized in L_p, $1<p\le 2$, real d-variate trigonometric system. Then Example 1q applies and gives for any $f_0\in L_p$

$$\|f_{C(t,p,d)m^{p'-1}\ln(m+1)}\|_p \le C\sigma_m(f_0,\Psi)_p. \tag{4.68}$$

We need the concept of cotype of a Banach space X. We say that X has cotype s if for any finite number of elements $u_i\in X$ we have the inequality

$$\left(\text{Average}_\pm\|\sum_i \pm u_i\|^s\right)^{1/s} \ge C_s\left(\sum_i\|u_i\|^s\right)^{1/s}.$$

It is known that the L_p spaces with $2 \le p < \infty$ have cotype $s = p$ and L_p spaces with $1 < p \le 2$ have cotype 2.

REMARK 4.69 *Suppose $\mathcal{D}$ is (K,K)-unconditional with a constant U. Assume that X is of cotype s with a constant C_s. Then $\mathcal{D}$ has the Nikol'skii-type $\ell_1 X$ property with parameters $K, 1-1/s$ and $C_1 = 2UC_s^{-1}$.*

Proof. Our assumption about (K,K)-unconditionality implies: for any A, $|A| \le K$, we have

$$\|\sum_{i\in A} \pm x_i g_i\| \le 2U \|\sum_{i\in A} x_i g_i\|.$$

Therefore, by s-cotype assumption

$$\|\sum_{i\in A} x_i g_i\|^s \ge (2U)^{-s} C_s^s \sum_{i\in A} |x_i|^s.$$

This implies

$$\sum_{i\in A} |x_i| \le |A|^{1-1/s} \left(\sum_{i\in A} |x_i|^s\right)^{1/s} \le 2UC_s^{-1} |A|^{1-1/s} \|\sum_{i\in A} x_i g_i\|.$$

Example 3. Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$ and with cotype s. Let Ψ be a normalized in X unconditional basis for X. Then $U \le C(X,\Psi)$ and Ψ satisfies **A2** with $D = \infty$ and any K.

By Remark 4.69 Ψ satisfies **A1** with $r = 1 - \frac{1}{s}$. Theorem 4.4.4 gives

$$\|f_{C(t,X,\Psi)m^{(1-1/s)q'}}\| \le C\sigma_m(f_0,\Psi) \tag{4.70}$$

under condition $(1-1/s)q' \ge 1$.

Example 4. Let Ψ be the normalized in L_p, $2 \le p < \infty$, multivariate Haar basis $\mathcal{H}_p^d = \mathcal{H}_p \times \cdots \times \mathcal{H}_p$. It is an unconditional basis. Also it is known that L_p space with $2 \le p < \infty$ has cotype $s = p$. Therefore, Example 3 applies in this case. We give a direct argument here. It is an unconditional basis and therefore $U \le C(p,d)$. Next, for any A by (4.37)

$$\|\sum_{i\in A} x_i H_{i,p}\|_p \ge C(p,d) \left(\sum_{i\in A} |x_i|^p\right)^{1/p} \ge C(p,d)|A|^{\frac{1}{p}-1} \sum_{i\in A} |x_i|.$$

Therefore, we can take $r = \frac{1}{p'}$. Theorem 4.4.4 gives

$$\|f_{C(t,p,d)m^{2/p'}}\|_p \le C\sigma_m(f_0,\mathcal{H}_p^d)_p. \tag{4.71}$$

Inequality (4.71) provides some progress in Open Problem 7.2 (p. 91) from [26] in the case $2 < p < \infty$.

Example 4q. Let Ψ be the normalized in L_p, $1 < p \le 2$, univariate Haar basis $\mathscr{H}_p = \{H_{I,p}\}_I$, where $H_{I,p}$ the Haar functions indexed by dyadic intervals of support of $H_{I,p}$ (we index function 1 by $[0,1]$ and the first Haar function by $[0,1)$). Then for any finite set A of dyadic intervals we have for $f = \sum_I c_I(f) H_{I,p}$

$$\sum_{I \in A} |c_I| = \langle f, f_A^* \rangle, \quad f_A^* := \sum_{I \in A} (\operatorname{sign} c_I(f)) H_{I,p} \|H_{I,p}\|_2^{-2}.$$

Therefore,

$$\sum_{I \in A} |c_I| \le \|f\|_p \|f_A^*\|_{p'}.$$

It is easy to check that

$$\|H_{I,p}\|_{p'} \|H_{I,p}\|_2^{-2} = |I|^{-1/p} |I|^{1/p'} |I|^{-(1-2/p)} = 1.$$

By Lemma 4.3.1 (see also Lemma 1.23, p. 28, from [28]) we get

$$\|f_A^*\|_{p'} \le C(p) |A|^{1/p'}.$$

Thus

$$\sum_{I \in A} |c_I| \le C(p) |A|^{1/p'} \|f\|_p.$$

This means that $\mathscr{H}_p$ satisfies **A3** with $V = C(p)$ and $r = 1/p'$. Also it is an unconditional basis and therefore satisfies **A2** with $U = C(p)$. It is known that L_p space with $1 < p \le 2$ has modulus of smoothness $\rho(u) \le \gamma u^p$. Therefore, Theorem 4.4.8 applies in this case and gives

$$\|f_{C(t,p)m}\|_p \le C \sigma_m(f_0, \mathscr{H}_p)_p. \tag{4.72}$$

Inequality (4.72) solves the Open Problem 7.2 (p. 91) from [26] in the case $1 < p \le 2$.

Example 5. Let X be a Banach space with $\rho(u) \le \gamma u^2$. Assume that Ψ is a normalized Schauder basis for X. Then for any $f = \sum_i c_i(f) \psi_i$

$$\sum_{i \in A} |c_i(f)| \le C(\Psi) |A| \|f\|.$$

This implies that Ψ satisfies **A3** with $D = \infty$, $V = C(\Psi)$, $r = 1$ and any K. Theorem 4.4.7 gives

$$\|f_{C(t,X,\Psi)m^2 \ln m}\| \le C \sigma_m(f_0, \Psi). \tag{4.73}$$

We note that the above simple argument still works if we replace the assumption that Ψ is a Schauder basis by the assumption that a dictionary $\mathscr{D}$ is $(1,D)$-unconditional with constant U. Then we obtain

$$\|f_{C(t,\gamma,U)K^2 \ln K}\| \le C \sigma_K(f_0, \Psi), \quad \text{for} \quad K + C(t,\gamma,U) K^2 \ln K \le D.$$

Example 5q. Let X be a Banach space with $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Assume that Ψ is a normalized Schauder basis for X. Then for any $f = \sum_i c_i(f)\psi_i$

$$\sum_{i\in A} |c_i(f)| \le C(\Psi)|A|\|f\|.$$

This implies that Ψ satisfies **A3** with $D = \infty$, $V = C(\Psi)$, $r = 1$ and any T. Theorem 4.4.7 gives

$$\|f_{C(t,X,\Psi)m^{q'}\ln m}\| \le C\sigma_m(f_0,\Psi). \tag{4.74}$$

We note that the above simple argument still works if we replace the assumption that Ψ is a Schauder basis by the assumption that a dictionary $\mathscr{D}$ is $(1,D)$-unconditional with constant U. Then we obtain

$$\|f_{C(t,\gamma,q,U)K^{q'}\ln K}\| \le C\sigma_K(f_0,\mathscr{D}), \quad \text{for} \quad K + C(t,\gamma,q,U)K^{q'}\ln K \le D.$$

We now discuss application of general results of Section 4.4 to quasi-greedy bases. We begin with a brief introduction to the theory of quasi-greedy bases. Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_m\}_{m=1}^\infty$ be a normalized basis for X. The concept of quasi-greedy basis was introduced in [13].

Definition 4.6.1 *The basis Ψ is called quasi-greedy if there exists some constant C such that*

$$\sup_m \|G_m(f,\Psi)\| \le C\|f\|.$$

Subsequently, Wojtaszczyk [34] proved that these are precisely the bases for which the TGA merely converges, i.e.,

$$\lim_{n\to\infty} G_n(f) = f.$$

The following lemma is from [4] (see also [6] and [8] for further discussions).

Lemma 4.6.1 *Let Ψ be a quasi-greedy basis of X. Then for any finite set of indices Λ we have for all $f \in X$*

$$\|S_\Lambda(f,\Psi)\| \le C\ln(|\Lambda|+1)\|f\|.$$

We now formulate a result about quasi-greedy bases in L_p spaces. The following theorem is from [30]. We note that in the case $p = 2$ Theorem 4.6.1 was proved in [34]. Some notations first. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^\infty c_k(f)\psi_k$$

and the decreasing rearrangement of its coefficients

$$|c_{k_1}(f)| \ge |c_{k_2}(f)| \ge \dots .$$

Denote

$$a_n(f) := |c_{k_n}(f)|.$$

Theorem 4.6.1 *Let $\Psi = \{\psi_m\}_{m=1}^\infty$ be a quasi-greedy basis of the L_p space, $1 < p < \infty$. Then for each $f \in X$ we have*

$$C_1(p) \sup_n n^{1/p} a_n(f) \leq \|f\|_p \leq C_2(p) \sum_{n=1}^\infty n^{-1/2} a_n(f), \quad 2 \leq p < \infty;$$

$$C_3(p) \sup_n n^{1/2} a_n(f) \leq \|f\|_p \leq C_4(p) \sum_{n=1}^\infty n^{1/p-1} a_n(f), \quad 1 < p \leq 2.$$

Example 6. Let Ψ be a normalized quasi-greedy basis for L_p, $2 \leq p < \infty$. Theorem 4.6.1 implies for any $f = \sum_i c_i(f)\psi_i$

$$\sum_{i\in A} |c_i(f)| \leq \sum_{n=1}^{|A|} a_n(f) \leq C_1(p)^{-1} \sum_{n=1}^{|A|} n^{-1/p} \|f\|_p \leq C(p)|A|^{1-1/p} \|f\|_p.$$

This means that Ψ satisfies **A3** with $D = \infty$, $V = C(p)$, $r = 1 - \frac{1}{p}$. Theorem 4.4.7 gives

$$\|f_{C(t,p)m^{2(1-1/p)}\ln(m+1)}\| \leq C\sigma_m(f_0, \Psi). \tag{4.75}$$

Example 6q. Let Ψ be a normalized quasi-greedy basis for L_p, $1 < p \leq 2$. Theorem 4.6.1 implies for any $f = \sum_i c_i(f)\psi_i$

$$\sum_{i\in A} |c_i(f)| \leq \sum_{n=1}^{|A|} a_n(f) \leq C_3(p)^{-1} \sum_{n=1}^{|A|} n^{-1/2} \|f\|_p \leq C(p)|A|^{1/2} \|f\|_p.$$

This means that Ψ satisfies **A3** with $D = \infty$, $V = C(p)$, $r = 1/2$. Theorem 4.4.7 gives

$$\|f_{C(t,p)m^{p'/2}\ln(m+1)}\| \leq C\sigma_m(f_0, \Psi). \tag{4.76}$$

Example 7. Let Ψ be a normalized uniformly bounded orthogonal quasi-greedy basis for L_p, $2 \leq p < \infty$. For existence of such bases see [21]. Then orthogonality implies that we can take $r = 1/2$. We obtain from Lemma 4.6.1 that Ψ is (K, ∞) unconditional with $U \leq C\ln(K+1)$. Theorem 4.4.8 gives

$$\|f_{C(t,p,\Psi)m\ln\ln(m+3)}\|_p \leq C\sigma_m(f_0, \Psi)_p. \tag{4.77}$$

Example 7q. Let Ψ be a normalized uniformly bounded orthogonal quasi-greedy basis for L_p, $1 < p \leq 2$. For existence of such bases see [21]. Then orthogonality implies that we can take $r = 1/2$. We obtain from Lemma 4.6.1 that Ψ is (K, ∞) unconditional with $U \leq C\ln(K+1)$. Theorem 4.4.8 gives

$$\|f_{C(t,p,\Psi)m^{p'/2}\ln\ln(m+3)}\|_p \leq C\sigma_m(f_0, \Psi)_p. \tag{4.78}$$

4.7 Discussion

We study sparse approximation. In a general setting we study an algorithm (approximation method) $\mathscr{A} = \{A_m(\cdot, \mathscr{D})\}_{m=1}^{\infty}$ with respect to a given dictionary $\mathscr{D}$. The sequence of mappings $A_m(\cdot, \mathscr{D})$ defined on X satisfies the condition: for any $f \in X$, $A_m(f, \mathscr{D}) \in \Sigma_m(\mathscr{D})$. In other words, A_m provides an m-term approximant with respect to $\mathscr{D}$. It is clear that for any $f \in X$ and any m we have

$$\|f - A_m(f, \mathscr{D})\| \geq \sigma_m(f, \mathscr{D}).$$

We are interested in such pairs $(\mathscr{D}, \mathscr{A})$ for which the algorithm $\mathscr{A}$ provides approximation close to best m-term approximation. We introduce the corresponding definitions.

Definition 4.7.1 *We say that $\mathscr{D}$ is a greedy dictionary with respect to $\mathscr{A}$ if there exists a constant C_0 such that for any $f \in X$ we have*

$$\|f - A_m(f, \mathscr{D})\| \leq C_0 \sigma_m(f, \mathscr{D}). \tag{4.79}$$

If $\mathscr{D}$ is a greedy dictionary with respect to $\mathscr{A}$ then $\mathscr{A}$ provides ideal (up to a constant C_0) m-term approximations for every $f \in X$.

Definition 4.7.2 *We say that $\mathscr{D}$ is an almost greedy dictionary with respect to $\mathscr{A}$ if there exist two constants C_1 and C_2 such that for any $f \in X$ we have*

$$\|f - A_{C_1 m}(f, \mathscr{D})\| \leq C_2 \sigma_m(f, \mathscr{D}). \tag{4.80}$$

If $\mathscr{D}$ is an almost greedy dictionary with respect to $\mathscr{A}$, then $\mathscr{A}$ provides almost ideal sparse approximation. It provides $C_1 m$-term approximant as good (up to a constant C_2) as an ideal m-term approximant for every $f \in X$. We also need a more general definition. Let $\phi(u)$ be a function such that $\phi(u) \geq 1$.

Definition 4.7.3 *We say that $\mathscr{D}$ is a ϕ-greedy dictionary with respect to $\mathscr{A}$ if there exists a constant C_3 such that for any $f \in X$ we have*

$$\|f - A_{\phi(m)m}(f, \mathscr{D})\| \leq C_3 \sigma_m(f, \mathscr{D}). \tag{4.81}$$

If $\mathscr{D} = \Psi$ is a basis, then in the above definitions we replace dictionary by basis. In the case $\mathscr{A} = \{G_m(\cdot, \Psi)\}_{m=1}^{\infty}$ is the TGA the theory of greedy and almost greedy bases is well developed (see [28]). We present two results on characterization of these bases. A basis Ψ in a Banach space X is called *democratic* if there is a constant $C(\Psi)$ such that

$$\|\sum_{k \in A} \psi_k\| \leq C(\Psi) \|\sum_{k \in B} \psi_k\| \tag{4.82}$$

if $|A| = |B|$. This concept was introduced in [13]. In [5] we defined a democratic basis as the one satisfying (4.82) if $|A| \leq |B|$. It is known that for quasi-greedy bases the above two definitions are equivalent. It was proved in [13] (see Theorem

1.15, p. 18, [28]) that a basis is greedy with respect to TGA if and only if it is unconditional and democratic. It was proved in [5] (see Theorem 1.37, p. 38, [28]) that a basis is almost greedy with respect to TGA if and only if it is quasi-greedy and democratic.

Example 4q is the first result about almost greedy bases with respect to WCGA in Banach spaces. It shows that the univariate Haar basis is an almost greedy basis with respect to the WCGA in the L_p spaces for $1 < p \le 2$. Example 1 shows that uniformly bounded orthogonal bases are ϕ-greedy bases with respect to WCGA with $\phi(u) = C(t,p,D)\ln(u+1)$ in the L_p spaces for $2 \le p < \infty$. We do not know if these bases are almost greedy with respect to WCGA. They are good candidates for that.

It is known (see [28], p. 17, and Theorem 4.3.1 of this paper) that the univariate Haar basis is a greedy basis with respect to TGA for all L_p, $1 < p < \infty$. Example 4 only shows that it is a ϕ-greedy basis with respect to WCGA with $\phi(u) = C(t,p)u^{1-2/p}$ in the L_p spaces for $2 \le p < \infty$. It is much weaker than the corresponding results for the $\mathscr{H}_p$, $1 < p \le 2$, and for the trigonometric system, $2 \le p < \infty$ (see Example 2). We do not know if this result on the Haar basis can be substantially improved. At the level of our today's technique we can observe that the Haar basis is ideal (greedy basis) for the TGA in L_p, $1 < p < \infty$, almost ideal (almost greedy basis) for the WCGA in L_p, $1 < p \le 2$, and that the trigonometric system is very good for the WCGA in L_p, $2 \le p < \infty$.

Example 2q shows that our results for the trigonometric system in L_p, $1 < p < 2$, are not as strong as for $2 \le p < \infty$. We do not know if it is a lack of appropriate technique or it reflects the nature of the WCGA with respect to the trigonometric system.

We note that properties of a given basis with respect to TGA and WCGA could be very different. For instance, the class of quasi-greedy bases (with respect to TGA) is a rather narrow subset of all bases. It is close in a certain sense to the set of unconditional bases. The situation is absolutely different for the WCGA. If X is uniformly smooth, then WCGA converges for each $f \in X$ with respect to any dictionary in X. Moreover, Example 5q shows that if X is a Banach space with $\rho(u) \le \gamma u^q$ then any basis Ψ is ϕ-greedy with respect to WCGA with $\phi(u) = C(t,X,\Psi)u^{q'-1}\ln(u+1)$.

It is interesting to compare Theorem 4.4.3 with the following known result. The following theorem provides rate of convergence (see [28], p. 347). As above we denote by $A_1(\mathscr{D})$ the closure in X of the convex hull of the symmetrized dictionary $\mathscr{D}^{\pm} := \{\pm g : g \in \mathscr{D}\}$.

Theorem 4.7.1 *Let X be a uniformly smooth Banach space with modulus of smoothness $\rho(u) \le \gamma u^q$, $1 < q \le 2$. Take a number $\varepsilon \ge 0$ and two elements f_0, f^ε from X such that*

$$\|f_0 - f^\varepsilon\| \le \varepsilon, \quad f^\varepsilon / A(\varepsilon) \in A_1(\mathscr{D}),$$

with some number $A(\varepsilon) > 0$. Then, for the WCGA we have

$$\|f_m^{c,\tau}\| \le \max\left(2\varepsilon, C(q,\gamma)(A(\varepsilon)+\varepsilon)(1+\sum_{k=1}^m t_k^{q'})^{-1/q'}\right).$$

Both Theorem 4.7.1 and Theorem 4.4.3 provide stability of the WCGA with respect to noise. In order to apply them for noisy data we interpret f_0 as a noisy version of a signal and f^ε as a noiseless version of a signal. Then, assumption $f^\varepsilon/A(\varepsilon) \in A_1(\mathscr{D})$ describes our smoothness assumption on the noiseless signal and assumption $f^\varepsilon \in \Sigma_K(\mathscr{D})$ describes our structural assumption on the noiseless signal. In fact, Theorem 4.7.1 simultaneously takes care of two issues: noisy data and approximation in an interpolation space. Theorem 4.7.1 can be applied for approximation of f_0 under assumption that f_0 belongs to one of interpolation spaces between X and the space generated by the $A_1(\mathscr{D})$-norm (atomic norm).

We now give some historical remarks on the Lebesgue-type inequalities for redundant dictionaries. By the Lebesgue-type inequality we mean an inequality that provides an upper bound for the error of a particular method of approximation of f by elements of a special form, say, form $\mathscr{A}$, by the best-possible approximation of f by elements of the form $\mathscr{A}$. In our paper the method of approximation is the WCGA which provides an m-term approximant after mth iteration. Thus, form $\mathscr{A}$ is a linear combination of at most m dictionary elements. Therefore, we compare an error of the WCGA after m iterations with the best m-term approximation. First Lebesgue-type inequalities for redundant dictionaries were proved for the Orthogonal Matching Pursuit (OMP), which is the WCGA in a Hilbert space with the weakness parameter $t = 1$, under incoherence assumption on the dictionary.

Denote

$$M(\mathscr{D}) := \sup_{g \neq h; g,h \in \mathscr{D}} |\langle g, h\rangle|$$

the coherence parameter of a dictionary $\mathscr{D}$. The first general Lebesgue-type inequality for the OMP for the M-coherent dictionary was obtained in [9]. The authors proved that for the residual f_m^o of the OMP after m iterations one has

$$\|f_m^o\| \leq 8m^{1/2}\sigma_m(f) \quad \text{for} \quad m < 1/(32M).$$

The constants in this inequality were improved in [32]:

$$\|f_m^o\| \leq (1+6m)^{1/2}\sigma_m(f) \quad \text{for} \quad m < 1/(3M).$$

Further results were obtained in [3]: Assume $m \leq 0.05M^{-2/3}$, then we have

$$\|f^o_{[m \log m]}\| \leq 24\sigma_m(f).$$

The following inequality was obtained in [31]. For any $\delta \in (0, 1/4]$ set $L(\delta) := [1/\delta] + 1$. Assume m is such that $20Mm^{1+\delta}2^{L(\delta)} \leq 1$. Then we have

$$\|f^o_{m(2^{L(\delta)+1}-1)}\| \leq \sqrt{3}\sigma_m(f).$$

Recently, the above Lebesgue-type inequality was improved in [16]:

$$\|f^o_{2m}\| \leq 3\sigma_m(f)$$

for $m \leq (20M)^{-1}$.

The incoherence assumption on a dictionary is stronger than the Restricted Isometry Property (RIP) assumption. The corresponding Lebesgue-type inequalities for the OMP under RIP assumption were not known for a while. As a result new greedy-type algorithms were introduced and exact recovery of sparse signals and the Lebesgue-type inequalities were proved for these algorithms: the Regularized Orthogonal Matching Pursuit (see [20]), Compressive Sampling Matching Pursuit (CoSaMP) (see [19]), and the Subspace Pursuit (SP) (see [2]). The OMP is simpler than CoSaMP and SP, however, at the time of invention of CoSaMP and SP these algorithms provided exact recovery of sparse signals and the Lebesgue-type inequalities for dictionaries satisfying the Restricted Isometry Property (RIP) (see [19] and [2]). The corresponding results for the OMP were not known at that time. Later, a breakthrough result in this direction was obtained by Zhang [35]. In particular, he proved that if $\mathscr{D} \in R(31K,\delta)$, $\delta < 1/3$, then the OMP recovers exactly all K-sparse signals within $30K$ iterations. In other words, $f^o_{30K} = 0$. It is interesting and difficult problem to improve the constant 30 to the optimal one. There are several papers devoted to this problem (see [10, 33], and [17]). In this paper we developed Zhang's technique to obtain recovery results and the Lebesgue-type inequalities in the Banach space setting.

Concluding, we briefly describe the contribution of [17] and [29]. First, we presented in [29] a study of the Lebesque-type inequalities with respect to the WCGA in Banach spaces with $\rho(u) \le \gamma u^q$, $1 < q \le 2$, under conditions **A1** and **A2**. In the case $q = 2$ it has been done in [17]. The case $1 < q < 2$ uses the same ideas as in [17]. Second, we introduced in [29] a new condition **A3** and studied the WCGA with respect to dictionaries satisfying either **A3** or **A2** and **A3**. Condition **A3** and a combination of **A2** and **A3** turn out to be more powerful in applications than **A1** combined with **A2**. Third, we applied in [29] the general theory (see Sections 4.4–4.5 above) for bases. Surprisingly, this technique works very well for very different bases. It provides first results on the Lebesque-type inequalities for the WCGA with respect to bases in Banach spaces. Some of these results (for the $\mathscr{H}_p$, $1 < p \le 2$, and for the $\mathscr{R}\mathscr{T}_p$, $2 \le p < \infty$) are strong. This demonstrates that the technique used is an appropriate and powerful method.

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Chapter 5
Results on Non-linear Approximation for Wavelet Bases in Weighted Function Spaces

Eugenio Hernández and Maria de Natividade

Abstract The material in this paper comes from various conferences given by the authors. We start with a brief survey of harmonic analysis methods in linear and non-linear approximation related to signal compression. Special emphasis is made on wavelet-based methods and some of the mathematical theory of wavelets behind them. We also present recent results of the authors concerning non-linear approximation in sequence spaces and the validity of Jackson and Bernstein inequalities in general smoothness spaces.

5.1 Introduction

Images from the real world can be mathematically described in various ways [3]. A simple model considers analog images as non-negative functions of two variables $f(x,y)$ supported in the unit square $[0,1]^2$, which physically may be interpreted as light intensity fields upon a given screen. For most pictures from the real world, a precise mathematical expression is not known. Analog images must be "digitized" in order to be stored and manipulated by computers. We briefly describe how to produce a digital version of $f(x,y)$ (we follow [11, p. 324]): a photometer averages the light intensity over small squares (of side length 2^{-M}) distributed dyadically

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along the picture frame $[0,1]^2$. So if M is large (typically, $M \geq 8$), we can codify the image as a sequence of 2^{2M} coefficients: for $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$,

$$p_{\mathbf{k}} = p_{\mathbf{k}}^{(M)} = \frac{1}{|I_{M,\mathbf{k}}|} \int\int_{I_{M,\mathbf{k}}} f(x,y)\,dxdy, \quad 0 \leq k_1, k_2 < 2^M, \tag{5.1}$$

where $I_{M,\mathbf{k}}$ denotes the dyadic square $[\frac{k_1}{2^M}, \frac{k_1+1}{2^M}] \times [\frac{k_2}{2^M}, \frac{k_2+1}{2^M}]$. These squares are usually called *pixels* (or *picture elements*) *located at positions* $2^{-M}\mathbf{k}$, and correspond to the number of "dots" that form a computer screen. To each of them we associate a single number $p_{\mathbf{k}}$ (typically a rounded integer between 0 and 2^8), which represents the "color level" of the picture at that point. In this way we have converted $f(x,y)$ into a "digital image" $\{p_{\mathbf{k}}\}$, a sequence of integer numbers which can be stored and processed by computers.

Once the image is digitally stored, it can be manipulated with mathematical tools. Given a sequence of integer numbers $\{p_{\mathbf{k}}\}$, we construct the *observed image* $f^{(o)}(x,y)$ by:

$$f^{(o)}(x,y) = \sum_{\mathbf{k}} p_{\mathbf{k}}\, \phi_{M,\mathbf{k}}(x,y), \tag{5.2}$$

where $\phi_{M,\mathbf{k}}(\mathbf{x}) = \phi(2^M\mathbf{x} - \mathbf{k})$ and the function ϕ may be simply chosen as $\chi_{[0,1]^2}$ or replaced by smoother versions such as splines or wavelet-type scaling functions. In general, when M is sufficiently large, $f^{(o)}(x,y)$ is an almost indistinguishable copy of $f(x,y)$, and thus can be identified for mathematical purposes with the original image. The compression problem then consists in representing $f^{(o)}(x,y)$ with much less than 2^{2M} coefficients without losing the visual resemblance with the original image.

With this example in mind, we can describe a general mathematical setting for compression problems, which is based in classical approximation theory. We are given a general class of functions $\mathscr{F}$ (typically a Banach space), endowed with a metric $d_{\mathscr{F}}$, and an increasing sequence of subsets $D_N \subset \mathscr{F}, N = 1,2,\ldots$ We define the *error of approximation of* $f \in \mathscr{F}$ *by* D_N to be

$$\sigma(f, D_N)_{\mathscr{F}} \equiv \inf_{g \in D_N} d_{\mathscr{F}}(f,g), \qquad N = 1,2,\ldots \tag{5.3}$$

Then, the following questions have attracted the interest of scientist:

1. Decide, depending on applications, what metric $d_{\mathscr{F}}$ and what classes D_N are suitable in order to approximate functions in $\mathscr{F}$.
2. Find simple and fast algorithms to produce approximations $f_N \in D_N$ which are close to realize the infimum described in (5.3).
3. Investigate the rate of decay of the approximation error $\sigma(f, D_N)_{\mathscr{F}}$. More precisely, given a prescribed rate, say $N^{-\varepsilon}$, determine the class of functions $f \in \mathscr{F}$ for which $\sigma(f, D_N)_{\mathscr{F}} \leq N^{-\varepsilon}$ for all $N = 1,2,\ldots$

In the above example of images, one can take $\mathscr{F} = L^2\big([0,1]^2\big)$ and let D_N be a subset of functions with at most N non-null coefficients in the expansion (5.2). Then, when $N \ll 2^{2M}$ the best approximation f_N can be seen as a "compressed version" of the original image $f(x,y)$, from which we have removed the less essential information in order to speed up transmissions or reduce storage memory.

Of course, this setting of approximation can also be applied to other situations, such as the processing of other types of signals (music, digital TV, etc.) or the numerical solutions of PDEs. In this last case f_N is an approximation by a certain numerical method of the (unknown) solution f. In all these cases it is essential that the compressed signal f_N is a faithful representation of the original f, for which often we do not know a precise expression or this cannot be measured in the whole continuous range of space.

One way of obtaining a compressed signal f_N which is a faithful representation of an original signal f is to use Linear approximation in Hilbert spaces. For this method, one of the oldest and best known, we need a Hilbert space $\mathscr{H}$ and an orthonormal basis $\{e_j : j = 1,2,\dots\}$ of $\mathscr{H}$, so that the original signal f is represented by $f = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j$. Select the linear subspaces $L_N = span\{e_1,\dots,e_N\}$ as approximating sets. Then, the infimum defined by $\sigma(f,L_N)_{\mathscr{H}}$ as in (5.3) is attained by the orthogonal projection of f onto L_N, that is

$$f_N = \sum_{j=1}^{N} \langle f, e_j \rangle e_j\,. \tag{5.4}$$

This gives a precise estimate of the error for $f \in \mathscr{H}$:

$$\sigma(f,L_N)_{\mathscr{H}} = \|f - f_N\|_{\mathscr{H}} = \Big(\sum_{j=N+1}^{\infty} |\langle f, e_j \rangle|^2 \Big)^{1/2}. \tag{5.5}$$

Since $\sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2 < \infty$, we always have $\lim_{N\to\infty} \sigma(f,L_N)_{\mathscr{H}} = 0$. The interesting question is to find subspaces of $\mathscr{H}$ for which the decay rate of $\sigma(f,L_N)_{\mathscr{H}}$ is prescribed.

The first positive answers to this question were given by D. Jackson and S.N. Bernstein for approximation of continuous periodic functions by trigonometric polynomials, using L^∞ norms rather than L^2 norms (see [10] for a detailed account of these type of inequalities).

Linear methods of approximation do a good job in analyzing signals with "uniform smoothness." Audio signals is an example, since they are only perceived in a range of harmonics smaller than 20 kHz, and therefore have a reasonably uniform smoothness over $\mathbb{R}$ [42, p. 49].

But, linear Fourier Approximation is however a bad model for images, due to the discontinuities that they often present at the edges of objects. Low dimensional discontinuities will produce a low exponent of global smoothness. For instance, if $f = \chi_{[a,b]}$ is the characteristic function of an interval in $\mathbb{R}$, then the error decay is like

$N^{-\frac{1}{2}}$ (see [42, p. 380]). The representation of such signals can be largely improved by using non-linear approximation and wavelet bases.

In this article we start reviewing some important results in non-linear Approximation methods developed in the last fifteen years and study the behavior of the thresholding greedy algorithm for wavelet basis in weighted Lebesgue, Triebel-Lizorkin, and Orlicz spaces.

5.2 Non-linear Approximation: Definitions and First Results

We start describing the general setting of non-linear approximation. A good reference for the results stated in this section is the monograph [57] written by V. N. Temlyakov. Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a quasi-Banach space. In some proofs we will use that there exists $\rho = \rho_{\mathbb{B}} \in (0,1]$ such that the following ρ-power triangle inequality holds:

$$\|x+y\|_{\mathbb{B}}^{\rho} \leq \|x\|_{\mathbb{B}}^{\rho} + \|y\|_{\mathbb{B}}^{\rho}, \qquad x, y \in \mathbb{B} \tag{5.6}$$

(see [5, Lemma 3.10.1]). Observe that if (5.6) holds for $\rho = \rho_{\mathbb{B}}$, it also holds for any μ with $0 < \mu \leq \rho_{\mathbb{B}}$. The case $\rho_{\mathbb{B}} = 1$ gives a Banach space.

A sequence of vectors $\mathscr{B} = \{e_j\}_{j=1}^{\infty}$ is a basis of $\mathbb{B}$ if every $x \in \mathbb{B}$ can be uniquely represented as $x = \sum_{j=1}^{\infty} c_j e_j$ for some scalars c_j, with convergence in $\|\cdot\|_{\mathbb{B}}$. The basis $\mathscr{B}$ is **unconditional** if the series converges unconditionally, or equivalently if there is some $K > 0$ such that

$$\Big\| \sum_{j=1}^{\infty} \lambda_j c_j e_j \Big\|_{\mathbb{B}} \leq K \Big\| \sum_{j=1}^{\infty} c_j e_j \Big\|_{\mathbb{B}} \tag{5.7}$$

for every sequence of scalars $\{\lambda_j\}_{j=1}^{\infty}$ with $|\lambda_j| \leq 1$ (see e.g., [30, Chapter 5]).

Let Σ_N, $N = 1, 2, 3, \cdots$, be the set of all $y \in \mathbb{B}$ with at most N non-null coefficients in the unique basis representation of y. For $x \in \mathbb{B}$, the **N-term error of approximation** with respect to $\mathscr{B}$ is defined by

$$\sigma_N(x) = \sigma_N(x; \mathscr{B}, \mathbb{B}) \equiv \inf_{y \in \Sigma_N} \|x - y\|_{\mathbb{B}}, \quad N = 1, 2, 3 \cdots \tag{5.8}$$

We also set $\Sigma_0 = \{0\}$ so that $\sigma_0(x) = \|x\|_{\mathbb{B}}$.

Given $x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B}$, let π denote any bijection of $\mathbb{N}$ such that

$$\|c_{\pi(j)} e_{\pi(j)}\| \geq \|c_{\pi(j+1)} e_{\pi(j+1)}\|, \quad \text{for all} \quad j \in \mathbb{N}. \tag{5.9}$$

A **thresholding greedy algorithm of step** N is a correspondence assigning

$$x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} \longmapsto G_N^{\pi}(x) \equiv \sum_{j=1}^{N} c_{\pi(j)} e_{\pi(j)}$$

for any π as in (5.9). The **error of the thresholding greedy approximation** at step N is defined by

$$\gamma_N(x) = \gamma_N(x;\mathscr{B},\mathbb{B}) \equiv \sup_{\pi} \|x - G_N^{\pi}(x)\|_{\mathbb{B}}. \tag{5.10}$$

Notice that $\sigma_N(x) \leq \gamma_N(x)$, but the reverse inequality may not be true in general. The following definition was given in [38].

Definition 5.1. A basis $\mathscr{B}$ is said to be **greedy** in the quasi-Banach space $\mathbb{B}$ if there is a constant $c \geq 1$ such that

$$\gamma_N(x;\mathscr{B},\mathbb{B}) \leq c\,\sigma_N(x;\mathscr{B},\mathbb{B}), \quad \forall\, x \in \mathbb{B},\ N = 1,2,3,\cdots$$

Since greedy bases are those for which the thresholding greedy algorithm behaves essentially as the best approximation, they do a good job of finding and approximation to a signal that faithfully resembles the original one.

Orthonormal wavelets is an example of bases that are greedy in some Lebesgue spaces. The first result in this direction was proved by V.N. Temlyakov [53] for the Haar basis in $L^p(0,1), 1 < p < \infty$, and for all orthonormal wavelet bases that are L^p-equivalent to the Haar wavelet. Although in [53] the results were stated and proved for the case $L^p(0,1), 1 < p < \infty$, with minor modifications in the proof they also hold for orthonormal wavelet families in $L^p(\mathbb{R}^d), 1 < p < \infty$. Thus, we will describe them in this more general situation.

Let $\mathscr{D}(d)$ be the collection of all cubes in $\mathbb{R}^d$ of the form $Q_{j,k} = 2^{-j}([0,1)^d + k)$, $j \in \mathbb{Z}, k \in \mathbb{Z}^d$. Observe that $|Q_{j,k}| = 2^{-jd}$, and all the cubes of the same level $j \in \mathbb{Z}$ are disjoint. Given $Q = Q_{j,k} \in \mathscr{D}(d)$ and ψ a function defined in $\mathbb{R}^d$, we denote by ψ_Q the function

$$\psi_Q(x) = 2^{jd/2}\psi(2^j x - k).$$

Definition 5.2 ([40]). A finite collection of functions $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subset L^2(\mathbb{R}^d)$ is said to be an **orthonormal wavelet family** if the set

$$\mathscr{W} = \{\psi^l_{Q_{j,k}}(x) = 2^{jd/2}\psi_l(2^j x - k) : Q_{j,k} \in \mathscr{D}(d), l = 1,2,\ldots,L\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$.

Wavelet theory has been developed in the last 30 years. There are by now good monographs describing both the mathematical theory of wavelets and its applications. We refer the reader to [16, 30, 42, 44].

As far as we know, the definition given above was first published in [40] in the year 1986. Nevertheless, as earlier as 1910, A. Haar described a simple system in one dimension that fits Definition 5.2 [28]. Taking

$$h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)(x)}, \qquad x \in \mathbb{R},$$

the system $\mathscr{H} = \{h_I(x) : I \in \mathscr{D}(1)\}$ is an orthonormal basis of $L^2(\mathbb{R})$ which is known as the Haar basis.

To define the Haar system in $\mathbb{R}^d$ we proceed as follows. Let $h_0(x) = \chi_{[0,1)}$ and $h_1(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$, $x \in \mathbb{R}$. Let E be the set of indices $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ such that $\varepsilon_j = 0, 1$ and not all ε_j are zero. The set E has $2^n - 1$ elements that correspond to the non-zero vertices of the d-dimensional unit cube. For each $\varepsilon \in E$ let

$$h^\varepsilon(x) = \prod_{j=1}^{d} h_{\varepsilon_j}(x_j), \qquad x = (x_1, \dots, x_d).$$

The collection,

$$\mathscr{H}^d = \{h_Q^\varepsilon : Q \in \mathscr{D}(d),\ \varepsilon \in E\}$$

is an orthonormal wavelet family of $L^2(\mathbb{R}^d)$, known as the d-dimensional **Haar system**.

Many other wavelets have been constructed since the beginning of wavelet theory: Lemarié-Meyer wavelets [40], spline wavelets, compactly supported wavelets [15] or Minimally Supported Frequency wavelets (MSF) [30]. Observe that our wavelet families will always be normalized in $L^2(\mathbb{R}^d)$.

It can be shown that $\mathscr{H}^d$ is a basis for $L^p(\mathbb{R}^d), 1 < p < \infty$. A wavelet family $\mathscr{W}$ as given in Definition 5.2, with the set of indexes $\{1, 2, \dots, L\}$ replaced by E, is said to be L^p-equivalent to $\mathscr{H}^d$ if there exist constants $0 < C_1 \le C_2 < \infty$ such that for any finite set $A \subset \mathscr{D}(d) \times E$ and any complex coefficients $\{c_Q^\varepsilon : (Q, \varepsilon) \in A\}$ we have

$$C_1 \Big\| \sum_A c_Q^\varepsilon \psi_Q^\varepsilon \Big\|_p \le \Big\| \sum_A c_Q^\varepsilon h_Q^\varepsilon \Big\|_p \le C_2 \Big\| \sum_A c_Q^\varepsilon \psi_Q^\varepsilon \Big\|_p. \tag{5.11}$$

Theorem 5.1. *Let $1 < p < \infty$ and let $\mathscr{W}$ be a wavelet family in $\mathbb{R}^d$ which is L^p-equivalent to the Haar basis $\mathscr{H}^d$. Then, there exists a constant $C < \infty$ such that for any $f \in L^p(\mathbb{R}^d)$ and any $N = 1, 2, 3, \dots$ we have*

$$\gamma_N(f; \mathscr{W}, L^p) = \sup_\pi \|f - G_N^\pi(f; \mathscr{W}, L^p)\|_p \le C \sigma_N(f; \mathscr{W}, L^p).$$

This result was proved by V.N. Temlyakov in [53] and the reader can also find a proof in the monograph [57]. We will prove a similar result for Triebel-Lizorkin spaces in Section 5.3, and this result includes Theorem 5.1 as a particular case.

A well-known basis of $L^p([0,1)^d), 1 \le p \le \infty$, is the trigonometric system $\mathscr{T}^d = \{e^{2\pi i x \cdot k} : k \in \mathbb{Z}^d\}$. Contrary to the case of Theorem 5.1, where it is shown that the

Haar basis is greedy in $L^p, 1 < p < \infty$, the trigonometric system is far from greedy in $L^p([0,1)^d), 1 < p < \infty$, unless $p = 2$. This is contained in the following result due to V.N. Temlyakov [54]:

Theorem 5.2. *For each $f \in L^p([0,1)^d)\,, 1 \le p \le \infty$, and $N = 1,2,3,\cdots$ we have*

$$\gamma_N(f;\mathscr{T}^d, L^p) = \sup_{\pi} \|f - G_N^{\pi}(f;\mathscr{T}^d, L^p)\|_p \le (1 + 3N^{h(p)})\sigma_N(f;\mathscr{T}^d, L^p)\,.$$

where $h(p) := |\frac{1}{2} - \frac{1}{p}|$, and this result is best possible.

Comparing Theorems 5.1 and 5.2 one can see the better performance of wavelet bases with respect to the thresholding greedy algorithm compared to the trigonometric system.

The other result that we consider fundamental in the theory of non-linear approximation with the thresholding greedy algorithm is the characterization of Greedy bases due to S.V. Konyagin and V.N. Temlyakov given in [38]. To state the result we need the concept of **democratic basis**. Given a basis $\mathscr{B} = \{e_j\}_{j=1}^{\infty}$ in a quasi-Banach space $\mathbb{B}$, the functions

$$h_r(N;\mathscr{B},\mathbb{B}) \equiv \sup_{|\Gamma|=N} \Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} \quad \text{and} \quad h_\ell(N;\mathscr{B},\mathbb{B}) \equiv \inf_{|\Gamma|=N} \Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}},$$

are called the **right and left democracy functions of** $\mathscr{B}$ (see also [17, 25, 33]). We shall omit $\mathscr{B}$ or $\mathbb{B}$ when these are understood from the context.

Definition 5.3. A basis $\mathscr{B} = \{e_j\}_{j=1}^{\infty}$ in a quasi-Banach space $\mathbb{B}$ is said to be **democratic** if there exists $D < \infty$ such that for all $N = 1,2,3,\ldots$ we have

$$h_r(N;\mathscr{B},\mathbb{B}) \le D\, h_\ell(N;\mathscr{B},\mathbb{B})\,.$$

Equivalently, the basis $\mathscr{B}$ is **democratic** in $\mathbb{B}$ if there exists $D < \infty$ such that for any two finite sets of indices $A, B \subset \mathbb{N}$ with the same cardinality $|A| = |B|$ we have

$$\Big\| \sum_{k\in A} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} \le D \Big\| \sum_{k\in B} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}.$$

This was the original definition given in [38].

The reason for introducing this concept is the following result proved originally in [38]:

Theorem 5.3. *A basis $\mathscr{B} = \{e_j\}_{j=1}^{\infty}$ in a quasi-Banach space $\mathbb{B}$ is* **greedy** *if and only if $\mathscr{B}$ is* **unconditional** *and* **democratic** *in $\mathbb{B}$.*

This result was originally stated for Banach spaces. The proof for quasi-Banach spaces requires minor modifications using the ρ-power triangle inequality (5.6). A proof of this result for quasi-Banach sequence spaces can be found in [24].

5.3 Weights in $\mathbb{R}^d$

In this section, we are going to collect all results that we need about weights in $\mathbb{R}^d$. We denote by $W = W(\mathbb{R}^d)$ the set of positive functions w defined on $\mathbb{R}^d$ such that $0 < w(x) < \infty$ almost everywhere $x \in \mathbb{R}^d$ and are locally integrable. A weight w belongs to the Muckenhoupt class A_p, $1 < p < \infty$ $(w \in A_p = A_p(\mathbb{R}^d))$, if there exists a constant $C_w < \infty$ such that

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \leq C_w,$$

for every cube $Q \subset \mathbb{R}^d$, where $|Q|$ denotes the Lebesgue measure of Q. The A_1 condition for weights can be seen as the limiting case of the A_p conditions when $p \downarrow 1$, that is,

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big) ess_Q \sup(w^{-1}) \leq C_w.$$

The smaller constant C_w for which these inequalities hold is known as the A_p constant of the weight w. We have $A_1 \subset A_{p_1} \subset A_{p_2}$ if $1 < p_1 < p_2 < \infty$ (see [14] Theorem 1.14). The class $A_\infty(\mathbb{R}^d) = A_\infty$ is defined by

$$A_\infty = \bigcup_{p>1} A_p.$$

For a weight $w(x)$ in $\mathbb{R}^d$, and a measurable set $A \subset \mathbb{R}^d$ we write

$$w(A) = \int_A w(x)dx.$$

The condition A_∞ can be characterized in the following way: $w \in A_\infty$ if and only if there exist $\delta > 0$ and $0 < C_w < \infty$ such that for every cube $Q \subset \mathbb{R}^d$

$$\frac{w(A)}{w(Q)} \leq C_w\Big(\frac{|A|}{|Q|}\Big)^{\delta}, \quad \forall A \subset Q,\ A \text{ measurable.} \tag{5.12}$$

(see Theorem 2.9, Chapter IV of [14]).

For the results that we are going to prove here it is enough to assume a weaker condition than the one in (5.12), namely that the inequality holds only for $Q' \subset Q$ with $Q' \in \mathscr{D}(d)$. Thus, we define the class A_∞^d as the set of all weights $w \in \mathbb{R}^d$ for which there exist $\delta > 0$ and $0 < C_w < \infty$ such that for all $Q \in \mathscr{D}(d)$

$$\frac{w(Q')}{w(Q)} \leq C_w\Big(\frac{|Q'|}{|Q|}\Big)^{\delta},\ \forall\, Q' \subset Q,\ Q' \in \mathscr{D}(d). \tag{5.13}$$

When $w \in A_p,\ 1 \le p < \infty$, there exists $C_w,\ 0 < C_w < \infty$, such that for all $Q \subset \mathbb{R}^d$

$$C_w \Big(\frac{|A|}{|Q|}\Big)^p \le \frac{w(A)}{w(Q)}, \quad \forall\, A \subset Q,\ A \text{ measurable} \tag{5.14}$$

(take $f(x) = \chi_A(x)$ in part (b) of Theorem 2.1 of Chapter IV in [14]). In some of our results it will be sufficient to assume the following weaker condition: $w \in B_p^d,\ 1 \le p < \infty$, if there exists $C_w,\ 0 < C_w < \infty$, such that for all $Q \in \mathscr{D}(d)$

$$C_w \Big(\frac{|Q'|}{|Q|}\Big)^p \le \frac{w(Q')}{w(Q)}, \quad \forall\, Q' \subset Q, \quad Q' \in \mathscr{D}(d). \tag{5.15}$$

Define

$$B_\infty^d := \cup_{p \ge 1} B_p^d.$$

For future use we prove the following result:

Proposition 5.1. *Let w be a weight in $A_\infty^d \cap B_\infty^d$ (in particular if $w \in A_\infty$). Then, for any given $\tau > 0$, there exists a sequence of disjoint cubes $\{R_j\}_{j=1}^\infty \subset \mathscr{D}(d)$, such that*

$$C\tau \le w(R_j) \le \tau, \quad \forall\ j = 1, 2, 3, \cdots,$$

where C is a positive constant independent of j and τ.

For the proof we will need the following result:

Lemma 5.1. *Let $w \in A_\infty^d(\mathbb{R}^d)$. If $\{Q_k\}_{k=-\infty}^\infty$ is a family of dyadic cubes such that $Q_k \subset Q_{k+1}$ and $|Q_{k+1}| = 2^d |Q_k|$ for all $k \in \mathbb{Z}$, then*

$$\lim_{k \longrightarrow \infty} w(Q_k) = \infty \quad \textit{and} \quad \lim_{k \longrightarrow -\infty} w(Q_k) = 0. \tag{5.16}$$

Proof. Since $w \in A_\infty^d$, if $k \ge 0$, we use (5.13) to obtain

$$\frac{w(Q_0)}{w(Q_k)} \le C_w \Big(\frac{|Q_0|}{|Q_k|}\Big)^\delta = C_w \Big(\frac{1}{2^{kd}}\Big)^\delta.$$

Hence, $w(Q_k) \ge (C_w)^{-1} 2^{kd\delta} w(Q_0)$ and $\lim_{k \longrightarrow \infty} w(Q_k) = \infty$. On the other hand, if $k \le 0$, from (5.13) we obtain

$$\frac{w(Q_k)}{w(Q_0)} \le C_w \Big(\frac{|Q_k|}{|Q_0|}\Big)^\delta = C_w 2^{kd\delta}.$$

Thus, $w(Q_k) \le C_w 2^{kd\delta} w(Q_0)$ and $\lim_{k \longrightarrow -\infty} w(Q_k) = 0$.

Proof. (**Proof of Proposition 5.1**) Let $Q_k = [0, 2^k)^d,\ k \in \mathbb{Z}$. By Lemma 5.1 there exists $k_1 \in \mathbb{Z}$ such that

$$w(Q_{k_1}) \leq \tau < w(Q_{k_1+1}). \tag{5.17}$$

Choose $R_1 = Q_{k_1}$. We have

$$w(R_1) = w(Q_{k_1}) \leq \tau.$$

On the other hand, from the B_p^d condition (see (5.15)) we deduce

$$\frac{w(Q_{k_1})}{w(Q_{k_1+1})} \geq C_w \Big(\frac{|Q_{k_1}|}{|Q_{k_1+1}|} \Big)^p = C_w 2^{-dp},$$

so that

$$w(R_1) = w(Q_{k_1}) \geq C_w 2^{-dp} w(Q_{k_1+1}) > C_w 2^{-dp} \tau.$$

Hence, we can take $C = C_w 2^{-dp}$ in this first step.

Suppose that we have chosen a collection of disjoint cubes $R_1, \cdots, R_{m-1}$ such that $C\tau < w(R_j) \leq \tau$ for all $j = 1, 2, \cdots, m-1$. Without loss of generality, we can assume that all cubes R_j are in the positive cone of $\mathbb{R}^d$, that is, the set of all points of $\mathbb{R}^d$ with non-negative coordinates.

Choose $Q_0 = 2^{k_m}[0,1)^d,\ k_m \in \mathbb{Z}$, such that $R_j \subset Q_0$ for all $j = 1, 2, \cdots, m-1$. Consider the increasing family of dyadic cubes given by $Q_k = 2^{k_m+k}[0,1)^d,\ k = 0, 1, 2, \cdots$. Let $\widetilde{Q}_k,\ k = 1, 2, \cdots$, be a dyadic cube contained in Q_k such that $|\tilde{Q}_k| = \frac{|Q_k|}{2^d}$ and $\tilde{Q}_k \cap Q_{k-1} = \emptyset$. If $w(\tilde{Q}_k) \leq \tau$ for all $k = 1, 2, 3, \cdots$, since $w \in B_p^d$, we obtain

$$\frac{w(\tilde{Q}_k)}{w(Q_k)} \geq C_w \Big(\frac{|\tilde{Q}_k|}{|Q_k|} \Big)^p = C_w 2^{-dp}.$$

Hence, contrary to Lemma 5.1, we have $w(Q_k) \leq (C_w)^{-1} 2^{dp} \tau$ for all $k = 1, 2, \cdots$,. Therefore, there exists $k_m^0 \in \mathbb{Z}$ such that $w(\widetilde{Q}_{k_m^0}) > \tau$. Now consider a family of decreasing dyadic cubes contained in $\widetilde{Q}_{k_m^0}$. By Lemma 5.1 there exists $\widetilde{Q}_{k_m} \supset \widetilde{\widetilde{Q}}_{k_m}$ such that

$$w(\widetilde{\widetilde{Q}}_{k_m}) \leq \tau < w(\widetilde{Q}_{k_m}) \tag{5.18}$$

and $|\widetilde{\widetilde{Q}}_{k_m}| = \frac{|\widetilde{Q}_{k_m}|}{2^d}$. Choose $R_m = \widetilde{\widetilde{Q}}_{k_m}$. Since (5.18) are the same inequalities as (5.17), we have

$$C_w 2^{-dp} \tau < w(R_m) \leq \tau.$$

Observe that R_m has been chosen in the positive cone of $\mathbb{R}^d$ and disjoint with $R_1, \cdots, R_{m-1}$.

We will also need the following result:

Lemma 5.2. *Let* $w \in A_r$ *a weight in* $\mathbb{R}^d$ *with* $r \geq 1$. *Let* $0 < \delta < 1$ *and* $u(x) = w(x)^\delta$. *We have that* $u \in A_r$ *and* $w_Q \approx (u_Q)^{\frac{1}{\delta}}$, *where*

$$w_Q = \frac{1}{|Q|} w(Q) = \frac{1}{|Q|} \int_Q w(x)dx.$$

Proof. When $r > 1$, since $w \in A_r$ and $0 < \delta < 1$, using Jensen's inequality, we deduce

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q u(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q u(x)^{1-r'}dx\Big)^{r-1} \\
&= \Big(\frac{1}{|Q|}\int_Q w(x)^\delta dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{\delta(1-r')}dx\Big)^{r-1} \\
&\leq \Big[\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-r'}dx\Big)^{r-1}\Big]^\delta \leq C_w^\delta.
\end{aligned}$$

Hence, $u \in A_r$. To show the equivalence $w_Q \approx (u_Q)^{\frac{1}{\delta}}$, by Jensen's inequality with $\delta < 1$, we have

$$(u_Q)^{\frac{1}{\delta}} = \Big(\frac{1}{|Q|}\int_Q w(x)^\delta dx\Big)^{\frac{1}{\delta}} \leq \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big) = w_Q.$$

The function $h(t) = t^{-(r-1)\delta}$, $t > 0$, is convex; using again Jensen's inequality we obtain

$$\Big(\frac{1}{|Q|}\int_Q w^{1-r'}(x)dx\Big)^{-(r-1)\delta} \leq \Big(\frac{1}{|Q|}\int_Q [w(x)]^\delta dx\Big) = u_Q.$$

Hence, for $w \in A_r$ we deduce

$$w_Q = \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big) \leq C_w\Big(\frac{1}{|Q|}\int_Q w^{1-r'}(x)dx\Big)^{-(r-1)} \leq C(u_Q)^{\frac{1}{\delta}}.$$

If $r = 1$, since $w \in A_1$, by Jensen's inequality, for almost every $x \in Q$ we have

$$\Big(\frac{1}{|Q|}\int_Q u(x)dx\Big) = \Big(\frac{1}{|Q|}\int_Q w^\delta(x)dx\Big) \leq \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)^\delta \leq Cw(x)^\delta = Cu(x).$$

Hence, $u \in A_1$. On the other hand, we can again use Jensen's inequality to obtain $(u_Q)^{\frac{1}{\delta}} \leq w_Q$. Moreover, the condition $w \in A_1$ implies that

$$\begin{aligned}
w_Q = \frac{1}{|Q|}\int_Q w(x)dx \leq C ess_Q \inf w = C(ess_Q \inf u)^{\frac{1}{\delta}} \\
\leq C\Big(\frac{1}{|Q|}\int_Q u(x)dx\Big)^{\frac{1}{\delta}} = C(u_Q)^{\frac{1}{\delta}}.
\end{aligned}$$

5.4 Thresholding Greedy Algorithm for the Haar Wavelet in Weighted Lebesgue Spaces

In this section we are going to prove that the d-dimensional Haar system is a greedy basis for the weighted Lebesgue space $L^p(w), 1 < p < \infty$, provided that $w \in A_p(\mathbb{R}^d)$. This result has been obtained as a corollary to more general results in a couple of recent papers. It is proved for $d = 1$ in [34] and for more general d in [47]. The latter is obtained as a corollary of results for weighted Orlicz spaces (see also Section 5.6 in this paper). The proof that we give here is direct and appears in the Ph.D thesis of M. de Natividade [46].

The Haar system $\mathscr{H}^d = \{h_Q^\varepsilon : Q \in \mathscr{D}(d), \varepsilon \in E\}$ (see section 5.2) is an unconditional basis of $L^p(w)$ when $w \in A_p(\mathbb{R}^d)$ (see theorem 6 in [1]). In view of Theorem 5.3, $\mathscr{H}^d$ will be greedy in $L^p(w)$ if we show that $\mathscr{H}^d$ is democratic in the same space. We recall the definition of the Haar system in $\mathbb{R}^d$ given in Section 5.2. Let $h^0(t) = \chi_{[0,1)}(t)$ and $h^1(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)$. Define E as the set of all sequences $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d)$ such that $\varepsilon_j = 0$ or 1 and not all $\varepsilon_j = 0$. For each $\varepsilon \in E$ let

$$h^\varepsilon(x) = \prod_{j=1}^d h^{\varepsilon_j}(x_j).$$

The collection $\mathscr{H} = \{h_Q^\varepsilon : Q \in \mathscr{D},\ \varepsilon \in E\}$ (where $h_{Q_{j,k}}^\varepsilon(x) = |Q_{j,k}|^{-\frac{1}{2}} h^\varepsilon(2^j x - k)$) is the **Haar system** in $\mathbb{R}^d$, normalized in $L^2(\mathbb{R}^d)$.

Lemma 5.3. *Let $1 \le p < \infty$ and w be a weight in $\mathbb{R}^d$ such that $w \in A_\infty(\mathbb{R}^d)$. There exists a constant $0 < C_{p,d} < \infty$ such that*

$$\Big\| \sum_{Q \in \Gamma} \frac{h_Q^\varepsilon}{\|h_Q^\varepsilon\|_{L^p(w)}} \Big\|_{L^p(w)} \le C_{p,d} |\Gamma|^{\frac{1}{p}} \tag{5.19}$$

for all $\Gamma \subset \prod_{\varepsilon \in E} \mathscr{D}_\varepsilon(d)$ (2^{d-1} copies of $\mathscr{D}(d)$) finite.

Proof. Fix $\varepsilon \in E$ and assume first that $\Gamma \subset \mathscr{D}_\varepsilon(d)$. We have

$$\|h_Q^\varepsilon\|_{L^p(w)} = \Big(\int_{\mathbb{R}} |h_Q^\varepsilon(x)|^p w(x) dx \Big)^{\frac{1}{p}} = w(Q)^{\frac{1}{p}} |Q|^{-\frac{1}{2}}.$$

Thus, we have

$$\begin{aligned} \Big\| \sum_{Q \in \Gamma} \frac{h_Q^\varepsilon}{\|h_Q^\varepsilon\|_{L^p(w)}} \Big\|_{L^p(w)} &= \Big(\int_{\mathbb{R}} \Big| \sum_{Q \in \Gamma} \frac{h_Q^\varepsilon(x)}{w(Q)^{\frac{1}{p}} |Q|^{-\frac{1}{2}}} \Big|^p w(x) dx \Big)^{\frac{1}{p}} \\ &\le \Big(\int_{\mathbb{R}} \Big[\sum_{Q \in \Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{1}{p}}} \Big]^p w(x) dx \Big)^{\frac{1}{p}}. \end{aligned} \tag{5.20}$$

Given $x \in \bigcup_{Q\in\Gamma} Q$, let Q_x be the smallest cube in Γ that contains x. There exists a sequence of dyadic cubes $Q_x = Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_j \subset \cdots$ such that $|Q_j| = 2^j|Q_x|$. Observe that the cubes in Γ that contain x must be contained in this sequence. Then, we have

$$\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{1}{p}}} \leq \sum_{j=0}^{\infty} \frac{\chi_{Q_x}(x)}{w(Q_j)^{\frac{1}{p}}}.$$

Using (5.13) we obtain

$$\frac{w(Q_x)}{w(Q_j)} \leq C_w\Big(\frac{|Q_x|}{|Q_j|}\Big)^{\delta} = C_w 2^{-j\delta}.$$

That is, $w(Q_j) \geq \frac{1}{C_w} 2^{j\delta} w(Q_x)$. Hence, we have

$$\begin{aligned}\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{1}{p}}} \leq \sum_{j=0}^{\infty} \frac{\chi_{Q_x}(x)}{w(Q_x)^{\frac{1}{p}}} C_w^{\frac{1}{p}} 2^{-j\delta/p} \leq C_p \frac{\chi_{Q_x}(x)}{w(Q_x)^{\frac{1}{p}}} \\ \leq C_p\Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)}\Big)^{\frac{1}{p}}, \end{aligned} \tag{5.21}$$

where the last inequality is due to the fact that the last sum contains the term $\frac{\chi_{Q_x}}{w(Q_x)^{\frac{1}{p}}}$. Replacing (5.21) in (5.20) we deduce

$$\Big\|\sum_{Q\in\Gamma} \frac{h_Q^{\varepsilon}}{\|h_Q^{\varepsilon}\|_{L^p(w)}}\Big\|_{L^p(w)} \leq C\Big[\int_{\mathbb{R}} \Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)}\Big) w(x)dx\Big]^{\frac{1}{p}} = C_p|\Gamma|^{\frac{1}{p}}. \tag{5.22}$$

If $\Gamma \subset \prod_{\varepsilon\in E} \mathscr{D}_{\varepsilon}(d)$, write $\Gamma_{\varepsilon} = \{Q \in \Gamma : Q \in \mathscr{D}_{\varepsilon}(d)\}$; then $|\Gamma| = \sum_{\varepsilon\in E} |\Gamma_{\varepsilon}|$. Applying (5.22) to each $\varepsilon \in E$ we deduce

$$\Big\|\sum_{Q\in\Gamma} \frac{h_Q^{\varepsilon}}{\|h_Q^{\varepsilon}\|_{L^p(w)}}\Big\|_{L^p(w)} \leq C_p \sum_{\varepsilon\in E} |\Gamma_{\varepsilon}|^{\frac{1}{p}} \leq C_{p,d}|\Gamma|^{\frac{1}{p}}$$

with $C_{p,d} \leq 2^{2^d(1-\frac{1}{p})} C_p$.

The following Lemma gives the reverse inequality to (5.19) using duality arguments when $1 < p < \infty$.

Lemma 5.4. *Suppose that for all $q \in (1,\infty)$ the inequality (5.19) holds. If $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$ (see definition in section 5.3), then we have*

$$\Big\|\sum_{Q\in\Gamma} \frac{h_Q^{\varepsilon}}{\|h_Q^{\varepsilon}\|_{L^p(w)}}\Big\|_{L^p(w)} \geq \frac{C_w}{C_{p',d}}|\Gamma|^{\frac{1}{p}} \tag{5.23}$$

for all finite set $\Gamma \subset \prod_{\varepsilon\in E} \mathscr{D}_{\varepsilon}(d)$.

Proof. Given $f \in L^p(w)$, by duality we have

$$\|f\|_{L^p(w)} = \sup\Big\{\Big|\int_{\mathbb{R}} f(x)g(x)dx\Big| : \|g\|_{L^{p'}(w^{-\frac{p'}{p}})} \le 1, g \neq 0\Big\}. \tag{5.24}$$

Since $w \in A_p(\mathbb{R}^d)$, we deduce $w^{-\frac{p'}{p}} \in A_{p'}(\mathbb{R}^d)$. Let

$$g(x) = \frac{1}{C_{p'}|\Gamma|^{\frac{1}{p'}}} \sum_{Q\in\Gamma} \frac{h_Q^\varepsilon(x)}{\|h_Q^\varepsilon(x)\|_{L^{p'}(w^{-\frac{p'}{p}})}}.$$

Using our hypothesis with $q = p'$ and $w^{\frac{-p'}{p}} \in A_{p'}$, we deduce

$$\begin{aligned}\|g\|_{L^{p'}(w^{-\frac{p'}{p}})} &= \Big\|\frac{1}{C_{p'}|\Gamma|^{\frac{1}{p'}}} \sum_{Q\in\Gamma} \frac{h_Q^\varepsilon(x)}{\|h_Q^\varepsilon(x)\|_{L^{p'}(w^{-\frac{p'}{p}})}}\Big\|_{L^{p'}(w^{-\frac{p'}{p}})} \\ &\le \frac{1}{C_{p'}|\Gamma'|^{\frac{1}{p'}}} C_{p'}|\Gamma|^{\frac{1}{p'}} = 1.\end{aligned}$$

From (5.24) we obtain

$$\begin{aligned}&\Big\|\sum_{Q\in\Gamma} \frac{h_Q^\varepsilon}{\|h_Q^\varepsilon\|_{L^p(w)}}\Big\|_{L^p(w)} \ge \Big|\int_{\mathbb{R}}\Big(\sum_{Q\in\Gamma} \frac{h_Q^\varepsilon(x)}{|Q|^{-\frac{1}{2}} w(Q)^{\frac{1}{p}}}\Big) g(x)dx\Big| \\ &= \Big|\int_{\mathbb{R}}\Big(\sum_{Q\in\Gamma} \frac{h_Q^\varepsilon(x)}{|Q|^{-\frac{1}{2}} w(Q)^{\frac{1}{p}}}\Big) \frac{1}{C_{p'}|\Gamma|^{\frac{1}{p'}}}\Big(\sum_{Q\in\Gamma} \frac{h_Q^\varepsilon(x)}{|Q|^{-\frac{1}{2}}[w^{-\frac{p'}{p}}(Q)]^{\frac{1}{p'}}}\Big)\Big|dx \\ &= \frac{1}{C_{p'}|\Gamma|^{\frac{1}{p'}}} \sum_{Q\in\Gamma} \frac{1}{w(Q)^{\frac{1}{p}}[w^{-\frac{p'}{p}}(Q)]^{\frac{1}{p'}}} \int_{\mathbb{R}} \chi_Q(x)dx \\ &= \frac{1}{C_{p'}|\Gamma|^{\frac{1}{p'}}} \sum_{Q\in\Gamma} \frac{|Q|}{w(Q)^{\frac{1}{p}}[w^{-\frac{p'}{p}}(Q)]^{\frac{1}{p'}}}.\end{aligned}$$

Since $w \in A_p$ we have

$$w(Q)^{\frac{1}{p}}[w^{-\frac{p'}{p}}(Q)]^{\frac{1}{p'}} = \Big(\int_Q w(x)dx\Big)^{\frac{1}{p}}\Big(\int_Q w^{-\frac{p'}{p}}(x)dx\Big)^{\frac{1}{p'}} \le C_w|Q|.$$

Hence,

$$\Big\|\sum_{Q\in\Gamma} \frac{h_Q^\varepsilon}{\|h_Q^\varepsilon\|_{L^p(w)}}\Big\|_{L^p(w)} \ge \frac{C_w}{C_{p'}} \frac{|\Gamma|}{|\Gamma|^{\frac{1}{p'}}} = \frac{C_w}{C_{p',d}}|\Gamma|^{\frac{1}{p}}.$$

From Lemmas 5.3 and 5.4 we deduce the following result:

Theorem 5.4. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$. For the Haar system $\mathscr{H}^d$ in $\mathbb{R}^d$ we have*

$$h_l(N;\mathscr{H}^d, L^p(w)) \approx h_r(N;\mathscr{H}`d, L^p(w)) \approx N^{\frac{1}{p}}.$$

Hence, the Haar basis is greedy in $L^p(w)$, when $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$.

To end this section we will show that Lemma 5.4 is not valid if $w \notin A_p(\mathbb{R}^d)$. We will consider $d = 1$. Let $w(x) = p|x|^{p-1} \notin A_p(\mathbb{R})$, and $I_j = [0, \frac{1}{2^j})$, $j = 0, 1, 2, \cdots$ Observe that

$$w(I_j) = \int_0^{\frac{1}{2^j}} px^{p-1}dx = \frac{1}{2^{jp}} = |I_j|^p.$$

Then,

$$\Big\| \sum_{j=0}^{N-1} \frac{h_{I_j}}{\|h_{I_j}\|_{L^p(w)}} \Big\|_{L^p(w)} = \Big(\int_{\mathbb{R}} \Big| \sum_{j=0}^{N-1} \frac{h_{I_j}(x)}{|I_j|^{-\frac{1}{2}} w(I_j)^{\frac{1}{p}}} \Big|^p w(x)dx \Big)^{\frac{1}{p}}$$

$$= \Big(\int_{\mathbb{R}} \Big| \sum_{j=0}^{N-1} |I_j|^{-\frac{1}{2}} h_{I_j}(x) \Big|^p w(x)dx \Big)^{\frac{1}{p}} = \Big(\int_0^1 \Big| \sum_{j=0}^{N-1} 2^j h(2^j x) \Big|^p w(x)dx \Big)^{\frac{1}{p}}.$$

But,

$$\sum_{j=0}^{N-1} 2^j h(2^j x) = \begin{cases} 1 + 2 + \cdots + 2^{N-1}, & \text{if } 0 \le x < \frac{1}{2^N} \\ -1, & \text{if } \frac{1}{2^N} \le x < 1. \end{cases}$$

Thus,

$$\Big\| \sum_{j=0}^{N-1} \frac{h_{I_j}}{\|h_{I_j}\|_{L^p(w)}} \Big\|_{L^p(w)}^p = \int_0^{\frac{1}{2^N}} (2^N - 1)^p px^{p-1}dx + \int_{\frac{1}{2^N}}^1 px^{p-1}dx$$

$$= \frac{(2^N - 1)^p}{2^{Np}} + \Big(1 - \frac{1}{2^{Np}}\Big) \le 2.$$

Hence, we can never have (5.23) for $w(x) = p|x|^{p-1}$.

5.5 Thresholding Greedy Algorithm for Wavelet Bases in Weighted Triebel-Lizorkin Spaces

Under certain decay and smoothness conditions, wavelet bases provide characterizations of classical Lebesgue and Sobolev spaces. For $p \in (1, \infty)$ (and wavelets with enough decay) it is known that

$$\|f\|_{L_p(\mathbb{R}^d)} \approx \Big\| \Big[\sum_{\ell=1}^{L} \sum_{Q\in\mathscr{D}(d)} \big(|Q|^{-\frac{1}{2}} |\langle f\,,\,\psi_Q^\ell\rangle| \chi_Q(\cdot)\big)^2 \Big]^{1/2} \Big\|_{L_p(\mathbb{R}^d)}. \tag{5.25}$$

Equivalence (5.25) reduces to Plancherel theorem for $p = 2$ and it is essentially a Littlewood-Paley characterization of Lebesgue spaces for $p \neq 2$. Conditions on $\Psi = \{\psi^1, \dots, \psi^L\}$ for which (5.25) holds can be found, for instance, in [30, 44] or [58]. A collection $\Psi = \{\psi^1, \dots, \psi^L\}$ of functions in $L_2(\mathbb{R}^d)$ for which (5.25) holds is called *admissible wavelet family* for $L_p(\mathbb{R})$.

When $p \in (1,\infty)$ and $s > 0$, Sobolev spaces on $\mathbb{R}^d$ are defined as

$$W_p^s(\mathbb{R}^d) = \Big\{ f \in L_p(\mathbb{R}^d) \ : \ \mathscr{F}^{-1}[(1+|\xi|^2)^{s/2}\hat{f}(\xi)] \in L^p(\mathbb{R}^d) \Big\}. \tag{5.26}$$

They also have a characterization using wavelets. When $\psi^\ell, \ell = 1,2,\dots,L$, have enough smoothness and decay it is known that

$$\|f\|_{W_p^s(\mathbb{R}^d)} \approx \Big\| \Big[\sum_{\ell=1}^{L} \sum_{D\in\mathscr{D}(d)} \big(|Q|^{-\frac{s}{d}-\frac{1}{2}} |\langle f\,,\,\psi_Q^\ell\rangle| \chi_Q(\cdot)\big)^2 \Big]^{1/2} \Big\|_{L_p(\mathbb{R}^d)}. \tag{5.27}$$

We refer to [30, 41, 44] for conditions under which the collection Ψ is an admissible wavelet family for $W_p^s(\mathbb{R}^d)$.

The above spaces are particular cases of a more general class called Triebel-Lizorkin spaces $F_{p,q}^s(\mathbb{R}^d)$. The Lebesgue spaces $L_p(\mathbb{R}^d)$ coincide with $F_{p,2}^0(\mathbb{R}^d)$ when $1 < p < \infty$. Also the Sobolev spaces $W_p^s(\mathbb{R}^d)$ coincide with $F_{p,2}^s(\mathbb{R}^d)$ when $1 < p < \infty$ and $s > 0$. To include all the previously known results, in this section we consider weighted Triebel-Lizorkin spaces, with weights in an appropriate $A_p(\mathbb{R}^d)$ class. The results that follow are part of the PhD. thesis of Maria de Natividade [46].

Let $\mathscr{A}_1$ be the set of functions φ in the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ such that

$$Supp\,\hat{\varphi} \subset \{\xi \in \mathbb{R}^d : \frac{1}{2} < |\xi| < 2\} \text{ and } |\hat{\varphi}| \geq C > 0 \text{ if } \frac{3}{5} < |\xi| < \frac{5}{3}. \tag{5.28}$$

We define $\mathscr{A}_0$ as the set of all functions $\Phi \in \mathscr{S}(\mathbb{R}^d)$ such that

$$Supp\,\hat{\Phi} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\} \text{ and } |\hat{\Phi}(\xi)| \geq C > 0 \text{ if } |\xi| \leq \frac{5}{3}. \tag{5.29}$$

Given a function φ defined on $\mathbb{R}^d$ we use the notation

$$\varphi_k(x) = 2^{kd}\varphi(2^k x).$$

Definition 5.4. Let $s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \leq \infty,\ \varphi \in \mathscr{A}_1,\ \Phi \in \mathscr{A}_0$ and w be a weight in $\mathbb{R}^d$.

i) The **homogeneous weighted Triebel-Lizorkin space** $\dot{F}^s_{p,q}(w)$ is the set of all the tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}$ (modulo polynomials) such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{ks}|\varphi_k * f|)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} < \infty.$$

ii) The **(non-homogeneous) weighted Triebel-Lizorkin space** $F^s_{p,q}(w)$ is the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{F^s_{p,q}(w)} = \|\Phi * f\|_{L^p(w)} + \Big\| \Big(\sum_{k=1}^{\infty} (2^{ks}|\varphi_k * f|)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} < \infty.$$

When $w = 1$, Definition 5.4 coincides with the definition of Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$ and $F^s_{p,q}$ given in [20]. As in Section 5 of [20] it can be proved that

$$\|f\|_{F^s_{p,q}(w)} \approx \|f\|_{L^p(w)} + \|f\|_{\dot{F}^s_{p,q}(w)}$$

when $s > 0,\ 1 \le p < \infty,\ 0 < q \le \infty$ and $w \in A_p(\mathbb{R}^d)$.

Definition 5.4 depends on $\varphi \in \mathscr{A}_1$ and $\Phi \in \mathscr{A}_0$, but it can be proved that different choices of these functions satisfying (5.28) and (5.29) give the same spaces with equivalent norms.

It is well known that the spaces $\dot{F}^s_{p,q}$ and $F^s_{p,q}$ are characterized in terms of wavelet coefficients and that such families of wavelets are unconditional bases of the Triebel-Lizorkin spaces. These wavelets must have some properties (such as smoothness and vanishing moments) depending on the space we want to characterize. For future reference we give the definition of the regularity class $\mathscr{R}^{r,M}$.

Definition 5.5. (**Regularity classes**) Let r be a non-negative integer and $M > d + r$. The regularity class $\mathscr{R}^{r,M}$ is the set of all functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ such that

i) $\int_{\mathbb{R}^d} x^\alpha f(x)dx = 0 \quad \forall\, \alpha \in \mathbb{N}^d,\ 0 \le |\alpha| \le r$

ii) $|f(x)| \le \frac{C}{(1+|x|)^M}, \quad \forall\, x \in \mathbb{R}^d$

iii) $|D^\alpha f(x)| \le \frac{C}{(1+|x|)^M}, \quad \forall\, x \in \mathbb{R}^d,\ \forall\, \alpha \in \mathbb{N}^d,\ 0 < |\alpha| \le r+1$

We now described the known results about characterization of weighted Lebesgue, Hardy and Triebel-Lizorkin spaces using wavelet coefficients.

For the case of weighted Lebesgue spaces, we have $L^p(w) = F^0_{p,2}(w)$ if $w \in A_p(\mathbb{R}^d),\ 1 < p < \infty$. The following result about characterization of weighted Lebesgue spaces with wavelet coefficients is given in [23] and [1].

Proposition 5.2. *Let* $1 < p < \infty$, $w \in A_p(\mathbb{R}^d)$ *and* $\Phi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^d)$ *with* $L = 2^d - 1$ *be a wavelet family obtained from a 1-regular Multiresolution Analysis*

(MRA) (see definition in [44] and [1]). Then, $\mathscr{W} = \{\psi_Q^l : Q \in \mathscr{D}(d), l = 1, 2, \cdots, L\}$ *is an unconditional basis for* $L^p(w)$ *(see also [22]) and we have the following characterization:*

$$\|f\|_{L^p(w)} \approx \Big\| \Big(\sum_{l=1}^{2^d-1} \sum_{Q \in \mathscr{D}(d)} (|\langle f, \psi_Q^l \rangle| |Q|^{-\frac{1}{2}} \chi_Q(\cdot))^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(w)}.$$

The result given in Proposition 5.2 is also true for the Haar wavelet in $\mathbb{R}^d$ when $w \in A_p^{dy}(\mathbb{R}^d)$ (see theorem 6 in [1]).

For the weighted Hardy spaces $H^p(w),\ 0 < p \leq 1$ (see [14]) defined in $\mathbb{R}$ with $w \in A_\infty(\mathbb{R})$ we consider the **critical index** of w as $q_w = \inf\{q > 1 : w \in A_q(\mathbb{R})\}$.

Proposition 5.3. *(See theorem 4.2 in [23]). Let* $0 < p \leq 1,\ w \in A_\infty(\mathbb{R})$ *and* q_w *be the critical index of* w*. Let* $L \geq 1$ *such that* $L \geq \frac{q_w}{p}$ *and* $\psi \in L^2(\mathbb{R})$ *and orthonormal wavelet with* $\psi \in \mathscr{R}^{0,L}$ *(see Definition 5.5). Then, the collection* $\mathscr{W} = \{\psi_Q : Q \in \mathscr{D}\}$ *is an unconditional bases of* $H^p(w)$ *and*

$$\|f\|_{H^p(w)} \approx \Big\| \Big(\sum_{Q \in \mathscr{D}(1)} (|\langle f, \psi_Q \rangle| |Q|^{-\frac{1}{2}} \chi_Q(\cdot))^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(w)}.$$

The following result is due to M. Izuki and Y. Sawano [32] and gives a characterization of weighted Triebel-Lizorkin spaces $F_{p,q}^s(w)$ with $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in A_\infty(\mathbb{R}^d)$. We need some notation before stating the result. For $w \in A_\infty(\mathbb{R}^d)$ let $q_w = \inf\{u \in [1, \infty) : w \in A_u(\mathbb{R}^d)\}$ be the critical index of w. Define

$$\sigma_p(w) := d\Big(\frac{q_w}{\min(p, q_w)} - 1\Big) + (q_w - 1)d, \quad \sigma_q := \sigma_q(1) = d\Big(\frac{1}{\min(q, 1)} - 1\Big)$$

and

$$\sigma_{p,q}(w) := \max\{\sigma_p(w), \sigma_q\}.$$

Proposition 5.4. *(See theorems 1, 14, and 15 in [32]). Let* $s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \leq \infty$ *and* $w \in A_\infty(\mathbb{R}^d)$*. Let* $r = (1 + [s])_+$ *and suppose that* $\widetilde{\Phi} = \{\psi^0, \psi^l : l = 1, 2, \cdots, L\} \subset C^r(\mathbb{R}^d)$ *contains a scaling function* ψ^0 *and a family of* d*-dimensional wavelets* ψ^l *with compact support [15]. Furthermore, assume that* $\int_{\mathbb{R}^d} x^\beta \psi^l(x) dx = 0$ *for all* $|\beta| \leq \max\{r, L_F\}$ *where* $L_F = [\sigma_{p,q}(w) - s],\ l = 1, 2, \cdots, L$*. Then*

$$\|f\|_{F_{p,q}^s(w)} \approx \Big\| \Big(\sum_{\substack{Q \in \mathscr{D} \\ |Q|=1}} |\langle f, \psi_Q^0 \rangle|^2 \chi_Q(\cdot) \Big)^{\frac{1}{2}} \Big\|_{L^p(w)} + \Big\| \Big[\sum_{l=1}^{L} \sum_{Q \in \mathscr{D}} (|Q|^{-\frac{s}{d} - \frac{1}{2}} |\langle f, \psi_Q^l \rangle| \chi_Q(\cdot))^q \Big]^{\frac{1}{q}} \Big\|_{L^p(w)}.$$

Moreover, if $0 < p < \infty$, $0 < q < \infty$, $\mathscr{W} = \{\psi^0_{0,k}, \psi^l_{j,k} : j \in \mathbb{N},\ k \in \mathbb{Z}^d,\ l = 1,2,\cdots,L\}$ *is an unconditional basis of* $F^s_{p,q}(w)$.

Remark. A similar result for the homogeneous case can be found in [7].

Taking into account Proposition 5.4 we define the following sequence spaces associated to the Triebel-Lizorkin spaces $F^s_{p,q}(w)$ and $\dot{F}^s_{p,q}(w)$. We will consider only the case $L = 1$ since the case $L > 1$ changes only the constants in the following results.

Recall that $\mathscr{D}(d)$ denotes the set of all dyadic cubes in $\mathbb{R}^d$. By $\mathscr{D}_0(d)$ we denote the set of all cubes $Q \in \mathscr{D}(d)$ such that $|Q| = 1$, and by $\mathscr{D}^+(d)$ we mean the set of all $Q \in \mathscr{D}(d)$ such that $|Q| \le 1$.

Definition 5.6. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < r \le \infty$ and w a weight in $\mathbb{R}^d$.

i) Define $\mathfrak{s}\dot{F}^s_{p,r}(w)$ as the space of all sequences of complex numbers $\mathbf{s} = \{s_Q : Q \in \mathscr{D}(d)\}$ such that

$$\|\mathbf{s}\|_{\mathfrak{s}\dot{F}^s_{p,r}(w)} = \Big\| \Big(\sum_{Q \in \mathscr{D}(d)} (|Q|^{-\frac{s}{d}-\frac{1}{2}} |s_Q| \chi_Q(\cdot))^r \Big)^{\frac{1}{r}} \Big\|_{L^p(w)} < \infty.$$

ii) Define $\mathfrak{s}F^s_{p,r}(w)$ as the space of all sequences of complex numbers $\mathbf{s} = \{s_Q : Q \in \mathscr{D}^0(d)\} \cup \{s_Q : Q \in \mathscr{D}^+(d)\}$ such that

$$\|\mathbf{s}\|_{\mathfrak{s}F^s_{p,r}(w)} = \Big[\sum_{Q \in \mathscr{D}_0(d)} |s_Q|^p w(Q) \Big]^{\frac{1}{p}}$$
$$+ \Big\| \Big(\sum_{Q \in \mathscr{D}^+(d)} (|Q|^{-\frac{s}{d}-\frac{1}{2}} |s_Q| \chi_Q(\cdot))^r \Big)^{\frac{1}{r}} \Big\|_{L^p(w)} < \infty.$$

It is not difficult to prove that $\mathfrak{s}\dot{F}^s_{p,r}(w)$ and $\mathfrak{s}F^s_{p,r}(w)$ are quasi-Banach spaces. For these sequence spaces we consider the element $\mathbf{e}_Q$ defined by

$$\mathbf{e}_Q = \begin{cases} 1, & \text{if } Q = Q' \\ 0, & \text{if } Q \ne Q'. \end{cases}$$

From Definition 5.6 we deduce that $\dot{\mathscr{B}}_c = \{\mathbf{e}_Q : Q \in \mathscr{D}(d)\}$ and $\mathscr{B}_c = \{e_Q : Q \in \mathscr{D}^0(d)\} \cup \{e_Q : Q \in \mathscr{D}^+(d)\}$ are unconditional bases of $\mathfrak{s}\dot{F}^s_{p,r}(w)$ and $\mathfrak{s}F^s_{p,r}(w)$, respectively. Therefore, according to Theorem 5.3, to show that wavelet bases are greedy in Triebel-Lizorkin spaces all we need to study is its democracy functions.

Lemma 5.5. *Let* $0 < p < \infty$, $0 < r \le \infty$, *and* $w \in A^d_\infty$ *(see (5.13)) be a weight in* $\mathbb{R}^d$. *For all* $\Gamma \subset \mathscr{D}(d)$ *finite we have*

$$\Big(\sum_{Q \in \Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{r}{p}}} \Big)^{\frac{1}{r}} \approx \frac{\chi_{Q_x}(x)}{w(Q_x)^{\frac{1}{p}}} \quad \forall\, x \in \bigcup_{Q \in \Gamma} Q$$

where Q_x *is the smallest cube in* Γ *that contains* x.

Proof. There exist a unique sequence of dyadic cubes $Q_x \equiv Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_j \subset \cdots$ such that $|Q_j| = 2^{jd}|Q_x|$. Since all the cubes in Γ containing x belong to this sequence, for $x \in \bigcup_{Q\in\Gamma} Q$ we have

$$\Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{r}{p}}}\Big)^{\frac{1}{r}} \leq \Big(\sum_{j=0}^{\infty} \frac{\chi_{Q_x}(x)}{w(Q_j)^{\frac{r}{p}}}\Big)^{\frac{1}{r}}.$$

Using condition (5.13) we obtain

$$\frac{w(Q_x)}{w(Q_j)} \leq C_w 2^{-j\delta d}.$$

Hence,

$$\begin{aligned}\Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{r}{p}}}\Big)^{\frac{1}{r}} &\leq C_w^{\frac{1}{p}} \Big(\sum_{j=0}^{\infty} \frac{\chi_{Q_x}(x)}{w(Q_x)^{\frac{r}{p}}} 2^{-jd\delta\frac{r}{p}}\Big)^{\frac{1}{r}} \\ &\leq C_{w,p} \frac{\chi_{Q_x}(x)}{w(Q_x)^{\frac{1}{p}}} \leq C_{w,p} \Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{r}{p}}}\Big)^{\frac{1}{r}}\end{aligned}$$

where the last equality is due to the fact that the last sum contains the term $\frac{\chi_{Q_x(x)}}{w(Q_x)^{\frac{1}{p}}}$.

For $\Gamma \subset \mathscr{D}(d)$ finite, we use the notation

$$\widetilde{1}_\Gamma = \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_E} \tag{5.30}$$

to denote the normalized characteristic function of Γ in the space $E = \mathfrak{s}F^s_{p,r}(w)$ or $E = \mathfrak{s}\dot{F}^s_{p,r}(w)$.

Theorem 5.5. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < r \leq \infty$ and $w \in A^d_\infty$ (see (5.13)) be a weight in $\mathbb{R}^d$.*

i) If $\Gamma \subset \mathscr{D}(d)$ is a finite set, we have

$$\|\widetilde{1}_\Gamma\|_{\mathfrak{s}\dot{F}^s_{p,r}(w)} \approx N^{\frac{1}{p}}$$

Hence,

$$h_l(N;\dot{\mathscr{B}}_c, \mathfrak{s}\dot{F}^s_{p,r}(w)) \approx h_r(N;\dot{\mathscr{B}}_c, \mathfrak{s}\dot{F}^s_{p,r}(w)) = N^{\frac{1}{p}}.$$

ii) If $\Gamma \subset \mathscr{D}_0(d) \cup \mathscr{D}^+(d)$ is a finite set, we have

$$\|\widetilde{1}_\Gamma\|_{\mathfrak{s}F^s_{p,r}(w)} \approx |\Gamma|^{\frac{1}{p}}.$$

Hence,

$$h_l(N;\mathscr{B}_c,\mathfrak{s}F^s_{p,r}(w)) \approx h_r(N;\mathscr{B}_c,\mathfrak{s}F^s_{p,r}(w)) = N^{\frac{1}{p}}.$$

Proof. i) For $Q \in \mathscr{D}(d)$, from Definition 5.6 we deduce

$$\|\mathbf{e}_Q\|_{\mathfrak{s}\dot{F}^s_{p,r}(w)} = |Q|^{-\frac{s}{d}-\frac{1}{2}} w(Q)^{\frac{1}{p}}.$$

Hence,

$$\|\widetilde{1}_\Gamma\|_{\mathfrak{s}\dot{F}^s_{p,r}(w)} = \Big\| \Big(\sum_{Q\in\Gamma} \frac{\chi_Q(\cdot)}{w(Q)^{\frac{r}{p}}} \Big)^{\frac{1}{r}} \Big\|_{L^p(w)}. \tag{5.31}$$

Let $S_\Gamma(x) = \Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)^{\frac{r}{p}}} \Big)^{\frac{1}{r}}$. From Lemma 5.5 we obtain $S_\Gamma(x) \approx \frac{\chi_{Q_x}(x)}{w(Q_x)^{\frac{1}{p}}}$. Define now $Z_\Gamma(x) = \sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)}$. Using Lemma 5.5 with $r = 1$ and $p = 1$ we obtain $Z_\Gamma(x) \approx \frac{\chi_{Q_x(x)}}{w(Q_x)}$. Thus, $(S_\Gamma(x))^p \approx Z_\Gamma(x)$ for every $x \in \bigcup_{Q\in\Gamma} Q$. From (5.31) we deduce

$$\begin{aligned}
&\|\widetilde{1}_\Gamma\|_{\mathfrak{s}\dot{F}^s_{p,r}(w)} \approx \|S_\Gamma(x)\|_{L^p(w)} = \Big(\int_{\mathbb{R}^d} (S_\Gamma(x))^p w(x)dx \Big)^{\frac{1}{p}} \\
&\approx \Big(\int_{\mathbb{R}^d} Z_\Gamma(x) w(x)dx \Big)^{\frac{1}{p}} = \Big(\int_{\mathbb{R}^d} \sum_{Q\in\Gamma} \frac{\chi_Q(x)}{w(Q)} w(x)dx \Big)^{\frac{1}{p}} \\
&\Big(\sum_{Q\in\Gamma} \int_{\mathbb{R}^d} \frac{\chi_Q(x)}{w(Q)} w(x)dx \Big)^{\frac{1}{p}} = |\Gamma|^{\frac{1}{p}}.
\end{aligned}$$

ii) Let $\Gamma = \Gamma^0 \cup \Gamma^+$ with $\Gamma^0 \subset \mathscr{D}_0(d)$ and $\Gamma^+ \subset \mathscr{D}^+(d)$, in such a way that $|\Gamma| = |\Gamma^0| + |\Gamma^+|$. An argument similar to the one above gives

$$\|\widetilde{1}_\Gamma\|_{\mathfrak{s}F^s_{p,r}(w)} \approx |\Gamma^0|^{\frac{1}{p}} + |\Gamma^+|^{\frac{1}{p}} \approx |\Gamma|^{\frac{1}{p}}.$$

The abstract transference framework designed in Section 6.2 of [24] together with Theorem 5.5 and Propositions 5.2, 5.3, and 5.4 allows us to write the following result:

Theorem 5.6. *Consider the following cases that appear in Propositions 5.2, 5.3, and 5.4.*

(A) $1 < p < \infty$, $w \in A_p(\mathbb{R}^d)$ *and* $\mathscr{W}$ *a 1-regular wavelet basis.*

(B) $0 < p \leq 1$, $w \in A_\infty(\mathbb{R})$ *and* $\mathscr{W}$ *a wavelet basis L-regular with* $L > \frac{q_w}{p}$ *and* q_w *the critical index of* w.

(C) $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $w \in A_\infty(\mathbb{R}^d)$ *and* $\mathscr{W}$ *a system of wavelets as in Proposition 5.4.*

Let $E(w) = L^p(w)$, $H^p(w)$ *or* $F^s_{p,q}(w)$ *according to each one of the cases* (A), (B) *or* (C). *Then,*

$$h_l(N;\mathscr{W},E(w)) \approx h_r(N;\mathscr{W},E(w)) = N^{\frac{1}{p}},$$

and these wavelet systems are democratic in the corresponding space. Consequently, due to Theorem 5.3, they are also greedy when they are unconditional in the corresponding space.

5.6 Thresholding Greedy Algorithm for Wavelet Bases in Weighted Orlicz Spaces

We will start recalling the definition of Orlicz spaces with a weight as well as their basic properties needed in the sequel. A Young function is a non-decreasing convex function $\Phi : [0,\infty) \longrightarrow [0,\infty]$ such that $\lim_{t\longrightarrow\infty} \Phi(t) = \infty$ and doubling, that is there exists $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t \in (0,\infty)$. In this paper we will assume that $\Phi(0) = 0$, Φ is strictly increasing and finite everywhere, so that Φ is a continuous bijection of $[0,\infty)$. We will write $\mathscr{Y}$ to denote the class of such functions.

Definition 5.7. Let $\Phi \in \mathscr{Y}$ be a Young function and $w \in \mathrm{W}(\mathbb{R}^d)$ be a weight in $\mathbb{R}^d$. The weighted Orlicz space $L^{\Phi}(w)$ is the class of all measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ such that $\Phi\Big(\frac{|f(x)|}{\lambda}\Big) \in L^1(w)$ for some $\lambda > 0$.

The spaces $L^{\Phi}(w)$ are Banach spaces with the Luxemburg norm (see [4]) given by

$$\|f\|_{L^{\Phi}(w)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^d} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)dx \leq 1\Big\}. \tag{5.32}$$

Lemma 5.6 ([4]). *Let* $\Phi \in \mathscr{Y}$ *be a Young function and* $w \in \mathrm{W}(\mathbb{R}^d)$ *be a weight in* $\mathbb{R}^d$. *Then, for any measurable set* $E \subset \mathbb{R}^d$ *we have*

$$\|\chi_E\|_{L^{\Phi}(w)} = \frac{1}{\Phi^{-1}(\frac{1}{w(E)})}.$$

Proof. The result is obtained directly from (5.32). For $\lambda > 0$ we have

$$\int_{\mathbb{R}^d} \Phi(\lambda\chi_E(x))w(x)dx = \Phi(\lambda)\int_E w(x)dx = \Phi(\lambda)w(E).$$

Hence, from (5.32) we deduce

$$\begin{aligned}\|\chi_E\|_{L^{\Phi}(w)} &= \inf\{\lambda^{-1} > 0 : \Phi(\lambda)w(E) \leq 1\}\\ &= \Big[\sup\Big\{\lambda > 0 : \Phi(\lambda) \leq \frac{1}{w(E)}\Big\}\Big]^{-1} = \frac{1}{\Phi^{-1}(\frac{1}{w(E)})}.\end{aligned}$$

The function $\varphi(t) := \frac{1}{\Phi^{-1}(\frac{1}{t})}$, $0 < t < \infty$, satisfies $\varphi(t) = \|\chi_E\|_{L^\Phi(w)}$ for any measurable set $E \subset \mathbb{R}^d$ such that $w(E) = t$, and is called **the fundamental function of** $L^\Phi(w)$. The function φ is a continuous bijection of $[0,\infty)$, and strictly increasing. Moreover, φ is quasi-concave, that is, $\frac{\varphi(t)}{t}$ is non-increasing (see [4], page 67), and, hence, φ is doubling.

The Boyd indices of $L^\Phi(w)$ can be computed directly from the Young function Φ or from the fundamental function φ. Define

$$h_\varphi^+(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)},\ 0 < t < \infty.$$

The Boyd indices of $L^\Phi(w)$ are the dilation indices of φ, that is,

$$I_\varphi = \lim_{t \longrightarrow 0} \frac{\log h_\varphi^+(t)}{\log t} = \sup_{0<t<1} \frac{\log h_\varphi^+(t)}{\log t}$$

and

$$i_\varphi = \lim_{t \longrightarrow \infty} \frac{\log h_\varphi^+(t)}{\log t} = \inf_{1<t<\infty} \frac{\log h_\varphi^+(t)}{\log t}$$

(see [4], page 277 or [39] page 54). It is known that

$$0 \le i_\varphi \le I_\varphi \le 1.$$

Assuming that $i_\varphi > 0$, it can be proved that

$$\varphi(st) \le C_\varepsilon \max\{s^{i_\varphi - \varepsilon}, s^{I_\varphi + \varepsilon}\}\varphi(t), \quad s,t > 0 \tag{5.33}$$

and

$$\varphi(st) \ge C_\varepsilon \min\{s^{i_\varphi - \varepsilon}, s^{I_\varphi + \varepsilon}\}\varphi(t), \quad s,t > 0 \tag{5.34}$$

for all $\varepsilon > 0$ and some $C_\varepsilon > 0$ (see [37], page 3). In what follows we will always consider weighted Orlicz spaces with non-trivial Boyd indices, that is, $0 < i_\varphi \le I_\varphi < 1$, since only in this case we have characterizations in terms of wavelets.

A particular case of weighted Orlicz spaces are the weighted Lebesgue spaces $L^p(w)$. When $\Phi(t) = t^p$, $1 \le p < \infty$, then $L^\Phi(w) = L^p(w)$, and $\varphi(t) = t^{\frac{1}{p}}$. Thus, $h_\varphi^+(t) = t^{\frac{1}{p}}$ and this implies that $i_\varphi = I_\varphi = \frac{1}{p}$.

The next result shows that wavelet bases that belong to the regularity class $\mathscr{R}^{0,M}$, $M > d$, are unconditional bases for the weighted Orlicz spaces $L^\Phi(w)$ (see also [45]) for an appropriate weight w, since the norm of these spaces can be characterized in terms of a quadratic function.

Theorem 5.7. *Let $L^{\Phi}(w)$ be a weighted Orlicz space with $w \in \mathrm{W}(\mathbb{R}^d)$ whose Boyd indices satisfy $0 < i_\varphi \le I_\varphi < 1$. Let $\mathscr{W} = \{\psi_Q^\varepsilon : Q \in \mathscr{D},\ \varepsilon \in E\}$ be a wavelet bases in $L^2(\mathbb{R}^d)$ with $\psi^\varepsilon \in \mathscr{R}^{0,M}$, $M > d$. If $w \in A_{p^\Phi}(\mathbb{R}^d)$, where $p^\Phi = \frac{1}{I_\varphi}$, then every function $f \in L^\Phi(w)$ can be written in the form*

$$f = \sum_{\varepsilon \in E} \sum_{Q \in \mathscr{D}} \langle f, \psi_Q^l \rangle \psi_Q^l \tag{5.35}$$

with convergence in $L^\Phi(w)$. Moreover, we have the equivalence

$$\|f\|_{L^\Phi(w)} \approx \|S_\psi f\|_{L^\Phi(w)}, \quad \text{for all } f \in L^\Phi(w), \tag{5.36}$$

where

$$S_\psi f(x) = \Big(\sum_{\varepsilon \in E} \sum_{Q \in \mathscr{D}} |\langle f, \psi_Q^l \rangle|^2 \chi_Q(x) |Q|^{-1} \Big)^{\frac{1}{2}}. \tag{5.37}$$

From (5.36) we deduce that the convergence in (5.35) is unconditional in $L^\Phi(w)$. This result is also true for the d-dimensional Haar wavelet.

For the proof we will use the following extrapolation result:

Theorem 5.8 ([13]). *Let $\mathscr{F}$ be a family of pairs of non-negative measurable functions (f,g). Suppose that for some $1 \le p_0 < \infty$, and every weight $w \in A_{p_0}$ we have*

$$\int_{\mathbb{R}^d} f(x)^{p_0} w(x) dx \le C \int_{\mathbb{R}^d} g(x)^{p_0} w(x) dx, \quad \text{for all } (f,g) \in \mathscr{F}.$$

Then, if $L^\Phi(w)$ is a weighted Orlicz space such that the Boyd indices satisfy $0 < i_\varphi \le I_\varphi < 1$ and $w \in A_{p^\Phi}$, $p^\Phi = \frac{1}{I_\varphi}$, we have

$$\|f\|_{L^\Phi(w)} \le C \|g\|_{L^\Phi(w)}, \quad \text{for all } (f,g) \in \mathscr{F}.$$

Proof. (**Proof of Theorem 5.7**) In [23] (see also [1]) it is proved that

$$\|f\|_{L^p(w)} \approx \|S_\psi(f)\|_{L^p(w)} \tag{5.38}$$

for all $1 < p < \infty$ and $w \in A_p$, when $\psi^\varepsilon \in \mathscr{R}^{0,M}$. Consider the family $\mathscr{F} = \{(|f|, S_\psi(f)) : S_\psi(f) \in L^p(w)\}$. From the equivalence (5.38), there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^d} |f(x)|^p w(x) dx \le C_1 \int_{\mathbb{R}^d} |S_\psi(f)|^p w(x) dx,$$

for all $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$. By Theorem 5.8 we obtain

$$\|f\|_{L^\Phi(w)} \le C_1 \|S_\psi(f)\|_{L^\Phi(w)} \tag{5.39}$$

when $w \in A_{p^{\Phi}}$. For the reverse inequality we take $\mathscr{F} = \{(S_\psi(f), |f|), f \in L^p(w)\}$. From equivalence (5.38) there exists $C_2 > 0$ such that

$$\int_{\mathbb{R}^d} |S_\psi(f)|^p w(x)dx \leq C_2 \int_{\mathbb{R}^d} |f(x)|^p w(x)dx$$

for all $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$. From Theorem 5.8 we obtain

$$\|S_\psi(f)\|_{L^{\Phi}(w)} \leq C_2 \|f(x)\|_{L^{\Phi}(w)} \tag{5.40}$$

when $w \in A_{p^{\Phi}}$. Inequalities (5.39) and (5.40) prove (5.36). To prove (5.35) we use the fact that the projection operators $P_j(f) = \sum_{s=-\infty}^{j} \sum_{k \in \mathbb{Z}} \langle f, \psi_{s,k} \rangle \psi_{s,k}$ are bounded in $L^{\Phi}(w)$. We could again apply Theorem 5.8 since it is known that P_j are bounded in $L^p(w)$ (see also [45]).

In view of Theorems 5.3 and 5.7, to study the greediness of wavelet bases in weighted Orlicz spaces we need to study its democracy functions. This will be computed for wavelets in $\mathscr{R}^{0,M}$ in terms of the fundamental function of $L^{\Phi}(w)$. The results proved in this section have appeared in [47].

We consider the case $L = 1$ since the finite sum that appears in $S_\psi(f)$ will only change the constants in the calculations. The equivalence given in (5.36) allows us to study the democracy functions of weighted Orlicz spaces studying the democracy functions of the corresponding sequence spaces.

Definition 5.8. Let $w \in \mathrm{W}$ and $\Phi \in \mathscr{Y}$. Define $\mathfrak{s}L^{\Phi}(w)$ as the set of all sequences $\mathbf{s} = \{s_Q : Q \in \mathscr{D}(d)\}$ ($\mathscr{D}(d)$ dyadic cubes in $\mathbb{R}^d$) of complex numbers such that

$$\|\mathbf{s}\|_{\mathfrak{s}L^{\Phi}(w)} = \Big\| \Big(\sum_{Q \in \mathscr{D}(d)} |s_Q|^2 \chi_Q(x) |Q|^{-1} \Big)^{\frac{1}{2}} \Big\|_{L^{\Phi}(w)} < \infty. \tag{5.41}$$

The space $\mathfrak{s}L^{\Phi}(w)$ is a normed space. By Theorem 5.7, when $w \in A_{p^{\Phi}}$, $p^{\Phi} = \frac{1}{I_\varphi}$, and $\{\psi_Q : Q \in \mathscr{D}(d)\}$ is a wavelet basis in $\mathscr{R}^{0,M}$, $M > d$, we have

$$\|f\|_{L^{\Phi}(w)} \approx \|\{\langle f, \psi_Q \rangle : Q \in \mathscr{D}\}\|_{\mathfrak{s}L^{\Phi}(w)},$$

and hence, the spaces $L^{\Phi}(w)$ and $\mathfrak{s}L^{\Phi}(w)$ are isomorphic via the application $f \longmapsto \{\langle f, \psi_Q \rangle\}_{Q \in \mathscr{D}}$. Due to this identification, it is enough to consider the canonical basis $\mathscr{B}_c = \{\mathbf{e}_Q : Q \in \mathscr{D}(d)\}$ in the sequence space $\mathfrak{s}L^{\Phi}(w)$.

Proposition 5.5. *Let $\mathfrak{s}L^{\Phi}(w)$ be a weighted Orlicz sequence space with $i_\varphi > 0$, let $w \in \mathrm{W}$ be a weight in $\mathbb{R}^d$, and $\Phi \in \mathscr{Y}$.*

i) If $\Gamma \subset \mathscr{D}(d)$ is a family of pairwise disjoint cubes, then

$$\Big\| \sum_{Q \in \Gamma} \frac{\boldsymbol{e}_Q}{\|\boldsymbol{e}_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} = \Big\| \sum_{Q \in \Gamma} \frac{\chi_Q(\cdot)}{\varphi(w(Q))} \Big\|_{\mathfrak{s}L^{\Phi}(w)}. \tag{5.42}$$

ii) Moreover, if $w \in A_\infty^d \cap B_\infty^d$, then for any $\tau > 0$ and any $N \in \mathbb{N}$ there exists a family of disjoint cubes $\Gamma \subset \mathscr{D}(d)$, with $|\Gamma| = N$, such that

$$\Big\| \sum_{Q\in\mathscr{D}} \frac{\boldsymbol{e}_Q}{\|\boldsymbol{e}_Q\|_{\mathfrak{s}L^\Phi(w)}} \Big\|_{\mathfrak{s}L^\Phi(w)} \approx \frac{\varphi(N\tau)}{\varphi(\tau)}. \tag{5.43}$$

Proof. For an element of the basis $\mathscr{B}_c$, by (5.41) and Lemma 5.6 we deduce that

$$\|\mathbf{e}_Q\|_{\mathfrak{s}L^\Phi(w)} = \Big\| \Big(\frac{\chi_Q(\cdot)}{|Q|}\Big)^{\frac{1}{2}} \Big\|_{L^\Phi(w)} = \frac{\|\chi_Q(\cdot)\|_{L^\Phi(w)}}{|Q|^{\frac{1}{2}}} = \frac{\varphi(w(Q))}{|Q|^{\frac{1}{2}}}.$$

Hence, using again (5.41) we obtain

$$\begin{aligned}\Big\| \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\mathfrak{s}L^\Phi(w)}} \Big\|_{\mathfrak{s}L^\Phi(w)} &= \Big\| \Big(\sum_{Q\in\mathscr{D}} \frac{|Q|\chi_Q(\cdot)|Q|^{-1}}{\varphi(w(Q))^2} \Big)^{\frac{1}{2}} \Big\|_{L^\Phi(w)} \\ &= \Big\| \sum_{Q\in\Gamma} \frac{\chi_Q}{\varphi(w(Q))} \Big\|_{L^\Phi(w)},\end{aligned}$$

where the last equality is due to the fact that the cubes in Γ are pairwise disjoint.

ii) Given $\tau > 0$, by Proposition 5.1 there exists a family of disjoint dyadic cubes $\Gamma = \{R_1, R_2, \cdots, R_N\}$ such that $w(R_j) \approx \tau,\ j = 1, 2, \cdots, N$. In this case, due to i) we have

$$\begin{aligned}\Big\| \sum_{j=1}^N \frac{\mathbf{e}_{R_j}}{\|\mathbf{e}_{R_j}\|_{\mathfrak{s}L^\Phi(w)}} \Big\|_{\mathfrak{s}L^\Phi(w)} &= \Big\| \sum_{j=1}^N \frac{\chi_{R_j}(\cdot)}{\varphi(w(R_j))} \Big\|_{L^\Phi(w)} \approx \frac{1}{\varphi(\tau)} \Big\| \chi_{\bigcup_{j=1}^N R_j} \Big\| \\ &= \frac{1}{\varphi(\tau)} \varphi\Big(w\Big(\bigcup_{j=1}^N R_j\Big)\Big) \approx \frac{\varphi(N\tau)}{\varphi(\tau)},\end{aligned}$$

where the last equality is due to Lemma 5.6.

Remark 5.44 *From part ii) of Proposition 5.5, when $w \in A_\infty^d \cap B_\infty^d$, we deduce*

$$h_r(N; \mathscr{B}_c, \mathfrak{s}L^\Phi(w)) \gtrsim \sup_{\tau>0} \frac{\varphi(N\tau)}{\varphi(\tau)} = h_\varphi^+(N)$$

and

$$h_l(N; \mathscr{B}_c, \mathfrak{s}L^\Phi(w)) \lesssim \inf_{\tau>0} \frac{\varphi(N\tau)}{\varphi(\tau)} := h_\varphi^-(N).$$

We will prove that the right and left democracy functions of $\mathfrak{s}L^\Phi(w)$ coincide with h_φ^+ and h_φ^- (see Remark 5.44) under some assumptions on $w \in$ W. We start linearizing the square function (see (5.37)) of $\widetilde{1}_\Gamma$ with $\Gamma \subset \mathscr{D}(d)$, that is

$$S_\psi(\widetilde{1}_\Gamma)(x) = \Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{\varphi(w(Q))^2} \Big)^{\frac{1}{2}}.$$

Lemma 5.7. *Let $w \in A^d_\infty(\mathbb{R}^d)$ (see Section 5.3) and φ increasing with $i_\varphi > 0$. If $\Gamma \subset \mathscr{D}(d)$ is finite, for $x \in \bigcup_{Q\in\Gamma} Q$ we have*

$$\Big(\sum_{Q\in\Gamma} \frac{\chi_Q(x)}{\varphi(w(Q))^2} \Big)^{\frac{1}{2}} \approx \frac{\chi_{Q_x}(x)}{\varphi(w(Q_x))}, \tag{5.45}$$

where Q_x denotes the smallest cube in Γ that contains x.

Proof. We have the following estimate:

$$\begin{aligned}\Big(\sum_{Q\in\Gamma} \varphi(w(Q))^{-2}\chi_{Q_x}(x) \Big)^{\frac{1}{2}} &\le \Big(\sum_{\substack{Q\supset Q_x\\ Q\in\mathscr{D}(d)}} \frac{1}{\varphi(w(Q))^2} \Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{j=0}^{\infty} \frac{1}{\varphi(w(Q^j_x))^2} \Big)^{\frac{1}{2}} \chi_{Q_x}(x)\end{aligned}$$

where Q^j_x denotes the unique cube of measure $2^{jd}|Q_x|$ which contains Q_x. Since $Q_x := Q^0_x \subset Q^1_x \subset Q^2_x \subset \cdots$ we can use the condition $w \in A^d_\infty$ (see (5.13)) to show that exists $\delta > 0$ such that

$$\frac{w(Q_x)}{w(Q^j_x)} \le C_w \Big(\frac{|Q_x|}{|Q^j_x|} \Big)^{\delta} = C_w 2^{-jd\delta}.$$

Thus, we have

$$w(Q^j_x) \ge C_w^{-1} w(Q_x) 2^{jd\delta}$$

and

$$\varphi(w(Q^j_x)) \ge \varphi(C_w^{-1} w(Q_x) 2^{jd\delta}).$$

Hence,

$$\Big(\sum_{j=0}^{\infty} \frac{1}{\varphi(w(Q^j_x))^2} \Big)^{\frac{1}{2}} \le \Big(\sum_{j=0}^{\infty} \frac{1}{\varphi((C_w)^{-1} w(Q_x) 2^{jd\delta})^2} \Big)^{\frac{1}{2}}.$$

Since $i_\varphi > 0$, by (5.34) we can choose ε such that $0 < \varepsilon < i_\varphi$ and find a constant $C_\varepsilon > 0$ such that $\varphi((C_w)^{-1} 2^{jd\delta} \varphi(w(Q_x)) \ge C_\varepsilon ((C_w)^{-1} 2^{jd\delta})^{(i_\varphi - \varepsilon)} \varphi(w(Q_x))$. Hence, we have

$$\begin{aligned}\Big(\sum_{j=0}^{\infty} \frac{1}{\varphi(w(Q^j_x))^2} \Big)^{\frac{1}{2}} &\le \Big(\sum_{j=0}^{\infty} \frac{1}{(C_\varepsilon((C_w)^{-1} 2^{jd\delta})^{i_\varphi - \varepsilon} \varphi(w(Q_x)))^2} \Big)^{\frac{1}{2}} \\ &= \frac{(C_w)^{(i_\varphi - \varepsilon)} C_\varepsilon^{-1}}{\varphi(w(Q_x))} \Big(\sum_{j=0}^{\infty} 2^{-2jd\delta(i_\varphi - \varepsilon)} \Big)^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\Big(\sum_{Q\in\Gamma}\frac{\chi_Q(x)}{\varphi(w(Q))^2}\Big)^{\frac{1}{2}}\lesssim\frac{\chi_{Q_x}(x)}{\varphi(w(Q_x))}\lesssim\Big(\sum_{Q\in\Gamma}\frac{\chi_Q(x)}{\varphi(w(Q))^2}\Big)^{\frac{1}{2}}, \tag{5.46}$$

where the last inequality in (5.46) is true because the sum on the right-hand side contains at least the cube Q_x.

To prove our results we need to explain the concepts of *light* and *shade* of a cube in a finite collection of dyadic cubes Γ and also the concepts of *shaded* and *lighted* cubes introduced in [25].

Let $\Gamma_{\min}$ be the set of all **minimal** cubes in Γ, that is

$$\Gamma_{\min}=\Big\{Q_x\in\Gamma : x\in\bigcup_{Q\in\Gamma}Q\Big\}.$$

From Lemma 5.7 we deduce

$$\Big(\sum_{Q\in\Gamma}\frac{\chi_Q(x)}{\varphi(w(Q))^2}\Big)^{\frac{1}{2}}\approx\Big(\sum_{Q\in\Gamma_{\min}}\frac{\chi_{Q_x}(x)}{\varphi(w(Q_x))^2}\Big)^{\frac{1}{2}}.$$

Hence, only the cubes of $\Gamma_{\min}$ are relevant for our purposes. Nevertheless, since the cubes in $\Gamma_{\min}$ are not necessarily pairwise disjoint, we need to do a further selection.

Given a finite set $\Gamma\subset\mathscr{D}(d)$ fixed, for any $Q\in\Gamma$ we define the *shade* of Q as the union of all the cubes in Γ strictly contained in Q, that is,

$$Shade(Q)=\bigcup\Big\{R : R\in\Gamma, R\subsetneq Q\Big\}.$$

We define the *Light* of Q as:

$$Light(Q)=Q\backslash Shade(Q).$$

It is clear that $Q\in\Gamma_{\min}$ if and only if *Light(Q)* $\neq\emptyset$, and

$$\bigcup_{Q\in\Gamma}Q=\bigcup_{Q\in\Gamma_{\min}}Light(Q).$$

The sets in the last union are pairwise disjoint. Hence, due to Lemma 5.7 we obtain

$$\Big(\sum_{Q\in\Gamma}\frac{\chi_Q(x)}{\varphi(w(Q))^2}\Big)^{\frac{1}{2}}\approx\sum_{Q\in\Gamma_{\min}}\frac{\chi_{Light(Q)}(x)}{\varphi(w(Q))}, \tag{5.47}$$

where the sum in the right-hand side contains at most one non-zero term for each $x\in\mathbb{R}^d$. A cube $Q\in\Gamma$ is called *shaded* if $|Shade(Q)|>\frac{2^d-1}{2^d}|Q|$ and we write Γ_S for the collection of all cubes in Γ that are shaded. A cube Q in Γ is called *lighted* if it is not shaded, that is, if $|Light(Q)| \geq \frac{1}{2^d}|Q|$. We write Γ_L to denote the collection of all lighted cubes in Γ.

REMARK 5.48 *It is clear that $\Gamma_L \subset \Gamma_{\min}$ and by lemma 4.3 in [25] we have*

$$\frac{2^d-1}{2^d}|\Gamma| \leq |\Gamma_L| \leq |\Gamma_{\min}| \leq |\Gamma|, \quad \forall\, \Gamma \subset \mathscr{D} \text{ finite}.$$

Proposition 5.6. *Let $\mathfrak{s}L^{\Phi}(w)$ be a weighted Orlicz sequence space such that $i_\varphi > 0$, let $w \in A_\infty^d$ be a weight in $\mathbb{R}^d$ and $\Phi \in \mathscr{Y}$. Then, for all $\Gamma \subset \mathscr{D}(d)$ finite, we have*

$$\Big\| \sum_{Q \in \Gamma} \frac{\boldsymbol{e}_Q}{\|\boldsymbol{e}_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} \lesssim h_\varphi^+(N), \quad N = |\Gamma|. \tag{5.49}$$

In particular

$$h_r(N; \mathscr{B}_c, \mathfrak{s}L^{\Phi}(w)) \lesssim h_\varphi^+(N).$$

Proof. Given a set of dyadic cubes $\Gamma = \{Q_1, Q_2, \cdots, Q_N\}$, using (5.47) we deduce

$$\Big\| \sum_{Q \in \Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} \approx \Big\| \sum_{Q \in \Gamma_{\min}} \frac{\chi_{Light(Q)}(x)}{\varphi(w(Q))} \Big\|_{\mathfrak{s}L^{\Phi}(w)}. \tag{5.50}$$

Thus, it is enough to estimate the right hand side of (5.50). Let $\lambda = h_\varphi^+(|\Gamma_{\min}|)$; we have $\varphi(w(Q)|\Gamma_{\min}|) \leq \lambda \varphi(w(Q))$ for all $Q \in \Gamma_{\min}$. Since the set $\{Light(Q) : Q \in \Gamma_{\min}\}$ is a collection of pairwise disjoint sets and Φ is increasing, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi\Big(\frac{\sum_{Q \in \Gamma_{\min}} \frac{\chi_{Light(Q)}(x)}{\varphi(w(Q))}}{\lambda}\Big) w(x)dx \\
&= \sum_{Q \in \Gamma_{\min}} \Phi\Big(\frac{1}{\lambda \varphi(w(Q))}\Big) w(Light(Q)) \\
&\leq \sum_{Q \in \Gamma_{\min}} \Phi\Big(\frac{1}{\varphi(w(Q)|\Gamma_{\min}|}\Big) w(Q) \\
&= \sum_{Q \in \Gamma_{\min}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{w(Q)|\Gamma_{\min}|}\Big)\Big) w(Q) = 1.
\end{aligned}$$

Using (5.50), Remark 5.48 and taking into account that the function h_φ^+ is non-decreasing we have

$$\Big\| \sum_{Q \in \Gamma} \frac{e_Q}{\|e_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} \lesssim h_\varphi^+(|\Gamma_{\min}|) \leq h_\varphi^+(|\Gamma|).$$

Proposition 5.7. *Let $\mathfrak{s}L^{\Phi}(w)$ be a weighted Orlicz sequence space such that $i_\varphi > 0$, let $w \in B_p^d$ be a weight in $\mathbb{R}^d$ and $\Phi \in \mathscr{Y}$. Then, for every set $\Gamma \subset \mathscr{D}(d)$ finite we have*

$$\Big\| \sum_{Q\in\Gamma} \frac{\boldsymbol{e}_Q}{\|\boldsymbol{e}_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} \gtrsim h_{\varphi}^{-}(N), \quad N=|\Gamma|. \tag{5.51}$$

In particular,

$$h_l(N;\mathscr{B}_c,\mathfrak{s}L^{\Phi}(w)) \gtrsim h_{\varphi}^{-}(N).$$

Proof. Using (5.47) together with $\Gamma_L \subset \Gamma_{\min}$, we can write

$$\begin{aligned}\Big\| \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} &\approx \Big\| \sum_{Q\in\Gamma_{\min}} \frac{\chi_{Light(Q)}(x)}{\varphi(w(Q))} \Big\|_{\mathfrak{s}L^{\Phi}(w)} \\ &\gtrsim \Big\| \sum_{Q\in\Gamma_L} \frac{\chi_{Light(Q)}(x)}{\varphi(w(Q))} \Big\|_{\mathfrak{s}L^{\Phi}(w)}.\end{aligned}$$

Let $\lambda < h_{\varphi}^{-}(2^{-dp}C_w|\Gamma_L|)$; we have $\lambda\varphi(w(Q)) < \varphi(w(Q)2^{-dp}C_w|\Gamma_L|)$ for every $Q\in \Gamma_L$. Using $w\in B_p^d$ and since $|Light(Q)| > 2^{-d}|Q|$ for $Q\in\Gamma_L$ we deduce

$$\begin{aligned}&\int_{\mathbb{R}^d} \Phi\Big(\frac{\sum_{Q\in\Gamma_L}\frac{\chi_{Light(Q)}(x)}{\varphi(w(Q))}}{\lambda}\Big) w(x)dx \\ &= \sum_{Q\in\Gamma_L} \Phi\Big(\frac{1}{\lambda\varphi(w(Q))}\Big) w(Light(Q)) \\ &> \sum_{Q\in\Gamma_L} \Phi\Big(\frac{1}{\varphi(2^{-dp}C_w w(Q)|\Gamma_L|)}\Big) C_w 2^{-dp} w(Q) \\ &= \sum_{Q\in\Gamma_L} \Phi\Big(\Phi^{-1}\Big(\frac{1}{2^{-dp}C_w w(Q)|\Gamma_L|}\Big)\Big) 2^{-dp} w(Q) C_w = 1.\end{aligned}$$

Using (5.32) and Remark 5.48 we obtain

$$\begin{aligned}&\Big\| \sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{\mathfrak{s}L^{\Phi}(w)}} \Big\|_{\mathfrak{s}L^{\Phi}(w)} \geq h_{\varphi}^{-}(2^{-dp}C_w|\Gamma_L|) \\ &\geq h_{\varphi}^{-}(C_w 2^{-dp}(1-2^{-d})|\Gamma|).\end{aligned}$$

By (5.34) we have

$$\begin{aligned}&h_{\varphi}^{-}(C_w(2^{-dp}(1-2^{-d})|\Gamma|) = \inf_{t>0} \frac{\varphi(C_w(2^{-dp}(1-2^{-d})|\Gamma|t)}{\varphi(t)} \\ &\geq C\inf_{t>0} \frac{\varphi(|\Gamma|t)}{\varphi(t)} = Ch_{\varphi}^{-}(|\Gamma|).\end{aligned}$$

This proves the result.

Combining Remark 5.44 and Propositions 5.6 and 5.7 we obtain the following result:

Theorem 5.9. *Let $\mathfrak{s}L^{\Phi}(w)$ be a weighted Orlicz sequence space such that $i_\varphi > 0$, $w \in A^d_\infty \cap B^d_p$ for some p, and $\Phi \in \mathscr{Y}$. Then, for every $N = 1,2,3,\dots$, we have*

$$h_r(N;\mathscr{B}_c,\mathfrak{s}L^{\Phi}(w)) \approx h^+_\varphi(N) \quad \text{and} \quad h_l(N;\mathscr{B}_c,\mathfrak{s}L^{\Phi}(w)) \approx h^-_\varphi(N).$$

Since the spaces $L^{\Phi}(w)$ and $\mathfrak{s}L^{\Phi}(w)$ are isomorphic, the abstract transference framework design in Section 6.2 of [24] allows us to write the previous Theorem in the following way, taking into account Theorem 5.7:

Theorem 5.10. *Let $L^{\Phi}(w)$ be a weighted Orlicz space such that $i_\varphi > 0$, $w \in A_{p^\Phi}$ be a weight in $\mathbb{R}^d$, $\Phi \in \mathscr{Y}$ and $\mathscr{W} = \{\psi^\varepsilon_Q : Q \in \mathscr{D}(d), \varepsilon \in E\}$ be a wavelet basis for the class $\mathscr{R}^{0,M}$ with $M > d$. Then, we have*

$$h_r(N;\mathscr{W},L^{\Phi}(w)) \approx h^+_\varphi(N) \quad \text{and} \quad h_l(N;\mathscr{W},L^{\Phi}(w)) \approx h^-_\varphi(N).$$

If $\Phi(t) = t^p$, from Theorem 5.10 we obtain that admissible wavelet bases are democratic in weighted Lebesgue spaces $L^p(w)$ if $w \in A_p$.

Corollary 5.1. *Let $\Phi(t) = t^p$, $1 < p < \infty$, $w \in A_p$ be a weight in $\mathbb{R}^d$ and $\mathscr{W} = \{\psi^\varepsilon_Q : Q \in \mathscr{D}(d),\ \varepsilon \in E\}$ be a wavelet basis in the class $\mathscr{R}^{0,M}$ with $M > d$. We have*

$$h_r(N;\mathscr{W},L^p(w)) \approx h_l(N;\mathscr{W},L^p(w)) \approx N^{\frac{1}{p}}. \tag{5.52}$$

REMARK 5.53 *It has been proved in Lemma 5.2 of [25] that if $h^+_\varphi(N) \le C_1 h^-_\varphi(N)$ $\forall N = 1,2,3,\cdots$, then $\varphi(t) = t^\alpha$ for some $\alpha \in (0,\infty)$ and every $t > 0$. Therefore, $L^{\Phi}(w) = L^p(w)$ with $1 < p < \infty$. Hence, the only weighted Orlicz spaces for which wavelet bases in the class $\mathscr{R}^{0,M}$ are greedy correspond to weighted Lebesgue spaces $L^p(w)$ for some $p \in (1,\infty)$.*

5.7 Approximation Spaces: General Results

Given a quasi-Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ and a basis $\mathscr{B} = \{e_j : j \in \mathbb{N}\}$ in $\mathbb{B}$, a fundamental question in the theory of non-linear approximation is to identify the approximation spaces $\mathscr{A}^\alpha_q(\mathscr{B},\mathbb{B})$, $0 < \alpha < \infty$, $0 < q \le \infty$. These spaces are defined as the set of all $x \in \mathbb{B}$ such that the quantity

$$\|x\|_{\mathscr{A}^\alpha_q(\mathscr{B},\mathbb{B})} := \begin{cases} \|x\|_{\mathbb{B}} + \left[\sum_{N=1}^\infty (N^\alpha \sigma_N(x;\mathscr{B},\mathbb{B}))^q \frac{1}{N}\right]^{\frac{1}{q}}, & 0 < q < \infty, \\ \|x\|_{\mathbb{B}} + \sup_{N\ge 1} N^\alpha \sigma_N(x;\mathscr{B},\mathbb{B}), & q = \infty, \end{cases}$$

is finite (recall that $\sigma_N(x;\mathscr{B},\mathbb{B})$ is the N-term error of approximation defined in (5.8)).

Since $\sigma_N(x;\mathscr{B},\mathbb{B}) = \sigma_N(x)_{\mathbb{B}}$ is non-increasing, an equivalent quasi-norm in the approximation spaces can be obtained with dyadic numbers that is, if $\alpha > 0$ and $0 < q \leq \infty$,

$$\|x\|_{\mathscr{A}_q^\alpha(\mathscr{B},\mathbb{B})} \approx \begin{cases} \|x\|_{\mathbb{B}} + \left[\sum_{k=0}^{\infty}(2^{k\alpha}\sigma_{2^k}(x)_{\mathbb{B}})^q\right]^{\frac{1}{q}}, & 0 < q < \infty, \\ \|x\|_{\mathbb{B}} + \sup_{k=0,1,2\dots} 2^{k\alpha}\sigma_{2^k}(x)_{\mathbb{B}}, & q = \infty. \end{cases}$$

The identification of approximation spaces (or inclusions, if such identifications are not possible) are often done in terms of "smoothness" classes of the sort

$$\mathfrak{b}(\mathscr{B},\mathbb{B}) := \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} \,:\, \{\|c_j e_j\|_{\mathbb{B}}\}_{j=1}^{\infty} \in \mathfrak{b} \right\},$$

where $\mathfrak{b}$ is a suitable sequence space whose elements decay at infinity, such as ℓ^τ or the discrete Lorentz classes $\ell^{\tau,q}$.

The simplest case is when $\mathscr{B} = \{e_k : k \geq 1\}$ is an orthonormal basis in a Hilbert space $\mathbb{H}$. In this case, S.B. Stechkin in 1955 [52] for the case $q = 1$, and R. A. DeVore and V. N. Temlyakov in 1996 [19] for q general, proved the following result:

$$\mathscr{A}_q^\alpha(\mathscr{B},\mathbb{H}) = \ell^{\tau,q}(\mathscr{B},\mathbb{H}),\ \alpha > 0,\ 0 < q \leq \infty, \tag{5.54}$$

for $\frac{1}{\tau} = \alpha + \frac{1}{2}$, with equivalent norms.

Let $\mathscr{B}$ be an unconditional basis in a quasi-Banach space $\mathbb{B}$ with the property that there exists $p \in (0,\infty)$ such that

$$\frac{1}{C}|\Gamma|^{\frac{1}{p}} \leq \Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} \leq C|\Gamma|^{\frac{1}{p}} \tag{5.55}$$

for every finite set $\Gamma \subset \mathbb{N}$. In 2004, G. Kerkyacharin and D. Picard [35] (see also E. Hernández and G. Garrigós [24]) proved the following characterization for the spaces $\mathscr{A}_q^\alpha(\mathscr{B},\mathbb{B})$:

$$\mathscr{A}_q^\alpha(\mathscr{B},\mathbb{B}) = \ell^{\tau,q}(\mathscr{B},\mathbb{B}) \tag{5.56}$$

where $\frac{1}{\tau} = \alpha + \frac{1}{p} > 0$, with equivalent norms.

Condition (5.55) is sometimes referred to as $\mathscr{B}$ having the p-Temlyakov property [35], or as $\mathbb{B}$ being a p-space [24, 31]. For instance, wavelet bases in L^p satisfy this property [53]. With our notation, condition (5.55) is equivalent to the democracy property of $\mathscr{B}$, with democracy functions equivalent to the function $h(N) = N^{1/p}$.

Still, a further generalization is proved in [27] (see also [36]). In [27], R. Gribonval and M. Nielsen proved the following inclusions for the spaces $\mathscr{A}_q^\alpha(\mathscr{B},\mathbb{B})$: let $\mathscr{B}$ be a basis such that there exist $1 \leq p \leq q \leq \infty$ and constants $0 < A \leq B < \infty$ such that for $x = \sum_{j\in\mathbb{N}} s_j e_j \in \mathbb{B}$ we have

$$A\|\{\|s_j e_j\|_{\mathbb{B}}\}\|_{\ell^{q,\infty}} \leq \|x\|_{\mathbb{B}} \leq B\|\|s_j e_j\|_{\mathbb{B}}\}\|_{\ell^{p,1}}; \tag{5.57}$$

then

$$\ell^{\tau_p,s}(\mathscr{B},\mathbb{B}) \hookrightarrow \mathscr{A}_s^{\alpha}(\mathscr{B},\mathbb{B}) \hookrightarrow \ell^{\tau_q,s}(\mathscr{B},\mathbb{B}),\ 0<\alpha<\infty,\ 0<s\leq\infty,$$

where $\frac{1}{\tau_p} = \alpha + \frac{1}{p} > 0$ y $\frac{1}{\tau_q} = \alpha + \frac{1}{q} > 0$.

Condition (5.57) is referred in [27] as $(\mathscr{B},\mathbb{B})$ having the (p,q) **sandwich property**, and can be shown to be equivalent to

$$A'|\Gamma|^{1/q} \leq \Big\|\sum_{k\in\Gamma} e_k\Big\|_{\mathbb{B}} \leq B'|\Gamma|^{1/p} \tag{5.58}$$

for all $\Gamma \subset \mathbb{N}$ finite. Observe that (5.58) coincides with (5.55) when $p = q$.

All these results are particular cases of the following results, which involve the left and right democracy functions of the basis (see Section 5.2 for the definition). To state the results we need to introduce weighted discrete Lorentz spaces.

Let $\eta = \{\eta(k) : k \geq 1\}$ be a non-decreasing sequence of real numbers such that $\lim_{k\longrightarrow\infty}\eta(k) = \infty$, and η is doubling, that is, there exists $C > 0$ such that $\eta(2k) \leq C\eta(k)$ for all $k \geq 1$. We denote by $\mathscr{E}$ the set of all real sequences having these properties.

For $\eta \in \mathscr{E}$ and $0 < q \leq \infty$, the **weighted discrete Lorentz spaces** are defined as the set of all sequences $\mathbf{s} = \{s_k\}_{k=1}^{\infty} \in \mathbf{c}_0$ (that is $\lim_{k\longrightarrow\infty}|s_k| = 0$) such that the expression

$$\|\mathbf{s}\|_{\ell^q_\eta} := \begin{cases} \Big[\sum_{k\geq1}(\eta(k)s_k^*)^q \frac{1}{k}\Big]^{\frac{1}{q}}, & 0<q<\infty, \\ \sup_k \eta(k)s_k^*, & q=\infty, \end{cases} \tag{5.59}$$

is finite, where $\{s_k^*\}$ denotes the non-increasing rearrangement of $|s_k|$.

The particular case $\{\eta(k) = k^{\frac{1}{\tau}}\}$ gives the classical discrete Lorentz spaces $\ell^{\tau,q}$, $0 < \tau < \infty$, defined as the set of all sequences $\mathbf{s} = \{s_k\}_{k=1}^{\infty} \in \mathbf{c}_0$ such that the expression

$$\|\mathbf{s}\|_{\ell^{\tau,q}} := \begin{cases} \Big[\sum_{k\geq1}(k^{\frac{1}{\tau}}s_k^*)^q \frac{1}{k}\Big]^{\frac{1}{q}}, & 0<q<\infty, \\ \sup_k k^{\frac{1}{\tau}}s_k^*, & q=\infty, \end{cases} \tag{5.60}$$

is finite.

The space ℓ^q_η is a rearrangement invariant quasi-Banach space (see [9], p. 28) and a Banach space when $q \geq 1$ and $\{\eta(k)^q/k\}$ is decreasing (see [9], p. 28).

The weighted discrete Lorentz spaces for a general weigh η, and in particular its interpolation properties, are studied, for example, in [9, 43, 49].

The spaces ℓ^q_η have been used in the study of the approximation spaces $\mathscr{A}^\alpha_q(\mathscr{H}^d, L^p(0,1)^d)$ for the multidimensional Haar system (see [33]) and in the study of the approximation spaces $\mathscr{A}^\alpha_q(\mathscr{W}, L^\Phi(\mathbb{R}^d))$ in [25]. In the results given below we will use the sequences $\eta(k) = k^\alpha h_r(k;\mathscr{B},\mathbb{B})$ and $\eta(k) = k^\alpha h_l(k;\mathscr{B},\mathbb{B})$, $k = 1,2,3,\dots,$ for appropriate $\alpha > 0$, in such a way that they will belong to the class $\mathscr{E}$. Recall that $h_r(N;\mathscr{B},\mathbb{B})$ and $h_l(N;\mathscr{B},\mathbb{B})$ are the right and left democracy functions of the basis $\mathscr{B}$ (see Section 5.2).

Definition 5.9. Given a quasi-Banach space $(\mathbb{B}, \|\cdot\|_\mathbb{B})$ with a basis $\mathscr{B} = \{e_j\}_{j\in\mathbb{N}}$ and $\eta \in \mathscr{E}$, define

$$\ell^q_\eta(\mathscr{B},\mathbb{B}) = \Big\{x = \sum_{k=1}^{\infty} c_k e_k \in \mathbb{B} : \|x\|_{\ell^q_\eta(\mathscr{B},\mathbb{B})} := \|\{\|c_k e_k\|_\mathbb{B}\}_k\|_{\ell^q_\eta} < \infty\Big\}. \quad (5.61)$$

When $\eta \in \mathscr{E}$ these spaces are quasi-norm spaces.

In some of our results we need a stronger condition on η. For a non-decreasing sequence η define

$$M_\eta(m) = \sup_{k\in\mathbb{N}} \frac{\eta(k)}{\eta(mk)}, \quad m = 1,2,3,\cdots$$

Since we are assuming that η is non-decreasing, it follows that $M_\eta(m) \le 1$. Define $\mathscr{E}_+$ as the class of all sequences $\eta \in \mathscr{E}$ such that for some integer $m_0 > 1$ we have $M_\eta(m_0) < 1$. This is equivalent to $i_\eta > 0$, where i_η is the lower dilation index, defined by

$$i_\eta := \sup_{m\ge 1} \frac{\log M_\eta(m)}{-\log m}.$$

When $\eta(k) = \{k^\alpha \log^\beta(k+1)\}$ we have $i_\eta = \alpha$ and, hence, $\eta \in \mathscr{E}_+$ if and only if $\alpha > 0$. In general, if η is obtained from an increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as $\eta(k) = \phi(ak)$, for some $a > 0$, then $i_\eta > 0$ if and only if $i_\phi > 0$, where $i_\phi > 0$ denotes the standard lower dilation index of ϕ (see [39] page 54).

Remark 5.62 *It is easy to see that if $\eta \in \mathscr{E}$, then $k^\alpha\eta(k) \in \mathscr{E}_+$ for all $\alpha > 0$. Also, if $\eta \in \mathscr{E}_+$, then $\eta^r \in \mathscr{E}_+$ for all $r > 0$ with the same m_0.*

We can now state the promised general results.

Theorem 5.11. ***(Theorem 3.4 in [26])****. Let $\mathscr{B} = \{e_j : j \in \mathbb{N}\}$ be an unconditional basis in a quasi-Banach space $(\mathbb{B}, \|\cdot\|)$. Let $\alpha > 0$ and $0 < q \le \infty$ fixed. Then, for any sequence $\{\eta(k)\}_{k=1}^\infty \in \mathscr{E}_+$, the following statements are equivalent:*

1. There exists $C > 0$ such that for every $N = 1, 2, 3, \cdots$, and every set $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$ we have

$$\Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} \le C\eta(N). \tag{5.63}$$

2. $\ell^q_{k^\alpha\eta(k)}(\mathscr{B}, \mathbb{B}) \hookrightarrow \mathscr{A}^\alpha_q(\mathscr{B}, \mathbb{B})$.

3. (Jackson type inequality for $\ell^q_{k^\alpha\eta(k)}(\mathscr{B}, \mathbb{B})$) There exists $C_{\alpha,q} > 0$ such that for every $N = 1, 2, \cdots$, we have

$$\sigma_N(x; \mathscr{B}, \mathbb{B}) \le C_{\alpha,q} N^{-\alpha} \|x\|_{\ell^q_{k^\alpha\eta(k)}(\mathscr{B},\mathbb{B})}, \quad \forall x \in \ell^q_{k^\alpha\eta(k)}(\mathscr{B}, \mathbb{B}).$$

REMARK 5.64 *Observe that if any of the statements (2) or (3) in the above Theorem are true for a fixed $\alpha > 0$ and $q \in (0, \infty]$, then the same statements are true for all α and q, since (1) is independent of these parameters.*

REMARK 5.65 *The implications $(1) \Longrightarrow (2) \Longrightarrow (3)$ hold assuming the weakest condition $\{k^\alpha \eta(k)\} \in \mathscr{E}_+$. On the other hand, the stronger hypothesis $\eta \in \mathscr{E}_+$ is crucial to obtain $(3) \Longrightarrow (1)$, and cannot be removed as it is shown in Example 5.6 of [26].*

The best choice for η in Theorem 5.11 is when $\eta(N) = h_r(N; \mathscr{B}, \mathbb{B})$ with $C = 1$ in (5.63). The right democracy function $h_r(N) := h_r(N; \mathscr{B}, \mathbb{B})$ is always doubling. In fact, for $N \in \mathbb{N}$, choose $\Gamma_o \subset \mathbb{N}$ with$|\Gamma_o| = 2N$ such that

$$\frac{1}{2} h_r(2N) \le \Big\| \sum_{k\in\Gamma_o} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}.$$

Write $\Gamma_o = \Gamma \cup \Gamma'$ with $\Gamma' \cap \Gamma = \emptyset$ and $|\Gamma'| = |\Gamma| = N$. Then, if $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ satisfies the ρ-triangular inequality with $0 < \rho \le 1$, we have

$$\begin{aligned}
\frac{1}{2} h_r(2N) &\le \Big\| \sum_{k\in\Gamma_o} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} = \Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} + \sum_{k\in\Gamma'} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} \\
&\le \Big(\Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|} \Big\|^\rho_{\mathbb{B}} + \Big\| \sum_{\Gamma'} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|^\rho_{\mathbb{B}} \Big)^{\frac{1}{\rho}} \le 2^{\frac{1}{\rho}} h_r(N),
\end{aligned}$$

which proves that h_r is doubling with $C = 2^{1+\frac{1}{\rho}}$. Therefore, the sequence $\{k^\alpha h_r(k)\}_{k\in\mathbb{N}}$, $\alpha > 0$, belongs to $\mathscr{E}_+$ by Remark 5.62. In view of Remark 5.65, Theorem 5.11 gives the following:

Corollary 5.2. ***(Corollary 3.9 in [26]).*** *Let $\mathscr{B} = \{e_j\}_{j\in\mathbb{N}}$ be an unconditional basis in a quasi-Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. Let $\alpha > 0$ and $0 < q \le \infty$ be fixed. Then,*

$$\ell^q_{k^\alpha h_r(k)}(\mathscr{B}, \mathbb{B}) \hookrightarrow \mathscr{A}^\alpha_q(\mathscr{B}, \mathbb{B}). \tag{5.66}$$

We now state results that characterize the reverse inclusions to that in (2) of Theorem 5.11. Results of this type are usually called Bernstein type inequalities in the theory of Approximation spaces.

Theorem 5.12. ***(Theorem 3.4 in [26])****. Suppose that $\mathscr{B} = \{e_j\}_{j=1}^{\infty}$ is an unconditional basis of a quasi-Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. Let $\alpha > 0$ and $0 < q \le \infty$ be fixed. Then, for any increasing sequence of positive numbers $\{\eta(k)\}_{k=1}^{\infty}$, having the doubling property, the following conditions are equivalent:*

1. *There exists $C > 0$ such that for all $N = 1, 2, 3, \cdots$ and $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$ we have*

$$\frac{1}{C}\eta(N) \le \Big\| \sum_{k\in\Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}. \tag{5.67}$$

2. *(Bernstein type inequality for $\ell^q_{k^\alpha \eta(k)}(\mathscr{B}, \mathbb{B})$) There exists $C_\alpha > 0$ such that*

$$\|x\|_{\ell^q_{k^\alpha \eta(k)}(\mathscr{B},\mathbb{B})} \le C_\alpha N^\alpha \|x\|_{\mathbb{B}}$$

   for all $x \in \Sigma_N$, $N = 1, 2, 3, \cdots$

3. *$\mathscr{A}^\alpha_q(\mathscr{B}, \mathbb{B}) \hookrightarrow \ell^q_{k^\alpha \eta(k)}(\mathscr{B}, \mathbb{B})$.*

REMARK 5.68 *Observe that if any of the statements (2) or (3) in the above Theorem are true for a fixed $\alpha > 0$ and $q \in (0, \infty]$, then the same statements are true for all α and q, since (1) is independent of these parameters.*

REMARK 5.69 *We remark that the implications $(3) \Longrightarrow (2) \Longleftrightarrow (1)$ can be proved with the weaker hypothesis $\{k^\alpha \eta(k)\} \in \mathscr{E}$.*

The best choice for η in Theorem 5.12 is when $\eta(N) = h_l(N; \mathscr{B}, \mathbb{B})$ with $C = 1$ in (5.67). Contrary to what happens with the right democracy function, which is always doubling (see the proof that follows Remark 5.65), the left democracy function of a basis, $h_l(N) := h_l(N; \mathscr{B}, \mathbb{B})$, is not always doubling. For an example of a Banach space and a basis for which the left democracy function is not doubling see [59]. Hence, in order to apply Theorem 5.12 we need to assume that h_l is doubling. We have the following result:

Corollary 5.3. ***(Corollary 4.4 in [26])****. Let $\mathscr{B}$ be an unconditional basis in a quasi-Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. Let $\alpha > 0$ and $0 < q \le \infty$ be fixed. If $h_l(N)$ is doubling, then*

$$\mathscr{A}^\alpha_q(\mathscr{B}, \mathbb{B}) \hookrightarrow \ell^q_{k^\alpha h_l(k)}(\mathscr{B}, \mathbb{B}).$$

When the unconditional basis $\mathscr{B}=\{e_j\}_{j=1}^{\infty}$ is democratic in $\mathbb{B}$, that is $h_l(N)\approx h_k(N):=h(N)$, we can apply both Corollaries 5.2 and 5.3 to obtain the identification

$$\mathscr{A}_q^{\alpha}(\mathscr{B},\mathbb{B})=\ell^q_{k^\alpha h(k)}(\mathscr{B},\mathbb{B}) \qquad \alpha>0,\ 0<q\le\infty, \tag{5.70}$$

with equivalent quasi-norms.

5.8 Approximation Spaces for Wavelet Bases in Weighted Spaces

Several results have been proved in the literature similar to (5.54) when $\mathbb{H}$ is replaced by a particular space, for example L^p, and the basis $\mathscr{B}$ is a particular one (say, a wavelet basis). We refer the reader to the articles [18, 55, 56].

In this section we are going to apply the results of the previous section to the problem of finding the approximation spaces of wavelet basis in weighted Triebel-Lizorking spaces (in particular weighted Lebesgue spaces), and give inclusions for the approximation spaces of weighted Orlicz spaces.

We start with wavelet basis in **weighted Triebel-Lizorkin spaces**. In this case, for wavelets with enough regularity and decay as stated in Proposition 5.4, wavelet bases are unconditional in the weighted Triebel-Lizorkin spaces $F^s_{p,q}(w)$, $s\in\mathbb{R}, 0<p<\infty, 0<q\le\infty$ and $w\in A_\infty(\mathbb{R}^d)$. Moreover, in this situation, by Theorem 5.6 (part (C)), wavelet bases $\mathscr{W}$ are democratic in $F^s_{p,q}(w)$ and

$$h_l(N;\mathscr{W},F^s_{p,r}(w))\approx h_r(N;\mathscr{W},F^s_{p,r}(w))\approx N^{1/p}.$$

Therefore, we are in the ideal situation that (5.70) holds, and we conclude

$$\mathscr{A}_q^{\alpha}(\mathscr{W},F^s_{p,r}(w))=\ell^q_{k^{\alpha+1/p}}(\mathscr{W},F^s_{p,r}(w)),\alpha>0,\ 0<r\le\infty,$$

with equivalent quasi-norms. As we have observed in (5.60), the space $\ell^q_{k^{\alpha+1/p}}(\mathscr{W},F^s_{p,r}(w))$ coincides with the discrete Lorentz space $\ell^{\tau,q}(\mathscr{W},F^s_{p,r}(w))$ with $\frac{1}{\tau}=\alpha+\frac{1}{p}$. Thus, we have

$$\mathscr{A}_q^{\alpha}(\mathscr{W},F^s_{p,r}(w))=\ell^{\tau,q}(\mathscr{W},F^s_{p,r}(w)), \tag{5.71}$$

$\alpha>0,\ 0<q\le\infty,\ s\in\mathbb{R},\ 0<p<\infty,\ 0<r\le\infty,\ \frac{1}{\tau}=\alpha+\frac{1}{p}$, with equivalent quasi-norms.

For particular values of the parameters, the discrete Lorentz spaces on the right-hand side of (5.71) can be identified with weighted Besov spaces. We recall the definition of weighted Besov spaces and its characterization using wavelet bases.

Lt $\mathscr{A}_1$ be the set of functions φ belonging to the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ such that

$$Supp\,\hat{\varphi} \subset \{\xi \in \mathbb{R}^d : \frac{1}{2} < |\xi| < 2\}\ ,\ |\hat{\varphi}| \geq C > 0\ \text{ if }\ \frac{3}{5} < |\xi| < \frac{5}{3}\ \}.$$

Define $\mathscr{A}_0$ as the set of all functions $\boldsymbol{\Phi} \in \mathscr{S}(\mathbb{R}^d)$ such that

$$Supp\,\hat{\boldsymbol{\Phi}} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\}\ ,\ |\hat{\boldsymbol{\Phi}}(\xi)| \geq C > 0\ \text{ if }\ |\xi| \leq \frac{5}{3}\ \}.$$

Given a function φ defined in $\mathbb{R}^d$ we use the notation

$$\varphi_k(x) = 2^{kd}\varphi(2^k x).$$

Definition 5.10. Let $\alpha > 0,\ 0 < p,q \leq \infty,\ \varphi \in \mathscr{A}_1,\ \boldsymbol{\Phi} \in \mathscr{A}_0$ and w a weight in $\mathbb{R}^d$.

i) The **homogeneous weighted Besov space** $\dot{B}^{\alpha}_{p,q}(w)$ is the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}$(modulo polynomials) such that

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(w)} = \Big[\sum_{k\in\mathbb{Z}} (2^{k\alpha}\|\varphi_k * f\|_{L^p(w)})^q\Big]^{\frac{1}{q}} < \infty. \tag{5.72}$$

ii) The **(non-homogeneous) weighted Besov space** $B^{\alpha}_{p,q}(w)$ is the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B^{\alpha}_{p,q}(w)} = \|\boldsymbol{\Phi} * f\|_{L^p(w)} + \Big[\sum_{k=1}^{\infty} (2^{k\alpha}\|\varphi_k * f\|_{L^p(w)})^q\Big]^{\frac{1}{q}} < \infty. \tag{5.73}$$

If $\|f\|_{\dot{B}^{\alpha}_{p,q}(w)} = 0$, we must have $\varphi_k * f \equiv 0$ for all $k \in \mathbb{Z}$. Taking Fourier transforms we deduce $\hat{\varphi}(2^{-k}\xi)\hat{f}(\xi) \equiv 0$ for all $k \in \mathbb{Z}$. Due to the definition of $\varphi \in \mathscr{A}_1$ this is equivalent to $Supp\hat{f} = \{0\}$ and hence f is a polynomial. This explains why is necessary to use equivalent classes modulo polynomials in the definition of homogeneous weighted Besov spaces. On the other hand, this is not necessary in the definition of non-homogeneous Besov spaces since $\hat{\boldsymbol{\Phi}}(0) \neq 0$.

When $w \equiv 1$, Definition 5.10 coincides with the definition of Besov spaces $\dot{B}^{\alpha}_{p,q}$ and $B^{\alpha}_{p,q}$ given in [21, 48] and [20]. The spaces $B^{\alpha}_{p,q}$ coincide with classical Besov spaces defined in terms of modulus of continuity when $\alpha > d(\frac{1}{p} - 1)_+$ (see [48]).

The weighted case has been considered by several authors. A complete study of the properties of weighted Besov spaces when $w \in A_p(\mathbb{R}^d)$ was done in [8], including interpolation results. S. Roudenko [50, 51] has considered these spaces in the context of matricial weights in the homogeneous case. D. Haroske and H. Triebel [29] have considered weights of the form $w_\alpha = (1+|x|^2)^{\frac{\alpha}{2}},\ x \in \mathbb{R}^d$, and other variants of similar nature. Finally, in [32] the spaces $B^{\alpha}_{p,q}(w)$ are considered when $w \in A^{loc}_{\infty}$. As in Section 5 of [20] one can prove the following equivalence

$$\|f\|_{B^{\alpha}_{p,q}(w)} \approx \|f\|_{L^p(w)} + \|f\|_{\dot{B}^{\alpha}_{p,q}(w)},$$

when $\alpha > 0,\ 1 \leq p < \infty$ y $0 < q \leq \infty$.

The definitions given in 5.10 depend on the election of $\varphi \in \mathscr{A}_1$ and $\Phi \in \mathscr{A}_0$, but it can be proved that different choices of these functions produced equivalent results in (5.72) and (5.73), so that the spaces that they define coincide (see, for example, theorem 1.8 in [50]). The spaces $\dot{B}^{\alpha}_{p,q}(w)$ and $B^{\alpha}_{p,q}(w)$ are quasi-Banach spaces with the quasi-norms given by (5.72) and (5.73) respectively (see the references given above). In some of these references more general classes $\mathscr{A}_1$ and $\mathscr{A}_0$ are considered, but the definitions given produce equivalent quasi-norms.

We now state the known results concerning characterizations of weighted Besov spaces using wavelet coefficients.

Proposition 5.8. *(see Theorem 10.2 in [50] or Theorem 6.2 in [51]) Let* $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ *and* $w \in A_p(\mathbb{R}^d)$. *Suppose that* $\Phi = \{\psi^l : l = 1,2,\cdots,L\}$ *is a family of d-dimensional wavelets of Lemarié-Meyer type. Then,*

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(w)} \approx \sum_{l=1}^{L} \Big[\sum_{j\in\mathbb{Z}} \Big(\sum_{|Q|=2^{-jd}} (|Q|^{-\frac{\alpha}{d}-\frac{1}{2}} |\langle f, \psi^l_Q\rangle| w(Q)^{\frac{1}{p}})^p \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}},$$

and the wavelet basis $\mathscr{W} = \{\psi^l_Q : l = 1,2,\cdots,L, Q \in \mathscr{D}(d)\}$ *is unconditional if* $q < \infty$.

For the notation used in the next result see the paragraph that proceeds the statement of Proposition 5.4.

Proposition 5.9. *(See Theorems 1, 14, and 15 in [32]). Let* $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ *and* $w \in A_\infty(\mathbb{R}^d)$. *Suppose that* $\widetilde{\Phi} = \{\psi^0, \psi^L : L = 1,2,\cdots,L\} \subset C^r(\mathbb{R}^d)$, $r = (1+[s])_+$, *are a scaling function* ψ^0 *and a family of d-dimensional wavelets with compact support [15] such that* $\int_{\mathbb{R}^d} x^\beta \psi^l(x)dx = 0$ *for all* $|\beta| \leq \max\{r, L_B\}$, *where* $L_B = [\sigma_p(w) - \alpha]$, $l = 1,2,\cdots,L$. *Then,*

$$\|f\|_{B^{\alpha}_{p,q}(w)} \approx \Big\| \Big(\sum_{\substack{Q\in\mathscr{D}\\|Q|=1}} |\langle f, \psi^0_Q\rangle|^2 \chi_Q(\cdot) \Big)^{\frac{1}{2}} \Big\|_{L^p(w)}$$
$$+ \sum_{l=1}^{L} \Big[\sum_{j\in\mathbb{N}_0} \Big(\sum_{|Q|=2^{-jd}} (|Q|^{-\frac{\alpha}{d}-\frac{1}{2}} |\langle f, \psi^l_Q\rangle|^2 w(Q)^{\frac{1}{p}})^p \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}.$$

If $0 < p < \infty$ *and* $0 < q < \infty$, *then* $\mathscr{W}^+ = \{\psi^0_{0,k}, \psi^l_{j,k} : j \in \mathbb{N}, k \in \mathbb{Z}^d, l = 1,2,\cdots,L\}$ *is an unconditional basis of* $B^{\alpha}_{p,q}(w)$.

REMARK 5.74 *Proposition 5.8 is proved in the context of matrix weights and Proposition 5.9 is proved for* A_∞ *local weights. A similar result to the one of Proposition 5.9 is proved in [6] for the case of homogeneous Besov spaces.*

Taking into account the results stated in the above two Propositions, we are going to define sequence spaces associated with weighted Besov spaces. We take $L = 1$, since $L > 1$ does not significantly change the constants in the computations below. Recall that $\mathscr{D}(d)$ is the set of all dyadic cubes in $\mathbb{R}^d$. We denote by $\mathscr{D}_0(d)$ the set of all $Q \in \mathscr{D}(d)$ such that $|Q| = 1$ and by $\mathscr{D}^+(d)$ the set of all $Q \in \mathscr{D}(d)$ such that $|Q| \leq 1$.

Definition 5.11. Let $0 < p,q \leq \infty$, $\alpha > 0$ and w be a weight in $\mathbb{R}^d$.

i) Define $\mathfrak{s}\dot{B}^{\Psi}_{p,q}(w)$ as the set of all sequences of complex numbers $\mathbf{s} = \{s_Q : Q \in \mathscr{D}(d)\}$ such that

$$\|\mathbf{s}\|_{\mathfrak{s}\dot{B}^{\alpha}_{p,q}(w)} = \Big[\sum_{j\in\mathbb{Z}}\Big(\sum_{|Q|=2^{-jd}}(|Q|^{-\frac{\alpha}{d}-\frac{1}{2}}|s_Q|w(Q)^{\frac{1}{p}})^p\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} < \infty.$$

ii) Define $\mathfrak{s}B^{\Psi}_{p,q}(w)$ as the set of all sequences of complex numbers $\mathbf{s} = \{s_Q : Q \in \mathscr{D}^0(d)\} \cup \{s_Q : Q \in \mathscr{D}^+(d)\}$ such that

$$\|\mathbf{s}\|_{\mathfrak{s}B^{\alpha}_{p,q}(w)} = \Big[\sum_{Q\in\mathscr{D}_0(d)}|s_Q|^p w(Q)\Big]^{\frac{1}{p}}$$
$$+\Big[\sum_{j=0}^{\infty}\Big(\sum_{|Q|=2^{-jd}}|Q|^{-\frac{\alpha}{d}-\frac{1}{2}}|s_Q|w(Q)^{\frac{1}{p}})^p\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} < \infty.$$

The spaces $\mathfrak{s}\dot{B}^{\alpha}_{p,q}(w)$ and $\mathfrak{s}B^{\alpha}_{p,q}(w)$ are quasi-norm spaces. In these sequence spaces we consider the canonical elements

$$\mathbf{e}_{Q'} = \begin{cases} 1, \text{ if } Q = Q', \\ 0, \text{ if } Q \neq Q'. \end{cases}$$

From Definition 5.11 we deduce that $\dot{\mathscr{B}}_c = \{\mathbf{e}_Q : Q \in \mathscr{D}(d)\}$ and $\mathscr{B}_c = \{\mathbf{e}_Q : Q \in \mathscr{D}^0(d)\} \cup \{\mathbf{e}_Q : Q \in \mathscr{D}^+(d)\}$ are unconditional bases for $\mathfrak{s}\dot{B}^{\alpha}_{p,q}(w)$, and $\mathfrak{s}B^{\alpha}_{p,q}(w)$ respectively, when $0 < p,q \leq \infty$. They will be called canonical basis of the corresponding space.

Theorem 5.5 gives the democracy functions of the canonical bases in the sequence spaces $\mathfrak{s}F^s_{p,r}(w)$ and $\mathfrak{s}\dot{F}^s_{p,r}(w)$. They are all equivalent to $h(N) = N^{1/p}$, $N = 0,1,2,\ldots$. Thus, by (5.70) we have the identifications

$$\mathscr{A}^{\alpha}_q(\dot{\mathscr{B}}_c, \mathfrak{s}\dot{F}^s_{p,r}(w)) = \ell^{\tau,q}(\dot{\mathscr{B}}_c, \mathfrak{s}\dot{F}^s_{p,r}(w)) \tag{5.75}$$

and

$$\mathscr{A}^{\alpha}_q(\mathscr{B}_c, \mathfrak{s}F^s_{p,r}(w)) = \ell^{\tau,q}(\mathscr{B}_c, \mathfrak{s}F^s_{p,r}(w)), \tag{5.76}$$

where $s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q,r \leq \infty$, $w \in A^d_\infty$ (see (5.13)) and $\frac{1}{\tau} = \alpha + \frac{1}{p}$, with equivalent quasi-norms.

For particular values of these parameters, the discrete Lorentz spaces that appear in (5.75) and (5.76) can be identified with weighted Besov sequence spaces, as we prove next.

Lemma 5.8. *Let* $\alpha > s,\ 0 < p < \infty,\ 0 < r \le \infty$ *and* $w \in A_\infty(\mathbb{R}^d)$. *Let* τ *be defined by* $\frac{\alpha}{d} - \frac{1}{\tau} = \frac{s}{d} - \frac{1}{p}$. *We have,*

$$i)\ \ell^{\tau,\tau}(\dot{\mathscr{B}}_c, \mathfrak{s}\dot{F}^s_{p,r}(w)) = \mathfrak{s}\dot{B}^\alpha_{\tau,\tau}(w^{\frac{\tau}{p}}) \quad \text{and} \quad ii)\ \ell^{\tau,\tau}(\mathscr{B}_c, \mathfrak{s}F^s_{p,r}(w)) = \mathfrak{s}B^\alpha_{\tau,\tau}(w^{\frac{\tau}{p}}).$$

Proof. i) Since $w \in A_\infty(\mathbb{R}^d)$, there exists $r \ge 1$ such that $w \in A_r$. Applying Lemma 5.2 with $\delta = \frac{\tau}{p} < 1$ we deduce that $u(x) = w(x)^{\frac{\tau}{p}} \in A_r$ and $|Q|^{-\frac{1}{p}} w(Q)^{\frac{1}{p}} \approx |Q|^{-\frac{1}{\tau}} u(Q)^{\frac{1}{\tau}}$. Hence, we have

$$\begin{aligned}
\|\mathbf{s}\|_{\ell^{\tau,\tau}(\mathfrak{s}\dot{F}^s_{p,r}(w))} &= \Big(\sum_{Q\in\mathscr{D}} \|s_Q \mathbf{e}_Q\|^\tau_{\mathfrak{s}\dot{F}^s_{p,r}(w)}\Big)^{\frac{1}{\tau}} = \Big(\sum_{Q\in\mathscr{D}} |s_Q|^\tau \|\mathbf{e}_Q\|^\tau_{\mathfrak{s}\dot{F}^s_{p,r}(w)}\Big)^{\frac{1}{\tau}} \\
&= \Big(\sum_{Q\in\mathscr{D}} (|Q|^{-\frac{s}{d}-\frac{1}{2}} |s_Q| w(Q)^{\frac{1}{p}})^\tau\Big)^{\frac{1}{\tau}} \\
&\approx \Big(\sum_{Q\in\mathscr{D}} (|Q|^{-\frac{\alpha}{d}-\frac{1}{2}} |s_Q| u(Q)^{\frac{1}{\tau}})^\tau\Big)^{\frac{1}{\tau}} = \|\mathbf{s}\|_{\mathfrak{s}\dot{B}^\alpha_{\tau,\tau}(w^{\frac{\tau}{p}})}
\end{aligned}$$

ii) The proof of case ii) is similar.

The Lemma we just proved allows us to characterize some of the approximation spaces generated by the canonical basis in weighted Triebel-Lizorkin sequence spaces as particular weighted Besov sequence spaces.

Corollary 5.4. *Let* $0 < p < \infty,\ \gamma > 0,\ 0 < r \le \infty$ *and* $w \in A_\infty(\mathbb{R}^d)$. *Let* τ *be defined by* $\frac{1}{\tau} = \frac{\gamma}{d} + \frac{1}{p}$. *We have*

$$i)\ \mathscr{A}^{\frac{\gamma}{d}}_\tau(\dot{\mathscr{B}}_c, \mathfrak{s}\dot{F}^s_{p,r}(w)) = \mathfrak{s}\dot{B}^{s+\gamma}_{\tau,\tau}(w^{\frac{\tau}{p}}) \quad \text{and} \quad ii)\ \mathscr{A}^{\frac{\gamma}{d}}_\tau(\mathscr{B}_c, \mathfrak{s}F^s_{p,r}(w)) = \mathfrak{s}B^{s+\gamma}_{\tau,\tau}(w^{\frac{\tau}{p}}).$$

Proof. i) From Lemma 5.8 we have $\ell^{\tau,\tau}(\mathfrak{s}F^s_{p,r}(w)) = \mathfrak{s}\dot{B}^\alpha_{\tau,\tau}(w^{\frac{\tau}{p}})$ if $\frac{\alpha}{d} - \frac{1}{\tau} = \frac{s}{d} - \frac{1}{p}$. Taking $\alpha = s + \gamma > s$, the result follows from (5.75) with $\tau = q$. The proof of ii) is similar.

The abstract transference framework design in Section 6.2 of [24] allows us to translate Corollary 5.4 in terms of weighted Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(w)$ and $F^s_{p,q}(w)$ when there exists a characterization of these spaces with wavelet coefficients (see Propositions 5.2, 5.3, and 5.4). We start with the result for $L^p(w)$.

Corollary 5.5. *Let* $1 < p < \infty,\ \gamma > 0,\ \frac{1}{\tau} = \frac{\gamma}{d} + \frac{1}{p},\ w \in A_\tau(\mathbb{R}^d)$ *and* $\mathscr{W}$ *be a d-dimensional wavelet basis of Lemarié-Meyer type. We have*

$$\mathscr{A}^{\frac{\gamma}{d}}_\tau(\mathscr{W}, L^p(w)) = \dot{B}^\gamma_{\tau,\tau}(w^{\frac{\tau}{p}}).$$

For the weighted Triebel-Lizorkin spaces $F^s_{p,q}(w)$ we use the characterization given in Proposition 5.4 together with the characterization of Besov spaces given in Proposition 5.9 to obtain the following result, that generalizes the previous Corollary.

Corollary 5.6. *Let* $0 < p < \infty$, $0 < q < \infty$, $\gamma > 0$, $\frac{1}{\tau} = \frac{\gamma}{d} + \frac{1}{p}$ *and* $w \in A_\tau(\mathbb{R}^d)$. *Let* $\mathscr{W}^+$ *be a d-dimensional wavelet basis with compact support as in Propositions 5.4 and 5.9. We have*

$$\mathscr{A}_\tau^{\frac{\gamma}{d}}(\mathscr{W}^+, F^s_{p,q}(w)) = B^{\gamma+s}_{\tau,\tau}(w^{\frac{\tau}{p}}).$$

We now consider briefly the case of **weighted Orlicz spaces**. Wavelet bases $\mathscr{W}$ that satisfy the conditions stated in Theorem 5.7 are unconditional in weighted Orlicz spaces $L^\Phi(w)$ if the Boyd indices i_φ and I_φ are between 0 and 1 and w is a weight in the class $A_{p^\Phi}(\mathbb{R}^d)$, where $p^\Phi = \frac{1}{I_\varphi}$. But wavelet bases are not democratic, hence not greedy, in weighted Orlicz spaces unless the Orlicz space coincides with a weighted Lebesgue space L^p for $1 < p < \infty$ (see [25]). By Theorem 5.10, for $N = 1, 2, 3, \ldots$ we have

$$h_r(N; \mathscr{W}, L^\Phi(w)) \approx h^+_\varphi(N) \quad \text{and} \quad h_l(N; \mathscr{W}, L^\Phi(w)) \approx h^-_\varphi(N),$$

where h^+_φ and h^-_φ are defined in Remark 5.44.

Since the fundamental function φ is doubling (see the paragraph that follows the proof of Lemma 5.6) the functions h^+_φ and h^-_φ are also doubling. Therefore, by Corollaries 5.2 and 5.3 we have the continuous inclusions:

$$\ell^q_{k^\alpha h^+_\varphi}(\mathscr{W}, L^\Phi(w)) \hookrightarrow \mathscr{A}^\alpha_q(\mathscr{W}, L^\Phi(w)) \hookrightarrow \ell^q_{k^\alpha h^-_\varphi}(\mathscr{W}, L^\Phi(w)), \tag{5.77}$$

where $\alpha > 0, 0 < q \le \infty$, Φ is a Young function as introduced in the first paragraph of Section 5.6, $\varphi(t) = 1/\Phi^{-1}(1/t)$, $0 < t < \infty$, the fundamental function of $L^\Phi(w)$, and $w \in A_{p^\Phi}(\mathbb{R}^d)$.

The inclusions 5.77 were proved for the case $w \equiv 1$ in [25] (see Corollary 6.15 for the case of sequence Orlicz spaces). When $w \equiv 1$ the spaces that appear in the left- and right-hand side of the inclusions given in 5.77 can be related to the so-called Besov spaces with generalized smoothness, introduced in [12, 43] in the context of real interpolation with a function parameter. See Section 6.5 of [25] for details, including the wavelet characterization of Besov spaces with generalized smoothness proved in [2].

The case of identifying the spaces that appear in the left-and right-hand side of the inclusions given in 5.77 for a general weight w in the appropriate A_p class is still open, since, to the best of our knowledge, a characterization of weighted Besov spaces with generalized smoothness in terms of wavelet coefficients is not known.

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Chapter 6
Consequences of the Marcus/Spielman/Srivastava Solution of the Kadison-Singer Problem

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Abstract It is known that the famous, intractable 1959 Kadison-Singer problem in C^*-algebras is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and Engineering. The recent surprising solution to this problem by Marcus, Spielman and Srivastava was a significant achievement and a significant advance for all these areas of research. We will look at many of the known equivalent forms of the Kadison-Singer Problem and see what are the best new theorems available in each area of research as a consequence of the work of Marcus, Spielman and Srivastava. In the cases where *constants* are important for the theorem, we will give the best constants available in terms of a *generic constant* taken from (A. Marcus, D. Spielman and N. Srivastava, *Interlacing families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem*, arXiv 1306.3969v4). Thus, if better constants eventually become available, it will be simple to adapt these new constants to the theorems.

6.1 Introduction

The famous 1959 Kadison-Singer Problem [31] has defied the best efforts of some of the most talented mathematicians of the last 50 years. The recent solution to this problem by Marcus, Spielman and Srivastava [36] is not only a significant mathematical achievement by three very talented mathematicians, but it is also a major advance for a dozen different areas of research in pure mathematics, applied mathematics and engineering.

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Kadison-Singer Problem (KS) Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space ℓ_2?

A **state** of a von Neumann algebra $\mathscr{R}$ is a linear functional f on $\mathscr{R}$ for which $f(I) = 1$ and $f(T) \geq 0$ whenever $T \geq 0$ (whenever T is a positive operator). The set of states of $\mathscr{R}$ is a convex subset of the dual space of $\mathscr{R}$ which is compact in the ω^*-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the **pure states** (of $\mathscr{R}$).

This problem arose from the very productive collaboration of Kadison and Singer in the 1950s which culminated in their seminal work on triangular operator algebras. During this collaboration, they often discussed the fundamental work of Dirac [24] on Quantum Mechanics. In particular, they kept returning to one part of Dirac's work because it seemed to be problematic. Dirac wanted to find a "representation" (an orthonormal basis) for a compatible family of observables (a commutative family of self-adjoint operators). On pages 74–75 of [24] Dirac states:

"To introduce a representation in practice

(i) We look for observables which we would like to have diagonal either because we are interested in their probabilities or for reasons of mathematical simplicity;
(ii) We must see that they all commute — a necessary condition since diagonal matrices always commute;
(iii) We then see that they form a complete commuting set, and if not we add some more commuting observables to make them into a complete commuting set;
(iv) We set up an orthogonal representation with this commuting set diagonal.

The representation is then completely determined $\cdots$ **by the observables that are diagonal** $\cdots$"

The emphasis above was added. Dirac then talks about finding a basis that diagonalizes a self-adjoint operator, which is troublesome since there are perfectly respectable self-adjoint operators which do not have a single eigenvector. Still, there is a *spectral resolution* of such operators. Dirac addresses this problem on pages 57–58 of [24]:

> "We have not yet considered the lengths of the basic vectors. With an orthonormal representation, the natural thing to do is to normalize the basic vectors, rather than leave their lengths arbitrary, and so introduce a further stage of simplification into the representation. However, it is possible to normalize them only if the parameters are continuous variables that can take on all values in a range, the basic vectors are eigenvectors of some observable belonging to eigenvalues in a range and are of infinite length$\cdots$"

In the case of $\mathbb{D}$, the representation is $\{e_i\}_{i\in I}$, the orthonormal basis of l_2. But what happens if our observables have "ranges" (intervals) in their spectra? This led Dirac to introduce his famous δ-function — vectors of "infinite length."

From a mathematical point of view, this is problematic. What we need is to replace the vectors e_i by some mathematical object that is essentially the same as the vector, when there is one, but gives us something precise and usable when there is only a δ-function. This leads to the "pure states" of $B(\ell_2)$ and, in particular, the (vector) pure states ω_x, given by $\omega_x(T) = \langle Tx, x\rangle$, where x is a unit vector in $\mathscr{H}$. Then, $\omega_x(T)$ is the expectation value of T in the state corresponding to x. This expectation is the average of values measured in the laboratory for the "observable" T with the system in the state corresponding to x. The pure state ω_{e_i} can be shown to be completely determined by its values on $\mathbb{D}$; that is, each ω_{e_i} has a *unique* extension to $B(\ell_2)$. But there are many other pure states of $\mathbb{D}$. (The family of all pure states of $\mathbb{D}$ with the w^*-topology is $\beta(\mathbb{Z})$, the β-compactification of the integers.) Do these other pure states have unique extensions? This is the Kadison-Singer problem (KS).

By a "complete" commuting set, Dirac means what is now called a "maximal abelian self-adjoint" subalgebra of $B(\ell_2)$; $\mathbb{D}$ is one such. There are others. For example, another is generated by an observable whose"simple" spectrum is a closed interval. Dirac's claim, in mathematical form, is that each pure state of a "complete commuting set" has a unique state extension to $B(\ell_2)$. Kadison and Singer show [37] that is *not so* for each complete commuting set other than $\mathbb{D}$. They also show that each pure state of $\mathbb{D}$ has a unique extension to the uniform closure of the algebra of linear combinations of operators T_π defined by $T_\pi e_i = e_{\pi(i)}$, where π is a permutation of $\mathbb{Z}$.

Kadison and Singer believed that KS had a negative answer. In particular, on page 397 of [31] they state: "We incline to the view that such extension is non-unique".

Over the 55-year history of the Kadison-Singer Problem, a significant amount of research was generated resulting in a number of partial results as well as a large number of equivalent problems. These include the **Anderson Paving Conjectures** [2–4], the **Akemann-Anderson Projection Paving Conjecture** [1], the **Weaver Conjectures** [42], the **Casazza-Tremain Conjecture** [21], the **Feichtinger Conjecture** [14], the **R_ε-Conjecture** [22], the **Bourgain-Tzafriri Conjecture** [21] and the **Sundberg Problem** [19]. Many directions for approaching this problem were proposed and solutions were given for special cases: All matrices with positive coefficients are pavable [27] as are all matrices with "small" coefficients [11]. Under stronger hypotheses, solutions to the problem were given by Berman/Halpern/Kaftal/Weiss [8], Baranov and Dyakonov [7], Paulsen [33, 37, 38], Lata [32], Lawton [34], Popa [40], Grochenig [26], Bownik/Speegle [12], Casazza/Christensen/Lindner/Vershynin [14], Casazza/Christensen/Kalton [15], Casazza/Kutyniok/Speegle [20], Casazza/Edidin/Kalra/Paulsen [16] and much more.

Our goal in this paper is to see how the solution of [36] to the Kadison-Singer Problem answers each of the above problems in a quantitative way and to compute the best available constants at this time.

The paper is organized as follows. In Section 6.2 we introduce the basics of Hilbert space frame theory which forms the foundation for producing equivalences of the Paving Conjecture. Next, in Section 6.3 we give the basic Marcus/Spielman/Srivastava result proving **Weaver's Conjecture**. In Section 6.4 we present their proof of the **Akemann-Anderson Projection Paving Conjecture** and the **Anderson Paving Conjecture**. In Section 6.5 we prove the **Casazza/Tremain Conjecture**, the **Feichtinger Conjecture**, the R_ε**-Conjecture**, and the **Bourgain-Tzafriri Conjecture**. In Section 6.6 we prove the Feichtinger Conjecture in **Harmonic Analysis** (the stronger form involving **syndetic sets**), and solve the **Sundberg Problem**. Section 6.7 contains equivalents of the Paving Conjecture for **Large and Decomposable subspaces** of a Hilbert space. Finally, in Section 6.9 we will trace some of the history of the Paving Conjecture.

6.2 Frame Theory

Hilbert space **frame theory** is the tool which is used to connect many of the equivalent forms of the **Paving Conjecture**. So we start with an introduction to this area. A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathscr{H}$ is a **Riesz basic sequence** if there are constants $A,B>0$ so that for all scalars $\{a_i\}_{i\in I}$ we have

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

We call A,B the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathscr{H}$, we call it a **Riesz basis** for $\mathscr{H}$. So $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathscr{H}$ means there is an orthonormal basis $\{e_i\}_{i\in I}$ so that the operator $T(e_i)=f_i$ is invertible. In particular, each Riesz basis is **bounded**. That is, $0<\inf_{i\in I}\|f_i\|\le \sup_{i\in I}\|f_i\|<\infty$.

Hilbert space frames were introduced by Duffin and Schaeffer [25] to address some very deep problems in nonharmonic Fourier series (see [44]). A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space $\mathscr{H}$ is called a **frame** for $\mathscr{H}$ if there are constants $0<A\le B<\infty$ (called the **lower and upper frame bounds**, respectively) so that for all $f\in\mathscr{H}$

$$A\|f\|^2 \le \sum_{i\in I}|\langle f,f_i\rangle|^2 \le B\|f\|^2. \tag{6.1}$$

If we only have the right-hand inequality in Equation 6.1, we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If $A=B$, we call this an A-**tight frame** and if $A=B=1$, it is called a **Parseval frame**. If all the frame elements have the same norm, this is an **equal norm** frame and if the frame elements are of unit norm, it is a **unit norm frame**. It is immediate that $\|f_i\|^2\le B$. If also $\inf\|f_i\|>0$, $\{f_i\}_{i\in I}$ is a **bounded frame**. The numbers $\{\langle f,f_i\rangle\}_{i\in I}$ are the **frame coefficients** of the vector $f\in\mathscr{H}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence, the **synthesis operator** for $\{f_i\}_{i\in I}$ is the bounded linear operator $T:\ell_2(I)\to\mathscr{H}$ given by $T(e_i)=f_i$ for all $i\in I$. The **analysis operator** for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f)=\sum_{i\in I}\langle f,f_i\rangle e_i$. In particular,

$$\|T^*f\|^2=\sum_{i\in I}|\langle f,f_i\rangle|^2,\ \text{ for all } f\in\mathscr{H},$$

and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2 = \|T\|^2$. Comparing this to Equation 6.1 we have

Theorem 6.1. *Let $\mathscr{H}$ be a Hilbert space and $T : \ell_2(I) \to \mathscr{H}$, $Te_i = f_i$ be a bounded linear operator. The following are equivalent:*

1. *$\{f_i\}_{i\in I}$ is a frame for $\mathscr{H}$.*
2. *The operator T is bounded, linear, and onto.*
3. *The operator T^* is an (possibly into) isomorphism.*

Moreover, if $\{f_i\}_{i\in I}$ is a Riesz basis, then it is a frame and the Riesz bounds equal the frame bounds.

It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto. The **frame operator** for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathscr{H} \to \mathscr{H}$. That is,

$$Sf = TT^*f = T\left(\sum_{i\in I}\langle f, f_i\rangle e_i\right) = \sum_{i\in I}\langle f, f_i\rangle Te_i = \sum_{i\in I}\langle f, f_i\rangle f_i.$$

In particular,

$$\langle Sf, f\rangle = \sum_{i\in I}|\langle f, f_i\rangle|^2.$$

It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A, B if and only if $A\cdot I \le S \le B\cdot I$. So $\{f_i\}_{i\in I}$ is a Parseval frame if and only if $S = I$. **Reconstruction** of vectors in $\mathscr{H}$ is achieved via the formula:

$$\begin{aligned} f = SS^{-1}f &= \sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i \\ &= \sum_{i\in I}\langle f, S^{-1/2}f_i\rangle S^{-1/2}f_i. \end{aligned}$$

Recall that for vectors $u, v \in \mathscr{H}$ the **outer product** of these vectors uv^* is the rank one operator defined by:

$$(uv^*)(x) = \langle x, v\rangle u.$$

In particular, if $\|u\| = 1$, then uu^* is the rank one projection of $\mathscr{H}$ onto span u. Also, $v^*u = \langle u, v\rangle$. The frame operator S of the frame $\{f_i\}_{i\in I}$ is representable as

$$S = \sum_{i\in I} f_i f_i^*.$$

The **Gram operator** of the frame $\{f_i\}_{i\in I}$ is $G = T^*T$ and has the matrix

$$G = (\langle f_i, f_j\rangle)_{i,j\in I} = (f_j^* f_i)_{i,j\in I}.$$

It follows that the non-zero eigenvalues of G equal the non-zero eigenvalues of S and hence $\|G\| = \|S\|$.

It also follows that $\{S^{-1/2} f_i\}_{i\in I}$ is a Parseval frame **isomorphic** to $\{f_i\}_{i\in I}$. Two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space are **isomorphic** if there is a well-defined invertible operator T between their spans with $Tf_i = g_i$ for all $i \in I$. We now show that there is a simple way to tell when two frame sequences are isomorphic.

Proposition 6.1. *Let $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ be frames for a Hilbert space $\mathscr{H}$ with analysis operators T_1 and T_2, respectively. The following are equivalent:*

(1) The frames $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are isomorphic.
(2) ker T_1 = ker T_2.

Proof. (1) $\Rightarrow$ (2): If $Lf_i = g_i$ is an isomorphism, then $Lf_i = LT_1e_i = g_i = T_2e_i$ quickly implies our statement about kernels.

(2) $\Rightarrow$ (1): Since $T_i|_{(ker\ T_i)^\perp}$ is an isomorphism for $i = 1, 2$, if the kernels are equal, then

$$T_2 \left(T_1|_{(ker\ T_2)^\perp} \right)^{-1} f_i = g_i$$

is an isomorphism. □

In the finite dimensional case, if $\{g_j\}_{j=1}^n$ is an orthonormal basis of ℓ_2^n consisting of eigenvectors for S with respective eigenvalues $\{\lambda_j\}_{j=1}^n$, then for every $1 \le j \le n$, $\sum_{i\in I} |\langle f_i, g_j\rangle|^2 = \lambda_j$. In particular, $\sum_{i\in I} \|f_i\|^2 =$ trace S ($= n$ if $\{f_i\}_{i\in I}$ is a Parseval frame). An important result is

Theorem 6.2. *If $\{f_i\}_{i\in I}$ is a frame for $\mathscr{H}$ with frame bounds A, B and P is any orthogonal projection on $\mathscr{H}$, then $\{Pf_i\}_{i\in I}$ is a frame for $P\mathscr{H}$ with frame bounds A, B.*

Proof. For any $f \in P\mathscr{H}$,

$$\sum_{i\in I} |\langle f, Pf_i\rangle|^2 = \sum_{i\in I} |\langle Pf, f_i\rangle|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2. \quad \square$$

A fundamental result in frame theory was proved independently by Naimark and Han/Larson [23, 30]. For completeness we include its simple proof.

Theorem 6.3. *A family $\{f_i\}_{i\in I}$ is a Parseval frame for a Hilbert space $\mathscr{H}$ if and only if there is a containing Hilbert space $\mathscr{H} \subset \ell_2(I)$ with an orthonormal basis $\{e_i\}_{i\in I}$ so that the orthogonal projection P of $\ell_2(I)$ onto $\mathscr{H}$ satisfies $P(e_i) = f_i$ for all $i \in I$.*

Proof. The "only if" part is Theorem 6.2. For the "if" part, if $\{f_i\}_{i\in I}$ is a Parseval frame, then the synthesis operator $T : \ell_2(I) \to \mathscr{H}$ is a partial isometry. So T^* is an isometry and we can associate $\mathscr{H}$ with $T^*\mathscr{H}$. Now, for all $i \in I$ and all $g = T^* f \in T^*\mathscr{H}$ we have

$$\langle T^* f, Pe_i\rangle = \langle T^* f, e_i\rangle = \langle f, Te_i\rangle = \langle f, f_i\rangle = \langle T^* f, T^* f_i\rangle.$$

It follows that $Pe_i = T^* f_i$ for all $i \in I$. □

For an introduction to frame theory we refer the reader to Han/Kornelson/Larson/Weber [29], Christensen [23] and Casazza/Kutyniok [19].

6.3 Marcus/Spielman/Srivastava and Weaver's Conjecture

In [36] the authors do a deep analysis of what they call *mixed characteristic polynomials* to prove a famous conjecture of Weaver [42] which Weaver had earlier shown is equivalent to the *Paving Conjecture* which Anderson [2] had previously shown was equivalent to the **Kadison-Singer Problem**. We will mearly state the main theorem from [36] here and use it to find the best constants in the various equivalent forms of the Kadison-Singer Problem.

Theorem 6.4 (Marcul/Spielman/Srivastava). *Let r be a positive integer and let $u_1, u_2, \cdots, u_m \in \mathbb{C}^d$ be vectors such that*

$$\sum_{i=1}^{m} u_i u_i^* = I,$$

and $\|u_i\|^2 \le \delta$ for all i. Then there is a partition $\{S_1, S_2, \cdots, S_r\}$ of $[m]$ such that

$$\left\|\sum_{i\in S_j} u_i u_i^*\right\| \le \left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2, \text{ for all } j = 1, 2, \cdots, r.$$

In [42], Weaver reformulated the Paving Conjecture into **Discrepancy Theory** which generated a number of new equivalences of the Paving Conjecture [18, 21] and set the stage for the eventual solution to the problem. Setting $r = 2$ and $\delta = 1/18$ [36], this implies the original **Weaver Conjecture** KS_2 [42] with $\eta = 18$ and $\theta = 2$.

Theorem 6.5 (Marcus/Spielman/Srivastava). *There are universal constants $\eta \ge 2$ and $\theta > 0$ so that the following holds. Let $u_1, u_2, \cdots, u_m \in \mathbb{C}^d$ satisfy $\|u_i\| \le 1$ for all i and suppose*

$$\sum_{i=1}^{M} |\langle u, u_i\rangle|^2 = \eta, \text{ for every unit vector } u \in \mathbb{C}^d.$$

Then there is a partition S_1, S_2 of $\{1, 2, \cdots, m\}$ so that

$$\sum_{i \in S_j} |\langle u, u_i \rangle|^2 \leq \eta - \theta,$$

for every unit vector $u \in \mathbb{C}^d$ and each $j \in \{1, 2\}$.
Moreover, $\eta = 18$ and $\theta = 2$ work.

To make Theorem 6.4 more usable later, we will reformulate it into the language of operator theory. Recall, for a matrix operator

$$T = (a_{ij})_{i,j=1}^m,$$

we let

$$\delta(T) = \max_{1 \leq i \leq m} |a_{ii}|.$$

Theorem 6.6. *Let r be a positive integer. Given an orthogonal projection Q on ℓ_2^m with $\delta(Q) \leq \delta$, there are diagonal projections $\{P_j\}_{j=1}^r$ with*

$$\sum_{j=1}^r P_j = I,$$

and

$$\|P_j Q P_j\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2, \textit{ for all } j = 1, 2, \cdots, r.$$

Proof. Letting $u_i = Qe_i$ for all $i = 1, 2, \cdots, m$, we have that $Q = (u_i^* u_j)_{i,j=1}^m$. Choose a partition $\{S_j\}_{j=1}^r$ of $[m]$ satisfying Theorem 6.4 and let P_j be the diagonal projection onto $\{e_i\}_{i \in S_j}$. For any $k \in [r]$ we have

$$\|P_k Q P_k\| = \|(u_i^* u_j)_{i,j \in S_k}\| = \|\sum_{i \in S_k} u_i u_i^*\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2.$$ □

6.4 Marcus/Spielman/Srivastava and the Paving Conjectures

Perhaps the most significant advance on the Kadison-Singer Problem occurred when Anderson [2] showed that it was equivalent to what became known as the **(Anderson) Paving Conjecture**. The significance of this was that it removed the Kadison-Singer Problem from the burden of being a very technical problem in C^*-Algebras which had no real life outside the field, to making it a highly visible problem in Operator Theory which generated a significant amount of research—and eventually led to the solution to the problem.

Definition 6.1. Let $T : \ell_2^n \to \ell_2^n$ be an operator. Given $r \in \mathbb{N}$ and $\varepsilon > 0$ we say that T can be (r,ε)**-paved** if there is a partition $\{A_j\}_{j=1}^r$ of $[n]$ so that if P_i is the coordinate projection onto the coordinates A_j so that

$$\|P_i T P_i\| \le \varepsilon \|T\|, \text{ for all } i = 1,2,\cdots,r.$$

Or equivalently, there are coordinate projections $\{P_i\}_{i=1}^r$ so that

$$\sum_{i=1}^r P_i = I \text{ and } \|P_i T P_i\| \le \varepsilon \|T\|.$$

A major advance was made on the Kadison-Singer Problem by Anderson [2].

Theorem 6.7 (Anderson Paving Conjecture). *The following are equivalent:*

(1) The Kadison-Singer Conjecture is true.
(2) For every $0 < \varepsilon < 1$ there is an $r = r(\varepsilon) \in \mathbb{N}$ so that every zero diagonal operator T on a finite or infinite dimensional Hilbert space is (r,ε)-pavable.
(3) For every $0 < \varepsilon < 1$ there is an $r = r(\varepsilon) \in \mathbb{N}$ so that every zero diagonal self-adjoint operator $T = T^*$ on a finite or infinite dimensional Hilbert space is (r,ε)-pavable.

This result became known as the **Anderson Paving Conjecture** and became a major advance for the field. For the next 25 years a significant amount of effort was directed at proving (or giving a counter-example to) the Paving Conjecture. We give a brief outline of the history of this effort in Section 6.9.

In 1991, Akemann and Anderson reformulated the Paving Conjecture for **operators** into a paving conjecture for **projections**. They also gave a number of conjectures concerning paving projections and the Paving Conjecture. This was a major advance for the area reducing the problem to a very special class of operators. Theorem 6.4 also implies the **Akemann-Anderson Projection Paving Conjecture** [1] which they showed implies a positive solution to the Kadison-Singer Problem.

Theorem 6.8. *Given $\varepsilon > 0$, choose $\delta > 0$ so that*

$$\left(\frac{1}{\sqrt{2}} + \sqrt{\delta}\right)^2 \le 1 - \varepsilon.$$

For any projection Q on ℓ_2^m of rank d there is a diagonal projection P on ℓ_2^m with $\delta(P) \le \delta$ so that

$$\|PQP\| \le 1-\varepsilon \text{ and } \|(I-P)Q(I-P)\| \le 1-\varepsilon.$$

Proof. This is immediate from Theorem 6.6 letting $r = 2$ and noting that $P_2 = (I - P_1)$. ☐

The authors [36] then give a quantitative proof of the original **Anderson Paving Conjecture** [2]. To do this, we will first look at an elementary way to pass between paving for operators and paving for projections introduced by Casazza/Edidin/ Kalra/Paulsen [16]. In [16] there is a simple method for passing paving numbers back and forth between operators and projections with constant diagonal $1/2$ (or $1/2^k$ if we iterate this result). This was a serious change in direction for the paving conjecture for projections. The earlier work of Akemann/Anderson [1] and Weaver [42] emphasized paving for projections with small diagonal while the results in [16] showed that it is more natural to work with projections with large diagonal. The proof is a direct calculation.

Theorem 6.9 (Casazza/Edidin/Kalra/Paulsen). *If T is a self-adjoint operator with $\|T\| \le 1$, then*

$$A = \begin{bmatrix} T & \sqrt{I-T^2} \\ \sqrt{I-T^2} & -T \end{bmatrix}$$

is an idempotent. I.e. $A^2 = I$.

Hence,

$$P = \frac{I \pm A}{2},$$

is a projection.

It follows that the paving numbers for self-adjoint operators are at most the square of the paving numbers for projections. Using Theorem 6.9, in [16] they prove the following (See also [36]):

Proposition 6.2 (Casazza/Edidin/Kalra/Paulsen). *Suppose there is a function $r : \mathbb{R}_+ \to \mathbb{N}$ so that every $2n \times 2n$ projection matrix Q with diagonal entries equal to $1/2$ can be $(r(\varepsilon), \frac{1+\varepsilon}{2})$-paved, for every $0 < \varepsilon < 1$. Then every $n \times n$ self-adjoint zero diagonal matrix T can be $(r^2(\varepsilon), \varepsilon)$-paved for all $0 < \varepsilon < 1$.*

Proof. Given Q as in the proposition, $Q = (u_i^* u_j)_{i,j \in [2n]}$ is the gram matrix of $2n$ vectors $u_1, u_2, \cdots, u_{2n} \in \mathbb{C}^n$ with $\|u_i\|^2 = 1/2 = \delta$. Applying Theorem 6.6 we find a partition $\{A_i\}_{i=1}^r$ of $[2n]$ so that if P_i is the diagonal projection onto the coordinates of A_i we have for $k \in [r]$,

$$\|P_k T P_k\| = \|(u_i^* u_j)_{i,j \in A_k}\| = \|\sum_{i \in A_k} u_i u_i^*\| \le \left(\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}}\right)^2 < \frac{1}{2} + \frac{3}{\sqrt{r}} < \frac{1+\varepsilon}{2},$$

if $r = (\frac{6}{\varepsilon})^2$. So every Q can be $(r, \frac{1+\varepsilon}{2})$-paved. □

It follows that every self-adjoint operator can be (r, ε)-paved for $r = (\frac{6}{\varepsilon})^4$, in either the real or complex case.

Using this and Theorem [21, 36] gives a quantitative proof of the **Anderson Paving Conjecture** and hence of the Kadison-Singer Problem.

Theorem 6.10. *For every $0 < \varepsilon < 1$, every zero-diagonal real (Resp. complex) self-adjoint matrix T can be (r, ε)-paved with $r = (6/\varepsilon)^4$ (Resp. $r = (6/\varepsilon)^8$).*

Important: For complex Hilbert spaces, given an operator T, we write it as $T = A + iB$ where A, B are real operators. Paving A, B separately and intersecting the paving sets, we have a paving of T but with the paving number squared.

Remark 6.1. In [16] it is shown that $1/\varepsilon^2 \le r$ in Theorem 6.10. We can compare this to the value $r = (\frac{6}{\varepsilon})^4$ we are getting from the theorem.

6.5 Equivalents of the Paving Conjecture

Casazza/Tremain reformulated the Paving Conjecture into a number of conjectures related to problems in engineering. They also gave several conjectures related to the Paving Conjecture. Theorem 6.5 answers the **Casazza-Tremain Conjecture** [21].

Theorem 6.11. *Every unit norm 18-tight frame can be partitioned into two subsets each of which has frame bounds* $2, 16$.

Proof. Let $\{u_i\}_{i=1}^{18d}$ be a unit norm 18-tight frame in $\mathbb{C}^d$. By Theorem 6.5, we can find a partition $\{S_1, S_2\}$ of $[18d]$ so that for all $\|u\| = 1$ we have

$$\sum_{i \in S_1} |\langle u, u_i \rangle|^2 \le 16.$$

Thus,

$$\begin{aligned} 18 &= \sum_{i \in S_1} |\langle u, u_i \rangle|^2 + \sum_{i \in S_2} |\langle u, u_i \rangle|^2 \\ &\le 16 + \sum_{i \in S_2} |\langle u, u_i \rangle|^2. \end{aligned}$$

It follows that

$$\sum_{i \in S_2} |\langle u, u_i \rangle|^2 \ge 2.$$

By symmetry,

$$\sum_{i \in S_1} |\langle u, u_i \rangle|^2 \ge 2.$$

□

In his work on time-frequency analysis, Feichtinger [21, 26] noted that all of the Gabor frames he was using had the property that they could be divided into a finite number of subsets which were Riesz basic sequences. This led to a conjecture known as the **Feichtinger Conjecture** [14]. There is a significant body of work on this conjecture [5, 6, 26] (see also [32] for a large listing of papers on the Feichtinger Conjecture in reproducing kernel Hilbert spaces and many classical spaces such as Hardy space on the unit disk, weighted Bergman spaces, and Bargmann-Fock spaces). The following theorem gives the best quantitative solution to the **Feichtinger Conjecture** from the results of [36].

Recall, if $\varepsilon > 0$ and $\{u_i\}_{i=1}^{\infty}$ is a unit norm Riesz basic sequence with Riesz bounds $A = 1-\varepsilon, B = 1+\varepsilon$ we call $\{u_i\}_{i\in I}$ an ε-**Riesz basic sequence**. This is now a special case of the Feichtinger Conjecture, Theorem 6.12

Theorem 6.12. *Every unit norm B-Bessel sequence can be partitioned into r-subsets each of which is a ε-Riesz basic sequence, where*

$$r = \left(\frac{6(B+1)}{\varepsilon}\right)^4 \text{ in the real case },$$

and

$$r = \left(\frac{6(B+1)}{\varepsilon}\right)^8 \text{ in the complex case }.$$

Proof. Fix $0 < \varepsilon < 1$ and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for ℓ_2. Let $T : \ell_2 \to \ell_2$ satisfy $\|Te_i\| = 1$ for all $i = 1,2,\cdots$. Let $S = T^*T$. Since S has ones on the diagonal, $I-S$ has zero diagonal and so by Theorem 6.10 there is an

$$r = \left(\frac{6(\|s\|+1)}{\varepsilon}\right)^4 \text{ in the real case },$$

and

$$r = \left(\frac{6(\|s\|+1)}{\varepsilon}\right)^8 \text{ in the complex case },$$

and a partition $\{S_j\}_{j=1}^{r}$ of $\mathbb{N}$ so that if Q_{S_j} is the diagonal projection onto $\{e_i\}_{i\in S_j}$, we have

$$\|Q_{S_j}(I-S)Q_{S_j}\| \le \frac{\varepsilon}{\|S\|+1}\|I-S\|.$$

Now, for all $j = 1,2,\cdots,r$ and all $u = \sum_{i\in S_j} a_i e_i$ we have

$$\begin{aligned}
\|\sum_{i\in S_j} a_i Te_i\|^2 &= \|TQ_{S_j}u\|^2 \\
&= \langle TQ_{S_j}u, TQ_{S_j}u\rangle \\
&= \langle T^*TQ_{S_j}u, Q_{S_j}u\rangle \\
&= \langle Q_{S_j}u, Q_{S_j}u\rangle - \langle Q_{S_j}(I-S)Q_{S_j}u, Q_{S_j}u\rangle \\
&\ge \|Q_{S_j}u\|^2 - \frac{\varepsilon}{\|S\|+1}\|I-S\|\|Q_{S_j}u\|^2 \\
&\ge (1-\varepsilon)\|Q_{S_j}u\|^2 \\
&= (1-\varepsilon)\sum_{i\in S_j}|a_i|^2.
\end{aligned}$$

Similarly,

$$\|\sum_{i\in S_j} a_i Te_i\|^2 \le (1+\varepsilon)\sum_{i\in S_j}|a_i|^2, \text{ for all } j = 1,2,\cdots,r.$$

□

This result also answers the **Sundberg Problem** [19]. The question was: Can every bounded Bessel sequence be written as the finite union of non-spanning sets? The answer is now yes. We just partition our Bessel sequence into Riesz basic sequences. It is clear that Riesz basic sequences can be partitioned into non-spanning sets. I.e. Take one vector as one set and the rest of the vectors as the other set. Neither of these can span the Hilbert space.

This result answers another famous conjecture known as the **R_ε-Conjecture**. This was introduced by Casazza/Vershynin (unpublished) and was shown to be equivalent to the Paving Conjecture.

Theorem 6.13. *For every $0 < \varepsilon < 1$ there is an $r \in \mathbb{N}$ so that every unit norm Riesz basic sequence with upper Riesz bound B is a finite union of ε-Riesz basic sequences, where r is as in Theorem 6.12.*

We note that Theorem 6.13 fails for equivalent norms on a Hilbert space. For example, if we renorm ℓ_2 by letting $|\{a_i\}| = \|a_i\|_{\ell_2} + \sup_i |a_i|$, then the R_ε-Conjecture fails for this equivalent norm. To see this, let $f_i = (e_{2i} + e_{2i+1})/(\sqrt{2}+1)$ where $\{e_i\}_{i\in\mathbb{N}}$ is the unit vector basis of ℓ_2. This is now a unit norm Riesz basic sequence, but no infinite subset satisfies theorem 6.13. To check this, let $J \subset \mathbb{N}$ with $|J| = n$ and $a_i = 1/\sqrt{n}$ for $i \in J$. Then,

$$|\sum_{i\in J} a_i f_i| = \frac{1}{\sqrt{2}+1}\left(\sqrt{2} + \frac{1}{\sqrt{n}}\right).$$

Since the norm above is bounded away from one for $n \geq 2$, we cannot satisfy the requirements of theorem 6.13.

In 1987, Bourgain and Tzafriri [10] proved a fundamental result in Operator Theory known as the **Restricted Invertibility Principle**.

Theorem 6.14 (Bourgain-Tzafriri). *There are universal constants $A, c > 0$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for $1 \leq i \leq n$, then there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ of cardinality $|\sigma| \geq cn/\|T\|^2$ so that for all $j = 1, 2, \cdots, n$ and for all choices of scalars $\{a_j\}_{j\in\sigma}$,*

$$\|\sum_{j\in\sigma} a_j Te_j\|^2 \geq A \sum_{j\in\sigma} |a_j|^2.$$

In a significant advance, Spielman and Srivastava [41] gave an algorithm for proving the restricted invertibility theorem. Theorem 6.14 gave rise to a problem in the area which has received a great deal of attention [11, 21] known as the **Bourgain-Tzafriri Conjecture**. No one really noticed that this result is the finite version of the **Feichtinger Conjecture**. This conjecture is now a theorem. The proof is identical to the proof of Theorem 6.12

Theorem 6.15. *For every $0 < \varepsilon < 1$ and for every $B > 1$ there is a natural number r satisfying: For any natural number n, if $T : \ell_2^n \to \ell_2^n$ is a linear operator with*

$\|T\| \le B$ and $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$, then there is a partition $\{S_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all choices of scalars $\{a_i\}_{i \in S_j}$ we have

$$(1-\varepsilon)\sum_{i\in S_j}|a_i|^2 \le \|\sum_{i\in S_j} a_i Te_i\|^2 \le (1+\varepsilon)\sum_{i\in S_j}|a_i|^2,$$

where

$$r = \left(\frac{6(B+1)}{\varepsilon}\right)^4 \text{ in the real case },$$

and

$$r = \left(\frac{6(B+1)}{\varepsilon}\right)^8 \text{ in the complex case }.$$

6.6 Paving in Harmonic Analysis

Recall the definition of Laurent Operator:

Definition 6.2. If $f \in L^\infty[0,1]$, the **Laurent operator with symbol f**, denoted M_f, is the operator of multiplication by f.

In the 1980s, a very deep study of the Paving Conjecture for Laurant Operators was carried out by Berman/Halpern/Kaftal/Weiss [8, 9, 27]. They produced a parade of new techniques and interesting results in this direction including the introduction of the notion of **uniform paving**. They also showed that matrices with positive coefficients are pavable.

We need the following notation.

Notation: If $I \subset \mathbb{Z}$, we let $S(I)$ denote the $L^2([0,1])$-closure of the span of the exponential functions with frequencies taken from I:

$$S(I) = \mathrm{cl}(\mathrm{span}\{e^{2\pi i n t}\}_{n\in I}).$$

A deep and fundamental question in Harmonic Analysis is to understand the distribution of the norm of a function $f \in S(I)$. It is known [8, 9, 27] that if $[a,b] \subset [0,1]$ and $\varepsilon > 0$, then there is a partition of $\mathbb{Z}$ into arithmetic progressions $A_j = \{nr + j\}_{n\in\mathbb{Z}}$, $0 \le j \le r-1$ so that for all $f \in S(A_j)$ we have

$$(1-\varepsilon)(b-a)\|f\|^2 \le \|f\cdot\chi_{[a,b]}\|^2 \le (1+\varepsilon)(b-a)\|f\|^2.$$

What this says is that the functions in $S(A_j)$ have their norms nearly uniformly distributed across $[a,b]$ and $[0,1] \setminus [a,b]$. The central question is whether such a result is true for arbitrary measurable subsets of $[0,1]$ (but it is known that the partitions can no longer be arithmetic progressions [12, 27, 28]). If E is a measurable subset of $[0,1]$, let P_E denote the orthogonal projection of $L^2[0,1]$ onto $L^2(E)$, that is, $P_E(f) = f \cdot \chi_E$. The fundamental question here for many years, is now answered by the following result which is an immediate consequence of Theorem 6.12.

Theorem 6.16: *If $E \subset [0,1]$ is measurable and $\varepsilon > 0$ is given, there is a partition $\{S_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for all $j = 1, 2, \cdots, r$ and all $f \in S(A_j)$*

$$(1-\varepsilon)|E|\|f\|^2 \leq \|P_E(f)\|^2 \leq (1+\varepsilon)|E|\|f\|^2,$$

where

$$r = \left(\frac{6(|E|+1)}{\varepsilon|E|}\right)^8.$$

Recall that $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1]$. If $E \subset [0,1]$ of positive Lebesgue measure and $L^2(E)$ denotes the corresponding Hilbert space of square-integrable functions on E, then $f_n(t) = e^{2\pi int}\chi_E$ for $n \in \mathbb{Z}$ is a Parseval frame for $L^2(E)$ called the **Fourier Frame** for $L^2(E)$. Much work has been expended on trying to prove the Feichtinger Conjecture for Fourier Frames.

If $f \in L^2[0,1]$ and $0 \neq a$, we define

$$(T_a f)(t) = f(t-a).$$

Casazza/Christensen/Kalton [15] showed that if $f \in L^2[0,1]$ then $\{T_n(f)\}_{n\in\mathbb{N}}$ is a frame sequence if and only if it is a Riesz basic sequence.

Halpern/Kaftal/Weiss [27] studied **uniform pavings** for Laurent operators. In particular, they asked when we can pave Laurent operators with arithmetic progressions from $\mathbb{Z}$? They showed that this occurs if and only if the symbol f is Riemann Integrable. As a consequence of [36], a result of Paulsen implies that we can at least pave all Laurent operators with subsets of $\mathbb{Z}$ which have **bounded gaps**.

Definition 6.3: A set $S \subset \mathbb{N}$ is called **syndetic** if for some finite subset F of $\mathbb{N}$ we have

$$\cup_{n\in F}(S-n) = \mathbb{N},$$

where

$$S-n = \{m \in \mathbb{N} : m+n \in S\}.$$

Thus syndetic sets have **bounded gaps**. I.e. There is an integer p so that $[a, a+1, \cdots, a+p] \cap S \neq \phi$ for every $a \in \mathbb{N}$. We will call p the **gap length**.

Paving by syndetic sets arose from the fact that the Grammian of a Fourier Frame is a Laurent matrix. Moreover, dividing frames into Riesz sequences is equivalent to paving their Grammian [17, 37, 38].

At GPOTS (2008) Paulsen presented a quite general paving result which included paving by syndetic sets. The idea was to work in $\ell^2(G)$ where G is a countable discrete group and G-invariant frames—i.e. frames which are invariant under the action of G. Fourier frames are thus $\mathbb{Z}$-invariant. Paulsen next shows that a frame is G-invariant if and only if its Grammian belongs to the group von Neumann algebra $VN(G)$. Paulsen then shows that an element of $VN(G)$ is pavable if and only if it is pavable by syndetic sets. Unraveling the notation, it follows that G-invariant frames which can be partitioned into Riesz sequences can also be partitioned with syndetic partitions. These results then appeared in his paper [37, 38]. Lawton [34] gave a direct proof of syndetic pavings for Fourier Frames. We now give this result with the constants available from [36].

Theorem 6.17: *The Fourier frame $\{e^{2\pi int}\chi_E\}_{n\in\mathbb{Z}}$ for $L^2(E)$ can be partitioned into r syndetic sets $\{S_j\}_{j=1}^r$ with gap length $p \le r$ so that*

$$\left\{e^{2\pi int}\chi_E\right\}_{n\in S_j} \text{ is a } \varepsilon\text{-Riesz sequence for all } j=1,2,\cdots,r,$$

where

$$r = \left(\frac{6(|E|+1)}{\varepsilon|E|}\right)^8.$$

6.7 "Large" and "Decomposable" Subspaces of $\mathscr{H}$

In this section we give some new theorems arising from [36] relating to **large** and **decomposable** subspaces of a Hilbert space. These ideas were introduced in [18]. Throughout this section we will use the notation:

Notation: If $E \subset I$, we let P_E denote the orthogonal projection of $\ell_2(I)$ onto $\ell_2(E)$. Also, recall that we write $\{e_i\}_{i\in I}$ for the standard orthonormal basis for $\ell_2(I)$.

For results on frames, see Section 6.2.

Definition 6.4: A subspace $\mathscr{H}$ of $\ell_2(I)$ is **A-large** for $A>0$ if it is closed and for each $i \in I$, there is a vector $f_i \in \mathscr{H}$ so that $\|f_i\| = 1$ and $|f_i(i)| \ge A$. The space $\mathscr{H}$ is **large** if it is A-large for some $A > 0$.

It is known that every frame is isomorphic to a Parseval frame. The next lemma gives an alternative identification of these Parseval frames and relies on Proposition 6.1.

Lemma 6.1: *Let $T^* : \mathscr{H} \to \ell_2(I)$ be the analysis operator for a frame $\{f_i\}_{i\in I}$ for $\mathscr{H}$ and let P be the orthogonal projection of $\ell_2(I)$ onto $\mathscr{H}$. Then $\{Pe_i\}_{i\in I}$ is a Parseval frame for $T^*(\mathscr{H})$ which is isomorphic to $\{f_i\}_{i\in I}$.*

Proof: Note that $\{Pe_i\}_{i\in I}$ is a Parseval frame (Theorem 6.3) with synthesis operator P and analysis operator T_1^* satisfying $T_1^*(\mathscr{H}) = P(\ell_2(I)) = T^*(\mathscr{H})$. By Proposition 6.1, $\{Pe_i\}_{i\in I}$ is equivalent to $\{f_i\}_{i\in I}$. □

Now we will relate large subspaces of a Hilbert space with the range of the analysis operator of some bounded frame. We also give a quantitative version of the result for later use.

Proposition 6.3: *Let $\mathscr{H}$ be a subspace of $\ell_2(I)$.*

(I) The following are equivalent:
(1) The subspace H is large.
(2) The subspace H is the range of the analysis operator of some bounded frame.
(II) The following are equivalent:

(3) The subspace $\mathscr{H}$ is A-large.
(4) If P is the orthogonal projection of $\ell_2(I)$ onto $\mathscr{H}$, then $\|Pe_i\| \geq A$, for all $i \in I$.

Proof: (I) (1) $\Rightarrow$ (2): Suppose $\mathscr{H}$ is large. So, there exists an $A > 0$ such that for each $i \in I$, there exists a vector $f_i \in \mathscr{H}$ with $\|f_i\| = 1$ and $|f_i(i)| \geq A$. Given the projection P of (2) we have

$$A \leq |f_i(i)| = |\langle e_i, f_i \rangle| = |\langle Pe_i, f_i \rangle| \leq \|Pe_i\| \|f_i\| = \|Pe_i\|.$$

Note that this also proves (II) (3) $\Rightarrow$ (4).

(2) $\Rightarrow$ (1): Assume $\{f_i\}_{i \in I}$ is a bounded frame for a Hilbert space $\mathscr{K}$ with analysis operator T^* and $T^*(\mathscr{K}) = \mathscr{H}$. Now, $\{Pe_i\}_{i \in I}$ is a Parseval frame for $\mathscr{H}$ which is the range of its own analysis operator. Hence, $\{f_i\}_{i \in I}$ is equivalent to $\{Pe_i\}_{i \in I}$ by Proposition 6.1. Since $\{f_i\}_{i \in I}$ is bounded, so is $\{Pe_i\}_{i \in I}$. Choose $A > 0$ so that $A \leq \|Pe_i\| \leq 1$, for all $i \in I$. Then

$$A \leq |\langle Pe_i, Pe_i \rangle| = |\langle Pe_i, e_i \rangle| = |Pe_i(i)|.$$

So $\mathscr{H}$ is a large subspace.

Note that this also proves (II) (4) $\Rightarrow$ (3). □

Now we need to learn how to decompose the range of the analysis operator of our frames.

Definition 6.5: A closed subspace $\mathscr{H}$ of $\ell_2(I)$ is **r-decomposable** if for some natural number r there exists a partition $\{S_j\}_{j=1}^r$ of I so that $P_{S_j}(\mathscr{H}) = \ell_2(S_j)$, for all $j = 1, 2, \cdots, r$. The subspace $\mathscr{H}$ is *finitely decomposable* if it is r-decomposable for some r.

For the next proposition we need a small observation.

Lemma 6.2: *Let $\{f_i\}_{i \in I}$ be a Bessel sequence in $\mathscr{H}$ having synthesis operator T and analysis operator T^*. Let $E \subset I$, and let $\{f_i\}_{i \in E}$ have analysis operator $(T|_E)^*$. Then*

$$P_E T^* = (T|_E)^*.$$

Proof: For all $f \in \mathscr{H}$,

$$P_E T^*(f) = P_E \left(\sum_{i \in I} \langle f, f_i \rangle e_i \right) = \sum_{i \in E} \langle f, f_i \rangle e_i = (T|_E)^*(f).$$

□

We now have

Proposition 6.4: *If* $\{f_i\}_{i\in I}$ *is a unit norm frame for* $\mathscr{K}$ *with analysis operator* T^**, then for any* $0<\varepsilon<1$*,* $T^*(\mathscr{K})$ *is r-decomposable for*

$$r=\left(\frac{6(\|T\|^2+1)}{\varepsilon}\right)^4,$$

(with power 8 in the complex case).

Proof: We can partition I into $\{S_j\}_{j=1}^r$ so that each $\{f_i\}_{i\in S_j}$ is a Riesz basic sequence where

$$r=\left(\frac{6(\|T\|^2+1)}{\varepsilon}\right)^2,$$

for any $0<\varepsilon<1$. Thus, (see the discussion after Theorem 6.1) $(T|_{S_j})^*$ is onto for every $j=1,2,\cdots,r$ and hence (by Lemma 6.2) $P_{S_j}T^*$ is onto for all $j=1,2,\cdots,r$. □

Now we can put this altogether.

Theorem 6.18: *For every* $0<A<1$ *and* $0<\varepsilon<1$*, there is a natural number*

$$r=\left(\frac{6(A^2+1)}{\varepsilon A^2}\right)^4,$$

(power 8 in the complex case) so that every A-large subspace of $\ell_2(I)$ *is r-decomposable.*

Proof: By Proposition 6.3, if P is the orthogonal projection of $\ell_2(I)$ onto H, then $\|Pe_i\|\geq A$, for all $i\in I$. Then

$$\{f_i\}_{i\in I}=\left\{\frac{Pe_i}{\|Pe_i\|}\right\}_{i\in I},$$

is a unit norm frame with Bessel bound $1/A^2$. So by Proposition 6.4, our subspace is r-decomposable for

$$r=\left(\frac{6(A^2+1)}{\varepsilon A^2}\right)^4,$$

for any $0<\varepsilon<1$. □

6.8 Open Problems

There are a number of important open problems which remain even after the work of [36].

Problem 6.1: Can the η and θ in Theorem 6.5 be improved?

Problem 6.2: Can the values of r in the various results be improved?

It has been shown [14, 21] that every unit norm two tight frame can be partitioned into two linearly independent sets. But, there do not exist universal constants A, B so that all such frames can be partitioned into two subsets each with Riesz bounds A, B [16, 17]. This result raises a fundamental problem.

Problem 6.3: Can every unit norm two tight frame be partitioned into three subsets each of which are Riesz basic sequences with Riesz bounds independent of the dimension of the space?

Perhaps the most important open problem:

Problem 6.4: Find an implementable algorithm for proving the Paving Conjecture.

The most important case is really finding an algorithm for proving the Feichtinger Conjecture. The Feichtinger Conjecture potentially could have serious applications if we could quickly compute the appropriate subsets which are Riesz basic sequences.

6.9 Acknowledgement

For many years the Kadison-Singer Problem was a major motivating force for many of us. Its challenges made every day an exciting event. It also brought together mathematicians from many diverse areas of research—especially as the "polynomial people" came in to give the solution. As we discovered more elementary formulations of the problem, it became clear that this problem represented a fundamental idea for finite dimensional Hilbert spaces which was not understood at all. This just made the problem even more interesting. Almost everyone believed that the problem had a negative answer—which probably contributed to the problem remaining open for so long since we were only looking for a counter-example. The solution to this problem by Marcus/Spielman/Srivastava was a major achievement of our time and earned them the **Polya Prize**, a trip to the **International Congress of Mathematicians** and recognition yet to be established.

The 54-year search for a solution to the Kadison-Singer Problem represented a large number of papers by many brilliant mathematicians which culminated in the solution to the problem by Marcus/Spielman/Srivastava. We enclose here a brief summary of the historical development of the Kadison-Singer Problem from the direction of the Paving Conjecture. Since the authors are not experts in C^*-Algebras, we have chosen not to trace the history of the problem from that direction. So we will start in 1979 with the introduction of the **Anderson Paving Conjecture**. Also, there are hundreds of papers here so we will just consider those papers which introduced new directions (equivalences) of the Paving Conjecture.

- [31] (1959) Kadison and Singer formulate the **Kadison-Singer Problem**.
- [2] (1979) Anderson reformulates the Kadison-Singer Problem into the **Anderson Paving Conjecture**. This was significant because it changed the Kadison-Singer Problem from being a specialized problem hidden in C^*-Algebras and opened it up to everyone in Analysis.
- [8, 9, 27, 28] (1986) Berman/Halpern/Kaftal/Weiss make a deep study of the Paving Conjecture for Laurant Operators. They introduced the notion of **uniform pavability** and they show that matrices with positive coefficients are **pavable**. They also give a positive solution for paving for the Schatten C_p-norms for $p = 4,6$.
- [10] (1989) Bourgain-Tzafriri prove the famous **Restricted Invertibility Theorem** which naturally leads to the **Bourgain-Tzafriri Conjecture**.
- [11] (1991) Bourgain-Tzafriri show that matrices with small entries are **pavable**.
- [1] (1991) Akemann and Anderson formulate the **Akemann-Anderson Projection Paving Conjecture** and show it implies a positive solution to the Kadison-Singer Problem. This was important since it reduced paving to paving for a much smaller class of operators—projections.
- [26](2003) Gröchenig shows that localized frames satisfy the **Feichtinger Conjecture**. This conjecture was formulated in [14] but appeared much later.
- [42, 43] (2003–2004) Weaver formulates the **Weaver Conjectures** and gives a counter-example to a conjecture of Akemann and Anderson which would have implied a positive solution to the Kadison-Singer Problem.
- [14] (2005) Casazza/Christensen/Lindner/Vershynan introduce the **Feichtinger Conjecture** and show it is equivalent to the **Bourgain-Tzafriri Conjecture**.
- [12] (2006) Bownik/Speegle make a detailed study of the Feichtinger Conjecture for Wavelet Frames, Gabor Frames and Frames of Translates and relate the Feichtinger Conjecture to Gowers' work on a generalization of Van der Waerdan's Theorem.
- [18, 21] (2006) Casazza/Tremain and Casazza/Fickus/Tremain/Weber show that the Kadison-Singer Problem is equivalent to the **Feichtinger Conjecture**, the **Bourgain-Tzafriri Conjecture**, the **R_ε-Conjecture**, the **Casazza/Tremain Conjecture**, and a number of conjectures in Time-Frequency Analysis, Frames of Translates and Hilbert Space Frame Theory.
- [39] (2008) Paulsen and Raghupathi show that **paving** (respectively, paving Toeplitz operators) is equivalent to paving upper triangular matrices (respectively, paving upper triangular Toeplitz operators).
- [16] (2009) Casazza/Edidin/Kalra/Paulsen show that paving projections with constant diagonal 1/2 is equivalent to the Paving Conjecture. This is a new direction for paving projections as all previous work involved paving projections with very small diagonals. They also show that the Paving Conjecture fails for 2-paving.
- [34] (see also [38]) (2010) Lawton and Paulsen independently showed that if the Feichtinger Conjecture holds for Fourier frames, then each set in the partition into

Riesz basic sequences can be chosen to be a **syndetic set**. Paulsen first presented this at GPOTS (2008).

- (2007–2010) A large number of papers on the Feichtinger Conjecture appeared. Too numerous to list here. See [32] for a somewhat complete list—especially for reproducing kernel Hilbert spaces and for classical spaces.
- [17] (2011) Casazza/Fickus/Mixon/Tremain give concrete constructions of non-2-pavable projections.
- [13] (2012) Casazza introduces the **Sundberg Problem** which is implied by the Paving Conjecture.
- [36] (2013) Marcus/Spielman/Srivastava surprise the mathematical community by giving a positive solution to the Kadison-Singer Problem.

Remark 6.2: We were recently made aware of the thesis [35] of Y. Lonke from 1993 which has a proof that BT is equivalent to KS. Since it was written in Hebrew, it seems to have been overlooked. We now have English translations [35].

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Chapter 7
Model Sets and New Versions of Shannon Sampling Theorem

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Abstract In these notes distinct approaches to define model sets/quasicrystals are discussed. We also discuss some improvements on Shannon sampling theorem obtained by using simple model sets/quasicrystals.

7.1 Introduction

These notes are written as a contribution to the Summer School entitled *New Trends in Applied Harmonic Analysis Sparse Representations, Compressed Sensing and Multifractal Analysis* which was organized by Akram Aldroubi, Carlos Cabrelli, Stéphane Jaffard, and Ursula Molter and took place in Mar del Plata, Argentina in August 2013. These notes are not an original contribution, they are based on the synthesis paper of Y. Meyer [29] and on my early works [20–22] and [23].

In the first part of these notes we try to give an answer to the question "What is a quasicrystal?" This question was raised by Yves Meyer in [29]. Are quasicrystals almost periodic patterns ? For answering this question we need to define quasicrystals and almost periodic patterns. In the second part of these notes we investigate the universal sampling property of simple quasicrystals.

These notes are divided into sixteen sections (including the introduction and the conclusion).

In the second section some standard facts on almost periodic functions or distributions are listed for the reader's convenience. In the third section *generalized almost periodic measures* will be defined and studied. Following [29], we aim at

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relating the arithmetical properties of a Delone set Λ to the analytical properties of the corresponding measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$. We say that Λ is an almost periodic pattern if the measure σ_Λ is a *generalized almost periodic measure* (Definition 13). Yves Meyer shows that *model sets are almost periodic patterns* [29]. This improves on a theorem by J. Lagarias [12]. The proof of this theorem will be fully detailed in these notes. Let Λ_θ, $\theta > 2$, be the set of all finite sums $\sum_{k\geq 0}\varepsilon_k\theta^k$ with $\varepsilon_k \in \{0,1\}$. Yves Meyer proved in [29] that Λ_θ is an almost periodic pattern if and only if θ is a Pisot-Vijayaraghavan number. We detail this result in the fifth section. Pisot and Salem numbers are defined in the fifth section.

Almost lattices are studied in the sixth and seventh sections following [29]. *Almost lattices* are sets $\Lambda\subset\mathbb{R}^n$ of points which generalize lattices. An almost lattice Λ is a Delone set such that $\Lambda-\Lambda\subset\Lambda+F$ where F is a finite set. If $F=\{0\}$, Λ is a lattice. Model sets and the *cut and projection* algorithm are presented in Section 7.7. The eighth and ninth sections are devoted to the properties of *harmonious sets*. The $\varepsilon-$dual Λ_ε^* of a harmonious set Λ is defined in Section 7.8 and Theorem 14 answers the issue of knowing whether the $\varepsilon-$dual of Λ_ε^* is Λ. The aim of Theorem 15 due to Y. Meyer [29] is to relate *almost lattices* to *model sets* and to *harmonious sets*. Every mean-periodic function f whose spectrum is contained in an almost lattice is in fact an almost periodic function. A stronger statement proved by Y.Meyer [29] is fully detailed in Section 7.10. Several Poisson type identities from [29] are recalled in Sections 7.11 and 7.12. Again these sections are not original and can be viewed as a survey of the remarkable achievements by J. Lagarias, R. Moody, and Y. Meyer.

The recently discovered sampling properties of simple quasicrystals are mentioned in Sections 7.13–7.14. In Section 7.15 we improve the main result of [21]. This last section of this essay contains original results.

We conclude with some comments on three tentative definitions of quasicrystals.

7.2 Almost Periodic Functions and Measures

Quasicrystals are almost periodic patterns. As it was stressed by J. Lagarias in [12] this statement cannot be true if a naïve definition of almost periodicity is being used. (see also [13–15] and [19]). The definition of almost periodic patterns will be unveiled in Section 7.3.

7.2.1 Almost Periodic Functions

The reader who is familiar with the theory of almost periodic functions in the sense of Bohr is urged to skip this subsection. The next one should also be skipped if one has previously read *La théorie des distributions* by Laurent Schwartz ([38]). In

Lagarias [12] almost periodic functions in the sense of Bohr are called “uniformly almost periodic functions.”

The Fourier transform $\mathscr{F}(f) = \hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) \;=\; \int_{\mathbb{R}^n} \exp(-ix\cdot\xi)\, f(x)\, dx \tag{7.1}$$

and the Fourier inversion formula reads

$$f(x) \;=\; (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(ix\cdot\xi)\, \hat{f}(\xi)\, d\xi\ . \tag{7.2}$$

A continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is almost periodic (in the sense given by Harald Bohr) if for each positive ε there exists a *relatively dense* set Λ_ε of *ε-almost periods* τ for f. These two concepts (relatively dense and ε-almost period) are now defined.

A subset $\Lambda \subset \mathbb{R}^n$ is *relatively dense* if there exists a positive R such that each ball with radius R (whatever be its center) contains at least a point λ in Λ. This definition was introduced by Besicovitch.

We say that τ is an *ε-almost period* of a function $f : \mathbb{R}^n \mapsto \mathbb{C}$ if

$$\sup_{x\in\mathbb{R}^n} |f(x+\tau) - f(x)| \;\le\; \varepsilon \tag{7.3}$$

The space of almost periodic functions on $\mathbb{R}^n$ equipped with the norm $\|f\|_\infty$ is a Banach space which will be denoted by $\mathscr{E}$. Here and in what follows, $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|$.

Let f be an almost periodic function. The orbit of f under translations is the collection $\mathscr{E}$ of all functions $f(\cdot - y), y \in \mathbb{R}^n$. The orbit $\mathscr{E}$ of an almost periodic function f is a precompact set for the topology of uniform convergence on $\mathbb{R}^n$. In other words for every sequence $x_j \in \mathbb{R}^n$ there exists a subsequence x_{j_k} such that $f(x - x_{j_k})$ converges to an almost periodic function g uniformly on $\mathbb{R}^n$.

Every finite trigonometric sum $P(x) = \sum_{\lambda\in S} c(\lambda)\exp(i\lambda\cdot x)$ is an almost periodic function (S being an arbitrary finite subset of $\mathbb{R}^n$). H. Bohr proved that a function $f : \mathbb{R}^n \to \mathbb{C}$ is almost periodic if and only if, for each $\varepsilon > 0$, there exists a finite trigonometric sum $P_\varepsilon(x) = \sum_{\lambda\in S(\varepsilon)} c(\lambda,\varepsilon)\exp(i\lambda\cdot x)$ such that $\|f - P_\varepsilon\|_\infty \le \varepsilon$.

The construction of $S(\varepsilon)$ and $P_\varepsilon(x)$ will be detailed below. A detour to the Bohr compactification of $\mathbb{R}^n$ is needed to better understand what an almost periodic function looks like.

The Bohr compactification of $\mathbb{R}^n$ is denoted by $\mathscr{G}_n$. It is the dual group (in the sense of Pontryagin duality) of the group $\mathbb{R}^n$ equipped with the discrete topology. The elements of the compact group $\mathscr{G}_n$ are the characters χ on $\mathbb{R}^n$ which are defined now.

Definition 1. A function $\chi : \mathbb{R}^n \mapsto \mathbb{T}$ is a character on $\mathbb{R}^n$ if it maps the additive group $\mathbb{R}^n$ to the multiplicative group $\mathbb{T}$ of complex numbers of modulus 1 and if it is a group homomorphism: $\chi(x+y) = \chi(x)\chi(y)\ \ (\forall x, y \in \mathbb{R}^n)$.

A character does not need to be continuous. Any product $\chi\chi'$ between two characters is still a character. As it was said above the group $\mathscr{G}_n$ is the multiplicative group of all such characters. Then $\mathbb{R}^n$ is a subgroup of $\mathscr{G}_n$ since every continuous character is a character. More precisely each $\omega \in \mathbb{R}^n$ will be identified to the character χ_ω defined by $\chi_\omega(x) = \exp(i\omega \cdot x)$. Moreover $\mathbb{R}^n$ is dense in $\mathscr{G}_n$. The canonical embedding of $\mathbb{R}^n$ into $\mathscr{G}_n$ will be denoted by $\mathscr{J}_n$. We obviously have $\mathscr{G}_n = \mathscr{G}_1 \times \cdots \times \mathscr{G}_1$. With these notations we have

Lemma 1. *Let F be a continuous function on $\mathscr{G}_n$. Then its restriction $f = F \circ \mathscr{J}_n$ to $\mathbb{R}^n$ is an almost periodic function. Conversely any almost periodic function f on $\mathbb{R}^n$ is the restriction to $\mathbb{R}^n$ of a continuous function F on $\mathscr{G}_n$. This F is unique and is the continuation of f to $\mathscr{G}_n$.*

Let f be an almost periodic function. The ball centered at x, with radius R is denoted by $B(x,R)$ and the constant c_n is the inverse of the volume of the unit ball. Then the limit

$$\mathscr{M}(f) \;=\; \lim_{R\to+\infty} c_n R^{-n} \int_{B(x,R)} f(y)\,dy\,. \tag{7.4}$$

is attained uniformly in x.

Moreover $\mathscr{M}(f) = \int_{\mathscr{G}_n} F(x)\,dx$ when f and F are related by Lemma 1.

Is the Bohr compactification of $\mathbb{R}^n$ actually needed in Lemma 1 ? Two definitions will be used to answer this question.

Definition 2. A compact abelian group G is a compactification of $\mathbb{R}^n$ if G is the dual group (in the sense of Pontryagin duality) of a dense subgroup $\Gamma \subset \mathbb{R}^n$. We then denote by $J : \mathbb{R}^n \mapsto G$ the canonical embedding.

In other words, G is the compact group of all weak characters $\chi : \Gamma \mapsto \mathbb{T}$. For answering our question we need to define the spectrum S of an almost periodic function f (see Definition 3 below). As in [29], it will be proved that the "smallest" group G on which f extends continuously is the dual of the additive group Γ generated by the spectrum S of f.

For each $\omega \in \mathbb{R}^n$, $\exp(-i\omega \cdot x) f(x)$ is also an almost periodic function. This remark paves the road to the definition of the Fourier coefficient of f at the frequency $\omega \in \mathbb{R}^n$. This Fourier coefficient is denoted by $\hat{f}(\omega)$ and is defined as

$$\hat{f}(\omega) \;=\; \mathscr{M}[\exp(-i\omega \cdot x) f(x)] \tag{7.5}$$

The notation $\hat{f}(\omega)$ is confusing since $\hat{f}(\omega)$ is not the value at ω of the distributional Fourier transform of f. However we have $\hat{f}(\omega) = \hat{F}(\omega)$ when $f = F \circ \mathscr{J}_n$ as in Lemma 1. Here $\hat{F}(\omega)$ is the ordinary Fourier coefficient at the frequency ω of the continuous function F.

If f is almost periodic, so is $|f|^2$, and one has

$$\mathscr{M}(|f|^2) = \sum |\hat{f}(\omega)|^2. \tag{7.6}$$

Definition 3. The set S of frequencies ω for which $\hat{f}(\omega) \neq 0$ is at most a numerable set. This set S is named the spectrum of f.

In a sense f is the sum of its Fourier series expansion:

$$f(x) \sim \sum_{\omega \in S} \hat{f}(\omega)\, e^{i\omega\cdot x}\,. \tag{7.7}$$

The Fourier series expansion (7.7) of f becomes an ordinary Fourier expansion when f is viewed as a continuous function F on $\mathscr{G}_n$.

The issue raised by the convergence of the series (7.7) needs to be elucidated. An almost periodic function f is uniquely defined by its Fourier coefficients and (7.7) becomes a true fact after some manipulations which will be detailed below. If $\sum_{\omega\in S} |\hat{f}(\omega)|$ is finite, the Fourier series expansion of f converges to f uniformly on $\mathbb{R}^n$. If it is not the case, some summation procedures generalizing Cesaro summation are needed to give a meaning to (7.7). H. Bohr proved the following theorem:

Theorem 1. *For each numerable set $S \subset \mathbb{R}^n$ and each $\varepsilon > 0$ there exist a finite subset $S(\varepsilon) \subset S$ and a family $\beta_S(\varepsilon,\omega)$ of weight factors with the following properties*

(a) $0 \le \beta_S(\varepsilon,\omega) \le 1$
(b) $\lim_{\varepsilon\downarrow 0} \beta_S(\varepsilon,\omega) = 1$ *for each $\omega \in S$,*
(c) $\beta_S(\varepsilon,\omega) = 0$ *if ω does not belong to the finite set $S(\varepsilon)$*
(d) and finally for every almost periodic function f whose spectrum is contained in S we have $\|f - P_\varepsilon\|_\infty \to 0$, $\varepsilon \to 0$, when

$$P_\varepsilon(x) = \sum_{\omega\in S(\varepsilon)} \beta_S(\varepsilon,\omega)\,\hat{f}(\omega)\,\exp(i\omega\cdot x) \tag{7.8}$$

As indicated by the subscript S, $\beta_S(\varepsilon,\omega)$ does not depend on f but only on the additive properties of the set S.

One is tempted to say that the distributional Fourier transform $\hat{f}$ of f is $(2\pi)^n \sum_{\omega\in S} \hat{f}(\omega)\delta_\omega$ where δ_ω is the Dirac mass at ω. This is not true at this naïve level. We cannot write $\hat{f} = (2\pi)^n \sum_{\omega\in S} \hat{f}(\omega)\delta_\omega$ since this sum of Dirac masses is not defined unless $\sum_{\omega\in S} |\hat{f}(\omega)|$ is finite. If it is the case, the two definitions of the Fourier transform of an almost periodic functions agree as indicated in the following lemma:

Lemma 2. *Let us assume that f is an almost periodic function and that $\sum_{\omega\in S} |\hat{f}(\omega)|$ is finite. Then the distributional Fourier transform of f is $(2\pi)^n \sum_{\omega\in S} \hat{f}(\omega)\delta_\omega$. Conversely if the distributional Fourier transform of a function $f \in L^\infty$ is a finite atomic measure $\sigma = \sum_{\omega\in S} c(\omega)\delta_\omega$ then f is an almost periodic function and its Fourier coefficients are $\hat{f}(\omega) = (2\pi)^{-n} c(\omega)$.*

We now answer the problem raised by Lemma 1. Let f be an almost periodic function. Let S be the spectrum of f and Γ be the additive sub-group of $\mathbb{R}^n$ generated by S. Let G be the dual group of Γ. Then $\mathbb{R}^n$ can be viewed as dense subgroup of G since every continuous character on $\mathbb{R}^n$ can be restricted to Γ. We denote by $J : \mathbb{R}^n \mapsto G$ this embedding. With these notations we have

Lemma 3. *The almost periodic function f can be uniquely written $f = F \circ J$ where F is continuous on G.*

Indeed the finite trigonometric sums

$$P_\varepsilon(x) \;=\; \sum_{\omega \in S(\varepsilon)} \beta_S(\varepsilon,\omega)\,\hat{f}(\omega)\,e^{i\omega\cdot x} \tag{7.9}$$

converge uniformly to f. These sums $P_\varepsilon(x)$ extend continuously to G since their frequencies belong to Γ. Therefore f extends continuously to G.

The convolution product between two almost periodic functions is defined by

$$(f \circledcirc g)(x) = \mathscr{M}[f(x-\cdot)g(\cdot)] \tag{7.10}$$

and we have

$$\mathscr{M}(f \circledcirc g) = \mathscr{M}(f)\mathscr{M}(g) \tag{7.11}$$

If f and g are two almost periodic functions on $\mathbb{R}^n$ and if F and G denote their continuation to $\mathscr{G}_n$, then the restriction of $F * G$ to $\mathbb{R}^n$ is $f \circledcirc g$. The identity (7.11) becomes obvious since $\int_{\mathscr{G}_n}(F*G)\,dx = (\int_{\mathscr{G}_n} F\,dx)(\int_{\mathscr{G}_n} G\,dx)$.

We now compute the Fourier coefficients of the convolution product between two almost periodic functions.

Lemma 4. *The convolution product $h = f \circledcirc g$ between two almost periodic functions is an almost periodic function and the Fourier coefficients of h are given by*

$$\hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega) \tag{7.12}$$

We write $f_\omega(x) = \exp(-i\omega\cdot x)f(x)$ and use the same notations for g and $h = f \circledcirc g$. Then it suffices to observe that $h_\omega = f_\omega \circledcirc g_\omega$ and to use (7.11). It is important to observe that the Fourier series of h is absolutely convergent: $\sum |\hat{h}(\omega)|$ is finite.

The following proposition will be used in this essay:

Proposition 1. *Let H be an additive subgroup of $\mathbb{R}^n$ and f be an almost periodic function on $\mathbb{R}^n$. Then there exists a unique almost periodic function g such that $\hat{g}(\xi) = \hat{f}(\xi)$ if $\xi \in H$ and $\hat{g}(\xi) = 0$ if $\xi \notin H$. Moreover $\|g\|_\infty \le \|f\|_\infty$.*

Let us sketch the proof of Proposition 1. Let $\mathscr{H}$ be the dual group of H and let F_j be a sequence of continuous functions on $\mathscr{H}$ which are an approximation to the identity. We can assume $F_j \ge 0$ and $\int_{\mathscr{H}} F_j(y)\,dy = 1$. Let $h : \mathbb{R}^n \mapsto \mathscr{H}$ be the canonical embedding. We then consider the almost periodic function $f_j = F_j \circ h$. We have

$\hat{f}_j(\xi) = 0$ if $\xi \notin H$ and $\hat{f}_j(\xi) = \hat{F}_j(\xi)$ if $\xi \in H$. We form the convolution products $g_j = f \circledcirc f_j$ in the sense of the convolution between almost periodic functions. The Fourier coefficients of g_j are $\hat{g}_j(\xi) = \hat{f}(\xi)\hat{f}_j(\xi)$. That is why $\hat{g}_j(\xi)$ vanishes outside H. Moreover $\|g_j\|_\infty \leq \|f\|_\infty$ by the properties of the convolution product. Finally $\hat{f}_j(\xi) \to 1$ as j tends to infinity. These convolution products converge to a function g uniformly on $\mathbb{R}^n$ and g possesses the required properties.

If $\Lambda \subset \mathbb{R}^n$ is a model set and ϕ a compactly supported continuous function, then $f(x) = \sum_{\lambda \in \Lambda} \phi(x - \lambda)$ is not an almost periodic function. This was observed by J. Lagarias and paves the road to the definitions which are given in Section 7.3.

7.2.2 Almost Periodic Measures

The reader who is familiar with the theory of distributions by Laurent Schwartz is invited to skip this subsection and to jump to the next one. Almost periodic measures are called "uniformly almost periodic measures" by J. Lagarias.

Schwartz proposed the following definition of an almost periodic distribution.

Definition 4. A distribution S is almost periodic if for every testing function $\phi \in \mathscr{D}$ the convolution product $S * \phi$ is an almost periodic function in the sense of Bohr.

This immediately extends to almost periodic measures. The only difference is that the class $\mathscr{D}$ of testing functions is replaced by the class $\mathscr{E}$ of compactly supported continuous functions. Yves Meyer proved in [29] that this definition of almost periodic measures is too demanding since the sum σ_Λ of Dirac masses on a model set Λ is not an almost periodic measure in general (see Theorem 18 below). This was already observed by J. Lagarias. In contrast σ_Λ is a *generalized almost periodic measure* which motivates the definitions given in Section 7.3. Lagarias proved that for every compactly supported continuous function ϕ the convolution product $\sigma_\Lambda * \phi$ is a Besicovitch almost periodic function. This can be found in [12] and this result was improved in [29]. In this subsection the well-known properties of almost periodic measures are listed for the reader's convenience.

Definition 5. A Borel measure μ on $\mathbb{R}^n$ is almost periodic if for every compactly supported continuous function g the convolution product $\mu * g = f$ is an almost periodic function.

If μ is an almost periodic measure, the closed graph theorem implies that

$$\sup_{x \in \mathbb{R}^n} \int_{B(x)} |d\mu| = C < \infty \tag{7.13}$$

Here $B(x)$ is the ball centered at x with radius 1. Following A. Hof [6] we say that μ is a *translation bounded measure*. We then have

Lemma 5. *A translation bounded measure μ is almost periodic if and only if μ is an almost periodic distribution.*

The Banach space of translation bounded measures is equipped with the norm $\sup_{x\in\mathbb{R}^n}|\mu|[B(x)]$. The weak convergence of a bounded sequence μ_j of translation bounded measures is defined by the duality with compactly supported continuous functions.

Here is an example of an almost periodic measure. Let θ be a real valued continuous function of $x\in\mathbb{R}$ and let us assume that $\theta(x+1)=\theta(x)$. Let Λ_θ consist of all real numbers $k+\theta(\sqrt{2}k),\ \ k\in\mathbb{Z}$. We define a measure σ_θ by $\sigma_\theta=\sum_{\lambda\in\Lambda_\theta}\delta_\lambda$ where δ_a is the Dirac mass at a.

Lemma 6. *The measure σ_θ is almost periodic.*

This example is playing a seminal role in the forthcoming analysis of almost periodic patterns and quasicrystals. The set Λ_θ is an almost periodic pattern (see Definition 13 below). However this set is not a quasicrystal in any sense of this word. For instance, Λ_θ is not an almost lattice (see Definition 22). The proof relies on the following Diophantine approximation property.

Lemma 7. *Let ε be a positive number. Then the set*

$$M_\varepsilon=\{\tau\in\mathbb{Z};\inf_{k\in\mathbb{Z}}|\sqrt{2}\tau-k|\le\varepsilon\}$$

is relatively dense.

This being recalled, let ϕ be a compactly supported continuous function and let $f(x)=\sum\phi(x-k-\theta(\sqrt{2}k))$. We need to show that $f(x)$ is an almost periodic function. If $\tau\in\mathbb{Z}$, we obviously have

$$f(x+\tau)=\sum_{k\in\mathbb{Z}}\phi(x-k-\theta(\sqrt{2}(k+\tau))).$$

But $|\theta(\sqrt{2}(k+\tau))-\theta(\sqrt{2}k)|\le\eta(\varepsilon)$ uniformly in k by the continuity of θ and for every $\tau\in M_\varepsilon$. It now suffices to observe that $\eta(\varepsilon)$ tends to 0 with ε and that the series defining $f(x)$ is locally finite. This concludes the proof.

The following observation is trivial but crucial in what follows.

Lemma 8. *Let μ be a translation bounded measure and χ_R be the indicator function of the ball $\{x;|x|\le R\}$. If $R\ge1$ and if f is a bounded Borel function we have*

$$c_nR^{-n}|\int f\chi_R\,d\mu|\le C\|f\|_\infty \tag{7.14}$$

We now define the mean value of an almost periodic measure.

Lemma 9. *If the closed ball centered at x with radius R is denoted by $B(x,R)$, then the mean value $\mathcal{M}(\mu)$ of an almost periodic measure μ is defined by*

$$\mathcal{M}(\mu) = \lim_{R\to+\infty} c_n R^{-n} \mu[B(x,R)] \tag{7.15}$$

and this limit is attained uniformly in x.

A proof of Lemma 9 is given for the reader's convenience. The indicator function of $B(x,R)$ is denoted by $\chi_{x,R}$. Next ϕ denotes a non-negative continuous function supported by $B(0,1)$ with $\int \phi(x)\,dx = 1$. Let us furthermore assume that ϕ is an even function. Then as R tends to infinity we have

$$\int \chi_{x,R}\,d\mu = \int (\chi_{x,R})(\mu * \phi)\,dx + O(R^{n-1}) \tag{7.16}$$

The proof of (7.16) is almost obvious. Since ϕ is even the integral in the RHS can be written $J = \int (\chi_{x,R} * \phi)(y)\,d\mu(y) = J_1 + J_2$ where J_1 is the integral over $|y-x| \le R-1$ and J_2 runs over $R-1 \le |y-x| \le R+1$. We split the LHS similarly as $I = I_1 + I_2$. If $|y-x| \le R-1$, we have $(\chi_{x,R} * \phi)(y) = 1$ and $I_1 = J_1$. But the error terms I_2 and J_2 are $O(R^{n-1})$ since μ is translation bounded.

If μ is an almost periodic measure, τ is a measure with a finite mass, and $I(\tau) = \int d\tau$, the convolution product $\mu * \tau$ is still an almost periodic measure and

$$\mathcal{M}(\mu * \tau) = \mathcal{M}(\mu) I(\tau) \tag{7.17}$$

We now prove the following lemma

Lemma 10. *If μ is an almost periodic measure and f is an almost periodic function, then the product $f\mu$ is an almost periodic measure.*

We first consider $\chi_\omega(x) = \exp(i\omega\cdot x)$ and we shall prove that $\chi_\omega\mu$ is an almost periodic measure. Let us assume that ϕ is continuous and compactly supported. We then have $[(\chi_\omega\mu) * \phi](x) = \int \chi_\omega(x-y)d\mu(x-y)\phi(y)\,dy = \chi_\omega(x)\int \phi(y)\chi_\omega(-y)d\mu$ $(x-y)\,dy = \chi_\omega(x)(\mu * \phi_\omega)(x)$ where $\phi_\omega(x) = \phi(x)\chi_\omega(-x)$. Finally $\mu * \phi_\omega$ is an almost periodic function and so is $(\mu * \phi_\omega)\chi_\omega$. By linearity Lemma 10 holds for every trigonometric polynomial and by density for every almost periodic function f.

The convolution product $\mu * g$ between g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and an almost periodic measure μ is an almost periodic function. Therefore the distributional Fourier transform of an almost periodic measure makes sense. The Fourier coefficients of an almost periodic measure μ are not defined through its distributional Fourier transform $\hat{\mu}$ but as follows.

Definition 6. The Fourier coefficients of an almost periodic measure μ are

$$\hat{\mu}(\omega) = \mathcal{M}[\mu(x)\exp(-ix\cdot\omega)],\ \omega \in \mathbb{R}^n. \tag{7.18}$$

An almost periodic measure is uniquely defined by its Fourier coefficients $\hat{\mu}(\omega)$ and one can write a formal expansion

$$d\mu(x) \sim \sum_{\omega\in S} \hat{\mu}(\omega)\, e^{i\omega\cdot x}\,. \tag{7.19}$$

as it was the case for almost periodic functions.

Lemma 11. *Let μ be an almost periodic measure and f an almost periodic function. Then we have for every compactly supported continuous function g*

$$\mathscr{M}[(f*g)\mu] = \mathscr{M}[(\mu*\check{g})f] \tag{7.20}$$

By definition $\check{g}(x) = g(-x)$. It suffices to check this property when $f(x) = \exp(i\omega\cdot x)$ where it is trivial. The following result will be needed below.

Proposition 2. *Let μ be an almost periodic measure such that $\mathscr{M}(f\mu) \geq 0$ for every non-negative almost periodic function f. Then μ is a non-negative measure.*

Let ϕ be a non-negative compactly supported function such that $\int \phi(x)\,dx = 1$. Since μ is translation bounded we have $\mathscr{M}[(f*\phi)\mu] = \mathscr{M}[f(\mu*\phi)]$. Since the LHS is non-negative so is the RHS. We then use the following lemma

Lemma 12. *Let F be a real valued almost periodic function such that for every non-negative almost periodic function f we have $\mathscr{M}(fF) \geq 0$. Then F is non-negative.*

The proof is straightforward. It suffices to consider the function $f = -\inf(F,0)$. We then have $\mathscr{M}(fF) \geq 0$ but $\mathscr{M}(fF) = -\mathscr{M}(|f|^2)$ which implies $\inf(F,0) = 0$. Returning to Proposition 2 we have $\mu*\phi \geq 0$ for every $\phi \geq 0$ which implies $\mu \geq 0$.

Lemma 13. *If μ and ν are two almost periodic measures, their convolution product $\tau = \mu \circledcirc \nu$ is still an almost periodic measure.*

We shall define this convolution product. As above the indicator function of the closed ball $B(0,R)$ centered at 0 with radius R is denoted by χ_R and we write $\mu_R = \mu\chi_R$. Next the measure τ_R is defined by

$$\tau_R = c_n R^{-n} \mu_R * \nu \tag{7.21}$$

The convolution product $\tau = \mu \circledcirc \nu$ is the limit in the distributional sense of τ_R. Lemma 14 yields a stronger result.

Lemma 14. *For every continuous and compactly supported function g the functions $h_R = \tau_R * g$ are almost periodic and converge uniformly on $\mathbb{R}^n$ to an almost periodic function h as R tends to infinity.*

The measures τ_R are uniformly translation bounded since the total mass of $c_n R^{-n}\mu_R$ does not exceed a constant C. It suffices to prove Lemma 14 for a dense collection of functions g to obtain the general case. Let us assume that $g = g_1 * g_2$ where g_1, g_2, are continuous and compactly supported. Then $\tau_R * g = c_n R^{-n}(\mu_R * g_1) * (\nu * g_2)$ and $\mu_R * g_1 = (\mu * g_1)\chi_R + O(R^{n-1})$ since μ is translation bounded. The two functions $h_1 = \mu * g_1$, $h_2 = \nu * g_2$ are almost periodic and $c_n R^{-n} h_1 \chi_R * h_2$ converges uniformly on $\mathbb{R}^n$ to the convolution product $h_1 \circledcirc h_2$.

Lemma 15. *If μ and ν are two almost periodic measures, we have*

$$\mathscr{M}(\mu \circledcirc \nu) = \mathscr{M}(\mu)\mathscr{M}(\nu) \tag{7.22}$$

The proof is left to the reader.

Here is an example of the convolution product between two almost periodic measures. We consider the Dirac comb $\mu = \sum_{k\in\mathbb{Z}} \delta_k$ and for some positive $\alpha \notin \mathbb{Q}$ we consider $\nu = \alpha \sum_{k\in\mathbb{Z}} \delta_{\alpha k}$. Then the convolution product $\mu \circledcirc \nu$ is the Lebesgue measure on the real line.

A specific example of an almost periodic measure will be detailed now. This example is aimed at proving the following fact

Proposition 3. *There exists an almost periodic measure μ such that $|\mu|$ is not an almost periodic measure.*

Two constructions of μ will be given. Here is the first one. We consider the set $\Gamma_0 = 2\mathbb{Z}+1$ of odd integers and write $\Gamma_j = 2^j\Gamma_0$, $j \in \mathbb{N}$. Then $\mathbb{Z}\setminus\{0\}$ is the disjoint union of Γ_j, $j \geq 0$. We let σ_j be the sum $\sum_{k\in\Gamma_j} \delta_k$ of Dirac masses at $k \in \Gamma_j$. Then σ_j is a $2^{j+1}-$periodic measure. Let τ be the Dirac comb $\sum_{k\in\mathbb{Z}} \delta_k$. Then $\sigma = \sigma_0 + \sigma_1 + \cdots = \tau - \delta_0$. Therefore σ is not an almost periodic measure.

Lemma 16. *Let $\mu_j = \sigma_j * (\delta_0 - \delta_{2^{-j-1}})$. Then the sum $\mu = \mu_0 + \mu_1 + \cdots$ is an almost periodic measure.*

We first observe that μ is translation bounded since $|\mu|([k,k+1]) = 2$ for $k \in \mathbb{Z}$. It then suffices to prove that $\mu * g$ is an almost periodic function for every smooth testing function g. We set $g_j = g * \mu_j$ and observe that g_j is $2^{j+1}-$periodic. Moreover $\|g_j\|_\infty \leq C2^{-j}$ which implies that $\sum_{j\geq 0} g_j$ is an almost periodic function.

Lemma 17. *The measure $|\mu|$ is not almost periodic.*

Indeed $|\mu| = \sum_{j\geq 0} |\mu_j| = \sum_{j\geq 0} \sigma_j * (\delta_0 + \delta_{2^{-j-1}})$. If $|\mu|$ was an almost periodic function, then the sum $\mu + |\mu| = 2\sigma$ would also be an almost periodic function which is not the case.

In the second construction we consider the set $\Gamma_0 = 3\mathbb{Z}+1$ and write $\Gamma_j = 3^j\Gamma_0$, $j \in \mathbb{N}$. We let σ_j be the sum $\sum_{k\in\Gamma_j} \delta_k$ of Dirac masses at $k \in \Gamma_j$. Then σ_j is a $3^{j+1}-$periodic measure.

Lemma 18. *The sum $\sigma = \sigma_0 + \sigma_1 + \cdots$ is not an almost periodic measure.*

The proof relies on the following observation.

Lemma 19. *Let $\Lambda \subset \mathbb{Z}$. Then $\sigma = \sum_{k\in\Lambda} \delta_k$ is an almost periodic measure if and only if Λ is a periodic set.*

The proof is left to the reader. In our case $\cup_{j\geq 0}\Gamma_j$ is not a periodic set. The measure $\sigma = \sigma_0 + \sigma_1 + \cdots$ will be an example of a generalized almost periodic measure. We proceed to the construction of μ.

Lemma 20. *Let $\mu_j = \sigma_j * (\delta_0 - \delta_{3^{-j-1}})$. Then the sum $\mu = \mu_0 + \mu_1 + \cdots$ is an almost periodic measure but $|\mu|$ is not an almost periodic measure.*

The proofs are the same as above.

The duality between almost periodic measures and almost periodic functions is defined by

$$< f,\mu >= \mathscr{M}(f\mu) \tag{7.23}$$

This makes sense since the product between an almost periodic measure and an almost periodic function is an almost periodic measure. In other words an almost periodic measure μ defines a Borel measure $\mathscr{J}(\mu)$ on the Bohr compactification $\mathscr{G}$ of $\mathbb{R}^n$ and the Fourier coefficients of μ are identical to the Fourier coefficients of $\mathscr{J}(\mu)$. The mapping $\mu \mapsto \mathscr{J}(\mu)$ from the space of almost periodic measures to the space of Borel measures on $\mathscr{G}$ is injective but is not onto. This motivates the second definition of almost periodic measures.

If a measure μ is a Poisson measure as defined below, then μ is an almost periodic measure and the distributional Fourier transform $\hat{\mu}$ agrees with the Fourier transform of μ.

Definition 7. A Poisson measure is an almost periodic measure μ whose distributional Fourier transform $\hat{\mu}$ is also an almost periodic measure.

Let μ be a Poisson measure and ϕ a function in the Schwartz class. Then $\tau = \hat{\mu}\hat{\phi}$ is a bounded measure and is the Fourier transform of $f = \mu * \phi$. Let us write τ as sum between a continuous component τ_1 and an atomic component τ_2. The inverse Fourier transform of τ is f. The inverse Fourier transform of τ_2 is an almost periodic function h with an absolutely convergent Fourier series. Let us show that $\tau_1 = 0$. We know that the inverse Fourier transform of τ_1 is the almost periodic function $g = f - h$. Since τ_1 is a continuous measure its inverse Fourier transform g satisfies $\mathscr{M}(|g|) = 0$. Since g is almost periodic we have $g = 0$ as announced. We just proved the following.

Lemma 21. *A Poisson measure and its distributional Fourier transform are purely atomic measures. Conversely let us assume that both μ and its distributional Fourier transform are translation bounded atomic measures. Then μ is an almost periodic measure.*

Let μ be a Poisson measure. We have $\hat{\mu} = \sum_{\omega\in S} c(\omega)\delta_\omega$. With an abuse of language we say that S is the support of $\hat{\mu}$. Then we have

Lemma 22. *If $\omega \notin S$, we have $\mathscr{M}[\exp(-i\omega\cdot x)\, d\mu(x)] = 0$. If $\omega \in S$, then $\mathscr{M}[\exp(-i\omega\cdot x)\, d\mu(x)] = \hat{\mu}(\{\omega\})$.*

In short one has $\hat{\mu}(\omega) = \hat{\mu}(\{\omega\})$ where the left-hand side is the Fourier coefficient of the Poisson measure μ at ω and the right-hand side is the mass of the atomic measure $\hat{\mu}$ at ω.

To prove Lemma 22 it suffices to consider a testing function ϕ whose Fourier transform satisfies $\hat{\phi}(\omega) = 1$ and to apply to $\phi * \mu$ the known properties of almost periodic functions with an absolutely convergent Fourier series.

A construction of Poisson measures will be given in Sections 7.11 and 7.12. The measure described in Lemma 6 is not a Poisson measure.

7.3 Generalized Almost Periodic Functions and Measures

Generalized almost periodic functions are defined now and pave the way to generalized almost periodic measures which are the right tool to analyze quasicrystals. The main theorem of [29] says that if Λ is a quasicrystal then the sum $\mu_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ of Dirac masses on Λ is a generalized almost periodic measure. Here we differ from Lagarias [12] and from Laurent Schwartz. The definition given by Schwartz is too restrictive. The measure μ_Λ is not an almost periodic distribution in the sense given by Schwartz. A generalized almost periodic measure is a Besicovitch almost periodic measure as defined by Lagarias in [12]. Therefore this result implies Lagarias theorem. Similarly for every compactly supported continuous function ϕ the function $f(x) = \sum_{\lambda\in\Lambda} \phi(x-\lambda)$ is a generalized almost periodic function. This function f is not an almost periodic function.

The notation $\mathscr{M}$ is defined by (7.4).

Definition 8. A real valued function f defined on $\mathbb{R}^n$ is a generalized almost periodic function if it is a Borel function and if for every positive ε there exist two almost periodic functions g_ε and h_ε such that

$$g_\varepsilon \leq f \leq h_\varepsilon \tag{7.24}$$

and

$$\mathscr{M}(h_\varepsilon - g_\varepsilon) \leq \varepsilon \tag{7.25}$$

It is easily checked that the collection of all g-a-p functions is a vector space $\mathscr{X}$. This vector space can be equipped with the L^∞ norm and is then a Banach space. A saddle issue will soon appear since some g-a-p functions shall be identified to 0 while they do not have a zero norm. For instance, any compactly supported continuous function f is a g-a-p function which should be identified to 0 while $\|f\|_\infty > 0$.

Returning to Definition 8 we set $\varepsilon = 1/j$ and with an obvious abuse of notations we have $g_j \leq f \leq h_j$ and $\varepsilon_j = \mathscr{M}(h_j - g_j) \to 0$. Replacing g_j by $\sup(g_1, \cdots, g_j)$

and h_j by $\inf(h_1, \cdots, h_j)$ we can further assume that g_j is an increasing sequence of almost periodic functions and that h_j is a decreasing sequence of almost periodic functions.

We now abbreviate "almost periodic" into a-p and "generalized almost periodic" into g-a-p. We have proved the following lemma.

Lemma 23. *A real valued Borel function f is a g-a-p function if and only if there exist an increasing sequence g_j of a-p functions and a decreasing sequence h_k of a-p functions such that*

(a) $g_j \le f \le h_k \quad (j, k \in \mathbb{N})$
(b) $\mathscr{M}(h_j - g_j)$ *tends to 0 as j tends to infinity.*

The reader is warned that we do not have in general $\lim_{j\to\infty} g_j(x) = f(x)$. If indeed $f(x)$ is a non-negative continuous function on the real line with a compact support, we can decide that $g_j = 0$ while g_j will be a suitable 2^j-periodic function. This example will be detailed below. Similarly we do not have in general $\lim_{k\to\infty} h_k(x) = f(x)$.

Note that a generalized almost periodic function f is not an almost periodic distribution in general. It is not true that $f * \phi$ is an almost periodic function when ϕ is a testing function. To give an example, we need to anticipate a little. Let Λ be a quasicrystal (see Section 7.3), and let us consider $f(x) = \sum_{\lambda\in\Lambda} \theta(x-\lambda)$ where Λ is a quasicristal which is not a lattice and θ is a testing function. Then $f * \phi$ cannot be almost periodic in the sense of Bohr.

A g-a-p function f belongs to $L^\infty(\mathbb{R}^n)$. Keeping the preceding notations we have $\|f\|_\infty \le \sup(\|g_1\|_\infty, \|h_1\|_\infty)$.

Theorem 2. *Let f be a g-a-p function. For every sequence $x_j \in \mathbb{R}^n$ there exist a subsequence x_{j_k} and a g-a-p function g such that*

(a) $f(x - x_{j_k}) \rightharpoonup g(x)$ *in the weak topology defined by the duality between L^∞ and L^1*
(b) $\mathscr{M}[|g(x) - f(x - x_{j_k})|] \to 0,\ k \to \infty.$

We know that $u_\varepsilon \le f \le v_\varepsilon$ where u_ε and v_ε are almost periodic functions and $\mathscr{M}(v_\varepsilon - u_\varepsilon) \le \varepsilon$. We extract subsequences in such a way that (a) $f(x - x_{j_k}) \rightharpoonup g(x)$ where the weak convergence is defined by the duality between L^∞ and L^1 together with (b) $u_\varepsilon(x - x_{j_k}) \to U_\varepsilon$ where the convergence is uniform on $\mathbb{R}^n$ and (c) $v_\varepsilon(x - x_{j_k}) \to V_\varepsilon$ where the convergence is also uniform. We still have $\mathscr{M}(V_\varepsilon - U_\varepsilon) \le \varepsilon$ and passing to the weak limits we obtain $U_\varepsilon \le g \le V_\varepsilon$. Therefore g is a g-a-p function. The proof of the second claim in Theorem 2 is immediate. We have

$$|g(x) - f(x - x_{j_k})| \le |g(x) - U_\varepsilon(x)| + |U_\varepsilon(x) - u_\varepsilon(x - x_{j_k})| + |u_\varepsilon(x - x_{j_k}) - f(x - x_{j_k})|.$$

But $|U_\varepsilon(x) - u_\varepsilon(x - x_{j_k})| \le \varepsilon$ if $k \ge k_0$ while the mean values of the two other terms in the RHS do not exceed ε. This ends the proof.

Almost periodic functions extend by continuity to the Bohr compactification $\mathscr{G}_n$ of $\mathbb{R}^n$. In a sense to be made precise the extension to $\mathscr{G}_n$ of a generalized almost periodic function is a Riemann integrable functions (see Theorem 4 below for a more precise statement). The definition of Riemann integrable functions is given for the reader's convenience:

Definition 9. A Borel function f on a compact abelian group G is Riemann integrable if and only if for every positive ε there exist two functions f_ε and g_ε which are continuous on G and satisfy the following:

(a) $f_\varepsilon \leq f \leq g_\varepsilon$ everywhere on G
(b) $\int_G (g_\varepsilon - f_\varepsilon)\,dx \leq \varepsilon$

The definition of Riemann integrable functions can be generalized to other contexts, a generalization which will not be used below. A function f is Riemann integrable if and only if the set of points $x \in G$ where f is not continuous is a null set. Returning to Definition 9 it suffices to check (a) and (b) when $\varepsilon = 1/j$, $j = 1, 2, \cdots$. Replacing f_j by $\sup(f_1, f_2, \cdots, f_j)$ we can assume that f_j is an increasing sequence of continuous functions. Similarly we can assume that g_k is a decreasing sequence of continuous functions and we have

(i) $f_j \leq f \leq g_j$ everywhere on G
(ii) $\int_G (g_j - f_j)\,dx \to 0$ as j tends to infinity.

In general we cannot expect f to be at the same time the pointwise limit of the sequence f_j and of the sequence g_j. More precisely if for every $x \in G$ we have $g_j(x) \to f(x)$ and $f_j(x) \to f(x)$, then f is a continuous function. The pointwise convergence of the sequence $f_j(x)$ to $f(x)$ holds only almost everywhere. This seems inconsistent with the fact that a Riemann integrable function is defined everywhere. If $f(x) = g(x)$ almost everywhere and f is Riemann integrable, this does not imply that g is Riemann integrable. Let us stress that a Riemann integrable function can be restricted to a set of measure 0 since it is defined everywhere.

In general a g-a-p function does not need to be Riemann integrable. A counterexample will be given below.

A complex valued function f is a generalized almost periodic function if $\Re f$ and $\Im g$ are generalized almost periodic functions. The vector space of generalized almost periodic functions is a Banach when equipped with the norm $\|f\|_\infty$. In the following lemma, $B(x,R)$ denotes the ball centered at x with radius R and c_n is the inverse of the volume of $B(0,1)$.

Lemma 24. *If f is a g-a-p function, the limit*

$$\mathscr{M}(f) = \lim_{R \to +\infty} c_n R^{-n} \int_{B(x,R)} f(y)\,dy \tag{7.26}$$

is attained uniformly in x.

The function f will be assumed to be real valued which suffices for proving the general case. One writes $M(f,R,x) = c_n R^{-n} \int_{B(x,R)} f(y)\,dy$. Then (7.4) and Lemma 23 imply

(a) $M(g_j,R,x) \le M(f,R,x) \le M(h_k,R,x)$
(b) $\lim_{R\to\infty} M(g_j,R,x) = \mathscr{M}(g_j)$
(c) $\lim_{R\to\infty} M(h_k,R,x) = \mathscr{M}(h_k)$
(d) $\mathscr{M}(h_k) - \mathscr{M}(g_j) \le \varepsilon$ if $j,k \ge j_0$.
(e) $\mathscr{M}(g_j)$ is an increasing sequence and $\mathscr{M}(h_k)$ is a decreasing sequence.

Properties (d) and (e) imply that the increasing sequence $\mathscr{M}(g_j)$ and the decreasing sequence $\mathscr{M}(h_j)$ are converging to the same limit λ and (a), (b), and (c) imply that $M(f,R,x)$ tends to λ uniformly in x as R tends to infinity.

Proposition 4. *If f is a g-a-p function, so is $|f|$. If f and g are two g-a-p functions, so are* $\sup(f,g)$ *and fg.*

The proof of the first statement is almost obvious. Let us assume that f is a g-a-p function and prove that $f^+ = \sup(f,0)$ is a g-a-p function. Indeed $g_j \le f \le h_j$ implies $g_j^+ \le f^+ \le h_j^+$ and $h_j^+ - g_j^+ \le h_j - g_j$. This yields the first claim and the second follows immediately.

Let us prove the last claim. Using the first or the second claim one can assume $f \ge 0$ and $g \ge 0$. Then we have with obvious notations $0 \le u_j \le f \le v_j$ and $0 \le g_j \le g \le h_j$ where u_j, v_j, g_j, h_j are a-p together with $\mathscr{M}(v_j - u_j) \le \varepsilon_j$, $\mathscr{M}(h_j - g_j) \le \varepsilon_j$. Then $u_j g_j \le fg \le v_j h_j$ and $v_j h_j - u_j g_j \le (v_j - u_j)\|h_1\|_\infty + (h_j - g_j)\|v_1\|_\infty$. Therefore $\mathscr{M}(v_j h_j - u_j g_j)$ tends to 0.

Corollary 1. *Let f and g be two g-a-p functions. Then the following limit exists*

$$\mathscr{M}(fg) \;=\; \lim_{R\to+\infty} c_n R^{-n} \int_{B(x,R)} f(y)g(y)\,dy \tag{7.27}$$

This follows from Lemma 24 and Proposition 4.

We need to define the Fourier coefficients of a g-a-p function f.

Definition 10. The Fourier coefficient of a g-a-p function f at the frequency $\omega \in \mathbb{R}^n$ is

$$\hat{f}(\omega) = \mathscr{M}[f(x)\exp(-i\omega\cdot x)] \tag{7.28}$$

A g-a-p function is not characterized by its Fourier coefficients. Indeed we have:

Lemma 25. *Any bounded Borel function θ with compact support is a g-a-p function. We have $\hat{\theta}(\omega) = 0$ identically.*

To prove the first claim it suffices to treat the case when θ is non-negative and vanishes outside $[-1,1]^n$. We can assume $0 \le \theta \le 1$. We simply decide that $g_\varepsilon = 0$ and construct a continuous function h_ε with the following properties:

(a) h_ε is $2/\varepsilon-$periodic in each variable
(b) $h_\varepsilon = 1$ on $[-1,1]^n$
(c) $0 \le h_\varepsilon \le 1$ and $h_\varepsilon = 0$ on $[-1/\varepsilon, 1/\varepsilon]^n \setminus [-2,2]^n$.

Then $\mathscr{M}(h_\varepsilon) \le \varepsilon^n$ which ends the proof. The second claim is obvious. This example is showing that a g-a-p function is not Riemann integrable in general. It is also showing that we cannot expect f to be the limit of g_ε or of h_ε.

The same argument shows that a continuous function vanishing at infinity is a g-a-p function. However there exists a continuous function f such that (a) f is uniformly bounded on the real line, (b) $\mathscr{M}[|f(x)|] = 0$ but (c) f is not a g-a-p function. An example will be given below.

Theorem 3. *Let f be a g-a-p function. Then for every positive ε there exists a relatively dense set Λ such that*

$$\tau \in \Lambda \Rightarrow \mathscr{M}(|f(\cdot + \tau) - f(\cdot)|) \le \varepsilon \tag{7.29}$$

The proof is straightforward. The notations of Definition 8 being kept, we have $f_\varepsilon(x) \le f(x) \le g_\varepsilon(x)$. The relatively dense set M_ε is defined by the two conditions

(a) $\|f_\varepsilon(x+\tau) - f_\varepsilon(x)\|_\infty \le \varepsilon$
(b) $\|g_\varepsilon(x+\tau) - g_\varepsilon(x)\|_\infty \le \varepsilon$

Here we use the fact that the vector valued function $(f_\varepsilon, g_\varepsilon)$ is almost periodic. We then have

$$|f(x+\tau) - f(x)| \le \sup(|g_\varepsilon(x+\tau) - f_\varepsilon(x)|, |f_\varepsilon(x+\tau) - g_\varepsilon(x)|) \tag{7.30}$$

But

$$|g_\varepsilon(x+\tau) - f_\varepsilon(x)| \le |g_\varepsilon(x+\tau) - g_\varepsilon(x)| + |g_\varepsilon(x) - f_\varepsilon(x)| \tag{7.31}$$

The first term in the RHS of (7.31) does not exceed ε as well as the mean value of the second term. The second term in the RHS of (7.30) is treated similarly.

Corollary 2. *Keeping the notations of Theorem 3 we have*

$$\tau \in \Lambda \Rightarrow \mathscr{M}(|f(\cdot + \tau) - f(\cdot)|^2) \le \varepsilon \tag{7.32}$$

Therefore a g-a-p function is almost periodic in the sense of Besicovitch.

The convolution product $f \circledcirc g$ between two g-a-p functions is defined by

$$(f \circledcirc g)(x) = \mathscr{M}[f(x - \cdot)g(\cdot)] \tag{7.33}$$

Corollary 3. *Let f and g be two g-a-p functions. Then their convolution product $f \circledcirc g$ is an almost periodic function in the sense of Bohr.*

Indeed $|(f \circledcirc g)(x+\tau) - (f \circledcirc g)(x)| \le \|g\|_\infty \mathscr{M}[|f(\cdot + \tau) - f(\cdot)|]$ and it suffices to apply Theorem 3.

Lemma 26. *Keeping the same notations we have*

$$\mathscr{M}(f \circledcirc g) = (\mathscr{M} f)(\mathscr{M} g) \tag{7.34}$$

By linearity we can assume $f, g \geq 0$. We have by Lemma 23, $u_j \leq f \leq v_j$ and $g_j \leq g \leq h_j$. Then

$$\mathscr{M}[u_j(x-\cdot)g_j(\cdot)] \leq \mathscr{M}[f(x-\cdot)g(\cdot)] \leq \mathscr{M}[v_k(x-\cdot)h_k(\cdot)].$$

In other words

$$u_j \circledcirc g_j \leq f \circledcirc g \leq v_k \circledcirc h_k.$$

Next

$$\int (u_j \circledcirc g_j)\chi_R \, dx \leq \int (f \circledcirc g)\chi_R \, dx \leq \int (v_k \circledcirc h_k)\chi_R \, dx.$$

Multiplying by $c_n R^{-n}$, passing to the limit as R tends to infinity and using (7.6) we obtain

$$\mathscr{M}(u_j)\mathscr{M}(g_j) \leq \mathscr{M}(f \circledcirc g) \leq \mathscr{M}(v_k)\mathscr{M}(h_k).$$

Finally

$$\mathscr{M}(v_k)\mathscr{M}(h_k) - \mathscr{M}(u_j)\mathscr{M}(g_j) \leq \varepsilon$$

if $j, k \geq j_0$. Indeed these four sequences are bounded and satisfy property (b) in Lemma 23. Then the proof ends as the one in Lemma 24.

Let us provide the reader with examples of generalized almost periodic functions.

The simplest example is the $2\pi-$periodic function $f = \text{sign}(\sin x)$ where $\text{sign}(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$. More generally a $T-$periodic Borel function is g-a-p if and only if it is Riemann integrable on $[0, T]$.

We continue with another example.

Lemma 27. *The function* $\Phi(x) = \text{sign}[\sin(x) + \sin(\sqrt{2}x)]$ *is a g-a-p function.*

Indeed we have $\text{sign}(x) \geq \theta_j(x)$ where

(a) $\theta_j(x) = -1$ if $x \leq 0$
(b) $\theta_j(x) = 1$ if $x \geq 1/j$
(c) $\theta_j(x) = -1 + 2jx$ if $0 \leq x \leq 1/j$.

This sequence θ_j is increasing. But we also have $\text{sign}(x) \leq \eta_j(x) = \theta_j(x + \frac{1}{j})$ and the sequence η_j is decreasing. Finally $\mathscr{M}[\eta_j(\sin(x) + \sin(\sqrt{2}x)) - \theta_j(\sin(x) + \sin(\sqrt{2}x))]$ tends to 0 as j tends to infinity. Indeed $\eta_j - \theta_j$ is supported by the interval $[-1/j, 1/j]$ and the support of $\eta_j(\sin(x) + \sin(\sqrt{2}x)) - \theta_j(\sin(x) + \sin(\sqrt{2}x))$ is $E_j = \{x; |\sin x + \sin(\sqrt{2}x)| \leq 1/j\}$. These E_j are a decreasing sequence of sets and their intersection $\cap_{j\in\mathbb{N}} E_j$ is the union of $(\sqrt{2}-1)2\pi\mathbb{Z}$ with $(\sqrt{2}+1)(\pi + 2\pi\mathbb{Z})$. Therefore the averaged measure of E_j tends to 0 as j tends to infinity and Φ is g-a-p.

This example can be generalized as follows. Let F be a Riemann integrable function on the two-dimensional torus $\mathbb{T}^2$. Then the function $f(t) = F(t, \sqrt{2}t)$ is g-a-p. Following [29], the proof will be given in a more general context. Let Γ be a dense subgroup of $\mathbb{R}^n$ and let G be the dual group of Γ. This dual group is the multiplicative group consisting of all characters on Γ. As it was said above, a character χ is a mapping $\Gamma \mapsto \mathbb{T}$ which satisfies the identity $\chi(x+y) = \chi(x)\chi(y)$ $(\forall x, y \in \Gamma)$. Then G is a compact abelian group. Each $y \in \mathbb{R}^n$ is a character on $\mathbb{R}^n$ defined by $\chi_y(x) = \exp(ix \cdot y)$. This remark implies that $\mathbb{R}^n$ can be viewed as a dense subgroup of the compact group G. Let us denote by J the canonical embedding of $\mathbb{R}^n$ into G defined by $J(y) = \chi_y$.

Lemma 28. *Let F be a Borel function on G. If F is Riemann integrable, then the function $F \circ J$ is g-a-p.*

Let us insist on the fact that a Riemann integrable function is defined everywhere. It is not a class of functions. Therefore $F \circ J$ makes sense.

We now prove Lemma 28. Definition 9 implies the following property. For every positive ε there exist two functions A_ε and B_ε such that (a), (b), and (c) hold:

(a) A_ε and B_ε are continuous on G
(b) $A_\varepsilon \leq F \leq B_\varepsilon$
(c) $\int_G (B_\varepsilon - A_\varepsilon)\, dx \leq \varepsilon$.

We set $g_\varepsilon = A_\varepsilon \circ J$ and $h_\varepsilon = B_\varepsilon \circ J$. These two almost periodic functions have the required properties and $F \circ J$ is g-a-p.

The converse statement is true. Let $\mathscr{G}_n$ be the Bohr compactification of $\mathbb{R}^n$ and $\mathscr{J}$ be the canonical embedding of $\mathbb{R}^n$ into $\mathscr{G}_n$.

Theorem 4. *Let f be a g-a-p function. Then f can be written $f = F \circ \mathscr{J} + r$ where*

(a) F is Riemann integrable on $\mathscr{G}_n$
(b) We have $\mathscr{M}(f) = \int_{\mathscr{G}_n} F(x)\, dx$
(c) More generally we have $\hat{f}(\omega) = \hat{F}(\omega)$ for every $\omega \in \mathbb{R}^n$
(d) r satisfies $\mathscr{M}(|r|) = 0$.

In the LHS of (c) $\hat{f}(\omega)$ denotes the Fourier coefficient of f viewed as a g-a-p function while $\hat{F}(\omega)$ is the ordinary Fourier coefficient of F defined on the compact abelian group $\mathscr{G}_n$. The function F is not unique.

Definition 11. With the notations of Theorem 4 we say that F is an extension of f to $\mathscr{G}_n$.

If F and F' are two such extensions of f, then $F = F'$ almost everywhere on $\mathscr{G}_n$.

If, for instance, f is a continuous function with compact support, $F = 0$ and $f = r$. For proving Theorem 4 we lift f to $\mathscr{G}_n$, i.e. we consider the auxiliary function $\tilde{f}$ defined on $\mathscr{J}(\mathbb{R}^n) \subset \mathscr{G}_n$ by $\tilde{f} \circ \mathscr{J} = f$. The a-p function f_j which is defined by (i) and (ii) of Lemma 23 extends to a continuous function F_j on $\mathscr{G}_n$ and similarly the function g_j which is defined by (i) and (ii) of Lemma 23 extends to a continuous function G_j on $\mathscr{G}_n$. We have $F_j \circ \mathscr{J} = f_j$ and $G_j \circ \mathscr{J} = g_j$. Therefore $f_j \le f \le g_k$ implies $F_j \le \tilde{f} \le G_k$ on $\mathscr{J}(\mathbb{R}^n)$. But $\mathscr{J}(\mathbb{R}^n)$ is dense in $\mathscr{G}_n$. Finally $F_j(x) \le G_k(x)$, $\forall x \in \mathscr{G}_n$. Let $F(x)$ be the pointwise limit of the increasing sequence $F_j(x)$ and let $G(x)$ be the pointwise limit of the decreasing sequence $G_k(x)$. We know that $\|G_j - F_j\|_1 = \mathscr{M}(g_j - f_j)$ tends to 0 as j tends to infinity. It implies that the function F is Riemann integrable on $\mathscr{G}_n$ and $F = G$ almost everywhere. We would have $F(x) = G(x)$ everywhere if and only if F was continuous on $\mathscr{G}_n$ and this happens if and only if the function f we started with is almost periodic in the sense of Bohr. Every point x_0 where $F(x_0) = G(x_0)$ is a point of continuity of F. Then $F(x_0)$ is the limit of $\tilde{f}(x)$ as x tends to x_0 in $\mathscr{G}_n$.

It remains to relate F to f which is not entirely obvious. The function $F \circ \mathscr{J}$ makes sense since F is defined everywhere on $\mathscr{G}_n$. We set $r = f - F \circ \mathscr{J}$ and claim that $\mathscr{M}(|r|) = 0$. Indeed $f_j \le f \le g_j$ and $f_j = F_j \circ \mathscr{J} \le F \circ \mathscr{J} \le G_j \circ \mathscr{J} = g_j$ imply $f_j - g_j \le r \le g_j - f_j$ and $|r| \le g_j - f_j$. The claim follows from $\mathscr{M}(g_j - f_j) \to 0$.

The reader has observed that in the proof the function G could have been used instead of F. Any Borel function U such that $F \le U \le G$ would have played the same role. We then say that U is an extension of f to $\mathscr{G}_n$. The proofs of (b) and (c) are straightforward. Indeed these properties hold for each pair (f_j, F_j) and it suffices to pass to the limit using Lebesgue's dominated convergence theorem on $\mathscr{G}_n$.

Lemma 29. *If f and g are two g-a-p functions and if F and G are their extensions to $\mathscr{G}_n$, then FG is an extension of fg to $\mathscr{G}_n$.*

It suffices to prove Lemma 29 when f and g are non-negative. Then the claim follows from the proof of Theorem 4.

Lemma 30. *Let f be a g-a-p function. Then we have*

$$\sum |\hat{f}(\omega)|^2 = \mathscr{M}[|f|^2] \tag{7.35}$$

The proof is straightforward. An extension of the g-a-p function $|f|^2$ is $|F|^2$ if F is an extension of f to $\mathscr{G}_n$. Property (b) in Theorem 4 implies that the RHS of (7.35) equals $\int_{\mathscr{G}_n} |F(x)|^2\, dx$ and it then suffices to use (c) and Plancherel theorem on $\mathscr{G}_n$.

Definition 12. The spectrum of a g-a-p function f is the set S of all ω such that $\hat{f}(\omega) \neq 0$.

We know from (7.35) that S is a numerable set.

Theorem 5. *Let f be a g-a-p function, let S be the spectrum of f and H be the additive subgroup of $\mathbb{R}^n$ generated by S. Then the conclusion of Theorem 4 is valid when $\mathscr{G}_n$ is replaced by the compact abelian group G which is the dual group of H.*

Let G_0 be the annihilator of H in $\mathscr{G}_n$. A character $\chi \in \mathscr{G}_n$ belongs to G_0 if and only if $\chi(x) = 1, \forall x \in H$. Then G is the quotient group $\mathscr{G}_n / G_0$. With an abuse of notation we write $\omega(\chi) = \chi(\omega)$ for every $\omega \in H$ and $\chi \in \mathscr{G}_n$. Then ω is a character on $\mathscr{G}_n$. Keeping this notation we have $\omega(x) = 1$, $(\forall x \in G_0)$, for every $\omega \in H$. For any function $u \in L^1(\mathscr{G}_n)$ we denote by $v \in L^1(\mathscr{G}_n)$ the function defined by $v(x) = \int_{G_0} u(x+y)\,dy$. For almost every $x \in \mathscr{G}_n$ we have $v(x+y) = v(x)$, $\forall y \in G_0$. Then $v(x)$ defines a function on $G = \mathscr{G}_n/G_0$. This function will also be denoted by v by an abuse of notations and we have

$$\int_{\mathscr{G}_n} u(x)\,dx = \int_G v(x)\,dx \tag{7.36}$$

If $\omega \in H$ and if $\omega(x)$, $x \in \mathscr{G}_n$, denotes the corresponding character on $\mathscr{G}_n$ we apply (7.36) to the auxiliary function $u(x)\omega(x)$. We observe that $\omega(x+y) = \omega(x)$, $(\forall y \in G_0)$ and we obtain

$$\hat{u}(\omega) = \hat{v}(\omega),\ (\forall \omega \in H) \tag{7.37}$$

We return to the proof of Theorem 5. The notations of Theorem 4 are kept and $F \in L^1(\mathscr{G}_n)$ is an extension of f. We know from (c) that the Fourier coefficients of F vanish outside H. We define three functions F'_j, F', G'_j by

(a) $F'(x) = \int_{G_0} F(x+y)\,dy$
(b) $F'_j(x) = \int_{G_0} F_j(x+y)\,dy$
(c) $G'_j(x) = \int_{G_0} G_j(x+y)\,dy$

This makes sense for every $x \in \mathscr{G}_n$. These three functions F'_j, F', and G'_j are G_0-invariant and are therefore defined on $G = \mathscr{G}_n/G_0$. We then have $\hat{F}(\omega) = \hat{F}'(\omega)$ by (7.37) for every $\omega \in H$. Moreover the functions F'_j and G'_j are continuous on G. Then $F_j(x) \le F(x) \le G_j(x)$ on $\mathscr{G}_n$ implies $F'_j(x) \le F'(x) \le G'_j(x)$ everywhere on G. Moreover $\int_G (G'_j - F'_j)\,dx \to 0$. It follows that F' is Riemann integrable on G. We set $f' = F' \circ J$ where $J : \mathbb{R}^n \mapsto G$ is the canonical embedding. It remains to prove that $\mathscr{M}(|f - f'|) = 0$. For proving this claim we observe that the two g-a-p functions f and f' have the same Fourier coefficients. This is obvious if $\omega \notin H$ since $\hat{f}(\omega) = 0 = \hat{f}'(\omega)$. If $\omega \in H$, we have $\hat{f}(\omega) = \hat{F}(\omega)$ by (c) and similarly $\hat{f}'(\omega) = \hat{F}'(\omega)$. But $\hat{F}(\omega) = \hat{F}'(\omega)$. Finally we apply the following lemma to $f - f'$.

Lemma 31. *If f is a g-a-p function and if $\hat{f}(\omega) = 0$ everywhere, then $\mathscr{M}(|f|) = 0$.*

Lemma 31 follows from Lemma 30 and concludes the proof of Theorem 5.

Lemma 32. *If f and g are two g-a-p functions and if F and G are their extensions to some compactification of $\mathbb{R}^n$, then $F * G$ is an extension of $f \circledcirc g$.*

The argument used in Lemma 29 applies here.

The conclusion of Lemma 4 is still valid for g-a-p functions as it will be proved now.

Lemma 33. *If f and g are two g-a-p functions, the Fourier coefficient of $f \circledcirc g$ at the frequency ω is the product $a(\omega)b(\omega)$ between $a(\omega) = \hat{f}(\omega)$ and $b(\omega) = \hat{g}(\omega)$.*

We write $f_\omega(x) = \exp(-i\omega \cdot x)$ and use the same notations for g and $h = f \circledcirc g$. Then it suffices to observe that $h_\omega = f_\omega \circledcirc g_\omega$ and to apply Lemma 26.

Corollary 4. *If f and g are two g-a-p functions, then the Fourier series of $f \circledcirc g$ is absolutely convergent.*

Indeed the Fourier coefficient of $f \circledcirc g$ at the frequency ω is the product $c(\omega)$ between $a(\omega) = \hat{f}(\omega)$ and $b(\omega) = \hat{g}(\omega)$. But $a, b \in l^2$ implies $c \in l^1$.

The error term r in Theorem 4 satisfies $\mathscr{M}[|r|] = 0$. This raises the following issue. Let f be a continuous function on the real line satisfying $\|f|\|_\infty \le C$ and $\mathscr{M}[|f|] = 0$. Is it a g-a-p function ? The answer is no, as the following example shows.

Proposition 5. *Let $\Lambda \subset \mathbb{R}$ be the increasing sequence $\lambda_k = k^2 + k\sqrt{2}$, $k \in \mathbb{N}$. Let ϕ be a non-negative compactly supported continuous function with $\phi(0) = 1$. Let us consider $f = \sum_{\lambda \in \Lambda} \phi(x - \lambda)$. Then f is not a g-a-p function.*

The proof relies on the fact that Λ is dense in the Bohr compactification $\mathscr{G}$ of $\mathbb{R}$. We will prove a stronger statement

Lemma 34. *For every almost periodic function u we have*

$$\sigma_N(u) = \frac{1}{N} \sum_{1 \le k \le N} u(\lambda_k) \to \mathscr{M}(u) \tag{7.38}$$

For proving Lemma 34, it suffices by density to treat the case where $u(x) = \exp(i\omega x)$. We use a theorem by van der Corput saying that if P is a polynomial with at least one irrational coefficient (other than the constant term) then the sequence $P(n)$ is uniformly distributed modulo 1. If $u(x) = \exp(i\omega x)$, we have $\sigma_N(u) = A_N(\omega)$ where $A_N(\omega) = \frac{1}{N}\sum_{1 \le k \le N} \exp(i\omega(k^2 + k\sqrt{2}))$. At least one among the two numbers $\omega, \omega\sqrt{2}$ is irrational and van der Corput's theorem applies. Then $A_N(\omega)$ tends to 0 for $\omega \neq 0$.

We now return to Proposition 5 and argue by contradiction. Let us assume $g_\varepsilon \le f \le h_\varepsilon$ with $\mathscr{M}[h_\varepsilon - g_\varepsilon] \le \varepsilon$. Then $h_\varepsilon \ge 1$ on Λ which implies $h_\varepsilon \ge 1$ everywhere by density. But $\mathscr{M}[h_\varepsilon - g_\varepsilon] \ge \mathscr{M}[h_\varepsilon - f] \ge \mathscr{M}[1 - f]$. We reach a contradiction since $\mathscr{M}[f] = 0$.

Here is a second counterexample. We denote by $\theta > 2$ a real number which is not a Pisot number. For instance, $\theta = 5/2$. We define Λ_θ as the set consisting of all finite sums $\sum_{k \ge 0} \varepsilon_k \theta^k$ where $\varepsilon_k \in \{0, 1\}$.

Lemma 35. *Λ_θ is dense in the Bohr compactification of $\mathbb{R}$ and the Beurling density of Λ_θ is 0.*

Lemma 35 will be proved below. Then the argument used for the sequence $k^2 + k\sqrt{2}$ works if we replace it by Λ_θ and Proposition 5 is still true when $\Lambda = \Lambda_\theta$.

7.3.1 Generalized Almost Periodic Measures

Definition 13. A real valued Borel measure μ is a generalized almost periodic measure on $\mathbb{R}^n$ (a g-a-p measure) if the following property holds:

For every $\varepsilon > 0$ there exist two almost periodic measures μ_ε and ν_ε such that

$$\mu_\varepsilon \leq \mu \leq \nu_\varepsilon \tag{7.39}$$

and

$$\mathscr{M}(\nu_\varepsilon - \mu_\varepsilon) \leq \varepsilon \tag{7.40}$$

We aim at relating the arithmetical properties of a Delone set $\Lambda \subset \mathbb{R}^n$ to the analytical properties of the corresponding measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$.

Definition 14. We say that Λ is an almost periodic pattern if the corresponding measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a *generalized almost periodic measure*.

Before moving further let us give a motivating example. Let $\theta > 2$ be a real number and let Λ_θ be the set consisting of all finite sums $\sum_{k\geq 0} \varepsilon_k \theta^k$ with $\varepsilon_k \in \{0,1\}$. With these notations we have

Theorem 6. *The set Λ_θ is an almost periodic pattern if and only if θ is a Pisot-Vijayaraghavan number.*

The definition of Pisot numbers will be given in the next section. Let us first prove the easy part of Theorem 6, following [29]. If θ is not a Pisot number, then the sequence $P_j(x) = \cos(x)\cos(x\theta)\cdots\cos(x\theta^{j-1})$ tends to 0 for every $x \neq 0$. This follows from the following theorem by Charles Pisot:

Theorem 7. *The following two properties are equivalent ones*

(a) There exists a real number $\alpha \neq 0$ such that $\alpha\theta^j = m_j + \varepsilon_j,\ j \in \mathbb{N}$, where $m_j \in \mathbb{Z}$ and $\varepsilon_j \in l^2$
(b) θ is a Pisot number.

Let $\mathscr{G}$ be the Bohr compactification of the real line and let $\tilde{\mu}_j$ be the image of the measure $2^{-j}\mu_j$ by the canonical embedding $\mathscr{J} : \mathbb{R} \mapsto \mathscr{G}$.

Lemma 36. *As j tends to infinity the measures $\tilde{\mu}_j$ weakly converge to the Haar measure on $\mathscr{G}$.*

Let $c_j(\omega)$ be the Fourier coefficients of $\tilde{\mu}_j$. We then have $|c_j(\omega)| = P_j(\omega/2)$. This remark and Theorem 7 imply the weak convergence of the probability measures $\tilde{\mu}_j$ to the Haar measure on $\mathscr{G}$. Lemma 36 is proved.

Therefore Λ_θ is dense in $\mathscr{G}$. We return to the proof of Theorem 6 and argue by contradiction. Let us assume that θ is not a Pisot number and that σ_θ is a g-a-p measure. We then follow the lines of the proof of Proposition 5. We denote by ϕ a non-negative continuous function supported by $[-1,1]$ such that $\phi(0) = 1$. Then $\sigma_\theta * \phi$ is a g-a-p function. There exist two a-p functions u_ε and v_ε such that

$u_\varepsilon \le \sigma_\theta * \phi \le v_\varepsilon$ with $\mathscr{M}(v_\varepsilon - u_\varepsilon) \le \varepsilon$. The definition of ϕ implies $\sigma_\theta * \phi \ge 1$ on Λ_θ. Since Λ_θ is dense in $\mathscr{G}$ we have $v_\varepsilon \ge 1$ everywhere. On the other hand, $u_\varepsilon \le \sigma_\theta * \phi$ which implies $u_\varepsilon^+ \le \sigma_\theta * \phi$. The density of Λ_θ is 0 which implies that the mean value of $\sigma_\theta * \phi$ is also 0. Therefore $\mathscr{M}(u_\varepsilon^+) = 0$. This yields the required contradiction since we have proved that $\mathscr{M}(v_\varepsilon - u_\varepsilon) \ge 1$.

Another proof is provided by the following lemma.

Lemma 37. *Let us assume that an almost periodic measure μ exists such that $\mu \ge \sigma_\theta$. Then θ is a Pisot number. Conversely if θ is a Pisot number such an almost periodic measure exists.*

If the hypothesis is relaxed and if one assumes that μ is a g-a-p measure such that $\mu \ge \sigma_\theta$, then the conclusion will be the same. Indeed there exists an almost periodic measure ρ such that $\rho \ge \mu \ge \sigma_\theta$.

We now prove Lemma 37. Once more let $\phi = \phi_\alpha$ be a continuous function of the real variable x supported by $[-\alpha, \alpha]$, such that $\phi(0) = 1$ and $0 \le \phi(x) \le 1$. We have $\mu \ge \sigma_\theta$ which implies $\mu * \phi(x) \ge \sum_{\lambda\in\Lambda_\theta} \phi(x-\lambda)$. Therefore $\mu * \phi(x) \ge 1$ on Λ_θ and $\mu * \phi(x) \ge 1$ everywhere if θ is not a Pisot number. Indeed $\mu * \phi$ is an almost periodic function and Λ_θ is dense in the Bohr compactification of the real line. We now let α tend to 0 and we have $\mu * \phi_\alpha(x) \to \mu\{x\}$ everywhere. We arrive at a contradiction since $\mu\{x\} \ge 1$ everywhere is impossible.

If θ is a Pisot number, then Λ_θ is contained in a model set M and the measure $\mu = \sum_{\lambda\in M} \delta_\lambda$ is a g-a-p measure (Corollary 13 of Theorem 24, Section 7.11). We obviously have $\mu \ge \sigma_\theta$.

We return to the proof of the second claim in Theorem 6. One has for every $j \ge 1$, $\Lambda_\theta = \theta^j \Lambda_\theta + F_j$ where the cardinality of F_j is 2^j. Let $\mu(x)$ be the almost periodic measure of Lemma 37. With an obvious abuse of notations we define $\tau_j(x) = \theta^{-j} \sum_{\lambda\in F_j} \mu(\theta^{-j}(x-\lambda))$. We then have $\tau_j \ge \sigma_\theta$ together with $\mathscr{M}(\tau_j) = C2^j\theta^{-j}$ which tends to 0 as j tends to infinity. Therefore the definition of a g-a-p measure is satisfied with $\tau_j \ge \sigma_\theta \ge 0$.

Let us return to the properties of g-a-p measures.

Lemma 38. *If μ is a g-a-p measure and f is an almost periodic function, the product $f\mu$ is a g-a-p measure.*

The proof is obvious if f is non-negative. The general case follows from the fact that any almost periodic function f can be written $f = f_1 - f_2$ where f_1 and f_2 are non-negative and almost periodic. This argument does not work if f is a g-a-p function.

Lemma 39. *If μ is a g-a-p measure, then*

$$\mathscr{M}(\mu) = \lim_{R\to+\infty} c_n R^{-n} \mu(B(x_0, R) \qquad (7.41)$$

exists uniformly in $x_0 \in \mathbb{R}^n$ and we have

$$\mathcal{M}(\mu) = \lim_{\varepsilon \to 0} \mathcal{M}(\mu_\varepsilon) \tag{7.42}$$

when μ_ε is defined by (7.39).

The proof is the same as the one of Lemma 24.

Corollary 5. *Let μ be a g-a-p measure. Then for every almost periodic function f the limit*

$$\mathcal{M}(f\mu) \;=\; \lim_{R \to +\infty} c_n R^{-n} \int_{B(x_0,R)} f\, d\mu \tag{7.43}$$

exists uniformly in $x_0 \in \mathbb{R}^n$.

It suffices to combine Lemma 38 and Lemma 39.

Definition 15. Let μ be a g-a-p measure. The Fourier coefficients $\hat{\mu}(\omega)$, $\omega \in \mathbb{R}^n$ are defined by $\mathcal{M}(e_\omega \mu)$ where $e_\omega(x) = \exp(-i\omega \cdot x)$.

The following remark will be used in Section 7.11. We have

Lemma 40. *With the notations of Definition 13 we have*

$$\hat{\mu}(\omega) = \lim_{\varepsilon \to 0} \hat{\mu}_\varepsilon(\omega) \tag{7.44}$$

If moreover the measures $\hat{\mu}_\varepsilon$ are Poisson measures, we have

$$\hat{\mu}(\omega) = \lim_{\varepsilon \to 0} \hat{\mu}_\varepsilon(\{\omega\}) \tag{7.45}$$

This follows from Lemma 39.

In Section 7.11 will be treated some examples where the measures μ_ε are Poisson measures and the computation of $\hat{\mu}_\varepsilon(\omega)$ will be trivial.

Lemma 41. *If μ is a g-a-p measure and if f is continuous with compact support, then the convolution product $\mu * f$ is a g-a-p function.*

The proof is immediate if f is non-negative. The general case follows since every compactly supported continuous function f is a difference $f = f_1 - f_2$ between two non-negative ones. This leads to the following definition.

Definition 16. A translation bounded measure μ is a weakly g-a-p measure if $\mu * f$ is a g-a-p function for every compactly supported continuous function f.

Proposition 6. *Let μ be a weakly g-a-p measure and let f be an almost periodic function. Then the limit*

$$\mathcal{M}(f\mu) \;=\; \lim_{R \to +\infty} c_n R^{-n} \int_{B(x_0,R)} f\, d\mu \tag{7.46}$$

exists uniformly in $x_0 \in \mathbb{R}^n$.

The proof of Proposition 6 relies on Corollary 6 of Lemma 42 which is stated now.

Lemma 42. *Let μ be a translation bounded measure, f be a function in $L^\infty(\mathbb{R}^n)$ and g be a continuous function with a compact support. For $R > 0$ we denote by μ_R the product $\mu\chi_R$ where χ_R is the indicator function of the closed ball $B_R = \{x; |x| \leq R\}$. Then*

$$\int (f * g)\, d\mu_R = \int (\mu * \check{g})\chi_R f\, dx + O(R^{n-1}) \tag{7.47}$$

Here $\check{g}(x) = g(-x)$. The LHS is $I = \int\int \chi_R(x) f(y) g(x-y)\, d\mu(x) dy$ and the integral in the RHS is $J = \int\int \chi_R(y) f(y) g(x-y)\, d\mu(x) dy$. We have $|x-y| \leq C_0$ which is given by the compact support of g. The triangle inequality yields

$$|I - J| \leq \int\int_{\{-C_0+R \leq |x| \leq C_0+R\}} |f(y) g(x-y)|\, d|\mu|(x) dy = O(R^{n-1}).$$

Corollary 6. *The notations of Lemma 42 being kept, if one of the limit exists we have*

$$\mathscr{M}[(f * g)\mu] = \mathscr{M}[(\mu * \check{g})f] \tag{7.48}$$

We return to the proof of Proposition 6. Almost periodic functions are uniform limits of finite trigonometric sums and a standard argument shows that it suffices to prove Proposition 6 when $f(x) = \exp(i\omega \cdot x)$. Then $f = f * \phi$ for a suitable testing function ϕ. We assume that ϕ is even and we use Corollary 6. We obtain $\mathscr{M}[(f * \phi)\mu] = \mathscr{M}[(\mu * \phi) f]$ if the RHS exists. But $\mu * \phi$ is a g-a-p function and it now suffices to use Corollary 5.

Here is an example of a weakly g-a-p measure which is not a g-a-p measure. We consider in one dimension the series $\sigma = \sum_{k \in \mathbb{Z}} (\delta_{k+\varepsilon_k} - \delta_k)$ where ε_k tends to 0 as k tends to infinity. We then have

Lemma 43. *The measure σ is a weakly g-a-p measure but is not a g-a-p measure.*

The measure σ is translation bounded. If ϕ is a testing function the convolution product $g = \phi * \sigma$ is a continuous function tending to 0 at infinity. Therefore g is a g-a-p function. We need to prove that σ is not a g-a-p measure. The first and immediate observation is $\mathscr{M}(f\sigma) = 0$ for every almost periodic function f. This is obvious since $f(k+\varepsilon_k) - f(k)$ tends to 0 as k tends to infinity. We then argue by contradiction and assume that $\mu_\varepsilon \leq \sigma \leq \nu_\varepsilon$ where μ_ε and ν_ε are close. If f is a non-negative almost periodic function we obtain $\mathscr{M}[f\mu_\varepsilon] \leq 0 \leq \mathscr{M}[f\nu_\varepsilon]$.

Lemma 12 implies that $\nu_\varepsilon \geq 0$ and $\mu_\varepsilon \leq 0$. Therefore $\sigma \leq \nu_\varepsilon$ implies $\tau = \sum_{k \in \mathbb{Z}} \delta_{k+\varepsilon_k} \leq \nu_\varepsilon$ while $\sigma \geq \mu_\varepsilon$ implies $\rho = -\sum_{k \in \mathbb{Z}} \delta_k \geq \mu_\varepsilon$. Finally $\mathscr{M}(\nu_\varepsilon - \mu_\varepsilon) \leq \varepsilon$ implies $\mathscr{M}(\tau - \rho) \leq \varepsilon$ which is not the case.

If μ is a weakly g-a-p measure, the linear functional $f \mapsto \mathscr{M}(f\mu)$ is continuous on the space of all almost periodic functions. Therefore this functional defines a Borel measure $\tilde{\mu}$ on the Bohr compactification $\mathscr{G}$ of $\mathbb{R}^n$. The mapping $\mu \mapsto \tilde{\mu}$ is not injective.

Definition 17. The Fourier coefficients $\hat{\mu}(\omega)$ of a weakly g-a-p measure μ are defined by (7.46) for $f(x) = \exp(-i\omega \cdot x), \forall \omega \in \mathbb{R}^n$. The spectrum of μ is the numerable set of $\omega \in \mathbb{R}^n$ such that $\hat{\mu}(\omega) \neq 0$.

A weakly g-a-p measure μ is not uniquely defined by its Fourier coefficients $\hat{\mu}(\omega)$. However these Fourier coefficients uniquely determine $\tilde{\mu}$ which is a Borel measure on the Bohr compactification $\mathscr{G}_n$ of $\mathbb{R}^n$. More precisely the Fourier series expansion of $\tilde{\mu}$ is given by

$$\tilde{\mu}(\chi) = \sum \hat{\mu}(\omega)\chi(\omega).$$

We now define the convolution product between two generalized almost periodic measures.

Lemma 44. *Let μ be a compactly supported measure and ν be a translation bounded measure. Let u and v be two compactly supported continuous functions. We then have*

$$\int (u*v)\, d(\mu*\nu) = \int (\check{u}*\mu)(\nu*\check{v})\, dx \tag{7.49}$$

The trivial identities $\int f(x)g(x)\, dx = (f*\check{g})(0)$ and $(f*g)(-x) = \check{f}*\check{g}$ are applied to $f = \check{u}*\mu$, $g = \nu*\check{v}$ and this ends the proof.

As above the product between the measure μ and the indicator function of the ball $B(0,R)$ is denoted by μ_R and c_n is the inverse of the volume of the unit ball $B_n \subset \mathbb{R}^n$. We then have

Theorem 8. *Let μ and ν be two weakly g-a-p measures. Then the convolution products $\tau_R = c_n R^{-n}\mu_R * \nu$ converge weakly to an almost periodic measure $\mu \circledcirc \nu$: for every compactly supported continuous function g, the convolution product $(\mu \circledcirc \nu)*g$ is an almost periodic function in the sense of Bohr.*

We first observe that the measures τ_R are uniformly translation bounded. To prove the weak convergence it suffices to replace g by an approximation $u*v$ where u and v are two compactly supported continuous functions. We then prove the convergence of $I(R) = \int (u*v)\, d\tau_R$ as R tends to infinity. Lemma 44 yields $I(R) = \int (\mu_R*\check{u})(\check{v}*\nu)\, dx$. Using Lemma 42 we have $I(R) = \int (\mu*\check{u})(\check{v}*\nu)\chi_R\, dx + O(R^{-n+1})$ and finally Proposition 6 implies the first claim. The proof of the second claim is similar. The measure $\mu \circledcirc \nu$ is translation bounded and it suffices to prove the claim when $g = u*v$. Then $(\mu \circledcirc \nu)*(u*v) = (\mu*u) \circledcirc (\nu*v)$ and Corollary 4 implies Theorem 8.

Model sets will be defined in Section 7.7. The following theorem will be proved in Section 7.11.

Theorem 9. *The sum $\sigma_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$ of Dirac masses on a model set Λ is a generalized almost periodic measure.*

Theorem 8 implies that the autocorrelation measure $\tau = \sigma_\Lambda * \tilde{\sigma}_\Lambda$ is an almost periodic measure. This measure τ is a Poisson measure which will be computed explicitly in Section 7.12.

7.4 Diffraction Measures

Physicists (see [41]) are not interested in the distributional Fourier transform of an almost periodic function f but rather in describing how the energy of f is distributed in the frequency domain. Before computing this distribution let us start with a few trivial remarks. If $f \in L^1(\mathbb{R}^n)$ and if $\tilde{f}(x) = \overline{f}(-x)$, the autocorrelation of f is defined as $(f * \tilde{f})(x) = \int_{\mathbb{R}^n} f(x+y)\overline{f}(y)\, dy$. Taking the Fourier transforms we then have

$$\mathscr{F}(f * \tilde{f}) = |\hat{f}|^2 \tag{7.50}$$

This convolution product $f * \tilde{f}$ does not make any sense if f is an almost periodic function. If, for instance, $f = 1$, $f * \tilde{f}$ is infinite. In the frequency domain it would amount to compute the square of the Dirac mass δ_0. That is why Nobert Wiener proposed a new definition of the autocorrelation $\phi = f * \tilde{f}$ when f belongs to $L^\infty(\mathbb{R}^n)$. In his remarkable book [42] Wiener focused on functions of one real variable and defined two classes $S \subset L^\infty(\mathbb{R})$ and $S' \subset L^\infty(\mathbb{R})$. A function f belongs to S if the limit

$$\phi(x) = \lim_{T \to +\infty} \frac{1}{2T} \int_{T}^{T} f(x+y)\overline{f}(y)\, dy \tag{7.51}$$

exists for every $x \in \mathbb{R}$ and f belongs to S' if moreover $\phi(x)$ is continuous on the real line. Wiener proved that whenever f belongs to S' the Fourier transform of ϕ is a non-negative measure σ with a finite mass. In other words $\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iux) d\sigma(u)$. Wiener writes:

> "As σ determines the energy distribution in the spectrum of f we may call it briefly the "spectrum" of f. The author sees no compelling reason to avoid a physical terminology in pure mathematics when a mathematical concept corresponds closely to a concept familiar in physics."

The definition of the Wiener spectrum given in [24] is incorrect and differs from the one given by Wiener. The spectrum in [24] was defined as the closure of the support of the measure σ. The spectrum of an almost periodic function f is the numerable set S of frequencies which are present in the Fourier series of f while it would be defined in [24] as the closure of S.

Wiener gave the following examples in [42], p. 151. If $f(x) = \exp(ix^2/2)$, then $\phi(0) = 1$ while $\phi(x) = 0$ when $x \neq 0$. Therefore $\exp(ix^2/2)$ belongs to S but not to S'. This example shows that the Fourier transform of f and the Fourier transform $\hat{\phi}$ of the autocorrelation of f are unrelated in general. When $f(x) = \exp(ix^2/2)$, we have $\hat{f}(\xi) = \sqrt{2\pi}\exp(-i\xi^2/2)$ while $\phi(x) = 0$ almost everywhere. Therefore $\hat{\phi}(\xi) = 0$, a result which differs from the expected $|\hat{f}(\xi)|^2 = 2\pi$. If $f(x) = \omega_k$ on $[k, k+1)$, $k \in \mathbb{Z}$ and if the random variables ω_k are i.i.d. and equidistributed in $[-1, 1]$, then $\phi(x) = 1 - |x|$ on $[-1, 1]$ and 0 outside. If $f(x) = \exp(i\sqrt{|x|})$, then $\phi(x) = 1$ identically.

Wiener did not treat the case where the limit in (7.51) does not exist. We follow [24] in the general case $f \in L^\infty(\mathbb{R}^n)$. We then define for $R > 0$, f_R by $f_R(x) = f(x)$ if $|x| \leq R$ and 0 otherwise. Let c_n be the inverse of the volume of the unit ball.

Definition 18. An autocorrelation function ϕ of f is defined as

$$\phi = \lim_{R_j \to \infty} c_n R_j^{-n} f_{R_j} * \tilde{f}_{R_j} \tag{7.52}$$

where R_j is any sequence for which the right-hand side converges in the topology $\sigma(L^\infty, L^1)$.

Let us observe that if $f \in L^\infty(\mathbb{R}^n)$, then $c_n R^{-n} \|f_R * \tilde{f}_R\|_\infty \leq \|f\|_\infty^2$. The unit ball of $L^\infty(\mathbb{R}^n)$ is compact when it is equipped with the topology $\sigma(L^\infty, L^1)$. This is why Definition 18 makes sense.

The limit would be the same in (7.52) if $F_R = f_{R_j} * \tilde{f}_{R_j}$ was replaced by $G_R = f_{R_j} * \tilde{f}$. Indeed $G_R - F_R$ converges to 0 uniformly on compact sets when R tends to infinity as the following lemma shows.

Lemma 45. *If u, v belong to $L^\infty(\mathbb{R}^n)$, if $u(x) = 0$ for $|x| \geq R$ and $v(x) = 0$ for $|x| \leq R$, then $|(u * v)(x)| \leq C_n |x| R^{n-1} \|u\|_\infty \|v\|_\infty$.*

We have $(u*v)(x) = \int_A u(x-y)v(y)\,dy$ where $A = \{y;\, |y| \geq R,\, |x-y| \leq R\}$. It implies $R - |x| \leq |y| \leq R$. The measure of A is $v_n(R^n - (R-|x|)^n = O(|x|R^{n-1})$ where v_n is the volume of the unit ball.

There are in general several autocorrelation functions of a given $f \in L^\infty(\mathbb{R}^n)$. Let us assume, for instance, $n = 1$ and let q_j, $j \in \mathbb{N}$, be an increasing sequence of integers such that $q_0 = 1$ and $q_{j+1}/q_j \to \infty$ as $j \to \infty$. Let $f(x)$ be the step function defined by $f(x) = 0$ when $q_{2j} \leq |x| < q_{2j+1}$ and $f(x) = 1$ when $q_{2j+1} \leq |x| < q_{2j+2}$. Then any constant $c \in [0, 1]$ is an autocorrelation function of f.

The convergence of the sequence of functions $c_n R_j^{-n} f_{R_j} * \tilde{f}_{R_j}$ for the topology $\sigma(L^\infty, L^1)$ implies the convergence in the distributional sense. Therefore the Fourier transforms $c_n R_j^{-n} |\hat{f}_{R_j}|^2$ also converge in the distributional sense. By construction we have $c_n R_j^{-n} \int_{\mathbb{R}^n} |\hat{f}_{R_j}|^2(\xi)\,d\xi \leq (2\pi)^n \|f\|_\infty^2$ which implies that the weak limit of the sequence $c_n R_j^{-n} |\hat{f}_{R_j}|^2$ is a non-negative Borel measure σ with a finite total mass.

Definition 19. Each such measure σ is a diffraction measure of f. The inverse Fourier transform of σ is ϕ.

In [24] Yves Meyer defines the spectrum S of f as being the closed support of the measure σ, which following Meyer [29] it's a mistake. Wiener did not make that mistake. Indeed if the Meyer definition was followed, the spectrum of an almost periodic function would not be a numerable set, but its closure. Let us be more specific. The autocorrelation of an almost periodic function f can be given three equivalent definitions. The first definition uses the definition of the convolution product between two almost periodic functions which is given by $\phi = f \circledcirc \tilde{f}$. The second definition is (7.52). A third definition is given by

$$\phi = \lim_{R\to\infty} (c_n R^{-n}) f_R * \tilde{f} \tag{7.53}$$

Then for an almost periodic function f the RHS of (7.53) converges uniformly to ϕ on $\mathbb{R}^n$ which would be impossible if (7.52) was used. The three definitions of ϕ are equivalent ones when f is almost periodic.

We then have

Proposition 7. *Let f be an almost periodic function and S denote the spectrum of f. The autocorrelation ϕ of f is an almost periodic function and its Fourier transform $\hat{\phi}$ is the atomic measure $\sigma = (2\pi)^n \sum_{\lambda\in S} |\hat{f}(\lambda)|^2 \delta_\lambda$ where δ_λ is the Dirac mass at λ. The total mass of σ is finite and equals $(2\pi)^n \mathscr{M}(|f|^2)$.*

We now turn to the autocorrelation $\mu \circledcirc \tilde{\mu}$ of a weakly g-a-p measure. Theorem 8 implies the following result:

Theorem 10. *The autocorrelation $\tau = \mu \circledcirc \tilde{\mu}$ of a weakly g-a-p measure μ is an almost periodic measure. The Fourier transform in the distributional sense of $\mu \circledcirc \tilde{\mu}$ is the non-negative atomic measure $(2\pi)^n \sum_{\omega\in S} |\hat{\mu}(\omega)|^2 \delta_\omega$ where $\hat{\mu}(\omega)$ are the Fourier coefficients of μ.*

The total mass of this atomic measure is infinite.

Let us treat a simple one dimensional example where λ_j, $j \in \mathbb{Z}$, is a strictly increasing sequence of real numbers. We denote by Λ the set $\{\lambda_j, j \in \mathbb{Z}\}$ and we consider the measure $\mu = \sum_{-\infty}^{+\infty} \delta_{\lambda_j}$. Is it a g-a-p measure ? What is its spectrum ? We will try to address these two issues.

It will be assumed that Λ is a Delone set : $\inf_{j\in\mathbb{Z}}(\lambda_{j+1} - \lambda_j) > 0$ and $\sup_{j\in\mathbb{Z}} (\lambda_{j+1} - \lambda_j) < \infty$. We then consider the partial sum $\mu_R = \sum_{|\lambda_j|\le R} \delta_{\lambda_j}$ and its Fourier transform $P_R(\omega) = \hat{\mu}_R(\omega) = \sum_{|\lambda_j|\le R} \exp(-i\omega\lambda_j)$. Let us study the sequence $\frac{|P_R(\omega)|^2}{2R}$. Since Λ is a Delone set we have for any interval I of length $|I| \ge 1$,

$$\int_I |P_R(\omega)|^2 \, d\omega \le C(I)R \tag{7.54}$$

where $C(I)$ only depends on the length $|I|$ of I. It implies that there exists a sequence R_k tending to infinity such that the measures σ_k defined by the densities $\frac{|P_{R_k}(\omega)|^2}{R_k} d\omega$ converge to a limit σ as k tends to infinity.

If μ is a g-a-p measure, the subsequence R_k is not needed to insure the required convergence. The spectrum of the g-a-p measure μ is the set S of frequencies ω such that $\hat{\mu}(\omega) = \lim_{R\to\infty} \frac{P_R(\omega)}{2R} \neq 0$. Then Theorem 10 implies that the weak limit of the sequence $\frac{|P_R(\omega)|^2}{2R}$ is the atomic measure $\sum_{\omega\in S} |\hat{\mu}(\omega)|^2 \delta_\omega$. Finally we found a necessary condition for μ to a g-a-p measure. Here is the algorithm. We simultaneously study the sequence $\eta_R(\omega) = \lim_{R\to\infty} \frac{P_R(\omega)}{2R}$ and the sequence $\theta_R(\omega) = \frac{|P_R(\omega)|^2}{2R}$. The sequence $\eta_R(\omega)$ shall converge pointwise to a limit $c(\omega)$. We denote by S the set of ω for which $c(\omega) \neq 0$. Then the sequence $\theta_R(\omega)$ shall converge weakly to the limit $\sum_{\omega\in S} |c(\omega)|^2 \delta_\omega$.

7.5 Pisot-Vijayaraghavan Numbers and Salem Numbers

If Λ is a quasicrystal and if for a real number $\theta > 1$ we have $\theta\Lambda \subset \Lambda$, then θ is a Pisot-Vijayaraghavan number or a Salem number. Following [29], this will be proved in Section 7.8, Theorem 17. For the time being let us define these remarkable algebraic integers.

Definition 20. A Pisot-Vijayaraghavan number is a real number $\theta > 1$ with the following two properties:

(a) θ is an algebraic integer of degree $n \geq 1$
(b) the $n-1$ conjugates $\theta_2, \cdots, \theta_n$ of θ satisfy

$$|\theta_2| < 1, \cdots, |\theta_n| < 1. \tag{7.55}$$

For example, the natural integers $2, 3, \cdots$ are Pisot-Vijayaraghavan numbers and condition (b) is vacuous in that case. When the degree n of θ exceeds 1, the minimal polynomial of θ is $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ where $a_1 \in \mathbb{Z}, \cdots, a_n \in \mathbb{Z}$. Then the conjugates $\theta_2, \cdots, \theta_n$ of θ are the other solutions to this equation and can be either real or complex numbers. Raphaël Salem proved that the set S of all Pisot numbers is closed. The smallest Pisot number $\rho = 1.324717\cdots$ is named the plastic number and is the real solution to the equation $x^3 - x - 1 = 0$. The two other solutions z_1 and z_2 to this equation are complex numbers. They satisfy $z_1 = \bar{z}_2$ and $z_1 z_2 = |z_1|^2 = |z_2|^2 = 1/\rho$ which is fully consistent with the fact that ρ is a Pisot number.

Salem numbers (see [37]) are defined in the same way as follows: in Definition 20, the condition (a) is unchanged and condition (b) is replaced by $|\theta_2| \leq 1, \cdots, |\theta_n| \leq 1$ with, at least, equality somewhere. Then the degree n of θ is even. Up to some permutation between the conjugates we always have $\theta_2 = \frac{1}{\theta}$ and $|\theta_3| = \cdots = |\theta_n| = 1$.

7.6 Additive Properties of Almost Lattices

The material which is presented now can either be found in the Meyer's book [27] or in some remarkable papers by F. Lagarias or R. Moody. See ([13–15, 30–32] and

[33]) Quasicrystals will be given several definitions in this essay. These definitions are closely related but are not equivalent ones. The first definition which can be found in [27] or [28] is stressing the connection between a quasicrystal and a lattice. As it was said above a lattice $\Lambda \subset \mathbb{R}^n$ is an additive discrete subgroup with a compact quotient. A subset Λ of $\mathbb{R}^n$ is a subgroup if and only if $\Lambda - \Lambda \subset \Lambda$ where $\Lambda - \Lambda$ is the collection of all $\lambda - \lambda'$, $\lambda \in \Lambda$, $\lambda' \in \Lambda$ and $\subset$ denotes inclusion. A lattice Λ is also a Delone set. Let us define Delone (or Delaunay) sets.

Definition 21. A subset Λ of $\mathbb{R}^n$ is a Delone set if there exist two radii $R_2 > R_1 > 0$ such that

(a) each ball with radius R_1, whatever be its location, shall contain at most one point in Λ
(b) each ball with radius R_2, whatever be its location, shall contain at least one point in Λ.

Given $R_2 > R_1 > 0$ the collection of all Delone sets satisfying (a) and (b) will be denoted by $\mathscr{D}(R_2, R_1)$.

The first requirement can be given the following equivalent formulation: there exists a positive number r such that $\forall \lambda \in \Lambda$, $\forall \lambda' \in \Lambda$ and $\lambda \neq \lambda'$ we have $|\lambda' - \lambda| \geq r$. A collection of points fulfilling the second condition is relatively dense.

If Λ is a Delone set and F is finite, it is not always true that $\Lambda + F$ is a Delone set. Let us discuss this property. Let Λ be a Delone set and for every $R > 0$ and $y \in \Lambda$ let $\Lambda_{y,R}$ be the intersection between $\Lambda - y$ and the closed ball $|x| \leq R$. These $\Lambda_{y,R}$ are named a $R-$patch of Λ and the corresponding $\Lambda^*_{y,R} = \Lambda_{y,R} - y$ is a centered $R-$patch [15]

Lemma 46. *The following two properties of a Delone set Λ are equivalent ones*

(a) For every finite set F, $\Lambda + F$ is a Delone set
(b) For each $R > 0$, there are only finitely many centered $R-$patches $\Lambda^*_{y,R}$, $y \in \Lambda$.

Property (b) is exactly the Penrose tiling property. There are finitely many local configurations in Λ only.

We now define almost lattices. Almost lattices were named "Meyer sets" by J.C. Lagarias in [11, 13].

Definition 22. An almost lattice is a subset Λ of $\mathbb{R}^n$ fulfilling the following two conditions:

(a) For every finite set F, $\Lambda + F$ is a Delone set
(b) There exists a finite set $F_0 \subset \mathbb{R}^n$ such that $\Lambda - \Lambda \subset \Lambda + F_0$.

J. C. Lagarias proved in [13] that the definition of almost lattices is be the same if (a) was replaced by the weaker condition that Λ is a Delone set and if (b) was replaced by the weaker condition that $\Lambda - \Lambda$ is a Delone set. See also [11] or [15].

This approach is great since it is elegant and concise. However an almost lattice in that sense may not have the other properties we would like to impose to quasicrystals. Indeed almost lattices are not almost periodic patterns since any subset of an almost lattice with a positive lower density is an almost lattice as the following lemma shows.

Lemma 47. *Let Λ be an almost lattice and M an arbitrary subset of Λ. Then the following properties are equivalent ones*

(a) M is an almost lattice
(b) M is a Delone set.

An almost lattice is a Delone set. Therefore $(a) \Rightarrow (b)$. Let us prove $(b) \Rightarrow (a)$. We use the following observation.

Lemma 48. *If $M \subset \Lambda$ is a Delone set, there exists a finite set A such that $\Lambda \subset M+A$.*

Let us prove this remark. Since M is a Delone set, there exists a constant R such that for every $\lambda \in \Lambda$ there exists a $m \in M$ such that $|\lambda - m| \leq R$. But $\lambda - m \in \Lambda - \Lambda \subset \Lambda + F$. The intersection $A = (\Lambda + F) \cap \{|x| \leq R\}$ is finite. Therefore $\lambda - m \in A$, as announced.

We return to the proof of Lemma 48 and write $M - M \subset \Lambda - \Lambda + A - A \subset \Lambda + F + A - A \subset M + A + A - A + F$ as announced.

Lemma 48 implies that an almost lattice can be extremely irregular. For instance, any set E of integers with a positive lower density is an almost lattice. We cannot expect the corresponding measure $\mu = \sum_{k \in E} \delta_k$ to be a generalized almost periodic measure.

7.7 Almost Lattices and Model Sets

A lattice $\Gamma \subset \mathbb{R}^N$ is a discrete subgroup with compact quotient. In other words $\Gamma = A(\mathbb{Z}^N)$ where A is an $N \times N$ invertible matrix. We define $\mathrm{vol}(\Gamma)$ as the volume of any fundamental domain V of Γ. A fundamental domain is any Borel set V such that $V + \gamma$, $\gamma \in \Gamma$, is a measurable partition of $\mathbb{R}^N$. Then $\mathrm{vol}(\Gamma) = |\det A|$. The dual lattice $\Gamma^* \subset \mathbb{R}^N$ is defined by $\exp(iy \cdot x) = 1$ for every $x \in \Gamma$ and every $y \in \Gamma^*$. We obviously have $\Gamma = (\Gamma^*)^*$.

The definition of a model set $\Lambda \subset \mathbb{R}^n$ is given now.

One starts with an integer $m \geq 1$, we set $N = n + m$, $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and consider a lattice $\Gamma \subset \mathbb{R}^N$. If $(x, y) = X \in \mathbb{R}^n \times \mathbb{R}^m$, we write $x = p_1(X)$ and $y = p_2(X)$.

Let us assume that $p_1 : \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and that $p_2(\Gamma)$ is a dense subgroup of $\mathbb{R}^m$. Let $\Gamma^* \subset \mathbb{R}^N$ be the dual lattice and p_1^*, p_2^* be defined as p_1, p_2. Then $p_1^* : \Gamma^* \to p_1^*(\Gamma^*)$ is a one-to-one mapping and $p_2^*(\Gamma^*)$ is a dense subgroup of $\mathbb{R}^m$. A set $K \subset \mathbb{R}^m$ is Riemann-integrable if its boundary has a zero Lebesgue measure. The boundary of K is $\overline{K} \setminus L$ where $\overline{K}$ is the closure of K and L

is the interior of K. The interior of K is the largest open set contained in K. If K is Riemann-integrable, then K has a positive measure if and only if K has a non-empty interior.

Definition 23. Let K be a Riemann-integrable compact subset of $\mathbb{R}^m$ with a positive measure. Then the model set Λ defined by Γ and K is

$$\Lambda = \{\lambda = p_1(\gamma); \gamma \in \Gamma, p_2(\gamma) \in K\} \tag{7.56}$$

A subset Λ of $\mathbb{R}^n$ is a model set if either Λ is a lattice or if one can find m, Γ, and K such that Λ is the model set defined by (7.56).

The compact set K is named the *window* of the model set Λ. The reader is referred to Definition 4, page 48 of [27]. But in this reference the compact window K is replaced by an open set Ω with a compact closure. For proving that a model set has a uniform density, we were assuming that the closure of Ω is Riemann integrable. Following [29], it will be proved (Theorem 14, Section 7.8) that the definition of a model set by an open window Ω is inconsistent with the definition of the dual model set Λ^*. The issue of replacing a Riemann integrable compact set by an arbitrary compact is addressed in [33]. If K is an arbitrary compact set, it may happen that Λ is the empty set while K has a large measure. Another issue is the problem of the uniqueness of the triple (m, Γ, K) for a given model set Λ. This will be discussed later on in these notes.

Let $\mathscr{P} : p_1(\Gamma) \mapsto p_2(\Gamma)$ be the mapping defined by $\mathscr{P}(p_1(\gamma)) = p_2(\gamma), \gamma \in \Gamma$. This mapping satisfies $\mathscr{P}(x+y) = \mathscr{P}(x) + \mathscr{P}(y), \forall x, \forall y \in p_1(\Gamma)$. It will be proved (see Corollary 14 of Theorem 24, Section 7.11) that $\mathscr{P}(\Lambda)$ is equidistributed in K.

A lattice cannot be defined by (7.56). If $\Lambda = \mathbb{Z}$ was defined by (7.56) then $L = \Phi(\mathbb{Z})$ would be contained in K. But $L = \alpha\mathbb{Z}$ for some $\alpha \in p_2(\Gamma)$. This forces $\alpha = 0$ and implies $K = \{0\}$ which is not permitted by Definition 24. The same argument shows that $\mathbb{Z} + \{0, \sqrt{2}\}$ is not a model set. J. Lagarias defines a perfect crystal as a sum $\Lambda = \Lambda_0 + F$ between a lattice and a finite set.

Proposition 8. *The class of model sets is not translation invariant.*

If Λ is a model set and if $\tau \in p_1(\Gamma)$, then $\Lambda + \tau$ is still a model set defined by the new window $K + \mathscr{P}(\tau)$. But this observation breaks down if $\tau \notin p_1(\Gamma)$. This is not a proof of the fact that $\Lambda + \tau$ is not a model set. The proof will be detailed below (see also [29]).

In [27] the definition of model sets was slightly more general than the one which is given here. Indeed $\mathbb{R}^m$ was replaced by a locally compact abelian group. Here is an example where such a generalization is relevant. Let $\mathbb{Z}_q$ be the ring of $q-$adic integers. Here q does not need to be a prime number. Then $\mathbb{Z}$ can be considered as a dense subgroup of $\mathbb{Z}_q$. Let $\Omega \subset \mathbb{Z}_q$ be an open set. Then $\Lambda = \Omega \cap \mathbb{Z}$ is a repetitive set of integers. An example is given by $\Lambda = \cup_1^\infty \Lambda_j$ where n_j is a given sequence of integers and $\Lambda_j = n_j + q^j\mathbb{Z}$. If Λ_{j+1} intersects the union $M_j = \Lambda_1 \cup \cdots \cup \Lambda_j$ for some j, then $\Lambda_{j+1} \subset M_j$. This remark implies the following: either $\Lambda = m_{j_0}$ for some j_0 or Λ is the disjoint union of a subsequence $\Lambda_{j_k}, k \in \mathbb{N}$. In the first case Λ is periodic and in the second one Λ is a repetitive set of integers which is not periodic. This repetitive set is not a model in the sense given by Definition 24.

Here is the simplest example of a model set. For a real number x we denote by $m = [x]$ the integral part of x, and by $\{x\}$ the fractional part of x. Then $x = [x] + \{x\}, m \in \mathbb{Z}, \{x\} \in [0,1)$. Let $\alpha > 0$ be irrational. Then

$$\Lambda_\alpha = \{k + \{k\alpha\},\ k \in \mathbb{Z}\} \tag{7.57}$$

is aimed to be the first example a model set. Unfortunately this example fails to satisfy the requirements of Definition 24. To prove this observation let the uniqueness of the triple (m, Γ, K) be taken as granted. Then $\lambda_k = k + \{k\alpha\}$ can also be written as $\lambda_k = k + (k\alpha - l) = k(1+\alpha) - l$ where $0 \le k\alpha - l < 1$. Then $\Gamma \subset \mathbb{R}^2$ is defined by $x_1 = k(1+\alpha) - l, x_2 = k\alpha - l, k, l \in \mathbb{Z}$. Here K is replaced by $[0,1)$ which is not compact. This is not a serious issue and the union $\Lambda_\alpha^* = \Lambda_\alpha \cup \{1\}$ is a model set. Should the compact set K be replaced by a bounded open set ? For answering this issue we now follow Lagarias [15] and introduce an interesting class of Delone sets.

Definition 24. A Delone set $\Lambda \subset \mathbb{R}^n$ is repetitive if for each finite subset $F \subset \Lambda$ there exists a relatively dense set M such that $F + M \subset \Lambda$.

We could think that every model set is repetitive. A weaker statement is not even true. The weaker statement says that for each finite subset F of a model set Λ there are infinitely many τ's such that $F + \tau \subset \Lambda$. If $F = \{0,1\}$, the configuration $\{x, x+1\}$ appears only when $x = 0$ in the model set $\Lambda_\alpha \cup \{1\}$ which is defined above. Therefore the model set $\Lambda_\alpha \cup \{1\}$ is not repetitive.

Lemma 49. *Let ∂K be the frontier of K. Then any model set Λ for which $\partial K \cap p_2(\Gamma) = \emptyset$ is repetitive.*

The proof can be found in [15].

The set Λ_α^* which was defined above can be given another definition.

Lemma 50. *Let $\Theta : \mathbb{R} \mapsto \mathbb{T}^2$ the continuous homomorphism defined by $\Theta(t) = (t, \alpha t)$. Let $V \subset \mathbb{T}^2$ be defined by*

$$V = \{x \in \mathbb{T}^2;\ x_1 = s, x_2 = (1+\alpha)s,\ 0 < s < 1\} \tag{7.58}$$

Then

$$\Lambda_\alpha^* = \{t \in \mathbb{R}; \Theta(t) \in V\} \tag{7.59}$$

The following Lemma will be proved below in a more general setting.

Lemma 51. *If $\lambda_k = k + \{k\alpha\}, k \in \mathbb{Z}$, then the sequence $\Theta(\lambda_k), k \in \mathbb{Z}$, is equidistributed in V.*

The construction given in Lemma 50 can be generalized. We keep the notations of Definition 24. As above $\Gamma \subset \mathbb{R}^N, N = n+m$, is a lattice and Γ^* is the dual lattice. Let $\Delta = \mathbb{R}^N/\Gamma$ be the compact quotient group which is isomorphic to $\mathbb{T}^N$ and let $\zeta : \mathbb{R}^N \mapsto \Delta = \mathbb{R}^N/\Gamma$ be the quotient map. The dual lattice Γ^* is the dual group of Δ. Let us identify $\mathbb{R}^n$ with $L = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^N$ and define $\Theta : \mathbb{R}^n \mapsto \Delta$ by $\Theta(x) = \zeta((x,0))$. Then Θ is injective if and only if $p_2 : \Gamma \mapsto \mathbb{R}^m$ is injective. The mapping ζ restricted to $\{0\} \times \mathbb{R}^m$ is one-to-one. Let W be the subgroup $\zeta[\{0\} \times \mathbb{R}^m]$ of Δ. As above $K \subset \mathbb{R}^m$ is a Riemann integrable compact set. We embed K into Δ into $V = \zeta[\{0\} \times (-K)]$. Let us stress that $V \subset W$. This will not be satisfied by $y + V, y \in \Delta$, unless $y \in W$. This observation will be seminal in the proof of Proposition 8. Let Λ be the model set defined by Γ and K. Then we have

Lemma 52. *With the preceding notations the model set Λ can be defined by*

$$\Lambda = \{x \in \mathbb{R}^n; \Theta(x) \in V\} \tag{7.60}$$

Indeed $x \in \Lambda$ reads $x = \gamma_1$ with $(\gamma_1, \gamma_2) \in \Gamma$, $\gamma_2 \in K$. Then $(\gamma_1, 0) - (\gamma_1, \gamma_2) = (0, -\gamma_2) \in \{0\} \times (-K)$. Therefore $\Theta(x) = \zeta(\gamma_1, 0) \in V$. Conversely $\Theta(x) \in V$ implies $(x,0) - (\gamma_1, \gamma_2) = (0, -y)$ for some $y \in K$. Therefore $x \in \Lambda$.

An equivalent version of Lemma 52 is needed in the proof of Proposition 8. Let us consider the subgroup $H = p_1^*(\Gamma^*) \subset \mathbb{R}^n$ equipped with the discrete topology and let us denote by $\tilde{\Delta}$ the dual group of H. Then $\tilde{\Delta}$ and Δ are isomorphical since $p_1^* : \Gamma^* \mapsto H$ is an isomorphism. Let us denote by $h : H \mapsto \Gamma^*$ the inverse of $p_1^* : \Gamma^* \mapsto H$. The isomorphism $\mathscr{I}$ between Δ and $\tilde{\Delta}$ maps a point $(x,y) \in \mathbb{R}^N/\Delta$ to a character $\chi_{(x,y)} : H \mapsto \mathbb{T}$. This character is defined by

$$\chi_{(x,y)} = \exp[i(x \cdot u + y \cdot p_2^*(h(u)))], \ \forall u \in H. \tag{7.61}$$

The following definition will clarify the discussion.

Definition 25. A model set is primitive if $p_2 : \Gamma \to p_2(\Gamma)$ is injective.

This is equivalent to the assumption that H is a dense subgroup of $\mathbb{R}^n$. Then the compact group $\tilde{\Delta}$ is a compactification of $\mathbb{R}^n$ as in Definition 2. A model set Λ is primitive if and only if $\Lambda + \tau = \Lambda$ implies $\tau = 0$. Generic model sets are primitive but model sets which are not primitive are not uninteresting, as it will be shown below. The proof of Proposition 8 does not depend on that assumption, even if it is much more appealing in the case of a primitive model set.

The map $J : \mathbb{R}^n \mapsto \tilde{\Delta}$ is dual to the natural embedding $H \subset \mathbb{R}^n$. This amounts to saying that $J(x), x \in \mathbb{R}^n$ is a character on $H = p_1^*(\Gamma^*)$ which is defined by

$$J(x) = \exp(ix \cdot u), \forall u \in H. \tag{7.62}$$

We observe that the definition of $\Theta : \mathbb{R}^n \mapsto \Delta$ is consistent with the definition of $J : \mathbb{R}^n \mapsto \tilde{\Delta}$. Indeed (7.62) implies

$$J = \mathscr{I} \circ \Theta \tag{7.63}$$

This mapping $J : \mathbb{R}^n \mapsto \tilde{\Delta}$ is injective if the model set Λ is primitive (see Lemma 53 below). The subgroup $\mathscr{I}(W)$ of $\tilde{\Delta}$ will be denoted by $\tilde{W}$. It consists of the characters $\chi_y = \chi_{(0,y)}$ defined by (7.61). More precisely $\chi_y : H \mapsto \mathbb{T}$ is indexed by $y \in \mathbb{R}^m$ and defined by

$$\chi_y(u) = \exp[i(y \cdot p_2^*(h(u)))],\ \forall u \in H \tag{7.64}$$

We have for every $x \in \mathbb{R}^n$

$$J(x) \in \tilde{W} \Longleftrightarrow x \in p_1(\Gamma) \tag{7.65}$$

Let us prove this assertion. If $x = p_1(\gamma_0)$, then $J(x)$ is a character on $H = p_1^*(\Gamma^*)$ defined by $\exp(ip_1(\gamma_0) \cdot p_1(\gamma^*)) = \exp(-ip_2(\gamma_0) \cdot p_2(\gamma^*))$. It implies $J(x) = \chi_y \in \tilde{W}$ with $y = -p_2(\gamma_0)$.

If conversely $J(x) \in \tilde{W}$, there exists a $y \in \mathbb{R}^m$ such that $\exp(ix \cdot u) = \exp[i(y \cdot p_2^*(h(u)))],\ \forall u \in H$. Since this hold for every $(u, p_2^*(h(u))) \in \Gamma^*$ it implies $(x, -y) \in \Gamma$ which concludes the proof of (7.65).

Finally we denote by $\tilde{V} \subset \tilde{\Delta}$ the image of V by $\mathscr{I} : \Delta \mapsto \tilde{\Delta}$ and (7.60) together with (7.63) yield

$$\Lambda = \{x \in \mathbb{R}^n; J(x) \in \tilde{V}\} \tag{7.66}$$

We now return to Proposition 8 and prove a more precise result.

Proposition 9. *Let Λ be a model set defined by a lattice $\Gamma \subset \mathbb{R}^N$. If $\tau \notin p_1(\Gamma)$, then $\Lambda + \tau$ is not a model set.*

Let us begin with a wrong proof. We observe that$J(\Lambda + \tau) = J(\Lambda) + J(\tau)$ which is dense in $\tilde{V} + J(\tau)$. But $\tilde{V} \subset \tilde{W}$ and $\tilde{V} + J(\tau)$ cannot be contained in the group $\tilde{W}$ unless $\tau \in p_1(\Gamma)$. Here we used (7.65). This implies that $J(\Lambda + \tau)$ cannot be described by (7.66).

The error in this proof is the following. We only proved that $\Lambda + \tau$ cannot be constructed with the same J and the same $\tilde{\Delta}$ as the ones used in the construction of Λ. Are there other choices ? We now answer this question by the negative and the wrong proof will be validated. Let us assume that $\Lambda' = \Lambda + \tau$ is a model set defined by $N' = n + m'$ and a lattice Γ'. We denote by Γ'^* the dual lattice and begin the proof with finding the structure of the subgroup $H' = p_2^*(\Gamma'^*)$.

Let us use Theorem 13 or Proposition 12, Section 7.11 to compute the mean value over Λ' of $\exp(i\omega \cdot x)$, $\omega \in \mathbb{R}^n$. Let us denote by $\chi_{\Lambda'}(\omega)$ this mean value. We obtain $\chi_{\Lambda'}(\omega) = 0$ if $\omega \notin H'$. Moreover Theorem 13 tells us that H' is the subgroup generated by the set $E = \{\omega; \chi_{\Lambda'}(\omega) \neq 0\}$. But $\chi_{\Lambda'}(\omega) = \exp(i\omega \cdot \tau)\chi_\Lambda(\omega)$. Therefore $H' = H$. Once H' is known so are the compact group $\tilde{\Delta}'$ (which is the dual group of H') and the map J' which is dual to the canonical embedding $H' \subset \mathbb{R}^n$. We have $\tilde{\Delta}' = \tilde{\Delta}$ and $J' = J$. Let us prove that the subgroup $\tilde{W}'$ of $\tilde{\Delta}' = \tilde{\Delta}$ which is used in the construction of the model Λ' coincides with the group $\tilde{W}$. Since Λ' is assumed to be a model set we know that $\tilde{V}' = J(\Lambda')$ is contained in the group $\tilde{W}'$. Moreover the subgroup of $\tilde{W}'$ generated by $\tilde{V}'$ is dense in the group $\tilde{W}'$. But $\tilde{V}' = J(\Lambda') = J(\Lambda) + J(\tau)$. which implies $\tilde{V}' - \tilde{V}' = J(\Lambda) - J(\Lambda)$. That forces

$\tilde{V}' - \tilde{V}'$ to be contained in the group $\tilde{W}$ and forces the group $\tilde{W}'$ to be contained in the group $\tilde{W}$. Therefore $J(\tau) \in \tilde{W}$. It implies $\tau \in p_1(\Gamma)$ by (7.65) and we have reached a contradiction.

In a remarkable contribution R. Moody studied the compactified orbit $\Lambda + \tau$, $\tau \in \mathbb{R}^n$, of a model set. For this purpose R. Moody introduced a topology on the collections $\mathscr{D}(R_2, R_1)$ of Delone sets defined in Definition 21. The reader is urged to consult [34].

What happens when $p_2 : \Gamma \to p_2(\Gamma)$ is not a one-to-one mapping ? We then define the subgroup $\Gamma_0 \subset \Gamma$ by $p_2(\gamma) = 0$. We obviously have $\Gamma_0 = H \times \{0\}$ where $H = p_1(\Gamma_0)$ is a subgroup of $\mathbb{R}^n$. Finally the model set Λ is $H-$periodic: $\Lambda = \Lambda + H$. Here is a simple example where this situation occurs. We return to Definition 24 and set $n = 2, m = 1, \Gamma = \{(k_1 + k_3\sqrt{2}, k_2 + k_3\sqrt{3}, k_1 - k_2\sqrt{2}); k_1, k_2, k_3 \in \mathbb{Z}\}$ with $K = [0,1]$. The model set Λ is defined by $\Lambda = \{\lambda = (k_1 + k_3\sqrt{2}, k_2 + k_3\sqrt{3}); k_1 - k_2\sqrt{2} \in [0,1]\}$. Then Λ is invariant by translation by the subgroup $H = (\sqrt{2}, \sqrt{3})\mathbb{Z} \subset \mathbb{R}^2$. This model set Λ is the direct sum $M + H$ where $M \subset \mathbb{Z}^2$ is the strip defined by $k_1 - k_2\sqrt{2} \in [0,1]$. Therefore we cannot expect to write Λ as a direct sum $\Lambda = L + H$ where L would be a one dimensional model set contained in a line. This example is generalized as follows.

Proposition 10. *Let $x_1, \cdots, x_m$ be m linearly independent vectors of $\mathbb{R}^n$ and let H be the subgroup $\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_m$. We assume that*

(a) $H \cap \mathbb{Z}^n = \{0\}$
(b) for every $k \in \mathbb{Z}^n$, the m conditions $k \cdot x_j = 0$ $(1 \leq j \leq m)$ imply $k = 0$.

Let $K \subset \mathbb{R}^m$ be a Riemann integrable compact set and let $M_K \subset \mathbb{Z}^n$ be the strip defined by $(k \cdot x_1, \cdots, k \cdot x_m) \in K$. Then $\Lambda = M_K + H$ is a model set.

The proof is trivial. We denote by $x_j(q)$, $1 \leq q \leq n$, the coordinates of x_j. The lattice $\Gamma \subset \mathbb{R}^N, N = n + m$, is the set of points $\gamma = (k_1 + l_1 x_1(1) + \cdots + l_m x_m(1), \cdots, k_n + l_1 x_1(n) + \cdots + l_m x_m(n); k \cdot x_1, \cdots, k \cdot x_m)$ where $k \in \mathbb{Z}^n, l_j \in \mathbb{Z}$. Then $p_1 : \Gamma \mapsto \mathbb{R}^n$ is given by $p_1(\gamma) = (k_1 + l_1 x_1(1) + \cdots + l_m x_m(1), \cdots, k_n + l_1 x_m(n) + \cdots + l_m x_m(n))$. This map is one-to-one since $H \cap \mathbb{Z}^n = \{0\}$. Similarly $p_2(\gamma) = (k \cdot x_1, \cdots, k \cdot x_m)$ has a dense range since $H \cap \mathbb{Z}^n = \{0\}$.

These remarks pave the road to the definition of a primitive model set.

The examples are showing that the study of model sets cannot be restricted to the study of primitive model sets.

The relevance of primitive model sets is coming from the following observation.

Lemma 53. *The mapping $J : \mathbb{R}^n \mapsto \tilde{\Delta}$ has a dense range. This mapping $J : \mathbb{R}^n \mapsto \tilde{\Delta}$ is injective if and only if $p_2 : \Gamma \mapsto p_2(\Gamma)$ is injective.*

The proof is obvious.

This section ends with a few more properties of model sets. Proposition 10 which is coming now will be proved in Section 7.11. Keeping the preceding notations we have:

Proposition 11. *Let f be an almost periodic function and let g be the projection of f on the frequencies belonging to the subgroup H. Let $\tilde{g}$ be the continuation of g to $\tilde{\Delta}$. Then the mean value of f on the model set Λ is $|\tilde{V}|^{-1}\int_{\tilde{V}}\tilde{g}(x)\,dx$.*

In other words $J(\Lambda)$ is equidistributed on $\tilde{V}$.

A model set is always an almost lattice. But an almost lattice cannot be a model set in general. Indeed the class of almost lattices is translation invariant while the class of model sets is not, as Proposition 9 is showing. These remarks explain the role of the finite set F in the following theorem.

Theorem 11. *A model set is an almost lattice. Conversely if Λ is an almost lattice there exist a model set M and a finite set F such that $\Lambda \subset M+F$.*

Theorem 11 is proved in [25, 27]. One can also consult [13] or [32]. The main result of [29] can be stated now

Theorem 12. *Let Λ be a model set. Then $\sigma = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a generalized almost periodic measure.*

This fundamental fact proved in [29] will be fully detailed in the Section 7.11.

7.8 Diophantine Approximations and Harmonious Sets

Harmonious sets are defined now and, as it will be shown, almost lattices are harmonious sets. Let $\Lambda \subset \mathbb{R}^n$ be an arbitrary set of real numbers and let $\Gamma(\Lambda)$ be the subgroup of $\mathbb{R}^n$ generated by Λ, equipped with the discrete topology. A weak character on Λ is the restriction to Λ of an algebraic homomorphism from $\Gamma(\Lambda)$ into the group $\mathbb{T}$ of complex numbers of modulus 1. No continuity property is required on this algebraic homomorphism χ. It satisfies $\chi(x+y)=\chi(x)\chi(y)$, $x,y\in\mathbb{R}^n$. A strong character h is a restriction to $\Gamma(\Lambda)$ of a continuous homomorphism from $\mathbb{R}^n$ to $\mathbb{T}$ and is therefore given by $h(x)=\exp(i\xi\cdot x)$, $\quad \xi\in\mathbb{R}^n$.

Definition 26. Let $\varepsilon\in(0,2)$. A set $\Lambda\subset\mathbb{R}^n$ is $\varepsilon-$harmonious if for every weak character χ, there exists a strong character h such that

$$\sup_{\lambda\in\Lambda}|\chi(\lambda)-h(\lambda)|\leq\varepsilon. \tag{7.67}$$

An equivalent definition is given now, without using weak characters.

Definition 27. Given a subset Λ of $\mathbb{R}^n$ and $\varepsilon\in(0,2)$, let $\Lambda_\varepsilon^*\subset\mathbb{R}^n$ be the set of all $\xi\in\mathbb{R}^n$ such that $\sup_\Lambda|\exp(i\xi\cdot\lambda)-1|\leq\varepsilon$. This set $\Lambda_\varepsilon^*\subset\mathbb{R}^n$ is named the ε-dual set of Λ.

We now have

Lemma 54. *Let $\varepsilon \in (0,2)$. A set $\Lambda \subset \mathbb{R}^n$ is $\varepsilon-$harmonious if and only if Λ_ε^* is relatively dense in the sense of Besicovitch.*

Definition 28. A subset Λ of $\mathbb{R}^n$ is harmonious if it is $\varepsilon-$harmonious for all $\varepsilon \in (0,2)$.

A subset of harmonious set is a harmonious set. Any finite set F is harmonious. If Λ is harmonious and if F is finite, then $\Lambda+F$ is harmonious. If Λ is harmonious, so is $\Lambda-\Lambda$. It implies that the union of two harmonious sets is not harmonious in general. To give an example in the one dimensional case $\Lambda_1 = \mathbb{Z}$ and $\Lambda_2 = \sqrt{2}\mathbb{Z}$ are harmonious. If $\Lambda_1 \cup \Lambda_2$ was harmonious, then $\Lambda_1 - \Lambda_2$ would also be a harmonious set. This is impossible since any harmonious set is uniformly discrete : there exists a positive β such that $\lambda \in \Lambda$, $\lambda' \in \Lambda$ and $\lambda' \neq \lambda$ imply $|\lambda'-\lambda| \geq \beta$.

We now prove that model sets are harmonious sets. From now on Λ will denote a model set defined by a lattice Γ and a Riemann-integrable compact set K. Replacing Λ by a larger set we can assume that K is connected and that 0 is an interior point in K. Let $\Gamma^* \subset \mathbb{R}^n \times \mathbb{R}^m$ be the dual lattice of Γ. It means that Γ^* is the collection of all $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $x\cdot\xi + y\cdot\eta \in 2\pi\mathbb{Z}$ for each $(x,y) \in \Gamma$. Then $p_1 : \Gamma^* \to p_1(\Gamma^*)$ is still a one-to-one mapping while $p_2(\Gamma^*)$ is still dense in $\mathbb{R}^m$. For proving that Λ is harmonious we need to compute its $\varepsilon-$dual Λ^*. Theorem 14 indicates that this $\varepsilon-$dual is still a model set. Therefore it is a Delone set and Λ is harmonious by Lemma 54. Lemma 55 and Theorem 13 are needed in the computation of the $\varepsilon-$dual. They are now stated without proof and will be proved in Section 7.11.

Lemma 55. *If Λ is a model set as above, then each ball $B(x,R)$ centered at x with radius R contains $cR^n + o(R^n)$ points in Λ where $c = \frac{c_n}{\operatorname{vol}\Gamma}$ and where the little o is uniform in x.*

Here c_n is the volume of the unit ball.

If c_λ, $\lambda \in \Lambda$, is any bounded sequence, its mean value over Λ (if it exists) is defined as the uniform limit as R tends to infinity of

$$\gamma(x,R) \sum_{\lambda \in B(x,R)\cap\Lambda} c_\lambda \tag{7.68}$$

where $\gamma(x,R)$ is the inverse of the cardinality of $\{B(x,R)\cap\Lambda\}$ in such a way that the mean value over Λ of 1 be 1.

With these definitions in mind, one can state the following theorem. This theorem will be proved in Section 7.11 (see Proposition 12). One can also consult [29] and [26], page 32, Proposition 1.

Theorem 13. *Let Λ be a model set as above. Let $\omega \in \mathbb{R}^n$ and let us assume that $\omega \notin p_1(\Gamma^*)$. Then the mean value over Λ of $e^{i\omega\cdot\lambda}$ is 0.*

If $\omega = p_1(\gamma^*)$ and $\gamma^* \in \Gamma^*$, then the mean value of $e^{i\omega\cdot\lambda}$ over Λ is given by $|K|^{-1}\hat{\mathbf{1}}_K(p_2(\gamma^*))$ where $\mathbf{1}_K$ is the indicator function of K, $\hat{\mathbf{1}}_K$ denotes the Fourier transform of this indicator function and $|K|$ is the Lebesgue measure of K.

Theorem 13 is needed to compute the ε-dual of Λ when $0 < \varepsilon < \sqrt{2}$. The compact set K is now assumed to be connected. Then $|e^{i\omega\cdot\lambda} - 1| \le \varepsilon$ uniformly over Λ implies $\Re(e^{i\omega\cdot\lambda}) \ge \beta(\varepsilon) > 0$. By convexity the same inequality will be valid for any average of $e^{i\omega\cdot\lambda}$, $\lambda \in \Lambda$. Therefore $\omega \in p_1(\Gamma^*)$ (otherwise the mean value of $e^{i\omega\cdot\lambda}$ would be 0).

If $\omega = p_1(\gamma^*)$ while $\lambda = p_1(\gamma)$, then

$$\omega\cdot\lambda + p_2(\gamma)\cdot p_2(\gamma^*) \in 2\pi\mathbb{Z} \tag{7.69}$$

which implies

$$\exp(i\omega\cdot\lambda) = \exp[-i(p_2(\gamma)\cdot p_2(\gamma^*))]\,. \tag{7.70}$$

Since $p_2(\Gamma)$ is dense in $\mathbb{R}^m$, so is $K\cap p_2(\Gamma)$ in K. Then $|\exp[i(p_2(\gamma)\cdot p_2(\gamma^*))] - 1| \le \varepsilon$ for $p_2(\gamma) \in K$ implies $|\exp[i(y\cdot p_2(\gamma^*))] - 1| \le \varepsilon$ for any $y \in K$. In other words for every $y \in K$ there exists an $m \in \mathbb{Z}$ such that $|y\cdot p_2(\gamma^*) - 2m\pi| \le \theta$ where $2\sin(\theta/2) = \varepsilon$, $0 < \theta < \pi/2$. For $m \in \mathbf{Z}$ let us define the strip S_m by

$$|y\cdot p_2(\gamma^*) - 2m\pi| \le \theta,\, m \in \mathbb{Z}. \tag{7.71}$$

Then K is contained in the union of the disjoint strips S_m. Since K is a connected set and $0 \in K$ we have $K \subset S_0$.

The preceding discussion yields the following.

Lemma 56. *Let K be a connected compact set with $0 \in K$. Let $0 < \varepsilon < \sqrt{2}$ and $2\sin(\theta/2) = \varepsilon$, $0 < \theta < \pi/2$. Let Λ_K the model set defined by K. Then $|e^{i\omega\cdot\lambda} - 1| \le \varepsilon < \sqrt{2}$, $\lambda \in \Lambda$, is equivalent to the following : there exists a $\gamma^* \in \Gamma^*$ such that*

(a) $\omega = p_1(\gamma^*)$, $\gamma^* \in \Gamma^*$
(b) If $2\sin(\theta/2) = \varepsilon$, $0 < \theta < \pi/2$, then for every $y \in K$ we have $|y\cdot p_2(\gamma^*)| \le \theta$.

Using the following definition, Lemma 56 will be rephrased.

Definition 29. If $K \subset \mathbb{R}^n$ is a set, then its dual convex set K^* is defined by

$$K^* = \{y;|x\cdot y| \le 1, x \in K\}.$$

We then have

Corollary 7. *Let K be a connected compact set with $0 \in K$. Let $0 < \varepsilon < \sqrt{2}$ and $2\sin(\theta/2) = \varepsilon, 0 < \theta < \pi/2$. Let Λ_K be the model set defined by K. Then $|e^{i\omega\cdot\lambda} - 1| \leq \varepsilon < \sqrt{2}$, $\forall\lambda \in \Lambda$, is equivalent to $\omega \in \Lambda^*_{\theta K^*}$ which is defined using the lattice Γ^* instead of Γ.*

If K is a compact and convex set and if we keep two times the same value of ε, then we obtain $(\Lambda^*)^* = \Lambda$. This argument breaks down if $\varepsilon \geq \sqrt{2}$.

Before reading the remarkable contribution [32] we did not know the optimal value of ε ensuring that $|e^{i\omega\cdot\lambda} - 1| \leq \varepsilon$ uniformly over a given model set Λ implies $\omega \in \Lambda^*$ where Λ^* is also a model set. Robert Moody proved this fact for any $\varepsilon \in (0,2)$.

Let us assume that K is symmetric with respect to the origin, convex and compact with a positive measure. We then have a full symmetry between the model set Λ defined by K and its ε-dual Λ^*_ε defined by $K^*_\varepsilon = \theta K^*$. This discussion is summarized in the following theorem.

Theorem 14. *Let K be a compact and convex set, symmetric with respect to the origin, with a positive measure. Let us consider the model set defined by $\Lambda = \{\lambda = p_1(\gamma)\,;\, \gamma \in \Gamma;\, p_2(\gamma) \in K\}$. Let us assume $0 < \varepsilon < \sqrt{2}$ and define θ by $2\sin(\theta/2) = \varepsilon, 0 < \theta < \pi/2$. Then the $\varepsilon-$dual*

$$\Lambda^*_\varepsilon = \{y \in \mathbb{R}^n\,;\, |e^{i\lambda\cdot y} - 1| \leq \varepsilon\,,\, \forall\lambda \in \Lambda\}$$

of Λ is also a model set defined by

$$\Lambda^*_\varepsilon \;=\; \left\{\lambda^* = p_1(\gamma^*)\,;\, \gamma^* \in \Gamma^*;\, p_2(\gamma^*) \in \theta K^*\right\}$$

and the ε-dual of Λ^*_ε is Λ.

Any model set is contained in a larger model set for which $K = \{x; |x| \leq R\}$. Then Theorem 14 applies. Therefore any model set is harmonious. The converse implication is provided by Theorem 15.

Theorem 15. *The three following properties of a set $\Lambda \subset \mathbb{R}^n$ are equivalent ones*

(a) Λ is an almost lattice
(b) $\Lambda \subset M + F$ where M is a model set and F is a finite set
(c) Λ is harmonious and relatively dense.

The implication $(a) \Rightarrow (b)$ is proved in Meyer's book [27] or in [11, 13, 15]. The implication $(b) \Rightarrow (c)$ is provided by Theorem 14. Let us prove $(c) \Rightarrow (a)$ and assume that Λ is harmonious. Then $\Lambda_2 = \Lambda - \Lambda$ is also harmonious as it was mentioned earlier and Λ_2 is uniformly discrete. Since Λ is a Delone set, it is relatively dense and so is Λ_2. Therefore Λ_2 is a Delone set. Finally Lagarias' theorem implies that Λ is an almost lattice.

This discussion ends with the following observation.

Lemma 57. *There exists an almost lattice* Λ *such that* $(\Lambda_\varepsilon^*)_\varepsilon^* \neq \Lambda$ *for every* $0 < \varepsilon \leq 1$.

The simplest example is given by $\Lambda = \mathbb{Z} + \{0, \sqrt{2}\}$. Then if $0 < \varepsilon \leq 1$, the ε-dual Λ_ε^* of Λ is $2\pi M_\varepsilon$ where $M_\varepsilon \subset \mathbb{Z}$ defined by $|\exp(2\pi i m\sqrt{2}) - 1| \leq \varepsilon$. This follows from an obvious lemma :

Lemma 58. *Let* x *be a real number. If* $|1 - \exp(2\pi i x k)| \leq 1$ *for every* $k \in \mathbb{Z}$, *then* $x \in \mathbb{Z}$.

We now compute the ε-dual of Λ_ε^*. We use the following lemma.

Lemma 59. *Let* θ *be a real number with the following property: there exists* $\varepsilon \in (0, 1]$ *such that for every* $k \in \mathbb{Z}$, $|\exp(2\pi i k\sqrt{2}) - 1| \leq \varepsilon \Rightarrow |\exp(2\pi i k\theta) - 1| \leq \varepsilon$. *Then* $\theta = \pm\sqrt{2} + m$, $m \in \mathbb{Z}$.

Let us denote by $\mathbb{T}$ the multiplicative group of complex numbers of modulus 1 and by Γ the dense subgroup of $\mathbb{T}$ consisting of the points $\exp(2\pi i k\sqrt{2})$, $k \in \mathbb{Z}$. Let $\chi : \Gamma \mapsto \mathbb{T}$ be the map defined by $\chi[\exp(2\pi i k\sqrt{2})] = \exp(2\pi i k\theta)$, $k \in \mathbb{Z}$. We first prove the continuity of χ at 1. We assume by contradiction that a sequence $z_j = \exp(2\pi i k_j\sqrt{2})$ exists such that $z_j \to 1$ while the distance from $\zeta_j = \chi(z_j)$ to 1 exceeds $\eta > 0$. Let $J \subset \mathbb{T}$ denote the interval defined by $|z - 1| \leq \varepsilon$. We consider the sequence z_j^m, $m \in \mathbb{N}$. Then $z_j^m \in J$ when $j \geq j_0(\varepsilon)$ and $0 \leq m \leq N_j$. Moreover N_j tends to infinity with j. The hypothesis says that $\chi(\Gamma \cap J) \subset J$. But ζ_j^m is forced to leave J for some $m \leq m(\eta)$ since $|\zeta_j - 1| \geq \eta$. We reach the required contradiction when $N_j > m(\eta)$. Therefore χ is continuous at 1. But χ is a morphism. Therefore χ is uniformly continuous on Γ. Finally χ can be extended by continuation on $\mathbb{T}$ which implies $\chi(z) = z^q$ for some integer q. It is now immediate to conclude to $q = \pm 1$. Indeed $\Gamma \cap J$ is dense in J which implies that z^q cannot remain in J if $|q| \geq 2$ and $z \in \Gamma \cap J$.

Lemma 59 implies $(\Lambda_\varepsilon^*)_\varepsilon^* = \mathbb{Z} \pm \{0, \sqrt{2}\}$. If we had begun with $\Lambda = \mathbb{Z} \pm \{0, \sqrt{2}\}$, then we would obtain the same ε-dual as above and $(\Lambda_\varepsilon^*)_\varepsilon^* = \Lambda$. This example raises the problem of characterizing the Delone sets for which $(\Lambda_\varepsilon^*)_\varepsilon^* = \Lambda$ when $\varepsilon > 0$ is small enough. This issue was not addressed in [28].

The relationship between Pisot numbers, Salem numbers, and harmonious sets is given by the two following theorems.

Theorem 16. *Let* $\theta > 1$ *be a real number and let us consider the set* $S = \{1, \theta, \theta^2, \theta^3, \cdots\}$. *Then* S *is harmonious if and only if* θ *is a Pisot or a Salem number.*

The proof is given in [26], p. 19, Proposition 3.

A simple corollary is the following observation.

Theorem 17. *Let $\Lambda \subset \mathbb{R}^n$ be a quasicrystal. If $\theta > 1$ and $\theta\Lambda \subset \Lambda$, then θ is either a Pisot number or a Salem number.*

Conversely for each dimension n and each Pisot or Salem number θ, there exists a quasicrystal $\Lambda \subset \mathbb{R}^n$ such that $\theta\Lambda \subset \Lambda$

Let us prove the first statement in Theorem 17.

We have $\theta^n \Lambda \subset \Lambda, n \in \mathbb{N}$. If $\lambda_0 \in \Lambda$, $\lambda_0 \neq 0$, and $S = \{1, \theta, \theta^2, \theta^3, \cdots\}$, then $\lambda_0 S$ is contained in Λ. But a quasicrystal is a harmonious set and any subset of a harmonious set is still a harmonious set. Therefore $\lambda_0 S \subset \mathbb{R}^n$ is a harmonious set. Finally S is harmonious and Theorem 16 can be used.

We now prove the second statement. Let $\mathscr{K}$ be an algebraic number field of degree n over $\mathbb{Q}$. The ring $\Omega_{\mathscr{K}}$ of algebraic integers of K is isomorphic to $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. In other words there exist n algebraic integers $\omega_1, \cdots, \omega_n$ which are linearly independent over $\mathbb{Q}$ and such that

$$\Omega_{\mathscr{K}} \; = \; \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \, . \tag{7.72}$$

Let $\sigma_1 : \mathscr{K} \to \mathbb{C}, \cdots, \sigma_n : \mathscr{K} \to \mathbb{C}$ be the $n-1$ embeddings of $\mathscr{K}$ into $\mathbb{C}$. It will be assumed that σ_1 maps $\mathscr{K}$ to $\mathbb{R}$. Then $\mathscr{K}$ will be identified with its image by this mapping.

Lemma 60. *The set M of all Pisot or Salem numbers belonging to $\mathscr{K}$ is a model set.*

Some definitions are needed to prove Lemma 60. If the algebraic number field $\mathscr{K}$ is totally real, we define Γ in $\mathbb{R}^n$ by

$$\Gamma = \{\sigma_1(\omega), \cdots, \sigma_n(\omega);\; \omega \in \Omega_{\mathscr{K}}\}.$$

If σ_j is complex valued on $\mathscr{K}$, then we can always assume that the complex conjugate of σ_j is indexed by $j+1$. We have $\sigma_{j+1} = \overline{\sigma}_j$ and σ_{j+1} will be deleted in the definition of Γ. Then Γ is a lattice. Let $\Lambda_{\mathscr{K}}$ be the collection of all algebraic integers $\lambda \in \Omega_{\mathscr{K}}$ such that $|\sigma_2(\lambda)| \leq 1, \cdots, |\sigma_n(\lambda)| \leq 1$. We can conclude.

Lemma 61. *With the preceding notations, $\Lambda_{\mathscr{K}}$ is a model set for which $m = n-1$, $p_2 = (\sigma_2, \cdots, \sigma_n)$ and p_1 is the identity mapping.*

Here is another proof of Theorem 17 when θ is a Pisot number. We consider the set $\Lambda_m = \left\{\sum_{k \geq 0} \varepsilon_k \theta^k \,;\, \varepsilon_k \in \mathbb{Z} \,,\, |\varepsilon_k| \leq m\right\}$, where $\varepsilon_k \neq 0$ for finitely many k's only and m is the integral part of θ. Then we obviously have $\theta\Lambda_m \subset \Lambda_m$. Moreover Λ_m is relatively dense. We have $\Lambda_m - \Lambda_m = \Lambda_{2m}$. Lagarias' theorem applies if we can show that a positive β exists such that $|\lambda' - \lambda| > \beta$ whenever λ and λ' belong to Λ_{2m} and $\lambda \neq \lambda'$. But the algebraic integer $r = \lambda' - \lambda$ belong to Λ_{4m} and its $n-1$ conjugates $r_2, \cdots, r-n$ satisfy

$$|r_2| \le \frac{4m}{1-|\theta_2|} \quad ,\cdots, \quad |r_n| \le \frac{4m}{1-|\theta_n|}\,.$$

Therefore $|r_2\cdots r_n| \le C$. But $|(\lambda' - \lambda)r_2\cdots r_n| \ge 1$. Indeed the product between an algebraic integers of degree n and its $n-1$ conjugates is an ordinary integer. It implies $|\lambda' - \lambda| \ge 1/C$ as announced. Lagarias' theorem yields the required conclusion.

We have obtained a one dimensional model set Λ such that $\theta\Lambda \subset \Lambda$. If one needs an n-dimensional model set, it suffices to consider $\Lambda \times \cdots \times \Lambda \subset \mathbb{R}^n$.

The sum $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ will be analyzed when Λ is a model set.

Theorem 18. *Let Λ be a model set. Then σ_Λ is not an almost periodic measure unless Λ is a lattice. However σ_Λ is a generalized almost periodic measure.*

The proof of Theorem 18 begins with a lemma:

Lemma 62. *If σ_Λ is an almost periodic measure, then for every positive ε there exists a relatively dense set $T(\varepsilon)$ of $\tau \in \mathbb{R}^n$ such that the Hausdorff distance between Λ and $\Lambda+\tau$ does not exceed ε.*

For $r > 0$ let $g_r(x) = (1-|x|/r)_+$ and consider $f_r = \sigma_\Lambda * g_r$. We have $f_r = 1$ on Λ. If σ_Λ is an almost periodic measure, f_r will be an almost periodic function. It can be assumed that $|\lambda - \lambda'| > 2r, \forall\lambda \ne \lambda', \lambda,\lambda' \in \Lambda$ if $r > 0$ is small enough. Let τ be an $r\varepsilon$ almost period of f_r. Then $|f_r(\lambda) - f_r(\lambda+\tau)| \le r\varepsilon$ implies $|f_r(\lambda+\tau)-1| \le r\varepsilon$ and $|\lambda+\tau-\lambda'| \le \varepsilon$ for some $\lambda' \in \Lambda$. Lemma 62 is proved.

We return to Theorem 18. Without losing generality we can assume $0 \in \Lambda$ since the hypothesis and the conclusion are invariant by translation. We consider two arbitrary λ_1, λ_2 in Λ. With the notations used in Lemma 62 we have $\lambda_1 + \tau = \lambda_1' + \varepsilon_1$ and $\lambda_2 + \tau = \lambda_2' + \varepsilon_2$ with $\lambda_1', \lambda_2' \in \Lambda$ and $|\varepsilon_1|, |\varepsilon_2| \le \varepsilon$. This implies $\lambda_3 = \lambda_1 - \lambda_2 - \lambda_1' + \lambda_2' = \varepsilon_1 - \varepsilon_2$ and $|\lambda_3| \le 2\varepsilon$. But λ_3 belongs to the model set $\Lambda \pm \Lambda \pm \Lambda \pm \Lambda$ and if ε is small enough we have $\lambda_3 = 0$. Therefore $\varepsilon_1 = \varepsilon_2$ and this is true for every pair of two elements λ_1, λ_2 in Λ. It implies that there exists an $\alpha \in \mathbb{R}^n$ such that $\varepsilon_1 = \alpha$ is for $\lambda_1 \in \Lambda$. We set $M = T - \alpha$ and we have $M + \Lambda \subset \Lambda$. Since $0 \in$ *Lambda* it implies $M \subset \Lambda$.

$$\Lambda + v \subset \Lambda,\, v \in M \tag{7.73}$$

But Λ is a model set and (7.73) can be lifted to the lattice Γ and then be projected on $\mathbb{R}^m$. We then have $v = p_1(\gamma_0)$, $\gamma_0 \in \Gamma$. We set $w = p_2(\gamma_0) = \Phi(v)$ and obtain by density $K + w \subset K$ which implies $w = 0$. Therefore M is contained in the kernel N of Φ and $0 \in K$. Since M was relatively dense so is N. Finally $N \subset \Lambda$ and N is a Delone set. This implies that N is a lattice. Moreover N is the largest lattice contained in Λ. Since Λ is a model set we have $\Lambda = N$.

The proof of the second statement is postponed to Section 7.11.

7.9 More on Harmonious Sets

The ε-dual Λ_ε^* of a set $\Lambda \subset \mathbb{R}^n$ is defined by Definition 27. The characterization of harmonious sets given by Lemma 54 leads us to raise to the following problem.

Conjecture 1 *There exists a positive number $\alpha \in (0,1)$ with the following property : any set Λ of real numbers such that Λ_α^* is relatively dense is harmonious.*

A counterexample is given by the following theorem.

Theorem 19. *Let $\alpha \in (0,1)$ and $\theta \geq 4\pi/\alpha$. Let us consider the set*

$$\Lambda = \{1, \theta, \theta^2, \theta^3, \cdots\}$$

of powers of $\theta > 1$. Then Λ_α^* is relatively dense.

To prove Theorem 19, we start with the trivial remark that for every real number t we have $|\exp(it) - 1| \leq |t|$. We will prove that every interval $[2k\pi - \alpha, 2k\pi + \alpha]$, $k \in \mathbb{Z}$, intersects Λ_α^*. Therefore Λ_α^* will be relatively dense. The required point $\lambda \in \Lambda_\alpha^*$ is constructed by successive approximations, as described by the following lemma:

Lemma 63. *There exists a sequence $p_j \in \mathbb{Z}$ of integers with the following property : for every $k \in \mathbb{Z}$ and each integer m there exists a $\lambda_m \in [2k\pi - \alpha, 2k\pi + \alpha]$ such that $|\lambda_m \theta^j - 2p_j\pi| \leq \alpha$ for $0 \leq j \leq m$.*

The key point is that the sequence p_j, $j \in \mathbb{N}$, of integers does not depend on m. The construction will show that λ_m is a Cauchy sequence. The limit λ of this sequence belongs to $[2k\pi - \alpha, 2k\pi + \alpha]$ and satisfies $|\lambda\theta^j - 2p_j\pi| \leq \alpha$, $j \in \mathbb{N}$. Therefore λ belongs to $[2k\pi - \alpha, 2k\pi + \alpha] \cap \Lambda_\alpha^*$ as announced. It then suffices to prove Lemma 63 to obtain Theorem 19. We now start the construction and decide that λ_0 is any point in $I_0 = [2k\pi - \alpha, 2k\pi + \alpha]$ and set $p_0 = k$. Since $\theta \geq 4\pi/\alpha$, the length of the inflated interval θI_0 exceeds 4π which implies that θI_0 contains a full interval $I_1 = [2p_1\pi - \alpha, 2p_1\pi + \alpha]$. We now freeze p_1 and replace λ_0 by $\lambda_1 \in \theta^{-1} I_1 \subset I_0$. We proceed and observe that the length of θI_1 exceeds 4π. Therefore there exists a full interval $I_2 = [2p_2\pi - \alpha, 2p_2\pi + \alpha] \subset \theta I_1$. We now freeze p_2 and replace λ_1 by $\lambda_2 \in \theta^{-2} I_2 \subset \theta^{-1} I_1 \subset I_0$. The construction proceeds easily and Theorem 19 is proved.

To obtain the required counterexample, we fix a transcendental number $\theta \geq 4\pi/\alpha$. Then Λ is not harmonious.

We return to Conjecture 1. Robert Moody proved the following

Theorem 20. *Let $\Lambda \subset \mathbb{R}^n$ be a Delone set and let $\alpha \in (0, 1/2)$. If the α-dual Λ_α^* of Λ is also a Delone set, then Λ is harmonious.*

The proof can be found in [11] where J. Lagarias is telling that this theorem is dued to R. Moody.

7.10 Coherent Sets of Frequencies

We are interested here in the behavior at infinity of mean-periodic functions. Let me briefly explain what mean-periodic functions are. A complex valued continuous function f defined on $\mathbb{R}^n$ is mean-periodic if the closed linear span of the translates $f(\cdot - y), y \in \mathbb{R}^n$, is not the space of all continuous functions on $\mathbb{R}^n$. Here the topology is defined by uniform convergence on compact sets. An equivalent definition is given by the following:

Definition 30. A mean-periodic function is a continuous solution f of a convolution equation

$$f * \tau = 0 \tag{7.74}$$

where τ is a compactly supported Borel measure.

The case $\tau = 0$ is obviously excluded. We now restrict the attention to the one-dimensional case. The Fourier-Laplace transform of τ is the entire function $F(z) = \int_{\mathbb{R}} \exp(-izx)\,d\tau$. Let Λ denote the zero set of $F(z)$ where each zero λ is counted in Λ as many times as its multiplicity m_λ indicates. Then $f(x) = P(x)\exp(i\lambda x)$ is a solution to (7.74) if and only if $\lambda \in \Lambda$ and $P(x)$ is a polynomial of degree less than or equal to $m_\lambda - 1$. A finite sum of these building blocks is still a solution to (7.74). Finally any solution f to (7.74) has a Fourier series expansion

$$f(x) \sim \sum_{\lambda \in \Lambda} P_\lambda(x)\exp(i\lambda x) \tag{7.75}$$

where $P_\lambda(x)\exp(i\lambda x)$ are the abovementioned elementary solutions. It means that any solution to (7.74) is the limit of a sequence of finite linear combinations of elementary solutions to (7.74). The convergence is uniform on compact intervals. This does not mean that f is the limit of the sequence of partial Fourier sums in (7.75). Some summation procedures are needed.

The set $\Lambda \subset \mathbb{C}$ is not arbitrary since it is the zero set of the Fourier-Laplace transform of a function with compact support. Such sets Λ have been characterized by A. Beurling and P. Malliavin. If $\Lambda \subset \mathbb{R}$ has this property, we denote by C_Λ the Frechet space consisting of all mean-periodic functions which are limits of finite trigonometric sums $\sum_{\lambda\in\Lambda} a_\lambda \exp(i\lambda x)$. The Frechet space C_Λ is equipped with the topology of uniform convergence on compact sets. By construction finite trigonometric sums $g(x) = \sum_{\lambda\in\Lambda} a_\lambda \exp(i\lambda x)$. are dense in C_Λ. From now on multiplicities will be excluded in Λ.

Is it possible to relate the arithmetical structure of the given set Λ to the growth at infinity of all $f \in C_\Lambda$? The most natural problem is given by the following definition.

Definition 31. With the preceding notations and definitions we say that $\Lambda \subset \mathbb{R}$ is a coherent set of frequencies if every $f \in C_\Lambda$ is an almost periodic function in the sense of Bohr. This cannot happen if polynomials are allowed in the definition of C_Λ.

The simplest case where this is true is given by $\Lambda = \mathbb{Z}$. Then every $f \in C_\Lambda$ is 2π-periodic. Another example is given when Λ is a finite set. The following theorem relates harmonious sets to coherent sets of frequencies.

Theorem 21. *For any subset $\Lambda \subset \mathbb{R}$ the following two properties are equivalent:*

(a) For each positive ε there exists a Delone set T_ε such that

$$\tau \in T_\varepsilon \;\Rightarrow\; \|f(x-\tau)-f(x)\|_\infty \le \varepsilon \|f\|_\infty \quad , \quad f \in C_\Lambda \,. \tag{7.76}$$

(b) Λ is harmonious.

If it is the case, then every $f \in C_\Lambda$ is an almost periodic function in the sense of Bohr.

If $f(x) = e^{i\lambda x}$, then (7.76) implies $|e^{i\lambda\tau} - 1| \le \varepsilon$ for each $\tau \in T_\varepsilon$. Therefore Λ is harmonious. Conversely let us assume that Λ is harmonious. Let Λ_ε^* be the ε-dual of Λ. The following lemma (N. Varopoulos, oral communication) will be used in the proof of Theorem 21.

Lemma 64. *Let $\eta \in (0, \pi/2]$ and $\theta(x)$ be the $2\pi-$periodic odd function of the real variable x defined by*

(a) $\theta(x) = \sin x$ if $|x - k\pi| \le \eta$, $k \in \mathbb{Z}$,
(b) $\theta(x) = \sin \eta$ if $\eta \le x \le \pi - \eta$.

Then the Fourier coefficients γ_k, $k \in \mathbb{Z}$, of $\theta(x)$ satisfy

$$\sum_{k \in \mathbb{Z}} |\gamma_k| \le C\eta \log(1/\eta) \tag{7.77}$$

where C is a numerical constant.

The proof is straightforward and will be omitted.

Corollary 8. *For every real number x one has*

$$|\sin x| \le \sin \eta \Rightarrow \sin x = \sum_{k \in \mathbb{Z}} \gamma_k \exp(ikx)$$

where the Fourier coefficients γ_k have a small l^1 norm as indicated in (7.77).

This immediately implies the following:

Corollary 9. *There exists a constant C such that for every $x \in \mathbb{R}$ and every $\varepsilon > 0$,*

$$|\exp(ix) - 1| \le \varepsilon \Rightarrow \exp(ix) - 1 = 2i \exp(ix/2) \sum_{-\infty}^{\infty} \gamma_k \exp(ikx/2) \tag{7.78}$$

where the Fourier coefficients γ_k satisfy (7.77) with $2\sin(\eta/2) = \varepsilon$.

One writes $\exp(ix)-1 = 2i\exp(ix/2)\sin(x/2)$ and uses Lemma 64. In what follows the factor $2i\exp(ix/2)$ in front of (7.78) will be incorporated in the Fourier series expansion which changes the meaning of the coefficients γ_k without any modification in the claim.

Corollary 10. *For every* $\varepsilon > 0$, $f \in C_\Lambda$ *and* $\tau \in \Lambda_\varepsilon^*$

$$\|f(x+\tau)-f(x)\|_\infty \leq C\varepsilon\log(1/\varepsilon)\|f\|_\infty \tag{7.79}$$

The proof is simple. We assume that f is a finite trigonometric sum and write $f(x) = \sum c(\lambda)\exp(i\lambda x)$. The definition of Λ_ε^* and Corollary 10 imply $f(x+\tau)-f(x) = \sum\sum c(\lambda)\exp(i\lambda x)[\exp(i\lambda\tau)-1] = \sum\sum c(\lambda)\gamma_k\exp(i\lambda(x+k\tau/2)) = \sum\gamma_k f(x+k\tau/2)$. Finally $\|f(x+\tau)-f(x)\|_\infty \leq \|f\|_\infty\sum|\gamma_k|$ which ends the proof of Corollary 11. Keeping the same notations we have

Corollary 11. *Let us assume that* $\kappa = C\varepsilon\log(1/\varepsilon) < 1$ *and that* Λ_ε^* *is a Delone set. Then* Λ *is a coherent set of frequencies.*

Indeed let $T > 0$ be defined by $[0,T]+\Lambda_\varepsilon^* = \mathbb{R}$. Therefore every $x \in \mathbb{R}$ can be written $x = y+\tau,\ \tau \in \Lambda_\varepsilon^*,\ y \in [0,T]$. Corollary 10 implies

$$|f(y+\tau)-f(y)| \leq \kappa\|f\|_\infty$$

for every $f \in C_\Lambda$. This yields

$$|f(x)| \leq \sup_{y\in[0,T]}|f(y)|+\kappa\|f\|_\infty.$$

Finally $\|f\|_\infty \leq \sup_{y\in[0,T]}|f(y)|+\kappa\|f\|_\infty$ and a simple bootstrap yields the required result since $\kappa < 1$. This proves Theorem 21.

This argument can be improved. We follow [27] and prove that Λ is a coherent set of frequencies under a weaker hypothesis than the one used in Corollary 11. An improved version of Lemma 64 will be used for achieving this goal. We first consider an auxiliary function $\omega(x)$ defined by the following properties

(a) $\omega(x)$ is $2\pi/5$-periodic
(b) $\omega(x)$ is an even function
(c) $\omega(x) = \sin(\pi/10-x)$ on $[0,\pi/5]$.

Then a brute force calculation yields

Lemma 65. *The Fourier series expansion of* $\omega(x)$ *is given by*

$$\omega(x) = \sum\alpha_k\exp(i5kx)$$

where the Fourier coefficients α_k *are non-negative. We have*

$$\sum\alpha_k = \sin(\pi/10) \tag{7.80}$$

To prove Lemma 65 it suffices to observe that

$$\omega(x) + (\frac{d}{dx})^2 \omega(x) = -2\cos(\pi/10)\delta_0 + 2\cos(\pi/10)\delta_{\pi/5}.$$

Therefore $2(\sum|\beta_k|)\|f\|_\infty$. The definition of β_k and Lemma 65 yields $\kappa = 2\sum|\beta_k| = 2\sum\alpha_k = 2\sin(\pi/10) = \frac{\sqrt{5}-1}{2} = 0.6180339\cdots < 1$. We then obtain

Theorem 22. *Let Λ be a set of real numbers. Let us assume that Λ^*_ε, $\varepsilon = \frac{\sqrt{5}-1}{2}$, is a Delone set. Then Λ is a coherent set of frequencies.*

We know from Section 7.8 that such sets Λ are not harmonious in general. It is likely that the critical value $\varepsilon = \frac{\sqrt{5}-1}{2}$ can be replaced by a larger one. If instead of ω one used the function ω_0 which is even and $2\pi/3$ periodic with $\omega_0 = \sin(\pi/6 - x)$ on $[0, \pi/3]$, the Fourier coefficients $\tilde{\alpha}_k$ of ω_0 would still be non-negative. But we have $\kappa = \sum\tilde{\alpha}_k = 2\sin(\pi/6) = 1$ which is forbidden by the bootstrap argument. Finally one observes that Theorem 22 is implied by Theorem 20 when Λ is a Delone set. Indeed $\sin(\pi/10) < 1/4$.

Let us now consider the case where we are given a real number $\theta > 1$ and where $\Lambda(\theta)$ is the set of all finite sums $\lambda = \sum_{k\geq 0}\varepsilon_k\theta^k$; $\varepsilon_k \in \{0,1\}$. This set played a key role in elucidating the problem of spectral synthesis for the Cantor type set constructed with the dissection ratio $1/\theta$ [27].

Theorem 23. *The following two properties are equivalent ones*

(a) every $f \in C_{\Lambda(\theta)}$ is an almost periodic function in the sense of Bohr
(b) θ is a Pisot number.

The implication (a)$\Rightarrow$ (b) is not difficult. We use the following lemma.

Lemma 66. *Let us assume that θ is not a Pisot number. Then $P_n(x) = \Pi_0^{n-1}$ $\cos(\theta^k x/2)$ tends to 0 uniformly on compact sets not containing 0.*

Indeed the sequence $|P_n(x)|$ is decreasing and it suffices to prove that $P_n(x_0) \to 0$ for every $x_0 \neq 0$ for obtaining the required uniform convergence. We use the simple observation that an infinite product $\Pi_0^\infty(1-\varepsilon_k)$ converges to 0 when $0 \leq \varepsilon_k < 1$ and $\sum_0^\infty \varepsilon_k = +\infty$. Then the required pointwise convergence is implied by a famous theorem dued to Charles Pisot. This theorem says that the following two properties are equivalent ones:

(a) θ is a Pisot number
(b) There exists a $\alpha \neq 0$ such that for $k \in \mathbb{N}$,

$$\alpha\theta^k = m_k + \eta_k,\ m_k \in \mathbb{N},\ \sum_k |\eta_k|^2 < \infty \tag{7.81}$$

We have $|P_n| = |Q_n|$ where $Q_n(x) = \Pi_0^{n-1}\left(\frac{1+\exp(i\theta^k x)}{2}\right)$. Therefore $Q_n \in C_{\Lambda(\theta)}$ converges uniformly to 0 on any compact set not containing 0 while $Q_n(0) = 1$.

Piling up some translates of Q_n it is quite easy to construct an unbounded $f \in C_{\Lambda(\theta)}$ of the form $f(x) = \sum_{k\geq 0} a_k Q_{n_k}(x - x_k)$.

For proving (b)⇒(a) in Theorem 23, we prove a stronger statement. In fact $\Lambda(\theta)$ is a harmonious set when θ is a Pisot number (see the second proof of Theorem 17) and every harmonious set is a coherent set of frequencies, as Theorem 21 indicates.

7.11 Poisson Summation Formula and Model Sets

Let Λ be a model set defined as above by a lattice $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ and a compact set $K \subset \mathbb{R}^m$. We let H denote the group $p_1(\Gamma^*)$ where Γ^* is the dual lattice of Γ. Let us assume K to be Riemann-integrable with a positive measure and let φ denote any $C_0^\infty(\mathbb{R}^m)$ function vanishing outside K.

The corresponding weight factors $w(\lambda)$, $\lambda \in \Lambda$, are defined on the model set Λ by $w(p_1(\gamma)) = \varphi(p_2(\gamma))$, $\gamma \in \Gamma$. If φ was the indicator function of K (this indicator function is not smooth), we would have $w(\lambda) = 1$ on Λ.

With these notations, one obtains

Theorem 24. *Let μ be the sum $\sum_{\lambda\in\Lambda} w(\lambda)\delta_\lambda$ of Dirac masses over Λ where the weight factors $w(\lambda)$ are defined as above. Then the distributional Fourier transform $\nu = \hat{\mu}$ of μ is the atomic measure ν defined by*

$$\nu = \sum_{\Gamma^*} \omega(p_2(\gamma^*))\,\delta_{p_1(\gamma^*)} \tag{7.82}$$

where the dual weights $\omega(p_2(\gamma^*))$ are

$$\omega(y) \;=\; \frac{(2\pi)^n}{\operatorname{vol}\Gamma}\,\hat{\varphi}(-y), \quad y = p_2(\gamma^*)\,,\ \gamma^* \in \Gamma^*\,. \tag{7.83}$$

For proving Theorem 24 it suffices to show that $\int \hat{u}\,d\mu = \int u\,d\nu$ for every testing function u. This reads

$$\sum_{\lambda\in\Lambda} w(\lambda)\hat{u}(\lambda) = \sum_{\Gamma^*} \omega(p_2(\gamma^*))\,u(p_1(\gamma^*)) \tag{7.84}$$

But $w(\lambda) = w(p_1(\gamma)) = \varphi(p_2(\gamma))$ and one can forget the restriction $\lambda \in \Lambda$ which is given for free by the support of φ. Then (7.58) follows from the ordinary Poisson formula applied to the lattice Γ and the dual lattice Γ^*.

If one could find a testing function $\varphi \in C_0^\infty(\mathbb{R}^m)$ with a compactly supported Fourier transform, $\hat{\mu}$ would then be supported by a variant of the dual quasicrystal Λ^* of Λ. This is not the case. But let us assume that K is a convex set, symmetric with respect to 0 and Riemann-integrable. Then the dual quasicrystal $\Lambda^* = \Lambda_1^*$ is defined by the convex set K^* which is the dual of K. For each integer $m \geq 1$, let us define Λ_m^* using the enlarged window mK^* instead of the window K^*, p_1, p_2 and Γ^* being kept fixed. This notation differs from the one we used in Section 7.8.

Then Theorem 24 yields $\hat{\mu} = \sum_{m\geq 1} \nu_m$ where ν_m is supported by Λ_m^* and where this series has a rapid decay. More precisely

$$|\nu_m|(B) \;=\; O(m^{-N}) \tag{7.85}$$

for every ball B and each integer N.

Corollary 12. *The measure μ defined by Theorem 24 is a Poisson measure.*

This is obvious. We have more

Corollary 13. *Let Λ be a model set. Then the measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a generalized almost periodic measure.*

For each positive ε we construct two testing functions u_ε and v_ε such that $0 \leq u_\varepsilon \leq \chi_K \leq v_\varepsilon$ and $\int (v_\varepsilon - u_\varepsilon)\,dx \leq \varepsilon$. We then construct the measures μ_ε and ν_ε in terms of u_ε and v_ε as in Theorem 24 and we obviously have $\mu_\varepsilon \leq \sigma_\Lambda \leq \mu_\varepsilon$. It suffices to observe that $\mathscr{M}[\nu_\varepsilon - \mu_\varepsilon] = \int (v_\varepsilon - u_\varepsilon)\,dx \leq \varepsilon$.

Proposition 12. *Let Λ be a model set. Then the Fourier coefficients of the generalized almost periodic measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ are given by*

(a) $\hat{\sigma}_\Lambda(\omega) \;=\; \frac{1}{\mathrm{vol}\,\Gamma} \int_K \exp(i\omega\cdot x)\,dx$ *if* $\omega = p_2(\gamma^*)$, $\gamma^* \in \Gamma^*$
(b) $\hat{\sigma}_\Lambda(\omega) = 0$ *if* $\omega \notin p_2(\Gamma^*)$.

Proposition 12 follows immediately from Lemma 40.

Corollary 14. *If Λ is a model set and f is an almost periodic measure whose spectrum is contained in $H = p_1(\Gamma^*)$. Let G be the dual group of H and F the continuation of f to G. Then we have*

$$\mathscr{M}[f\mu] = \int_K F(x)\,dx \tag{7.86}$$

We now return to the notations used in Section 7.8. The role played by the ε-dual in the diffraction pattern of a model set Λ can be explained by the following observation. Let $f(x) = \sum_{\lambda\in\Lambda} w(\lambda)e^{i\lambda\cdot x}$ where the weights $w(\lambda)$, $\lambda \in \Lambda$, are nonnegative and $\sum_{\lambda\in\Lambda} w(\lambda) = 1$. Then we obviously have $|f(x)| \leq 1 = f(0)$.

Lemma 67. *The continuous function $|f(x)|$ almost attains its maximum on the dual model set. Indeed we have*

$$x \in \Lambda_\varepsilon^* \Rightarrow \Re f(x) \geq 1 - \frac{\varepsilon^2}{2}.$$

This observation explains why the Fourier transform of the measure σ_Λ peaks on the dual model set Λ_ε^*.

7.12 Algebras of Generalized Almost Periodic Measures

The notations of the preceding sections being kept we define a functional space $\mathscr{B}$ of Riemann integrable functions. A real valued function φ belongs to $\mathscr{B}$ if and only if for every positive ε there exist two functions u_ε and v_ε in the Schwartz class $\mathscr{S}(\mathbb{R}^m)$ such that $u_\varepsilon \leq \varphi \leq v_\varepsilon$ and $\int (v_\varepsilon - u_\varepsilon)\, dx \leq \varepsilon$. Let, for instance, K be a compact subset of $\mathbb{R}^m$ and let us assume that the boundary of K has a zero Lebesgue measure. Then the indicator function of K belongs to $\mathscr{B}$. It is easily proved that $\varphi \in \mathscr{B}$ implies $|\varphi| \in \mathscr{B}$. Moreover $\mathscr{B}$ is closed under convolution. A complex valued function φ belongs to $\mathscr{B}$ if and only if its real and imaginary parts belong to $\mathscr{B}$. If $\varphi \in \mathscr{B}$, we define the atomic measure

$$\mu_\varphi = \sum_{\gamma \in \Gamma} \varphi(p_2(\gamma))\delta_{p_1(\gamma)} \tag{7.87}$$

Lemma 68. *The atomic measure μ_φ is a generalized almost periodic measure and its Fourier coefficients are*

$$\hat{\mu}_\varphi(y) \;=\; \frac{1}{\operatorname{vol}\Gamma}\,\hat{\varphi}(-y), \quad y = p_2(\gamma^*)\,,\, \gamma^* \in \Gamma^* \tag{7.88}$$

The proof is the same as in Corollary 13. We denote by $\mathscr{A}$ the collection of these measures μ_φ. Then $\mathscr{A}$ is closed under the convolution product between g-a-p measures and the mapping $T : \varphi \mapsto \mathscr{A}$ satisfies $T(f * g) = T(f) \circledcirc T(g)$. Finally we define $\tilde{f}$ by $\tilde{f}(x) = \overline{f}(-x)$ when f is a function or a measure and we have $T\tilde{\varphi} = \tilde{T}(\varphi)$. This applies to the calculation of the autocorrelation $\mu \circledcirc \tilde{\mu}$ when $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ where Λ is a model set. We then have $\mu = \mu_{\chi_K}$ which implies $\nu = \mu \circledcirc \tilde{\mu} = \mu_\theta$ where $\theta = \chi_K * \tilde{\chi}_K$. This continuous function θ is compactly supported and its Fourier transform belongs to L^1. Therefore ν is a Poisson measure.

7.13 Model Sets and Irregular Sampling

Let $K \subset \mathbb{R}^n$ be a compact set and $E_K \subset L^2(\mathbb{R}^n)$ be the translation invariant subspace of $L^2(\mathbb{R}^n)$ consisting of all $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{f}(\xi) = \int \exp(-2\pi i x \cdot \xi) f(x)\, dx$ is supported by K. Note the 2π in the definition of the Fourier Transform. Let us assume that the measure of the compact set K does not exceed some fixed small constant β. The problem which is addressed here is to take advantage of this property in order to sample f efficiently.

H.J. Landau in [16] studies the sampling and the interpolation of functions $f \in E_K$. He introduce in [16] the definitions of stable sampling and stable interpolation. We now follow [16].

Definition 32. A set $\Lambda \subset \mathbb{R}^n$ is a set of stable sampling for E_K if there exists a constant C such that

$$f \in E_K \Rightarrow \|f\|_2^2 \leq C \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \tag{7.89}$$

A set $\Lambda \subset \mathbb{R}^n$ is a set of stable interpolation for E_K if every sequence $c(\lambda) \in l^2(\Lambda)$ is the restriction to Λ of a function $f \in E_K$.

H.J. Landau [16] discovered necessary conditions for stable sampling and stable interpolation. These conditions are relating the upper or lower density of Λ to the measure of K. Let us recall their definitions.

Definition 33. If $\Lambda \subset \mathbb{R}^n$ is a collection of points, the upper density of Λ is defined as

$$\overline{\mathrm{dens}}\Lambda = \overline{\lim}_{R\to\infty} c_n R^{-n} \sup_{x\in\mathbb{R}^n} \#\{\Lambda \cap B_{x,R}\} \tag{7.90}$$

where $B_{x,R}$ is the ball centered at x with radius R, c_n is the inverse of the volume of the unit ball, and $\#E$ denotes the cardinality of E. In this definition balls could be replaced by cubes as well [16]. The lower density is defined similarly :

$$\underline{\mathrm{dens}}\Lambda = \underline{\lim}_{R\to\infty} c_n R^{-n} \inf_{x\in\mathbb{R}^n} \#\{\Lambda \cap B_{x,R}\} \tag{7.91}$$

A collection Λ of points has a uniform density (denoted by $\mathrm{dens}\,\Lambda$) if the upper density and the lower density of Λ coincide.

Here is Landau's theorem.

Theorem 25. *If Λ be a set of stable sampling for E_K, then*

$$\underline{\mathrm{dens}}\Lambda \geq |K|. \tag{7.92}$$

Similarly if Λ be a set of stable interpolation for E_K, then

$$\overline{\mathrm{dens}}\Lambda \leq |K|. \tag{7.93}$$

These necessary conditions are not sufficient. The main result of this section shows that Landau's necessary conditions are sufficient for some sets of points Λ. To this end, we also need the following.

Definition 34. A model set or quasicrystal is simple if $m = 1$ and $K = I$ is an interval in Definition 24.

Note that now we fix the definition of a model set. In the sequel a model set is defined by a cut and project scheme. In cite[8] the main result is the following.

Theorem 26. *Let $\Lambda \subset \mathbb{R}^n$ be a simple model set and $K \subset \mathbb{R}^n$ be a compact set. Then*

(a) $|K| < \mathrm{dens}\,\Lambda$ implies that Λ is a set of stable sampling for E_K.
(b) If K is Riemann integrable, then $|K| > \mathrm{dens}\,\Lambda$ implies that Λ is a set of stable interpolation for E_K.

A lattice cannot be a simple model set. This is fortunate since a lattice cannot be a "universal sampling set," see the results obtained by A. Olevskii and A. Ulanovskii [35] and [36]. A model set is not a random set, this is again fortunate since a random set cannot be a "universal sampling set," see the results of G. Chistyakov, Y Lyubarskii, R. F. Bass, and K. Gröchenig [1] and [3].

In this essay we will not detail the proof of this result. The interested reader can consult [21] and [22]. A similar problem was studied in periodic setting in [23].

Let us indicate the strategy of the proof of (a) in Theorem 26. It will be assumed that Λ is symmetric which simplifies some calculations. The interested reader will have no difficulty to extend the proof to the general case. The proof of (a) is using two ingredients, namely (i) Ingham inequality and (ii) transference methods. Ingham inequality gives a solution to the problem of stable interpolation in the one dimensional case. We wanted to stress this surprising fact and that explains why we recall now the famous result which was proved by A. Ingham in 1936 [7] and which says the following :

Theorem 27. *Let $\beta > 0$ and Λ be an increasing sequence $\lambda_j,\ j \in \mathbb{Z}$, of real numbers such that $\lambda_{j+1} - \lambda_j \geq \beta,\ j \in \mathbb{Z}$. Let I be any interval with length $|I| > 1/\beta$. Then we have*

$$C\sum |c_j|^2 \leq \int_I |\sum c_j \exp(2\pi i \lambda_j t)|^2\, dt \tag{7.94}$$

where $C = \frac{2}{\pi}(1 - \frac{1}{|I|^2\beta^2})$.

This estimate of the constant C is not optimal. The condition $|I| > 1/\beta$ cannot be replaced by $|I| < 1/\beta$ and Theorem 27 does not tell anything in the limiting case $|I| = 1/\beta$.

Theorem 27 was generalized in 1950 by A. Beurling (see [2]) who proved the following :

Proposition 13. *Let $\beta > 0,\ \alpha > 0$ and $\Lambda = \{\lambda_j,\ j \in \mathbb{Z}\}$ be an increasing sequence of real numbers fulfilling the following two conditions*

(a) $\lambda_{j+1} - \lambda_j \geq \alpha > 0$
(b) there exists an integer $T \geq 1$ such that for every $j \in \mathbb{Z}$ we have $\lambda_{j+T} - \lambda_j \geq \beta T$.

Let I be any interval with length $|I| > 1/\beta$. Then we have

$$\sum |c_j|^2 \leq C \int_I |\sum c_j \exp(2\pi i \lambda_j t)|^2\, dt \tag{7.95}$$

where $C = C(\beta, \alpha, T, |I|)$.

Here the requirement on the length of I only depends on the averaged distance between λ_{j+1} and λ_j. Let us observe that (a) is a necessary condition for (7.95). Using the general tools introduced by Jean-Pierre Kahane (see [8]), Beurling's theorem can be deduced from Ingham's inequality. This deduction is explained in Kahane's paper. The result which is used in the proof of Theorem 26 is the following :

Theorem 28. *Let Λ be an increasing sequence $\lambda_j,\ j\in\mathbb{Z}$, of real numbers such that $\lambda_{j+1}-\lambda_j\ge\alpha>0$ and let*

$$\overline{\mathrm{dens}}\,\Lambda=\lim_{R\to\infty}R^{-1}\sup_{x\in\mathbb{R}}\mathrm{card}\{\Lambda\cap[x,x+R]\}\tag{7.96}$$

be the upper density of Λ. Then for any interval I fulfilling $|I|>\overline{\mathrm{dens}}\,\Lambda$ there exists a constant C such that

$$\sum|c_j|^2\le C\int_I|\sum c_j\exp(2\pi i\lambda_j t)|^2\,dt\tag{7.97}$$

Theorem 28 easily follows from Proposition 13.

Corollary 15. *If $|I|>\overline{\mathrm{dens}}\,\Lambda$, there exists a constant C such for any square summable sequence $c_j,\ j\in\mathbb{Z}$, there exists a function $f\in L^2(\mathbb{R})$ which is supported by the interval I such that $\hat f(\lambda_j)=c_j,\ j\in\mathbb{Z}$ and*

$$\|f\|_2\le C(\sum_{-\infty}^{+\infty}|c_j|^2)^{1/2}.\tag{7.98}$$

The constant C in (7.98) depends not only on α and $\overline{\mathrm{dens}}\,\Lambda$ but also on the speed at which the upper density is reached in (7.96).

The second ingredient is the transference method of Coifman and Weiss (see [4]) which permits to transfer L^2 estimates on an interval I to l^2 estimates for the quasicrystal defined by I. This piece of the argument is fully detailed in [21]. More precisely, let $\Gamma\subset\mathbb{R}^n\times\mathbb{R}$ be a lattice and if $(x,t)\in\mathbb{R}^n\times\mathbb{R}$, let us write $p_1(x,t)=x$, $p_2(x,t)=t$. We now assume that p_1 once restricted to Γ is an injective mapping onto $p_1(\Gamma)=\Gamma_1$. We make the same assumption on p_2. We furthermore assume that $p_1(\Gamma)$ is dense in $\mathbb{R}^n$ and $p_2(\Gamma)$ is dense in $\mathbb{R}$. The dual lattice of Γ is denoted Γ^* and is defined by $x\cdot y\in\mathbb{Z}$, $x\in\Gamma$, $y\in\Gamma^*$. We use the following notations. For $\gamma=(x,t)\in\Gamma$ we write $t=\tilde x$, $\ \tilde t=x$ and similarly for a function f defined on $p_2(\Gamma)$, we write $\tilde f(x)=f(\tilde x)$. The same notations are used for the two components of $\gamma^*\in\Gamma^*$. If $I=[-\alpha,\alpha]$, let

$$\Lambda_I=\{p_1(\gamma);\gamma\in\Gamma,\,p_2(\gamma)\in I\}.\tag{7.99}$$

Let us define

$$M_K=\{p_2(\gamma^*);\gamma^*\in\Gamma^*,\,p_1(\gamma^*)\in K\}.\tag{7.100}$$

Following Meyer's theory for model sets, the density of Λ_I is uniform and is given by $c|I|$ where $c=c(\Gamma)$ and similarly the density of M_K is $c'|K|$ (see [26] and [27]). After that, we sort the elements of M_K in increasing order and denote the corresponding sequence by $\{m_k;k\in\mathbb{Z}\}$. Then we have

Lemma 69. *The sequence $\{\tilde m_k;k\in\mathbb{Z}\}$ is equidistributed on K.*

We now sketch the proof of the first claim of the Theorem 26.
We replace K by a larger compact set still denoted by K which is Riemann integrable and satisfies $|K| < \operatorname{dens}\Lambda$. By density, we only consider the case $\hat{f} \in \mathscr{C}_0^\infty(K)$. As a consequence of Lemma 69 we get

$$\frac{1}{|K|}\|\hat{f}\|_2^2 = \lim_{T\to\infty}\frac{1}{2T}\sum_{k=-T}^{T}|\hat{f}(\tilde{m}_k)|^2. \tag{7.101}$$

The right-hand side in (7.101) is given by

$$\lim_{\varepsilon\downarrow 0}\varepsilon\sum_{k\in\mathbb{Z}}|\varphi(\varepsilon m_k)|^2|\hat{f}(\tilde{m}_k)|^2, \tag{7.102}$$

where $\varphi \in \mathscr{S}(\mathbb{R})$ and $\varphi(0) = 1$. At this stage we use the auxiliary function of the real variable x defined as

$$F_\varepsilon(x) = \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}}\varphi(\varepsilon m_k)\hat{f}(\tilde{m}_k)\exp(2\pi i m_k x). \tag{7.103}$$

We denote by ϕ the Fourier transform of φ. We will suppose that $\phi \in \mathscr{C}_0^\infty([-1,1])$ is a positive and even function. But $|K| < \operatorname{dens}\Lambda_I = c_0|I|$ implies $|I| > \operatorname{dens}M_K$. Therefore Beurling's theorem applies to the interval I, to the set of frequencies M_K and to the trigonometric sum defined in (7.103). Then one has

$$\varepsilon\sum_{k\in\mathbb{Z}}|\varphi(\varepsilon m_k)|^2|\hat{f}(\tilde{m}_k)|^2 \le C\int_I |F_\varepsilon(x)|^2 dx. \tag{7.104}$$

Let us compute the limit as $\varepsilon \to 0$ of the term in the right-hand side of (7.104). To this aim, we use the definition of M_K and write

$$F_\varepsilon(x) = \sqrt{\varepsilon}\sum_{\gamma^*\in\Gamma^*}\varphi(\varepsilon p_2(\gamma^*))\hat{f}(p_1(\gamma^*))\exp(2\pi i p_2(\gamma^*)x). \tag{7.105}$$

The Poisson identity reads $\sum_{\gamma\in\Gamma} u(\gamma) = c(\Gamma)\sum_{\gamma^*\in\Gamma^*}\hat{u}(\gamma^*)$ where $u \in \mathscr{S}(\mathbb{R}^n\times\mathbb{R})$ and yields

$$F_\varepsilon(x) = c(\Gamma)\frac{1}{\sqrt{\varepsilon}}\sum_{\gamma\in\Gamma}\phi\left(\frac{x-p_2(\gamma)}{\varepsilon}\right)f(p_1(\gamma)). \tag{7.106}$$

It remains to calculate

$$\lim_{\varepsilon\downarrow 0}\int_I |F_\varepsilon(y)|^2 dy, \tag{7.107}$$

where F_ε is given by (7.106). To this end, we notice that all terms in the right-hand side of (7.106) such that $|p_1(\gamma)| \ge \alpha+\varepsilon$ vanish on $I = [-\alpha,\alpha]$. Indeed the support of ϕ is contained in $[-1,1]$. We can restrict the summation to the set $\Lambda_{I,\varepsilon} = \{p_1(\gamma); \gamma\in\Gamma, |p_2(\gamma)| \le \alpha+\varepsilon\}$. For $0 \le \varepsilon \le 1$ we have

$$\lim_{\varepsilon\to 0}\Lambda_{I,\varepsilon} = \Lambda_I \text{ and } \Lambda_{I,\varepsilon} \subset \Lambda_{I,1}. \tag{7.108}$$

We split F_ε into a sum $F_\varepsilon = F_\varepsilon^N + R_N$ where

$$F_\varepsilon^N(x) = \frac{1}{\sqrt{\varepsilon}} \sum_{\gamma\in\Gamma, |p_1(\gamma)|\leq N, |p_2(\gamma)|\leq \alpha+\varepsilon} \phi\left(\frac{x-p_2(\gamma)}{\varepsilon}\right) f(p_1(\gamma)), \tag{7.109}$$

and

$$R_N(x) = \frac{1}{\sqrt{\varepsilon}} \sum_{\gamma\in\Gamma, |p_1(\gamma)|> N, |p_2(\gamma)|\leq \alpha+\varepsilon} \phi\left(\frac{x-p_2(\gamma)}{\varepsilon}\right) f(p_1(\gamma)). \tag{7.110}$$

The triangle inequality yields $\|R_N\|_2 \leq \varepsilon_N \|\phi\|_2$ with

$$\varepsilon_N = \sum_{\gamma\in\Gamma, |p_1(\gamma)|> N, |p_2(\gamma)|\leq \alpha+1} |f(p_1(\gamma))|^2. \tag{7.111}$$

For the term (7.109) the estimations are more involved. Since $|p_1(\gamma)| \leq N$, the points $p_2(\gamma)$ appearing in (7.109) are separated by a distance $\geq \beta_N > 0$. If $0 < \varepsilon < \beta_N$, the different terms in (7.109) have disjoint supports which implies

$$\|F_\varepsilon^N\|_{L^2(I)} \leq \sigma(N,\varepsilon)\|\phi\|_2 \tag{7.112}$$

where

$$\sigma(N,\varepsilon) = \sum_{\gamma\in\Gamma, |p_1(\gamma)|\leq N, |p_2(\gamma)|\leq \alpha+\varepsilon} |f(p_1(\gamma))|^2.$$

If ε is small enough, we have

$$\{\gamma \in \Gamma, |p_1(\gamma)| \leq N, |p_2(\gamma)| \leq \alpha+\varepsilon\} = \{\gamma \in \Gamma, |p_1(\gamma)| \leq N, |p_2(\gamma)| \leq \alpha\}.$$

and $\sigma(N,\varepsilon) = \sigma_{N,0}$. Therefore

$$\lim_{\varepsilon\to 0} \int_I |F_\varepsilon(y)|^2 dy \leq \sum_{\lambda\in\Lambda_I} |f(\lambda)|^2 + \eta_N, \tag{7.113}$$

and letting $N \to \infty$ we obtain the first claim.
This sketch of the proof of Theorem 26 explains why we have been unable to extend the argument to general quasicrystals. Indeed Ingham theorem is missing in several dimensions. The proof of (b) is simpler and is given in [21]. Here also the proof relies on the fact that the problem of stable sampling is solved for intervals. Once more the transference method relates sampling and interpolation. In the periodic case which was solved in [23] the proofs are much simpler since the complement of a set Λ of stable sampling for E_K is a set M of stable interpolation for E_U where $U = \mathbb{T}^n \setminus K$.

Finally we investigate the issue of retrieving a function f from its samples on a quasicrystal Λ when we only know that the measure of the closed support of $\hat{f}$ is less than β, this support being unknown. Partial answers will be given in 7.12.

7.14 An Open Problem

Let us fix a parameter $\beta \in (0,1/2)$ and define a collection $\mathbf{M}_\beta$ of functions $f \in L^2(\mathbb{R}^n)$ as follows: we write $f \in \mathbf{M}_\beta$ if $\hat{f}$ is supported by a compact set $K \subset \mathbb{R}^n$ whose measure $|K|$ does not exceed β. This compact set K depends on f and $\mathbf{M}_\beta$ is not a vector space. If f, g belong to $\mathbf{M}_\beta$, then $f+g$ belongs to $\mathbf{M}_{2\beta}$, a situation which is classical in nonlinear approximation.

Lemma 70. *For every $\beta \in (0,1/2)$ and every $d > 2\beta$ there exists a set $\Lambda \subset \mathbb{R}^n$ with the following properties :*

(a) density $\Lambda = d$
(b) the mapping $\Phi : \mathbf{M}_\beta \mapsto \ell^2(\Lambda)$ defined by $\Phi(f) = (f(\lambda))_{\lambda\in\Lambda_\alpha}$ is one-to-one.

Any $f \in \mathbf{M}_\beta$ can be retrieved from the information given by the "irregular sampling" $f(\lambda) = a(\lambda)$, $\lambda \in \Lambda$.

The open problem is to achieve this retrieving by some efficient and fast algorithm. We now prove Lemma 70.

We set $\alpha = \frac{d}{2\Gamma}$ and let $\Lambda = \Lambda_\alpha$ be a simple quasicrystal as in Theorem 26. If $f, g \in \mathbf{M}_\beta$ and $f = g$ on Λ, then $f - g \in \mathbf{M}_{2\beta}$ and $f - g = 0$ on Λ. Then Theorem 26 implies $f = g$.

The condition $d > 2\beta$ in Lemma 70 is sharp. Indeed we have

Lemma 71. *If $\beta > d/2$, there exist two distinct functions in M_β which coincide on Λ.*

The proof of this remark can be found in [21]. We now return to the main problem in this section and formulate the following:

Theorem 29. *Let Λ be a simple symmetric quasicrystal as above with density d.*

(a) Let us assume $f \in \mathbf{M}_\beta$ with $\beta < d/2$ and $F = \hat{f} \geq 0$. If a function $g \in L^2(\mathbb{R}^n)$ satisfies $\hat{g} \geq 0$ and $g(\lambda) = f(\lambda)$, $\lambda \in \Lambda$, then $g = f$.
(b) As in (a) let us assume $f \in \mathbf{M}_\beta$ with $\beta < d/2$ and $F = \hat{f} \geq 0$. Then F is the unique solution u of the following variational problem

$$\inf\{\|u\|_1 ; u \in L^1(\mathbb{R}^n),\ \hat{u}(\lambda) = \hat{F}(\lambda),\ \lambda \in \Lambda\}. \tag{7.114}$$

In (a) we do not assume $g \in \mathbf{M}_\beta$. There are no restrictions in (b) on the sign of u and we do not assume $u \in \mathbf{M}_\beta$. Since $\Lambda = -\Lambda$ we can use in (b) the Fourier transform $\hat{u}$ of u instead of the inverse Fourier transform. The fully detailed proof of this result can be found in [21]

7.15 An Improvement

If Λ is a set of stable sampling for E_K any "band-limited" $f \in E_K$ can be reconstructed from its samples $f(\lambda)$, $\lambda \in \Lambda$, and this reconstruction can be achieved by a linear algorithm. Let us now prove this remark. Let $L^2(K)$ be the space of all restrictions to K of functions in $L^2(\mathbb{R}^n)$. The norm of $f \in L^2(K)$ is $(\int_K |f(x)|^2\,dx)^{1/2}$. Then $\Lambda \subset \mathbb{R}^n$ is a set of stable sampling for E_K if and only if the family of functions $\exp(2\pi i\lambda\cdot x), \lambda \in \Lambda$, is a frame of $L^2(K)$. Frames were defined by R. J. Duffin and A. C. Schaeffer in [5]. We list some properties in the general context of Hilbert spaces.

Definition 35. Let H be a Hilbert space and let us consider a family $\mathscr{C} = \{e_\lambda,\ \lambda \in \Lambda\}$ of vectors of H indexed by a set Λ. We say that $\mathscr{C}$ is a frame if the two following properties are satisfied

(a) The mapping $J : l^2(\Lambda) \mapsto H$ defined by $J((c_\lambda)) = \sum_{\lambda\in\Lambda} c_\lambda e_\lambda$ is continuous.
(b) This mapping J is onto.

In other words every $x \in H$ can be written as $x = \sum_{\lambda\in\Lambda} \alpha(\lambda) e_\lambda$ where $\sum_{\lambda\in\Lambda} |\alpha(\lambda)|^2 \le C\|x\|^2$. This decomposition is not unique in general. If the mapping $J : l^2(\Lambda) \mapsto H$ is an isomorphism, the family $\{e_\lambda,\ \lambda \in \Lambda\}$ is a Riesz basis of H. An equivalent definition is given by the following lemma.

Lemma 72. *A family $\mathscr{C} = \{e_\lambda,\ \lambda \in \Lambda\}$ of vectors of H indexed by a set Λ is a frame if and only if there exist two constants $C_2 > C_1 > 0$ such that for every $x \in H$*

$$C_1\|x\|^2 \le \sum_{\lambda\in\Lambda} |< x, e_\lambda >|^2 \le C_2\|x\|^2 \tag{7.115}$$

Here is a counterexample. Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of a Hilbert space H. Then this basis is obviously a frame but the family

$$\{e_0, e_1, e_0, e_1, e_2, e_0, e_1, e_2, e_3, e_0, e_1, e_2, e_3, e_4, e_0, \cdots\}$$

is not a frame of H. Indeed each e_k is repeated infinitely many times. This will not happen below and in all the examples which are discussed in this essay, the mapping $\lambda \mapsto e_\lambda$ from Λ to the set $\{e_\lambda,\ \lambda \in \Lambda\}$ is 1 to 1.

If 7.89 holds the operator $\mathscr{I} = JJ^* : H \mapsto H$ is an isomorphism. We set $f_\lambda = \mathscr{I}^{-1} e_\lambda,\ \lambda \in \Lambda$, and we have for every $x \in H$,

$$x = \sum_{\lambda\in\Lambda} < x, e_\lambda > f_\lambda = \sum_{\lambda\in\Lambda} < x, f_\lambda > e_\lambda \tag{7.116}$$

This family $f_\lambda,\ \lambda \in \Lambda$ is named the dual frame.

We now return to the property of stable sampling. The Hilbert space H is $L^2(K)$ and the operator $J : l^2(\Lambda) \mapsto L^2(K)$ is defined by

$$J(c(\lambda) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x). \tag{7.117}$$

Then we have

Lemma 73. *With the preceding notations Λ is a set of stable interpolation if and only if the family of functions* $\exp(2\pi i\lambda\cdot x),\ \lambda \in \Lambda$, *is a frame of* $L^2(K)$.

Let L^2_K denote the Hilbert space of all square integrable functions supported by K. The adjoint operator J^* maps $F \in L^2_K$ to the sequence of Fourier coefficients $\{\hat{F}(\lambda),\ \lambda \in \Lambda.\}$ This observation and 7.116 imply the following lemma.

Lemma 74. *If Λ is a set of stable sampling, there exists a dual frame $\phi_\lambda \in E_K$, $\lambda \in \Lambda$, such that for every $f \in E_K$ we have*

$$f(x) = \sum_{\lambda \in \Lambda} f(\lambda)\phi_\lambda(x). \tag{7.118}$$

Note that if Λ is a set of stable interpolation there shall exist a positive number α such that $|\lambda - \lambda'| \geq \alpha$ if $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Lambda$. We conclude this discussion with the following remark, if Λ is both a set of stable sampling and of stable interpolation for E_K, then the family of functions $\exp(2\pi i\lambda \cdot x), \lambda \in \Lambda$, is a Riesz basis of $L^2(K)$.

7.15.1 Wealthy Frames

As it was said before a frame $\mathscr{C}$ is a family of vectors which span the Hilbert space H. Moreover the energy of the decomposition of $x \in H$ is controlled by the norm of x. In most cases this family $\mathscr{C}$ is overcomplete. This leads to the following definition:

Definition 36. A family e_λ, $\lambda \in \Lambda$, of vectors of H is a wealthy frame if for every finite $F \subset \Lambda$, the subfamily e_λ, $\lambda \in \Lambda, \lambda \notin F$, is still a frame of H.

An overcomplete frame is not wealthy in general. Let us consider, for example, the union $\mathscr{C}$ between an orthonormal basis of a Hilbert space H and a finite set $E \subset H$. Then $\mathscr{C}$ is an overcomplete frame which is not a wealthy frame.

The following theorem yields a characterization of wealthy frames.

Theorem 30. *Let e_λ, $\lambda \in \Lambda$, be a frame of H, let $F \subset \Lambda$ be finite and $\Lambda' = \Lambda \setminus F$. Then the following properties are equivalent*

(a) The set $\{e_\lambda,\ \lambda \in \Lambda'\}$ is total in H.
(b) If $y \in H$ is orthogonal to every vector e_λ, $\lambda \in \Lambda'$, then $y = 0$.
(c) There exists a constant C such that for every $x \in H$ one has

$$\|x\|^2 \leq C \sum_{\lambda \in \Lambda'} |< x, e_\lambda >|^2$$

(d) $\{e_\lambda,\ \lambda \in \Lambda'\}$ is a frame of H.
(e) For every $\lambda_0 \in F$ there exists a sequence $\alpha(\lambda, \lambda_0) \in l^2(\Lambda')$ such that

$$e_{\lambda_0} = \sum_{\lambda \in \Lambda'} \alpha(\lambda, \lambda_0) e_\lambda$$

The equivalence between (a) and (b) is trivial. For proving $(a) \Rightarrow (c)$ we argue by contradiction and assume that a sequence $x_j \in H$ exists such that $\|x_j\| = 1$ and $\sum_{\lambda \in \Lambda'} |< x_j, e_\lambda >|^2 \leq j^{-2}$. Extracting a subsequence one can assume that $x_j \rightharpoonup y$. Then we have $< y, e_\lambda >= 0$ for every $\lambda \in \Lambda'$. This implies $y = 0$ by (b). Therefore for every $\lambda_0 \in F$ we have $< x_j, e_{\lambda_0} >\to 0$. We then use the Lemma 72 and conclude to $\|x_j\| \to 0$. The implication (c)$\Rightarrow$ (d) is also given by Lemma 72 and (e) is a rephrasing of (d). Finally (e)$\Rightarrow$ (a) is obvious.

The first statement in Theorem 26 can be improved.

Corollary 16. *If $F \subset \Lambda$ is any finite set, then $\Lambda \setminus F$ is still a set of stable sampling for E_K.*

This will be proved by induction on the cardinality m of F. More precisely we fix a simple quasicrystal Λ and denote by $\mathscr{P}_m$ the following property:

For every finite set F of cardinality not exceeding m and every compact set K whose measure $|K|$ satisfies $|K| < \operatorname{dens}\Lambda$ the set $\Lambda \setminus F$ is a set of stable sampling for E_K.

Property $\mathscr{P}_0$ is given for free by Theorem 26 We now prove that $\mathscr{P}_m \Rightarrow \mathscr{P}_{m+1}$. Let us assume that the cardinality of F equals $m+1$. Translating Λ if needed we can assume $0 \in F$. We write $F' = F \setminus \{0\}$, $\Lambda_m = \Lambda \setminus F'$, and $\Lambda_{m+1} = \Lambda \setminus F$. Let us observe that $\Lambda_{m+1} = \Lambda_m \setminus \{0\}$. Theorem 30 will be applied to the Hilbert space L^2_K of square integrable functions supported by K and to the frame of H consisting of the functions $\exp(2\pi i\lambda \cdot x)$, $\lambda \in \Lambda_m$. To prove that the family of functions $\exp(2\pi i\lambda \cdot x)$, $\lambda \in \Lambda_{m+1}$, is still a frame of H it suffices to prove that the only function $g \in L^2_K$ which is orthogonal to $\exp(2\pi i\lambda \cdot x), \lambda \in \Lambda_{m+1}$ is 0. We consider $h(x) = g(x) - g(x-y)$ where $|y| < \varepsilon$. Then the Fourier transform of h vanishes on Λ_m. Indeed $\hat{h}(0) = 0$. But if ε is small enough the compact set $K_\varepsilon = K + B(0,\varepsilon)$ still satisfies $|K_\varepsilon| < \operatorname{dens}\Lambda$. Since h is supported by K_ε we have $h = 0$ by the induction hypothesis. It implies that g is a periodic function which contradicts the assumption that g is supported by K. Therefore $g = 0$ as claimed.

7.16 Conclusion

The most elegant definition of a quasicrystal is the one using Diophantine approximations. If this is accepted, a tentative definition of a quasicrystal Λ is given by the following conditions:

Definition 37. We say that Λ is a quasicrystal if it satisfies the following three properties

(a) Λ is a Delone set
(b) For every $\varepsilon \in (0,1)$ the ε-dual set Λ^*_ε is also a Delone set.
(c) The ε-dual set of Λ^*_ε is Λ.

As above the ε-dual set Λ^*_ε is defined by $\{y \in \mathbb{R}^n : |e^{iy\cdot\lambda} - 1| \leq \varepsilon,\, \lambda \in \Lambda\}$.

This definition has some advantages and some drawbacks. Let us begin with the good news. The Diophantine approximation definition of quasicrystals explains the role played by Pisot and Salem numbers as *inflation factors* of quasicrystals [27], [28].

Moreover the Diophantine approximation characterization is exactly what is needed for proving that quasicrystals are coherent sets of frequencies.

Here comes the bad news. The weak point of this definition is the absence of characterization of such sets. As noticed above, $M = \mathbb{Z} + \{0, \sqrt{2}\}$ does not satisfy the third condition while $\Lambda = \mathbb{Z} + \{0, \pm\sqrt{2}\}$ does. But Λ is not a model set. Therefore a quasicrystal in that sense is not a model set in general. A model set does not in general fulfil the third requirement (see Corollary 7). Finally the class described by Definition 37 is not translation invariant, as Corollary 7 is showing. If Λ is defined by a convex and symmetric window, then it is a quasicrystal but it suffices to move Λ by $\tau = p_1(\gamma)$ with $p_2(\gamma)$ small enough to produce a counterexample.

A second tentative definition of a quasicrystal is given by the "cut and projection" algorithm (model sets). The following example shows how much this definition is natural in some cases. Let $\mathscr{K} \subset \mathbb{R}$ be an algebraic number field and let us consider the set S of all Pisot or Salem numbers in $\mathscr{K}$. This set S is a model set closed under multiplication: if θ and θ' belong to S, so does $\theta\theta'$. This example fits quite naturally in the framework of the "cut an projection" definition of a quasicrystal. But this approach is suffering from a severe drawback. The class of model sets is not translation invariant.

Almost lattices are providing a third tentative definition of quasicrystals. Almost lattices are defined by the two conditions

(a) $\Lambda - \Lambda \subset \Lambda + F$
(b) Λ is a Delone set.

Here as above, F is finite. This approach is striking by its elegant simplicity. But the diffraction pattern of an almost lattice does not have the expected properties which are the distinctive feature of a "true quasicrystal" (see Lemma 47 of Section 7.6).

We end this essay by mentioning some recent result in the world of quasicrystals. Recent results was obtained in the construction of Riesz Basis [10, 17] and [9], for the Poisson formula in [18].

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Chapter 8
Stylometry and Mathematical Study of Authorship

Xianfeng Hu, Yang Wang, and Qiang Wu

Abstract Inspired by various authorship attribution problems in the history of literature and the application of machine learning in the study of literary stylometry, we develop a rigorous new method for the mathematical analysis of authorship by testing for a so-called chrono-divide in writing styles. Our method incorporates some of the latest advances in the study of authorship attribution, particularly techniques from support vector machines. By introducing the notion of relative frequency as a feature ranking metric our method proves to be highly effective and robust.

Applying our method to the Cheng-Gao version of *Dream of the Red Chamber* has led to convincing if not irrefutable evidence that the first 80 chapters and the last 40 chapters of the book were written by two different authors.

Applying our method to the novel *Micro*, we are able to confirm the existence of the chrono-divide and identify its location so that we can differentiate the contribution of Michael Crichton and Richard Preston.

We have also tested our method to the other three Great Classical Novels in Chinese. As expected no chrono-divides have been found. This provides further evidence of the robustness of our method.

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8.1 Introduction

Did Francis Bacon or Christopher Marlowe write some of the plays that were attributed to William Shakespeare? Did Cao Xueqin write the last forty chapters of *Dream of the Red Chamber*, one of the greatest masterpieces in the history of Chinese literature? Who wrote the *Federalist* papers? Did Bill Ayers write Obama's autobiography *Dreams From My Father*? There have been no shortages of authorship controversies throughout the history, and these are just a few of them. These questions have often generated fierce debates among historians and literary scholars. Historically connoisseurship, disciplinary knowledge, and background history have been central in such debates. In recent years, however, the "unemotional approach" using quantitative analysis based on mathematics, statistics, and computation has gained prominence in the study of authorship attribution.

Broadly speaking, the authorship attribution problem is to determine the authorship of a given sample of text. There are two classes of problems in authorship attribution [7]. The most common one is the *close class*, where it is known that the sample of text is by one of the authors from a given set, usually a very small set with two to three authors. For example, *Federalist* papers mentioned earlier is a typical closed class problem, where the authors are confined to Alexander Hamilton, James Madison, and John Jay. Far more challenging is the *open class* problem, where the sample text may come from a far larger set of authors. An example of an open class problem is to determine the authorship of books written anonymously or under pseudonyms, e.g. *The Primary Color* or *Imperial Hubris: Why the West is Losing the War on Terror*. The study of authorship attribution has seen rapid growth in recent years. Still, there remains many challenges even for the closed class problem. Accuracy is one of the most prominent concerns. For the open class problem the challenge is far greater, and so far there have been few attempts and none of the known techniques can be reliably scaled to handle the open class problem.

There is yet another important class of authorship questions, namely to determine whether a body of text is written by a single author. Such a study is valuable in a number of ways. For example, it is widely speculated among the Shakespearean scholars that some of Shakespeare plays were in fact collaborate efforts between Shakespeare and others, e.g. Middleton and Fletcher. It is of scholarly interest to not only confirm it but also to find out exactly which parts of the plays were written by playwrights other than Shakespeare. There were also historical controversies involving suspected fraud, as in the case of *Dream of the Red Chamber*, where a particular book or sequel attributed to certain well-known author might in fact be perpetrated by someone else. A related practice was for a well-known and prolific author to write only the first few chapters of a book and then pass it on to a ghost writer. There are clear benefits both ethically and scholarly to detect frauds and ghost writings. Modern days see an explosion of coauthored books and articles. It would be interesting to detect stylistic inconsistencies among parts of such books. As we shall see, mathematics can play a central role in the study of authorship, leading to the rapid growth of the field *stylometry*.

The problem of style quantification and authorship attribution in the literature goes at least as far back as 1854 by the English mathematician Augustus De Morgan [3], who in a letter to a clergyman on the subject of Gospel authorship, suggested that the lengths of words might be used to differentiate authors. In 1897 the term *stylometry* was coined by the historian of philosophy, Wincenty Lutaslowski, as a catch-all for a collection of statistical techniques applied to questions of authorship and evolution of style in the literary arts (see, e.g., [9]). Today, literary stylometry is a well-developed and highly interdisciplinary research area that draws extensively from a number of disciplines such as mathematics and statistics, literature and linguistics, computer science, information theory, and others. It is a central area of research in statistical learning (see, e.g., [5]). A popular classic technique for stylometric analysis of authorship involves comparing frequencies of the so-called *function words*, a class of words that in general have little content meaning, but instead serve to express grammatical relationships with other words within a sentence. Although this technique is still widely used today, the field of literary stylometry has seen impressive advances in recent years, with more and more new and sophisticated mathematical techniques as well as softwares being developed. We shall not focus on these advances here. Instead we refer all interested readers to the excellent survey articles by Juola [7] and Stamatatos [11] for a comprehensive discussion of the latest advances in the field.

This paper focuses on the detection of multiple authors with a given body of text. We use two case studies to show ways it can be done, and to illustrate the effectiveness and robustness of our methods. The first case study is the classical Chinese novel *Dream of the Red Chamber*. The controversy surrounding the book was well known and intensely debated in the Chinese literary circle for over 250 years. The second case study is the book *Micro* written by Michael Crichton and Richard Preston. The research for the first case study has been done in our work [6], and much of what we present here is reproduced from [6]. All materials in the second case study are new.

The rest of the paper is organized as follows: In Section 8.2 we detail our methodology for detecting multiple authors, which includes feature construction and selection techniques in machine learning. In Section 8.3 we present the case study for *Dream of the Red Chamber*. In Section 8.4 we analyze other books as a comparison. We also present an alternative technique for detecting multiple authors, and applying it for our second case study of the book *Micro*. Finally in Section 8.5 we close with our conclusions.

8.2 Chrono-Divide and Methodology

The main idea behind statistically or computationally supported authorship attribution is that by measuring some textual features we can distinguish between texts written by different authors. Nearly a thousand different measures including sentence length, word length, word frequencies, character frequencies, and vocabulary richness functions had been proposed thus far [10] over the years. Some of these measures, such as frequencies of function words, have proven effective while others, such as length of words, have proven less effective [7]. The field of literary stylometry has seen impressive advances over the years, and has become an increasingly important research field in the digital age with the explosion of texts online.

This paper focuses on a particular class of authorship controversies, in which there is a suspected change of authorship at some point of a book. In other words, one suspects that the first X chapters of a book were written by one author while the remaining Y chapters were written by another. Clearly, the authorship controversy for *Dream of the Red Chamber* falls into this category. Since no two authors have exactly the same writing style, no matter how similar they might be, a book written in such a fashion would have a stylistic discontinuity going from Chapter X to Chapter $X+1$. If we can quantify the styles of the two authors by a stylometric function $S(n)$ (a classifier) where n denotes chapters, or chronologically ordered samples, of the book in question, this stylistic discontinuity will appear as a dividing point in the stylometric function $S(n)$ going from $n = X$ to $n = X + 1$. Because the samples are ordered by time, we shall call this divide in the stylometric function $S(n)$ a *chrono-divide in style*, or simply a chrono-divide. This paper develops a technique for verifying and detecting chrono-divides in books or other body of texts. Knowing X and Y, as it is the case with *Dream of the Red Chamber*, can help validating the conclusion but is not always necessary for our method. Our method does not apply to any body of texts where two authors share the writing in an interwoven way without a chrono-divide.

The underlying principle of our study is that if a book is in fact written by two authors A and B, then there should exist a group of features that characterize the difference of their respective styles. These features will lead to a stylometric function that separates the book into two different classes. In the rest of the paper we shall use the more conventional term *classifier* for such a stylometric function. The foundational principle for literary stylometry is built around finding such classifiers. Suppose that a chrono-divide in style exists. Then an effective classifier will show a break point somewhere in the middle of the book, before and after which the classifier gives positive values and negative values, respectively. Thus in analyzing a book suspected to be written by two authors with a chrono-divide, one can look for a classifier that gives rise to such a break point. The existence of such a classifier will provide strong support for the two-author hypothesis. Conversely, if such a classifier cannot be found, then we can confidently reject the two-author with a chrono-divide hypothesis.

We use function characters and words to build and select a group of stylometric features having the highest discriminative power, and from which we construct our classifier. We shall detail our method in the following subsections.

8.2.1 Initial Stylometric Feature Extraction

Suppose the book in question is suspected to be written by two authors. For simplicity we shall call the part written by author A *Part A* and the part written by author B *Part B*. In many cases, such as with *Dream of the Red Chamber*, both Part A and part B are known. In some cases, they are not precisely known. However, for books suspected to have a chrono-divide from authorship change, there is usually a good estimate for where the divide is. Typically the first few chapters can be confidently attributed to A and the last few chapters to B.

We begin by choosing a feature set consisting of the kinds of features that might be used consistently by a single author over a variety of writings. Typically, these features include the frequencies of words (or characters for books in Chinese), phrases, mean and variation of sentence length, and frequencies of direct speeches and exclamations, and others. In our analysis, to get a better understanding of an author's writing style, we first find the most frequently used characters and words in the book, e.g. we would find the 500 most frequently used characters in the whole book, from which we pick out only, say, n function characters. We choose m words (combinations of characters) among the 300 most frequently used words in the same way. An important point is that by selecting only function characters and words we obtain a selection of characters and words that are *content independent*. This leads to an initial set of features consisting of the frequencies of the n characters and the m words, plus the mean and variance of sentence length as well as the frequencies of direct speeches and exclamations. These features will be computed over given sample texts of the book (e.g., chapters). We normalize each sample text in the following way: set the median of the mean and variation of sentence length and the frequencies of direct speeches, exclamations, n characters, and m words in each work of A and B to be 1. For each sample, we now get $n+m+4$ features.

8.2.2 Data Preparation

Having constructed the appropriate feature vectors, we build a distinguishing model through a machine learning algorithm. To do so requires careful data preparation. Since we usually have in hand only limited samples while the number of features will be very large, building a model directly on the entire book will easily lead to over-fitting. To overcome the over-fitting problem, we use the standard technique of separating the whole data into samples consisting of training data and test data. Our model will be established based only on the training data while its performance is tested over the independent test data. If we know Part A and Part B already, then a subset of each can be designated as training data. For books suspected to have a chrono-divide in style, the training data will consist of the first few chapters and the last few chapters. The rest of the book will be used as test data.

In order to obtain more training sets and testing sets we shall chunk the book in question into smaller pieces of sample texts of relatively uniform size and style.

In all the books we have studied, we have kept the sample texts to be at least 1000 characters long. In the case of *Dream of the Red Chamber* each sample text is a chapter.

8.2.3 Feature Subset Selection

When we build authorship analysis the model using the training data only, we do not use all the features ($n+m+4$ features). Instead we start out with all of them, but eventually select a subset of features that achieves the highest discriminative powers. Feature subset selection has been well understood for high dimensional data analysis in the machine learning context. First, the number of discriminative features may be small because the number of features an author uses in a consistently different way from others is usually not very big. Moreover, the classifier can perform very poorly if too many irrelevant features are included into the model. In this paper we will use Support Vector Machines Recursive Feature Elimination (SVM-RFE) introduced in [4] to realize feature selection.

SVM-RFE is a feature ranking method. Given a set of samples we can use linear SVM to build a linear classifier. It ranks the importance of the features according to their weights. As mentioned above, because of large feature size and small sample size, the classifier might not be robust. In addition, the high correlation between features may result in small weights for relevant features. Thus the ranking by SVM classifier directly may be inaccurate. In order to refine the ranking, the least important feature is removed and the linear SVM classifier is retrained. This new classifier provides a refined ranking for the remaining features. The process is then repeated until the ranking of all features are refined. This is the SVM-RFE method introduced in [4]. The idea underlying SVM-RFE is that in each repeat, although the overall ranking may be poor, the least important feature is very unlikely a relevant one. By iteratively eliminating the least important features the new classifiers will become more and more reliable and hence will provide better and better ranking. In the application of gene expression data analysis SVM-RFE has been proven to be substantially superior to the SVM direct ranking without RFE.

However in general SVM-RFE is not stable under the perturbation of samples. A small change in samples may result in very different feature ranking. There are two possible reasons. One is that the highly correlated variables are too sensitive and may be ranked in different orders by different classifiers. Another is that, due to the randomness, some subset of samples might be singular in the sense that they are less representative for the whole data structure. As a result the SVM classifiers are over-fitting and the feature ranking by SVM-RFE is therefore unreliable. The first situation is less harmful for classification performance while the second is vital. To overcome this phenomenon and guarantee the stability of the ranking, we use a pseudo-aggregation technique. We randomly choose a subset of training samples to run SVM-RFE to select the top important features. This process is repeated tens or hundreds times and only those features that appear important very frequently are deemed as truly important ones. This removes the randomness and results in a much more reliable ranking.

With this ranking of features, we can conclude which statistics are useful for quantifying the writing style. We use cross validation to select the number of features included in the final classification model. This group of features is a stable and most discriminative subset of features. A final classifier is built to classify the test data.

8.2.4 Data Analysis

The classifier we have built is used to analyze the authorship question. We examine the discriminative power of the classifier on the training data. If it cannot even reliably classify the training data, we can convincingly reject the two-author hypothesis. Even if it can the telling story will be whether it can classify, or detect a chrono-divide, from the test data. If it fails, then again we should reject the two-author hypothesis. On the other hand, if the classifier classifies the training data, and it can also classify the test data accurately or detect a clear chrono-divide, we can then convincingly conclude that the book does contain two different writing styles and can therefore be confidently attributed to two different authors. Moreover, the feature subset and the classifier describe the difference of the two authors' writing styles.

8.2.5 The Algorithm

In the following we summarize the process of our algorithm:

1. Initialize the data (the book), which contains parts A and B suspected to be written by two different authors.
2. Split part A and part B into many sections and extract the features for each section as described in Section 8.2.1. This forms the whole data set D, containing D_A and D_B.
3. Choose a portion (e.g., 20%–30%) of D_A and D_B, respectively, to form the test data set and leave the remaining as the training data set. The test data will not be used until the final model is built.
4. Randomly choose a subset from the training data as modeling data and the rest (again 20%–30%) as the validation data. Run SVM-RFE on the modeling data and use the validation data to determine all the parameters used. This provides a ranking of all the $n+m+4$ features extracted in step 2.
5. For d range from 1 to $n+m+4$, build a classifier using only the top d features and evaluate their performance on the validation data. The best model is the one

with minimal validation error and minimal number of top features. The feature subset of this best model is recorded.
6. Repeat T times step 4 and step 5 to obtain T best models and T subsets of corresponding important features. We recommend T to be larger than 50. Rank all the features in these subsets according to their appearance frequency. Denote N as the total number of features included.
7. For $d = 1, \cdots, N$, using cross validation to select the number of features that should be included in the final classifier. Denote it by d_*. Note we require both the cross validation error and the number of features to be as small.
8. Retrain the model using the whole training set based on this top d_* important features.
9. Using the classifier to classify the test data. Draw the conclusion according to the performance.

Since our ranking process involves aggregation of large number of models that are trained using SVM-RFE based on different subsets of the same data source, we refer to our approach as pseudo-aggregation SVM-RFE method.

8.3 Case Study: Analysis of *Dream of the Red Chamber*

8.3.1 Background

Dream of the Red Chamber (红 楼 梦) by Cao Xueqin (曹雪芹) is one of China's Four Great Classical Novels. For more than one and a half centuries it has been widely acknowledged as the greatest literary masterpiece ever written in the history of Chinese literature. The novel is remarkable for its vividly detailed descriptions of life in the 18th century China during the Qing Dynasty and the psychological affairs of its large cast of characters. There is a vast literature in *Redology*, a term devoted exclusively to the study of *Dream of the Red Chamber*, that touches upon virtually all aspects of the book one can imagine, from the analysis of even minor characters in the book to in-depth literary study of the book. Much of the scope of Redology is outside the focus of this paper.

The original manuscript of *Dream of the Red Chamber* began to circulate in the year 1759. The problems concerning the text and authorship of the novel are extremely complex and have remained very controversial even today, and they remain an important part of Redology studies. Cao, who died in 1763–4, did not live to publish his novel. Only hand-copied manuscripts – some 80 chapters – had been circulating. It was not until 1791 the first printed version was published, which was put together by Cheng Weiyuan (程伟元) and Gao E (高鄂) and was known as the *Cheng-Gao version*. The Cheng-Gao version has 120 chapters, 40 chapters more than various hand-copied versions that were circulating at the time. Cheng and Gao claimed that this "complete version" was based on previously unknown working papers of Cao, which they obtained through different channels. It was these last 40 chapters that were the subject of intense debate and scrutiny. Most modern scholars believe that these 40 chapters were not written by Cao. Many view those late additions as the work of Gao E. Some critics, such as the renowned scholar Hu Shi (胡适), called them forgeries perpetrated by Gao, while others believe that Gao was duped into taking someone else's forgery as an original work. There is, however, a minority of critics who view the last 40 chapters as genuine.

The analysis of the authenticity of the last 40 chapters has largely been based on the examination of plots and prose style by Redology scholars and connoisseurs. For example, many scholars consider the plotting and prose of the last 40 chapters to be inferior to the first 80 chapters. Others have argued that the fates of many characters in the end were inconsistent with what earlier chapters have been foreshadowing. A natural question is whether a mathematical stylometry analysis of the book can shed some light on this authenticity debate.

Although there is a vast Redology literature going back over 100 years, the number of studies of the book based on mathematical and statistical techniques is surprisingly small, particularly in view of the fact that such techniques have been used widely in the West for settling authorship questions. There have been some notable efforts however. Cao [1] meticulously broke down a number of function characters and words into classes according to their functions. By analyzing their frequencies Cao concluded that the first 80 chapters and the last 40 chapters were written by different authors. Zhang & Liu [14] examined the occurrence of characters in the book that are outside the GB2312 encoding system, and found that the vast majority of them appeared exclusively in the first 80 chapters. Yue [13] studied the authorship by combining both historical knowledge and statistical tools. By focusing his study on a few stylometric peculiarities such as the frequencies of 5 particular function characters and the proportion of texts to poems in each chapter, he concluded that it is unlikely that the first 80 chapters and the last 40 chapters were written by the same author. In the opposite direction, the studies of Chan [2] and Li & Li [8] concluded that the entire book was likely written by a single author. The study [8] focused on the usage of functional characters while [2] examined the usage of some eighty thousand characters. Both studies tabulated the frequencies of the selected characters, which led to a frequency vector for each of the first 40 chapters, the middle 40 chapters, and the last 40 chapters. The correlations of these frequency vectors were computed. In [8] the correlations were deemed large enough to conclude that the entire 120 chapters of the book were written by the same author. In [2] a fourth frequency vector using parts of a different book *The Gallant Ones* (儿女英雄传) was added for comparison. The correlations are significantly higher among the first three frequency vectors. This led to the same author conclusion in [2].

Although some of these aforementioned studies are impressive in their scopes, missing conspicuously from the Redology literature are studies based on the latest advances in literary stylometry, particularly some of the new and powerful methods from machine learning theory. While comparing the frequencies of function characters and words is clearly a viable way to analyze the authorship question, care needs to be taken to account for random fluctuations of these frequencies, especially when some of the function characters and words used for comparison have limited occurrences overall in the book and sometimes not at all in some chapters. None of the aforementioned studies employed cross validation to address random fluctuations. We have substantial reservations about drawing conclusions from correlations alone as in the studies of Chan [2] and Li & Li [8], because the differentiating power of any single variable such as correlation is rather limited. It would be interesting to see a more comprehensive study of correlations on a large corpus of texts in Chinese to determine its effectiveness as a metric for authorship attribution, something the authors failed to do in both studies. The use of the book *The Gallant Ones* in [2] for benchmark comparison is curious to us in particular, especially considering that the author did not limit to just function characters. The two books are of two different genres and are different in their respective background settings. It is possible that the correlation computed in [2] tells more about the genre than the authorship of the books.

Having established a rigorous protocol for finding chrono-divides, we are now in position to apply this protocol to investigate the authorship controversy of the Cheng-Gao version of *Dream of the Red Chamber*. In particular we investigate the existence of a chrono-divide at Chapter 80.

8.3.2 Separability of the Chapters by Cao and Gao

The book is first divided into samples. To balance the number of samples, we generate one sample for each of the first 80 chapters while using the conventional practice of duplicating each of the last 40 chapters into two chapters to obtain 80 samples. From those samples we extract the features by calculating the statistics proposed in Section 8.2.1. These features are then normalized for fair comparison. In total we have 196 variables. They are the 144 characters and 48 words, the normalized mean and variation of sentence length, and the frequencies of direct speeches and exclamations.

To investigate the authorship controversy we perform three separate tests. First we build a classifier for the whole book and look for the existence of a chrono-divide at Chapter 80. For added robustness and reliability we also perform the same tests only on the first 80 chapters and the last 40 chapters.

In the first experiment we apply our method to the whole Chen-Gao version of *Dream of the Red Chamber*. Samples from the first 60 chapters are designated as training samples for one class while samples from the last 30 chapters are designated as training samples for another class. The remaining samples, from Chapter 61 to 90, are held out as testing samples. The training samples are further randomly split into modeling data of 80 samples and validation data of 40 samples. The SVM-RFE is repeated 100 times and d_* is chosen using 50 cross validation runs. We have the following observations.

Instability of SVM-RFE. The randomness of the modeling set has resulted in very substantial fluctuations in the number of features selected as well as feature rankings. The resulted classifier may also perform quite differently. Table 8.1 lists the features selected using two different modeling data sets. One selects 11 features and the other selects only 4, with only one feature in common. The classifiers also perform differently. The experiments clearly establish the instability of SVM-REF.

Given such instability one cannot reliably draw any conclusion from any single run. For example, if a modeling data set separates the training data well, it might be due to over-fitting. Conversely, if it separates poorly, it might be due to under-fitting. This problem is overcome with our Pseudo Aggregate SVM-RFE method.

Modeling set	Features Selected	Validation Error
1	去，得，就，回，知，到，时，呢，倒，别，作	5/40
2	回，方，没，好些	1/40

Table 8.1 The features and validation errors of the classifiers obtained from two randomly selected modeling subsets.

Stability of Pseudo Aggregate SVM-RFE. Our pseudo aggregate SVM-RFE approach repeats SVM-RFE 100 times using randomized data sets. The data set from each repeat is used to select a set of features, from which a classifier is being built. For simplicity we shall refer to the data set, features, and the resulting classifier together from a repeat as a *model*. To counter random fluctuations we consider important features to be those that appear frequently among the 100 classifiers. This reduced the instability caused by randomness. In fact, our belief is as follows: if the two classes are well separated, there should exist a set of features that help to build a good classifier. Most modeling subsets should be able to select these features out and only a limited number of modeling sets might be singular and miss them. Conversely, if the two classes cannot be well separated, no consistently discriminative features exist. Different modeling set may lead to totally different feature subset. As a result, no feature appears with high frequency in all 100 models. This philosophy, however, is only partially true. When the two classes cannot be separated, the modeling process sometimes can overfit the data by selecting a lot of variables which results in high absolute frequencies for some less important or irrelevant features. Such a phenomenon is usually accompanied by a large number of variables and low validation accuracy. To improve the process we propose a more appropriate metric, which we call *relative frequency*. In relative frequency we weight the frequency by two criteria. In the first criterion a variable appearing in short models is weighted more than the variables appearing in long models. This leads to a weight of $h(n_j)$ for a variable in the j-th model, with n_j being the number of variables in the j-th model. In the second criterion a variable in a model with high predictive accuracy is weighted more than a variable with poor predictive accuracy. This provides another weight $g(A_j)$ for a variable in the j-th model, where A_j denotes the accuracy of the j-th model computed from the validation process. Mathematically the relative frequency for a variable x_i in a test run of M repeats is defined as

$$rf(x_i) = \frac{1}{M}\sum_{j=1}^{M} g(A_j)h(n_j)\mathbf{1}(x_i \text{ appears in model } j). \tag{8.1}$$

In our study we always set $M = 100$. For the weight $g(A_j)$, we wish it is equal to 1 if the model is very effective (i.e., $A_j = 100\%$) and decays fast to 0 if the model is not effective (i.e., A_j is close to or less than 50%). This motivated us to set $g(A_j) = \exp(\frac{A_j-1}{[2A_j-1]_+})$ where $[t]_+ = \max\{0,t\}$. The weight $h(n_j)$ is designed to decrease linearly as n_j increases and reaches zero when n_j is unreasonably high. This leads to $h(n_j) = [1-cn_j]_+$ for some constant c. We have chosen $c = 1/30$. It indicates the model becomes unreliable when more than 30 variables are selected and seems to work well in our experiments.

Our experiments show that features yielded from relative frequency rankings are very stable and consistent. We have performed runs of 100 repeats using different random seeds in MATLAB, and the results are always similar. An additional benefit of using relative frequency instead of absolute frequency is that the existence of an effective classifier is typically accompanied by high relative frequencies for the top features, while low relative frequencies for the top features usually imply poor separability. Hence we can use relative frequency as a simple guide on the separability of the samples. We will show some examples in the next section.

Results and conclusion. In Experiment 1 we have performed a run of 100 repeats on the entire Cheng-Gao version of *Dream of the Red Chamber*. Altogether 70 features have appeared in at least one model. However, of those only a small number of them have appeared with high enough frequency to be viewed as being important. We apply cross validation to select the number of features, and the mean cross validation error rate against different number of features is plotted in Figure 8.1 (a). The figure tells us that 10 to 50 features are enough to tell the style difference between the two parts. Using less characters and words is insufficient, while using more degrades the performance also by bringing in too much noise. The small cross validation error rate is encouraging, and it is already hinting a strong possibility that the two training sample sets have significant stylistic differences to support the two-author hypothesis.

To settle the two-author hypothesis more definitively we apply our classifier on the test data, which until now have never been used during the feature selection and classifier modeling process. In particular we investigate the existence of a chrono-divide in the values obtained through classifier. Figure 8.1 (b), which plots these values, clearly shows a chrono-divide at Chapter 80: For Chapter 81–90 the classifier yields all negative values while for Chapters 61–80 the classifier yields all positive values with the exception of Chapter 67. Allowing some statistical abberations to occur, our results provide an extremely convincing if not irrefutable evidence that there exist clear stylometric differences between the writings of the first 80 chapters and the last 40 chapters. This difference strongly supports the two-author hypothesis for *Dream of the Red Chamber*. We also note that our investigation did not need to assume that the knowledge that the stylistic change should be at Chapter 80. The fact that the chrono-divide we have detected is indeed at Chapter 80 lends even stronger support to the two-author hypothesis.

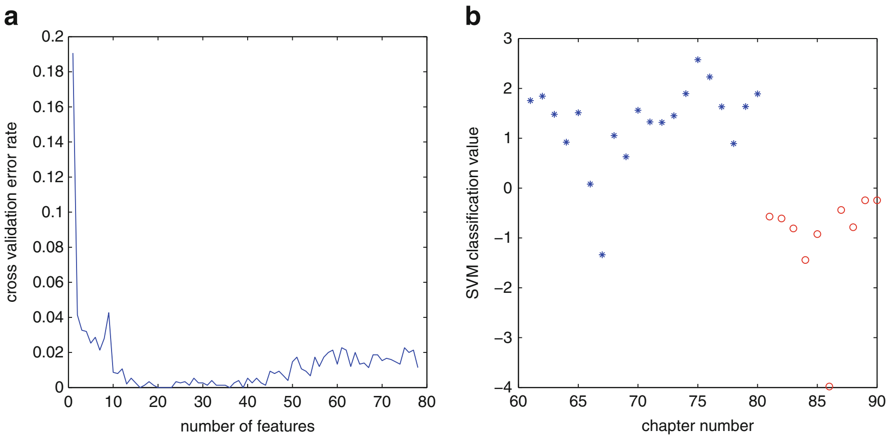

Fig. 8.1 Experiment 1: (a) Mean cross validation error rate; (b) Values of SVM classifier on chapters 61–90.

Interestingly, the fact that Chapter 67 appeared as an "outlier" in our classification serves as further evidence to the validity of our analysis. It was only after the tests we realized that the authorship of Chapter 67 itself is one of the controversies in Redology. Unlike the main controversy about the authorship of the first 80 chapters and the last 40 chapters, experts are less unified in their positions here. Again, our results strongly suggest that Chapter 67 is stylistically different from the rest of the first 80 chapters, and it may not be written by Cao. Our finding is consistent with the conclusion of [12].

8.3.3 Non-separability of the First 80 Chapters

To further validate our method we apply the same tests to the first 80 chapters of *Dream of the Red Chamber* to see whether we can get a chrono-divide (Experiment 2). We use the first 30 and last 30 chapters as the training data and leave chapters 31–50 as the test data. Figure 8.2 shows the mean cross validation error and the values of SVM classifier on the test data (chapters 31–50). The experiment shows much more features have been selected in the 100 repeats, implying the difficulty of find a consistent subset of discriminative features. The large errors on the training data also indicate the difficulty for separation. When the classifier is applied to the test data, there is clearly no chrono-divide. This suggests that our method yields a conclusion that is completely consistent with what is known.

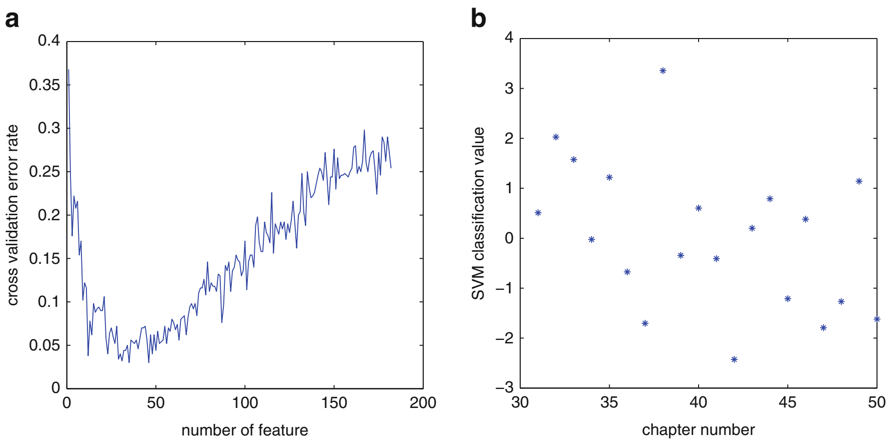

Fig. 8.2 Experiment 2: (a) Mean cross validation error rate; (b) Values of SVM classifier on chapters 31–50. Note there is no chrono-divide.

8.3.4 Analysis of Chapters 81–120: Style Change over Time

We next apply our method to the last 40 chapters (Experiment 3). Our first experiment has already confirmed that they are unlikely to be written by Cao. However, there are still debates on whether these were written entirely by one author (most likely Gao himself), or by more than one author. Our mathematical analysis may offer some insight here.

We split the 40 chapters into two subsets as before. The training data include Chapters 81–95 as one class and Chapters 106–120 as another. The test data are the middle 10 chapters. Because of the relatively small number of samples we have subdivided each chapter into 2 sections to increase the sample size. As a result we now have 60 samples in the training data and 20 in test data, with 2 samples corresponding to one chapter. The mean cross validation error of the final classifier and its classification values on the test samples are shown in Figure 8.3 (a) and (b), respectively.

In this experiment we observe that the performance in terms of both the classifier and feature ranking is noticeably worse than that in Experiment 1 but substantially better than that in Experiment 2. Furthermore, unlike the results from the first two experiments, the values from the classifier show an interesting trend. Compared with Figure 8.2 (b) where the values appeared to lack any order, the values here exhibit a clear gradual downward shift. On the other hand, compared to Figure 8.1 (b) the values plotted in Figure 8.3 (b) do not show a clear sharp chrono-divide, even though the values change gradually from being positive to being negative. What it tells us is that the writing style of the last 40 chapters had undergone a graduate change, but this change is unlikely to be due to change of authorship.

Our results here could be subject to several interpretations. One plausible interpretation is that Gao might indeed obtained some incomplete set of manuscripts by Cao, and tried to complete the novel based on what he had obtained. The style change is a result of the lack of genuine work by Cao as the story developed. A more plausible interpretation is that the last 40 chapters were written by someone such as Gao trying to imitate Cao's style, and over time the author became sloppier and returned more and more to his own style.

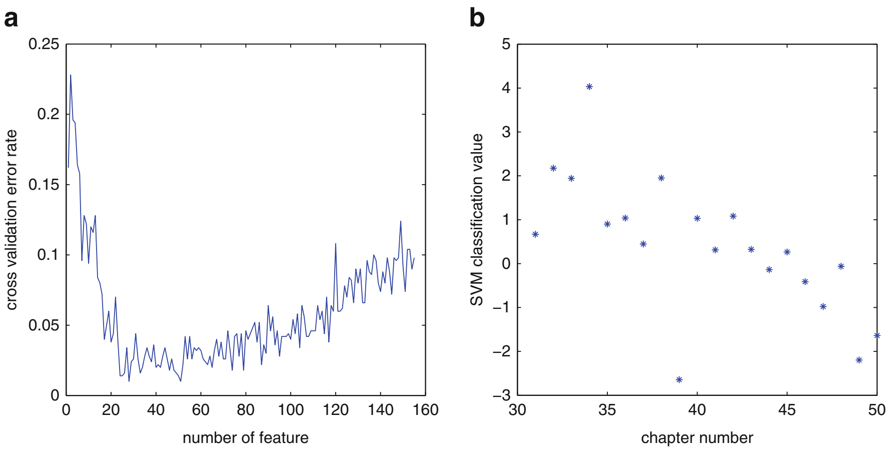

Fig. 8.3 Experiment 3: (a) Mean cross validation error rate; (b) Values of SVM classifier on chapters 96–105, which correspond to the samples 31–50 in all 80 samples. Note two samples come from one chapter in this experiment.

8.3.5 Comparison with Continued Dream of the Red Chamber

It is worth mentioning that there are several other attempts to complete *Dream of the Red Chamber* from its first 80 chapters, among them is *Continued Dream of the Red Chamber* (红楼梦续) by Qi Zichen (秦子忱). Using the same features for building the classifier in Experiment 1, we can compute the Euclidean distances between all chapters and their distances of chapters from *Continued Dream of the Red Chamber*, see Figure 8.4. Surprisingly, although these features are obtained in favor of the differences between Cao and Cheng-Gao, they lead to even larger distance between the first 80 chapters and those chapters of *Continued Dream of the Red Chamber*. It obviously implies that the style of the 40 chapters by Cheng-Gao is more similar to the 80 chapters by Cao compared to *Continued dream of the Red Chamber*. Maybe that's why the Cheng-Gao version is more popular than other versions.

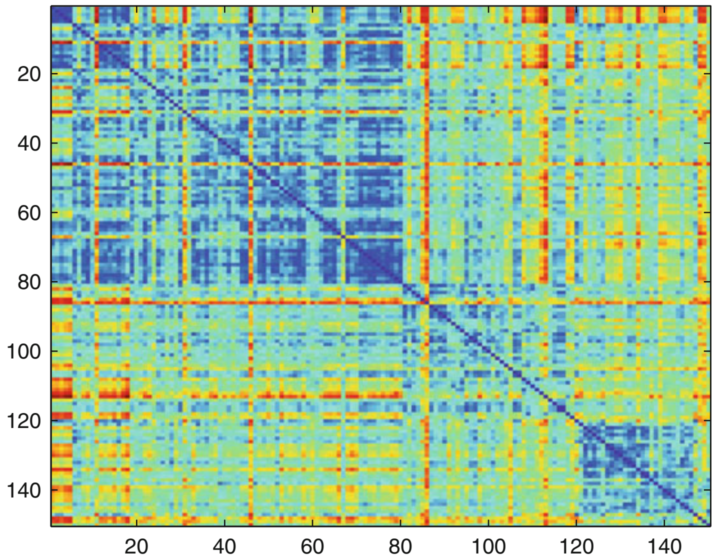

Fig. 8.4 Distances between the first 80 chapters of the Cheng-Gao version, the last 40 chapters of the Cheng-Gao version, and 30 chapters of *Continued Dream of the Red Chamber.*

8.4 Analysis of *Micro* and Other Books

To further bolster the validity of our approach we test our method on another book *Micro* by Michael Crichton and Richard Preston, which is known to have a chrono-divide, as well as the other three Great Classical Novels in Chinese literature, *Romance of the Three Kingdoms* (三国演义), *Water Margin* (水浒传), and *Journey to the West* (西游记), which, unlike *Dream of the Red Chamber*, do not have authorship controversy. Thus if our method is indeed robust we should expect a positive answer for *Micro* for the two-author hypotheses and negative answers for the other three.

8.4.1 Chrono-Divide of Micro

Micro, a techno-thriller published posthumously in 2011, is Michael Crichton's final novel. It was found on his computer upon his death in 2008 as an unfinished manuscript. HarperCollins commissioned science-writer Richard Preston to complete the novel from Crichton's notes and research. Although *Dream of the Red Chamber* is in Chinese, the principle of our method should apply to books in other languages. *Micro* thus provides us with a good test example. In this case study, we will use our approach to confirm and detect the chrono-divide of *Micro*. We will also perform a different new test using classifiers built directly from other books written by Crichton and Preston for comparison. The new test serves both as a validation of our method and as a comparison. Note that the second option is not available for *Dream of the Red Chamber*.

In the direct classifiers test we use the other books written by Crichton and Preston to generate the training data. A total of 17 books written by Crichton and 2 books by Preston were used for training. The initial features consist of the frequency of 241 most frequently used words in these books. To build classifiers each book was divided into multiple pieces with each piece containing approximately 2000 words. The frequencies of the 241 selected words of each piece form a sampling point. Overall 782 data points for Crichton and 104 data points from Preston were generated. To overcome the imbalance of the sampling points for Crichton and Richard, we only used 728 samples for Crichton and they are split into 7 subsets. Each subset is combined with the samples for Preston to form a training data set, from which we build a linear classifier. So totally 7 classifiers were constructed. Applying the classifiers to detect the chrono-divide in *Micro*, we chunk the book into 56 parts, each containing about 2000 words. Each part provides a testing sample point. We applied the 7 classifiers to this testing data. The average of the 7 classifiers are plotted in Figure 8.5 (a). The result shows a break point at around the 15th-16th sample points.

We can now compare the above method to the earlier method for *Dream of Red Chamber*. We assume that, compared with the overall style of an author across multiple books, the style of the author in a single book would be more consistent. As a result we divided *Micro* into 112 parts of approximately 1000 words each. Note that here we use less words for each piece in order to get enough training samples. The most frequently used 265 content independent words from the book were used as the initial features. We use the first 22 sample points and the last 22 sample points as training and validation data and the middle 68 sample points as test data. The classification results are shown in Figure 8.5 (b). A clear break point can be seen around the 29th–30th parts.

These two experiments confirm the existence of a chrono-divide in *Micro*, and provide further evidence of the validity of our original approach for discovering and locating chrono-divides. As a by-product, our results show that the change of authorship for *Micro* had occurred between 1/4 and 1/3 of the book. This is consistent with what Richard Preston had indicated in several interviews about the book.

8.4.2 Analysis of the Three Chinese Classical Novels

For the analysis of the other three Chinese novels, as with *Dream of the Red Chamber* we split each into training samples and test samples. Both *Romance of the Three Kingdoms* and *Water Margin* have 120 chapters. In both cases we designate the first 30 chapters and the last 30 chapters as the two classes of training data, and the middle 60 chapters as test data. For *Journey to the West* the two classes of training data are the first and last 25 chapters, respectively, with the middle 50 chapters as test data.

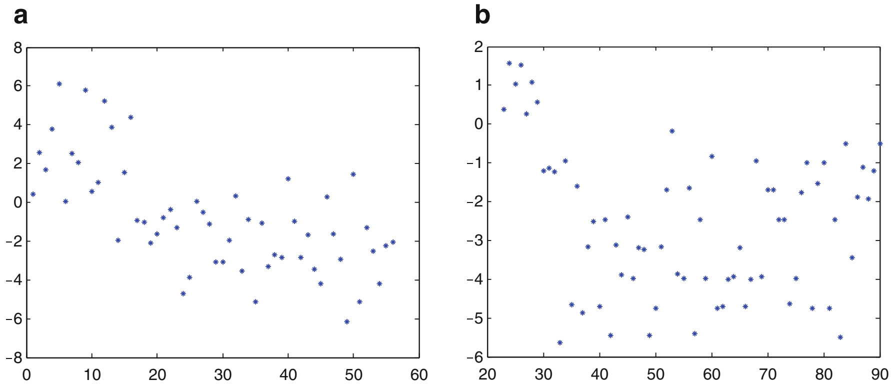

Fig. 8.5 *Micro*: (a) Average values of the 7 classifiers obtained by using the books of Crichton and Preston other than *Micro* as training samples. (b) Classification result by the classifier obtained using the first 22 parts and the last 22 parts of the book *Micro* as training samples.

We use the same procedure to test for chrono-divides on the three novels. Compared to *Dream of the Red Chamber*, the selected features show much lower relative frequencies, indicating difficulty in differentiating between the writing styles. Table 8.2 shows the relative frequencies (with $c = 1/30$) of the top 8 features for each of the four Great Classical Novels. Also of note is that in the case of *Water Margin*, 51 features are used to build a classifier from the 60 training data, which is clearly another strong indication of the difficulty.

Novel	Relative frequencies of the top 8 features							
Dream of the Red Chamber	0.57	0.46	0.43	0.36	0.31	0.30	0.29	0.19
Romance of the Three Kingdoms	0.31	0.27	0.26	0.25	0.23	0.22	0.17	0.15
Water Margin	0.18	0.17	0.16	0.16	0.14	0.11	0.11	0.10
Journey to the West	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02

Table 8.2 Relative frequencies of the top ranked 8 features for each of the four Chinese Great Classical Novels.

Figure 8.6 plots the values from the classifiers for all three novels. In all cases the values fluctuate in such a way that it is quite clear that no chrono-divides exist, as expected. This analysis shows that our approach can reliably reject the two-author hypothesis when it is false, lending further support to the effectiveness and robustness of our method.

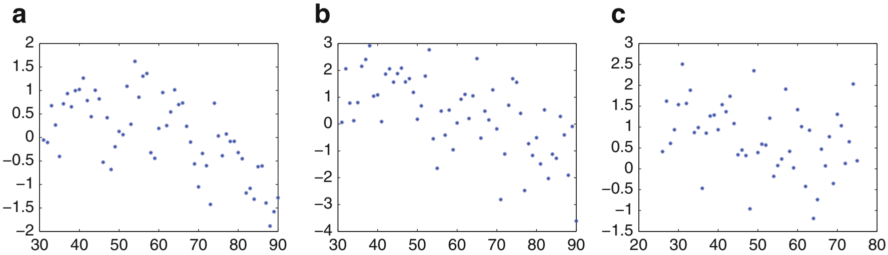

Fig. 8.6 Classification results on the testing samples of the other three classical novels: (a) *Romance of the Three Kingdoms*; (b) *Water Margin*; (c) *Journey to the West.*

8.5 Conclusions

Inspired by authorship controversy of *Dream of the Red Chamber* and the application of SVM in the study of literary stylometry, we have developed a mathematically rigorous new method for the analysis of authorship by testing for a chrono-divide in writing styles. We have shown that the method is highly effective and robust.

Applying our method to the Cheng-Gao version of *Dream of the Red Chamber* has led to convincing if not irrefutable evidence that the first 80 chapters and the last 40 chapters of the book were written by two different authors. Furthermore, our analysis has unexpectedly provided strong support to the hypothesis that Chapter 67 was not the work of Cao Xueqin either.

Applying our method to *Micro*, we are able to confirm the existence of chrono-divide and identify its location. It provides strong evidence for us to attribute the first 1/4 of the work to Michael Crichton and the left 3/4 to Richard Preston.

The robustness of our approach is also evidenced by its ability to reject the multiple author hypothesis when there is no chrono-divide, as we have done for the other three classical Chinese novels.

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Chapter 9
Thoughts on Numerical and Conceptual Harmonic Analysis

Hans G. Feichtinger

Abstract If one compares the literature on abstract Harmonic Analysis or publications with the word “Fourier Analysis” in the title with books describing the foundations of digital signal processing or systems theory one may easily get the impression that they describe two different worlds having little in common except for some vocabulary, indicating their joint roots.
It is the purpose of this article to shed some light on the existing and sometimes buried interconnections between these two branches of Fourier Analysis. We suggest to take a broader perspective on Harmonic Analysis, re-emphasizing how they are tied together in many ways and how numerical experiments may help to understand concepts of Harmonic Analysis. Based on our experience we firmly believe that Fourier Analysis provides opportunities to relate relevant mathematical theorems to valuable algorithms, which also have to be properly implemented (typically making use of the FFT, the Fast Fourier Transform). Thus this note tries to bridge the gap between pure mathematics and real-world engineering applications.
In a previous paper (H. G. Feichtinger, *Elements of Postmodern Harmonic Analysis*, pages 1 – 27, Springer, 2015) the author has already outlined some ideas in this direction, by introducing the idea of *Conceptual Harmonic Analysis*, as a link between the two worlds. The best way to describe the connections between the two worlds is by means of *generalized functions* (often called *distributions*). We will indicate that the *Banach Gelfand triple* $(S_0, L^2, S_0')(\mathbb{R}^d)$, which is based on a particular Banach algebra of continuous functions, namely the Segal algebra $\big(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\big)$, provides convenient vehicle to express many otherwise “soft transition” in a distributional sense. In the context of the space $S_0(\mathbb{R}^d)$ of test functions approximation by finite sequences makes perfect sense and one can justify many transitions between the two worlds (continuous versus finite signals) in a clear mathematical context (see H. G. Feichtinger and N. Kaiblinger, Quasi-interpolation in the Fourier algebra, *J. Approx. Theory*, 144(1):103–118, 2007).

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Applied and Numerical Harmonic Analysis, DOI 10.1007/978-3-319-27873-5_9

The current text also contains various considerations of a general nature, which are supposed to *stimulate discussions* and reflection of the work done in the field of abstract and applied Harmonic Analysis in general.

9.1 Classical Fourier Analysis Seen Critical

So far *Abstract Harmonic Analysis* (AHA) and what we would like to call it *numerical Fourier Analysis* are quite separated branches of mathematics. First of all, AHA has been the subject of controversial discussions in the past (involving among others my advisor Hans Reiter, who disliked the term). Following Edwin Hewitt (see his monumental books on the subjects [38, 39]) Abstract Harmonic Analysis is concerned with the generalization of results known in the Euclidean setting to the realm of *locally compact Abelian groups*[1]. In contrast, my advisor Hans Reiter kept teaching his students the spirit of the work of Andre Weil: The natural setting for *Harmonic Analysis* (viewed as "Fourier Analysis in the proper setting") is to study functions over locally compact Abelian group $\mathscr{G}$, without making use of the structure theory of such a group, making simply use of the basic facts concerning such groups and their dual groups (see [68]).

The existence of a translation invariant *Haar measure* on such a group $\mathscr{G}$ together with the existence of sufficiently many *characters* (i.e. continuous homomorphism from the group $\mathscr{G}$ into the torus group $\mathbb{T}$) provides a solid basis of this approach. They form the dual group $\widehat{\mathscr{G}}$, and this allows us to define the Fourier transform (and its inverse). One can prove the convolution theorem and Plancherel's theorem in this context. Thanks to Pontryagin's duality theorem one has a complete symmetry between time and frequency variables. This setting is well described in several mathematical books and at different levels of easiness (not to say difficulty) of reading, see [14, 15, 28, 46, 50, 51, 53].

There is also a long list of books on Fourier Analysis which view the field from a more concrete aspect, such as [33–35, 42, 48, 58, 66, 67], or Stein [57], Butzer [9], Ramanathan [48], and Trigub [60]. They work in the concrete setting of $\mathbb{R}^d$, endowed with the Lebesgue measure. Hence the Banach algebra $\big(L^1(\mathbb{R}^d), \|\cdot\|_1\big)$ (with respect to convolution) is well defined, and the Fourier transform, typically given by the integral formula

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i s\cdot t}\,dt \tag{9.1}$$

is well justified in this setting. For the inversion the usual summability tricks have to be applied, except for the case of "nice functions." Only if $\hat{f} \in L^1(\mathbb{R}^d)$, we can expect to recover the values of f from $\hat{f}$ in a symmetric way:

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(s)e^{+2\pi i s\cdot t}\,ds. \tag{9.2}$$

[1] http://at.yorku.ca/t/o/p/d/07.dir/german.htm

This pair of equations is often described as a continuous analogue of the expansion of a vector in the Euclidean space (or Hilbert space) with respect to an orthonormal basis, except that there are various problems involved (even if one has already understood how to interpret *Plancherel's Theorem*), allowing to view the Fourier transform as a unitary automorphims of $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$: According to (9.2) the function f can be interpreted as a *superposition* of pure frequencies, and the amplitudes (coefficients) are obtained by taking scalar products, i.e. by taking the forward Fourier transform (9.1). So in a way this looks like a continuous variant of the Fourier expansion of vectors, viewed as functions over finite Abelian groups, where the characters form a complete orthonormal system (maybe up to a suitable normalization factor).

It is true that the fact that the parameter s is in $\mathbb{R}^d$ (and not in a discrete set) causes some problems, because one has to think in terms of a "continuous superposition," but it is more disturbing that in fact the members of this orthonormal system, namely the *pure frequencies* $\chi_s : t \to e^{2\pi i s \cdot t}$ do not even belong to the Hilbert space $L^2(\mathbb{R}^d)$, and therefore it is not so easy to think of them as a "continuous" orthonormal basis.

Still another group of books describes Fourier Analysis as it is used in the application areas, among them most prominently in communication theory. Here the concept of *translation invariant systems* (TILS) is predominant. They are usually characterized as convolution operators, where the convolution kernel is called the *impulse response* of the system, and its Fourier transform is the so-called *transfer function*. Unfortunately the explanation of the corresponding claims is often quite vague and mathematically less rigorous.

Aside from the use of complex-valued functions, mysterious objects such as the Dirac δ-"function" appear, and divergent integrals are given—with some handwaving—a meaning which allows to manipulate certain integrals to reach the "desired results." Even if one finds occasionally some hints that "mathematicians know how to interpret such concepts properly" most of the students and readers of such books may be left with the feeling that this is a complicated matter which has little to do with the numerical practice. After all, sets of measure zero are of course mathematically relevant, but rarely encountered in real life, and even the perfect Lebesgue integral is not used when it comes to the numerical approximation of values of integrals. And it certainly does not help at all when it comes to integrate over a pure frequency $\chi_s(t) := e^{2\pi i s \cdot t}$!

Although important for applications the correspondence between sampling and periodization in the Fourier domain is not a standard part of modern books on Fourier Analysis, but it is receiving more and more attention, partially with the advent of wavelet theory and time-frequency analysis (see, e.g., the books [3, 26, 70]).

Since the discovery of the FFT (Fast Fourier Transform, see [12]) it has become a habit to mention the numerical computation of the DFT (Discrete or Finite Fourier Transform) as a way to efficiently compute this discrete transform, but users are often left alone with their possible questions concerning the connection between this discrete version of the Fourier transform and the corresponding integral transform. We think that this is a crucial point for the understanding of Fourier Analysis, both from a theoretical and the applied point of view.

Whatsoever is the frame work for (abstract or concrete) Fourier Analysis, the essence is the decomposition of signals, functions or distributions into building blocks which are invariant under translations. It turns out that the only natural building blocks are the characters (resp. *pure frequencies* in an engineering terminology) on the underlying group. For a (post) modern perspective see [17].

9.2 Sociology of Fourier Users

The key idea behind [16] is the principle that it might be helpful to find out which function spaces are useful for which purpose. Within Classical Fourier Analysis this question appears to be well settled since the advent of Lebesgue's integration theory in 1904 (Lecons sur l'intégration et la Recherche des Fonctions Primitives (1904), reprint [45]). In this setting (and more generally then using the Haar measure over LCA groups) it appears to be very natural to treat the Fourier transform as a mapping defined on $\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)$ resp. $\left(L^1(G), \|\cdot\|_1\right)$. This is a nice Banach space, the Fourier transform is a well-defined and in fact continuous function vanishing at infinity (i.e., belonging to $C_0(\mathbb{R}^d)$), according to the Riemann-Lebesgue Lemma. Moreover $L^1(G)$ appears to be a very natural domain for the definition of convolution, if one wants to define it pointwise (a.e.) and via the usual integral definition. Convolution (based again on the Lebesgue integral) gives us (commutative) "multiplication" on $\left(L^1(G), \|\cdot\|_1\right)$, turning it into a Banach algebra, and the convolution theorem is valid, namely

$$\widehat{(f*g)} = \hat{f}\cdot\hat{g}, \quad f,g \in L^1(G). \tag{9.3}$$

Finally we have $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which is the appropriate domain to formulate Plancherel's theorem, which tells us that the Fourier transform "is" (can be modified to be considered as) a unitary automorphism of this Hilbert space. Engineers say, it is energy preserving, which in fact is a very important property. It is a historical fact that the scale of L^p-spaces was then introduced, allowing us to formulate the Hausdorff-Young theorem: $\mathscr{F}L^p(G) \subset L^q(\widehat{G})$, for $1 \le p \le 2, 1/p + 1/q = 1$.

But if we ask ourselves, whether this is all we have to say about the Fourier transform, we have to take a closer look on the *user community* of scientists involved. As a matter of fact it consists of a number of separate groups who use the Fourier transform, in a significantly diverse way, with different tools and motivations, and surprisingly little interaction among these groups. Thus we are asking: *Who are the people using the Fourier transform and how can they make use of this mathematical tool? How does it help them to solve their problems?*

Let us try to mention various perspectives taken by the different communities:

1. Pure mathematicians may start from the Fourier transform showing up naturally as an *integral transform* over $\mathbb{R}^d$, hence it is natural to care of the most general form of an integral (they found that the Lebesgue integral is the perfect answer

to this question). Since the Fourier inversion formula is not valid in the expected form, all kinds of summability methods have been developed. On the other hand one can point to some fine investigations which allow to conclude essentially from smoothness of a function f to the integrability of its Fourier transform $\hat{f}$, which in turn implies that the inverse Fourier transform can be understood in the pointwise sense: For every $t \in \mathbb{R}^d$ it is possible to obtain $f(t)$ via (9.2).

2. One can take this approach further and extend these methods to the realm of LCA groups. The Lebesgue measure is replaced by the Haar measure on $\mathscr{G}$ and $\widehat{\mathscr{G}}$, respectively. One has again the Plancherel theorem, and various abstract rules (the dual of G/H is identified with $H^{\perp}$), or Weil's formula. One has to settle measure theoretical questions in order to avoid problems, but this can be done (and makes up a major part of [50] resp. [51]).
3. Colleagues interested in PDEs or even pseudo-differential operators make use of Fourier analytic methods in a rather different setting. For them it is important that Fourier Analysis allows to properly define Sobolev spaces of fractional order (not just the Sobolev spaces of integer order), to introduce the *Schwartz* space $\mathscr{S}(\mathbb{R}^d)$ of rapidly decreasing functions and hence the dual space, called the space $\mathscr{S}'(\mathbb{R}^d)$ of *tempered distributions*. This is not only a fairly large domain for the Fourier transform, but also—at least in principle—a setting which allows to deal with discrete and continuous, with periodic and non-periodic "signals" in a unified way and thus also to address questions of mutual approximation (see [22]).
4. Ever since the work of Cooley-Tukey [12] colleagues in numerical analysis have taken a pride in developing a variety of fast algorithms for the realization of a huge variety of versions of the discrete (finite) Fourier. A good and positive example in this direction is the development of FFTW (the *fastest Fourier transform in the West* by M. Frigo and S.S. Johnson [29–31], or [49]).
5. For engineers, especially in communication theory, a central question is the understanding of *translation invariant linear systems* (TILS). For them it is important to represent such a linear operator, commuting with group translations, as a convolution operator by some *impulse response*. On the Fourier transform side one expects a realization of such an operator via multiplication by the *transfer function*, which is supposedly the Fourier transform of the so-called impulse response. Depending on the author the so-called *Dirac function* (resp. Dirac-measure or Dirac-distribution) is introduced in an intuitive way. Usually it is mentioned that it is "not really a function, but can be used in the context of integrals as if it was a function." In other cases it is introduced as a "magic object" whose existence is taken for granted with certain the rules for computations. even if it is hard to explain it. The burden of showing that it is possible to establish a theory justifying these relatively formal manipulations involving the Dirac function or Dirac combs can safely be left to the (pedantic) mathematicians. Unfortunately such an approach does not save the engineering community from ignoring that the derivations presented in many well-known books represent a "*scandal in systems theory*," because it is not true that an arbitrary BIBOS (bounded input bounded output system) *is a convolution operator* (see Sandberg [54–56]).

6. For a majority of engineers the following viewpoint appears to be fully legitimate: Paraphrasing only slightly what one reads in engineering books we state: *In practice every signal is of finite energy, hence it is justified to work with the signal space* $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$. *Moreover, whenever it comes to computations one can handle only finite vectors. Hence one has to work with samples (typically regular samples taken at equi-distant nodes) of a continuous function. As a consequence, whenever a Fourier transform has to be computed it is natural to replace it by the corresponding discrete, i.e. fast Fourier transform.*

9.3 What We Find in the Applied Literature

After so many decades of separate development, it seems that the separation is not seen anymore as problematic by the different communities, engineers and pure mathematicians have settled in different worlds. Some see it as a natural situation, or an unavoidable consequence of the different mindset, also due to the distinct goals of these disciplines. But I find it sad (and worth formulating it) and problematic, that we do not even discuss this matter in our communities. So we should try our best to bridge the gap. Let me illustrate the problem by reviewing statements found in the applied literature.

A common formulation in the applied literature reads like the following sentence: *"Since in the computer only finite data (of finite precision) can be handled, it is necessary to work with (finitely many) samples of a given function instead of the continuous functions."* As a consequence expressions involving integrals have to be replaced by corresponding summation terms (i.e., summing over all the terms), among them the all important convolution integrals.

In this sense it is more than plausible that Fourier integrals are replaced by the FFT, and in fact there are good heuristic ways to indicate that "in some sense" Fourier integrals approximated by Riemannian sums have an appearance close to the DFT (hence the FFT). On the other hand, the definition of the Fourier transform (as an integral transform, creating for every $f \in L^1(\mathbb{R}^d)$ some bounded and continuous function over the continuous frequency domain) and often also the Fourier inversion formula appear as non-periodic limiting cases of the classical, periodic Fourier transform. But in which sense can we expect these mutual approximations to take place? Is there any chance for pointwise convergence, or at least almost everywhere? For all functions (whatever this means) or just certain "nice" functions?

According to most of the treatments in the current literature or also in the courses (e.g., those which can be watched from MIT or Standford) offered in the internet (via Youtube) this—in our view crucial—question is not well discussed in a mathematical satisfactory way, at least from the viewpoint of a (pedantic?) mathematician.

Of course, by ignoring to acknowledge that there might be a problem may cause some unpleasant situations, because it is hard for most applied people to distinguish between heuristic considerations and well-settled mathematical claims. Many of these problems are in fact model errors which are put under the rug when computations are carried out, and fortunately it is usually no problem for the daily practice. Nevertheless one has to warn to be not too careless, because there is the danger of making claims which are only valid under certain restrictions. If used under unfavorable circumstances, i.e. whenever the range of applications of a formula is stretched too much, the user may find himself way outside of the range of validity of such a formula.

There is another aspect to this question. For heuristic considerations it may suffice to know that eventually the discrete (sometimes finite) model will approximate well the continuous limit. But for an actual computation one would like to have a good approximation with as little computational costs as possible, and this may require a completely different approach. Such questions are typical for experts in numerical analysis, but not so much for engineers, nor for those coming from abstract Harmonic Analysis and like to work in the most general setting.

Let us give a concrete example of a similar situation: We all know that the very general definition of a Riemannian integral allows to show (via the concept of uniform continuity) that for *every* continuous function f on $[a,b]$ the integral $\int_a^b f(x)dx$ is well defined. But that Riemannian sums are not an efficient way to actually compute this integral, while good integration rules allow to compute an approximate numerical value of such an integral in a much more efficient way, but maybe with a little be less of robustness (which respect to the choice of f). In fact, typical estimates for such algorithms make use of size estimates of certain derivatives of f.

Similar things are also true in Fourier Analysis, but less common in the perception of users. So in fact, the community is challenged to provide more of such efficient algorithms. In addition, one may want to find out under which conditions they are applicable, and—in ideal cases—demonstrate the optimality of such statements, leaving the user with the comfortable feeling to be sure to have made a good choice whenever such a highly efficient algorithm can be used in a concrete situation.

To convince yourself, each of our readers is welcome and in fact encouraged to collect further evidence of the poor reception of mathematical concepts in the applied community, to point out misunderstandings and also to contact the author of this article who tries to get an overall picture of the situation, with the slight hope to improve teaching and communication between disciplines, once the problems are recognized and mathematically valid alternative approaches are offered.

Hence readers should consult not only mathematical books but also "practical guides" to the use of the Fourier transform, above all in order to see with their own eyes how discrete Fourier transform is seen (and taught) by highly experienced authors in the application areas, who know many practical tricks and pitfalls (to use their own words). We will not discuss here all the ways to avoid such pitfalls, but have to mention that some of them arise only because the finite vectors are not properly identified with functions on finite groups, and the "center of a vector" is not always easy (and the usual plotting routines are not a good guide!).

As books in this category let us mention the book *Mastering the discrete Fourier transform in one, two or several dimensions. Pitfalls and artifact*, by Isaac Amidror, see [1] or *Computational Fourier optics. A MATLAB tutorial* by David Voelz [65], an SPIE publication. Another book of this kind is entitled *Fourier approach to digital* *holography*, by R. Jozweicki [52]. In such books, addressing the applied community the authors show great experience with practical issues related to the discrete Fourier transform, but at the same time reveal that they are not familiar (as the majority of their peers) with some guiding principles which could be offered by modern Harmonic Analysis.

Of course books vary greatly in their attempt to either ignore theoretical concepts (which are viewed as a burden to practical people), or relate the reader to the mathematical literature, where the interested reader (but why should she/he be interested in the pedantic explanation of irrelevant technical details and strange concepts?) could find explanations (or what mathematicians consider an explanation) of concepts which for an engineer have anyway a "natural interpretation and meaning."

There is not only the *scandal in systems theory*, but there are also some curious inventions showing up in the applied area, e.g. the so-called *epsilon-distribution*, as the "anti-thesis" to the Dirac distribution (see [61–64]).

Concerning *Dirac sequences*, i.e. sequences of ordinary function approximating the Dirac "function" one can read in Kanwal's book (see p.9 of [43]): *It is not true that one has* $lim_{n\to\infty} f_n(0) = \infty$ *for any such Dirac sequence, it may equally be the case that one has* $lim_{n\to\infty} f_n(0) = -\infty$*!* Such statements leave the reader with the impression that one may expect in any case $lim_{n\to\infty}|f_n(0)| = \infty$. But this is of course wrong, because one can find other Dirac sequences which have any given limit, or those where this sequence is divergent. *So the correct conclusion would be:* It is not meaningful to talk about the value of δ at $x = 0$! More generally, generalized functions do not provide pointwise information, but rather average information.

A comparatively harmless lack of mathematical strength (or should one say even mathematical correctness) can be spotted in the otherwise quite elaborate book of Bracewell (([4]), p.27): *Of course* (?) convolution is associative, i.e. satisfies $f * (g * h) = (f * g) * h$, whenever the involved convolutions make sense. But there is no mathematical proof and thus *evidence* is only a wishful thinking (highly plausible however, if one considers the "typical situations," where in fact such an associativity law is valid). But wouldn't it be natural to slightly overinterpret this claim and expect that it remains valid for the case of generalized functions (distributions)? But then it suddenly would be wrong, because one can find examples of distributions, where the claimed associativity law is violated! In fact, if convolution is defined in an individual manner in the context of generalized functions, this associativity rule may even *fail to be valid* (cf.[2], p. 111/112).

9.4 A Set of Questions

It is one of the goals of this note to address some of the following basic questions (which according to our understanding are basic principles and cornerstones of what we have called *Conceptual Harmonic Analysis* in [16]):

- How can Fourier Analysis over finite groups be realized using FFTs?
- How can one approximate the continuous setting by finite sequences, in order to carry out approximate computations?
- How can one use distribution theory based on $(S_0, L^2, S_0')(\mathbb{R}^d)$ in order to come up with the most general setting?
- How can one learn from numerical experiments and how should simulations be set up?
- How much can/should one trust (qualitatively, quantitatively) the figures obtained by such computations?
- Are we sure that we have a good implementation or is there room for further improvement?
- What are open questions in this field?

This list of question is of course only a fragmented and subjective one, and each reader should start a little reflection concerning her/his own questions. Even this short list will not be answered point by point, but we will give some exemplary answers to some of them, to convey the spirit in which we want to see (and in fact stimulate) answers in the literature to come in the future.

9.5 Experimental Mathematics

In this subsection we want to share some ideas concerning another meeting ground between abstract and applied Harmonic Analysis. We call it *experimental Harmonic Analysis*. Looking back on the last 25 years this author observes that this part of his work has been a very important one. It simply did not exist (maybe with a few exceptions) before. Only in the early 90th it became possible to develop code easily in a high level language, and spending time on experiments instead of coding. Colleagues from our technical university reported, that a single master student was able to convert existing C-code realized by a whole team within a couple of months into MATLAB code. It became possible to run simulations, to test hypotheses by doing numerical computations, or to observe certain phenomena which appear only when systematic investigations are performed with variable parameters.

This experimental work (finally leading to the creation of NuHAG) started approximately 25 years ago, in the autumn of 1989, during a one-year visit to John Benedetto, at the University of Maryland, at College Park. At that time our papers [20] and [19] had been already published, but we had not done any algorithmic work. In fact, the reviewing process for the first paper (which was also finished first) was endless, for various reasons, which are also worthwhile being mentioned here: The reviewing process revealed that the pure mathematical reviewers thought that the general possibility of recovering a band-limited function from irregular samples (under the given conditions) was known already. But this was true only in an abstract sense and in the setting of the Hilbert space $L^2(\mathbb{R})$, without any indication of a *constructive* recovery of f from the given, potentially irregular samples. It was also clear that the standard building blocks (e.g., in the case of regular sampling) would not be useful for $L^1(\mathbb{R})$ or weighted versions of this space, because they would suggest to use a series expansions using SINC-like functions, which are not integrable. So we had to explain that our paper was about a concrete algorithm and guaranteed convergence rates in weighted L^p-spaces, under favorable conditions, i.e. whenever the samples were given at a set of good density (better than the Nyquist rate).

But we also received critical comments from the applied reviewers, who simply did not appreciate the use of weighted L^p-spaces, because they were viewed as a kind of *abstract non-sense*. But why should even a practically minded user be content with convergence in the simple-minded quadratic mean (i.e., in the L^2-sense) if the sampled functions show good decay at infinity. Shouldn't we expect to have better convergence, or more correctly, convergence at a given rate, for a variety of useful function spaces norms, depending on the input?

It turned out to be even more important to point out that this convergence is guaranteed to occur at a given rate, even if the user (or the person providing the data) *does not know* that one has samples coming from some $f \in L^p_w(\mathbb{R}^d)$. It was certainly during this period that the relevance of having *families of alike function spaces* was recognized, which until now (e.g., in the context of coorbit theory) is one of the central themes.

There are many similar stories to be told, therefore let us just present them in the form of a list. Summarizing this author's use of MATLAB over the years one can say that there is quite a diversity of applications and settings where numerical experiments have been helpful and sometimes enlightening:

1. Illustration of mathematical claims, e.g. inspection of the structure of a Gabor frame operator, visualization of the concentration of the spreading distribution of such an operator, time-frequency behavior of a signal, etc.
2. Carrying out demonstration of mathematical principles, such as a "visual proof" of Shannon's sampling theorem.
3. Development of efficient algorithms, ideally useful for applied scientists, e.g. in the field of irregular sampling or for applications in mobile communication (channel identification and channel decoding) or astronomical image processing (in our ESO project, where we have been dealing with hyperspectral images).
4. Performing simulations which should give some indications, e.g. about the dependence of a dual Gabor window on the Gabor lattice, estimates on the condition number of Gabor frames, etc.

In the long run such experiments gave us the feeling that such simulations, if done properly, allow to approximate very well the continuous limit. But this situation also quickly brought up another set of questions: *Can we be sure about this "intuitive approach"?* Are we better prepared to defend claims derived from numerical experiments than engineers, who often use finite, discrete computations to support claims made in the continuous context (and vice versa).

It is natural (but by no means trivial) to expect that the transition from small to large versions of a problem with increasing signal size and matrix dimensions, should be discussed in detail using appropriate function spaces. Only then one can understand this transition to the continuous limit qualitatively and then start to treat the transition in a quantitative way, instead of just watching (in good hope) certain phenomena. In such a way, observations made during extensive numerical simulations may become the basis for the formulation of a theorem, or an approximation theoretic theorem may be relevant in guaranteeing certain rates of convergence, and so on.

There is even the chance to combine numerical and theoretical estimates properly. For example, one might verify the invertibility of some operator (or an estimate for the size of its spectrum) by computing numerically the properties of a "approximating matrix," combined with a guaranteed a priori estimate on the approximation error.

9.5.1 Learning Fourier Analysis

In addition to testing hypotheses, developing efficient algorithms for signal processing applications or carrying out numerical computations software packages, for example MATLAB^{TM}, can be quite useful, but they are also extremely important and helpful in supporting the learning process. As we all know, "showing a few pictures" is more instructive than a long explanation of the same situation.

This raises some new questions, which I would like to summarize as well:

1. Given a continuous problem, e.g. the fact that the sampling of a signal corresponds to the periodization of its Fourier transform. This is easily implemented and also verified, so to say manually, even if it realized in a relatively naive way. The only thing to be observed is the necessity that the sampling rate has to be a divisor of the signal length, if one wants to have perfect results. For example, for $N = 480$ one could subsample by a factor of $3, 4, 6, 8, 10$, or 12 (for example).
2. Such demonstration case can also be used to train the group theoretical terminology. The mentioned divisibility problem corresponds exactly to the fact that the sampling set is a subgroup (with respect to addition) of the cyclic group of order N. For each sampling rate (e.g., 6) the length of the period on the Fourier transform side is given by the complementary divisor, which is $N/6$ in this case, thus describing the so-called orthogonal subgroup, a subgroup of the frequency group.
3. In some cases it will not only be necessary to explain, by which analogy the presentation of a finite dimensional example is derived from the continuous setting, but it may also be of (hopefully great) interest to discuss the kind of errors introduced by the finite dimensional approximation, and how it can be controlled (e.g., via a priori estimates, or by making the convergence claims for suitable classes of functions, in order to be able to guarantee some rate of convergence for a whole class of functions, and not just individual functions).

One simple way to understand periodic functions f is to interpret them as function over the unit circle. So the graph of a real-valued function can be considered as a subset of the cylinder (the direct product of the circle with the real line), with the value of f being marked in the vertical direction (Fig. 9.1).

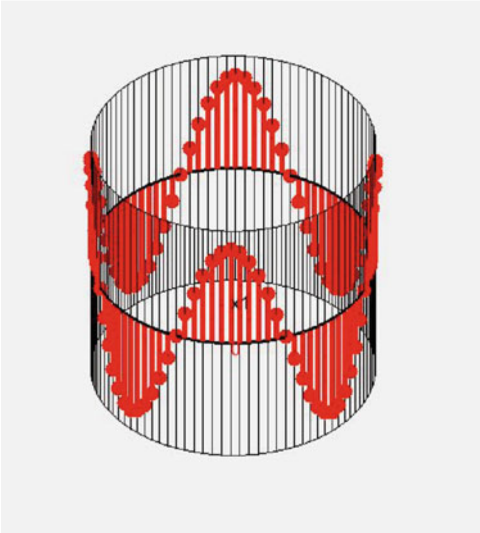

Fig. 9.1 The graph of a periodic function on the cylinder.

9.6 Fourier Analysis over Finite Groups

Since we discuss aspects of AHA (Commutative Harmonic Analysis over general LCA groups) it is natural to start with functions on a finite, Abelian group G. Possible references are the books by A. Terras ([59]), Luong ([47]), or Wong ([69]).

Of course, on any such group G one has the translation operators $T_x, x \in G$, given by $T_x f(y) = f(y-x), x, y \in G$. They form a collection of $N = \#(G)$ (cardinality of G) commuting operators, and their closed linear span is some N-dimensional *commutative algebra* of $N \times N$-matrices. For this commutative C^*-algebra one can find an orthonormal basis of *joint eigenvectors* for $\mathbb{C}^N = \ell^2(G)$, consisting of N so-called pure frequencies. In fact, the translation invariance together with the fact $T_x^N = Id_G$ for any $x \in G$ implies that the only possible eigenvalues are the unit roots of order N, which we want to denote by $\mathbb{U}_N$.

In this first technical section let us describe the transition from a purely mathematical description of signals over finite groups to the setting of MATLAB^{TM}.

If we accept for now the fact that it is a valid claim that a computer can only work with finite data, we should first understand the problem of Fourier Analysis over finite groups. Clearly a *complex-valued function* on a finite group H is nothing but a vector $(v_h)_{h \in H}$ indexed by the group elements.

To make it more concrete (and already point to a problem which I would like to call a psychological problem or a problem of indexing) let us consider two cases. In applications we may take an audio signal, say one second of a piece of sound. Sampled at the well-known HiFi rate of 44100 samples per second, we get a vector in $\mathbb{R}^N, N = 44100$ and not a priori a function on any group H. A simple digital camera will produce a pixel image, let us assume it is a gray-level image of format 300×400, which can be loaded into MATLAB as a matrix A of size 300×400 with 8-bit binary entries, i.e. with values in the set $\{0, 1, \cdots, 255\} \subset \mathbb{R}$. Here the binary (8-bit) value describes the level of brightness, i.e. 0 corresponds to complete darkness (black pixels), while large values are displayed close to white in the standard display format.

In a next step it is clear and quite plausible to look out for *finite Abelian groups* with the right cardinality, in other words to choose a cyclic group of order $N = 44100$ as the domain of the audio signal, and something like $\mathbb{Z}_{300} \times \mathbb{Z}_{400}$, i.e. a product of cyclic groups as the domain for the pixel image data.

But before simply applying the `FFT` resp. `FFT2` routine to the signal (which has to come in a suitable format in order to be loaded into MATLAB) let us verify how the identification with the group indices has to be done. Clear enough, $\mathbb{Z}_N$, for any value of $N \in \mathbb{N}$ has a neutral element with respect to addition (the group law), which is the zero-element, *but a naturally occurring vector does not have any component indexed by zero!* So we have to see how to convert the finite sequence (engineers would call it a finite, discrete signal) into a finite, periodic signal, i.e. an element of the group algebra $\ell^1(\mathbb{Z}_N)$, consisting of all sequences $\mathbf{x} := \sum_{k=0}^{N-1} c_k \delta_k$, or using different symbols $\overrightarrow{x} = \sum_{k=1}^{N} c_k \overrightarrow{e}_k$. Moreover instead of just sampling a function from left to right (the most left sampling point provides the first component in the sampling sequence) it should be taken care that the sampling value of f at zero is identified with the component in the unit circle which corresponds to the neutral element there, which is of course (the multiplicatively neutral element) $1 = \omega^0$, whatever the primite unit root of order N, denoted by ω, may be. Failing to choose this correct assignment (with the slight advantage that functions are nicely plotted) results in *artificial extra tricks* which are, however, not necessary if the assignment is done properly.

9.6.1 DFT in Mathematical Notation

Let us recall the usual description of the DFT (discrete Fourier transform). Starting from a finite sequence of length $L \in \mathbb{N}$, $f = [f(0), \cdots, f(L-1)] \in C^L$, resp. an L-periodic signal f its DFT as described in the book [5] by Briggs, Henson, and Van Emden, entitled *"The DFT. An Owner's Manual."* The Discrete Fourier Transform is defined by

$$\hat{f}(k) = \sum_{l=0}^{L-1} f(l) e^{-\frac{2\pi i k \cdot l}{L}} \qquad k = 0, \cdots, L-1, \tag{9.4}$$

the inverse transform (named IDFT) can be obtained via

$$f(k) = \frac{1}{L} \sum_{l=0}^{L-1} \hat{f}(l) e^{\frac{2\pi i k \cdot l}{L}} \qquad k = 0, \cdots, L-1. \tag{9.5}$$

From the point of view of realization within, e.g., MATLAB^{TM} one can establish the matrix which corresponds to this linear mapping. It is enough to apply the FFT-routine (which works column-wise) to the unit matrix, i.e. the collection of unit vectors in $\mathbb{C}^N$ (Fig. 9.2):

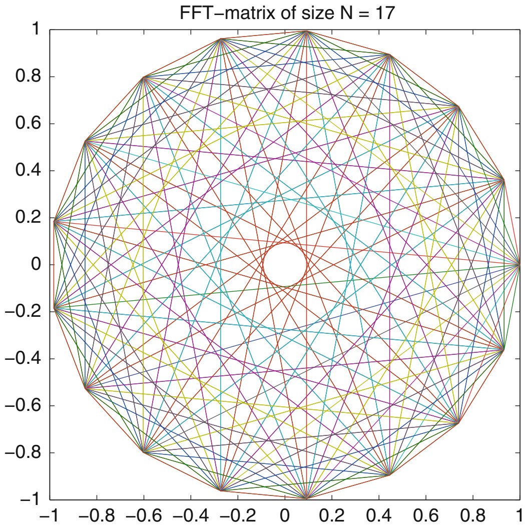

Fig. 9.2 A plot of the FFT-matrix, obtained by the command: `plot(fft(eye(17)));`

Since MATLAB does not allow to label the coordinates with an index zero (in fact: any "physical vector" has a first and a last coordinate, but no "zeroth coordinate") the more natural description (in terms of actual coordinates, used as running index in these sums, now called $k = 1, \cdots, L$) is to write:

$$\hat{f}(k) = \sum_{l=1}^{L} f(l) e^{-\frac{2\pi i (k-1)\cdot(l-1)}{L}} \qquad k = 1, \cdots, L, \tag{9.6}$$

and correspondingly

$$f(k) = \frac{1}{L} \sum_{l=1}^{L} \hat{f}(l) e^{\frac{2\pi i (k-1)\cdot(l-1)}{L}} \qquad k = 1, \cdots, L. \tag{9.7}$$

Of course these formulas are the routine realized via any mathematical software program providing the FFT of a given sequence. It is also not difficult to verify that (up to a normalizing factor $\sqrt{L}$) the DFT is a unitary mapping and the IDFT (as given by the above formula) is in fact the inverse operator. In other words, the matrix realizing the DFT as a linear mapping (which can be obtained in MATLAB by the command `F = fft(eye(L))/sqrt(L)` is unitary, and consequently the DFT just represents a change of bases, from the standard basis of unit vectors to the Fourier basis, consisting of (discrete) pure frequencies (or the same: the monomials restricted to the unit roots of order L).

9.6.2 Comparison with Polynomial Evaluation

A good way to understand some of the amazing properties (both conceptually and numerically) can be derived from the observation that the Fourier matrix is a Vandermonde matrix. More precisely, the Fourier matrix describes the transition between the coefficient of a polynomial with L (possibly zero) complex coefficients, described in the usual *mathematical order*, i.e. with respect to the basis $\{1,t,t^2,\cdot,t^{L-1}\}$, to the L points of the cyclic group $\mathbb{U}_L$ of order L, which are presented in their natural clockwise order. This means that we consider a vector $\mathbf{c} \in \mathbb{C}^L$ as the coefficients of a polynomial and then look at the values of

$$p_c(z) = \sum_{k=1}^{L} c_{k-1} z^k \quad \text{at the sequence} \quad \left(\omega^{-(k-1)}\right)_{k=1}^{L},$$

where $\omega = \omega_L = exp(2\pi i/L) = \cos(360/L) + i \cdot \sin(360/L)$ is the *primitive unit root* of order L. In this interpretation we see that the DFT/FFT of a sequence $\mathbf{c} \in \mathbb{C}^N$ can be described as the values of a polynomial with these coefficients (from constant to highest order $= N-1$) evaluated at the unit roots of order L starting from $\omega^0 = 1$, in the *clockwise sense*, resp. the mathematical negative sense.

One of the simple consequences of this identification is the fact that it is clear that one can use the FFT for the *fast realization of the Cauchy product*, i.e. the operation of convolution, which describes the coefficients of a product polynomial.

Given two polynomials the usual form of multiplications (corresponding to the usual multiplication of natural numbers that pupils are already learning in elementary school) requires to a carry out a number of multiplications which is more or less the products of the two orders of the polynomial factors. In other words, the computation of the square of a polynomial of degree 99 requires 10.000 multiplications (plus some reshuffling in between and at the end of the process).

The FFT allows to do the same thing considerably faster: We just have to consider the fact that a product of two polynomials of degree 99 will certainly be of degree < 255, hence it is enough to know it on the unit roots of order $256 = 2^8$ (a power of two makes a particularly fast FFT!) in order to find its coefficients by the (equally efficient) IFFT algorithms (which is just the FFT plus some minor modifications). So given the coefficients of the two polynomials it is enough to first extend the given (100) coefficients via zero-padding to the length $L = 256$. This procedure *does not change* the value of the polynomial at any point in the complex plane, but *forces* the FFT-routine to sample the polynomial not only at the unit roots of order 100 but at the unit roots of order 256). The pointwise multiplication is then quite cheap and allows to obtain the coefficient of the product polynomial, still over the unit roots of order $N = 256$, from which the coefficients are easily obtained using the IFFT routine. Here we are making use of the so-called *convolution theorem* for the Fourier transform, one of the highlights of Fourier Analysis and systems's theory, in a very transparent way.

Of course this approach also opens the way to an efficient multiplication of large natural numbers, because at the end in our usual decimal description of natural numbers every natural number is just the value of a polynomial at $x = 10$. Just to give an example: everybody knows that 1234 is just the value of the polynomial with the coefficient sequence $[1,2,3,4]$ at $x = 10$. In fact, the convention of MATLAB for the evaluation of polynomials is based on the same idea is the representation of number in our daily life. The length of the string 1234 tells us that the leading term must go along with the power t^3, or represents the "thousands," while the following digits represent the lower powers (down to the constant term, corresponding to the last digit, i.e. 4 in this case).

A simple MATLAB experiment verifying some of the claims made above can be realized numerically as follows:

```
L = 5; aa = rand(1,L);
u5 =  exp(2*pi*i*[0 : 1/L : (L-1)/L]);
norm(polyval(fliplr(aa),conj(u5)) - fft(aa)),
% the difference of normalized versions = 3.0405e-16
```

9.7 Poisson's Formula

One of the formulas which are both at the heart of Harmonic Analysis seen as a mathematical discipline, but also at the foundation of Digital signal processing is Poisson's formula, which has been also a crucial object of investigations in the work of Paul Butzer, who is certainly one of the pioneers advocating cooperation between mathematicians (he is mostly an approximation theory person) and engineers. He has addressed Poisson's formula in a series of papers, such as [6–8, 10].

The usual way of formulating Poisson's summation formula is the following one:

Theorem 9.1.

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \hat{f}(n), \tag{9.8}$$

but mathematicians found out that it is "not always valid." In other words one has to make some extra assumptions in order to guarantee that not only both sides are convergent, but in fact that the two infinite sums are equal. Katznelson's book [44] provides a counterexample. There is a more detailed discussion of this problem, evaluating under which extra conditions such counterexamples are still possible, despite a little bit of decay of both the function and its Fourier transform (however in a somewhat balanced way), see [40].

The usual proof goes like this: Given an integrable function $f \in L^1(\mathbb{R})$ one first observes that it can be periodized, i.e. one takes $f_{per}(x) := \sum_{n\in\mathbb{Z}} T_n f(x)$ and then the Fourier coefficients of this new periodic functions, usually denoted by $(\hat{f}_{per})(n)$ coincide with the samples of the continuous Fourier transform of f at n, i.e. with $\hat{f}(n)$, $n \in \mathbb{Z}$. At the heart of the proof are elementary properties of the exponential function. The "story" goes on by representing the periodic function f_{per} via its Fourier series expansion, i.e. using the fact that

$$f_{per}(t) = \sum_{n \in \mathbb{Z}} \hat{f}_{per}(n) e^{2\pi i n t}.$$

It seems to suffice to set $t = 0$ in this equation in order to obtain the Poisson formula, but there is a serious problem: it may well be that the Fourier series is convergent for $t = 0$, i.e. the right-hand side is then of course $\sum_{n \in \mathbb{Z}} \hat{f}(n)$, *but without representing* the actual value of $f_{per}(0)$.

For engineers the value of Poisson's formula is more in a related situation. Given a function f on the real line $\mathbb{R}$, one can expect that a (regularly) sampled of the function has a (generalized) Fourier transform (resp. in the usual sense, as the FT of a discrete signal on $\mathbb{Z}$), and that this function on the torus is in fact the same as the periodized version of $\hat{f}$. Changing the roles of the time and the frequency variable one can say: If we periodize a function f, then the Fourier coefficients of the periodized version of f are just the samples of $\hat{f}$ (taken at the lattice $\Lambda^{\perp}$, associated in a natural way with the periodization lattice). (See [26] for a recent paper in this direction).

In the setting of $S_0(\mathbb{R}^d)$ we have the following valid statements:

Theorem 9.2. *For $f \in S_0(\mathbb{R}^d)$ one has*

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n), \tag{9.9}$$

the sum being absolutely convergent on both sides.

There are of course similar versions for general lattices (using the orthogonal lattice on the Fourier side then), also valid for all $f \in S_0(\mathbb{R}^d)$. For a proof of the above formula (9.8) one simply has to set $t = 0$ (in the current context this is justified!) in the statement below:

Theorem 9.3. *For $f \in S_0(\mathbb{R}^d)$ one has: The periodized version f_{per} belongs to the algebra $\big(A(\mathbb{T}), \|\cdot\|_A\big)$ of absolutely convergent Fourier series, hence f_{per} is pointwise represented by the uniformly and absolutely convergent series*

$$f_{per}(t) = \sum_{n \in \mathbb{Z}^d} \hat{f}_{per}(n) e^{2\pi i n t} = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{2\pi i n t}, \quad t \in \mathbb{R}, \tag{9.10}$$

the sum being absolutely and uniformly convergent.

In the current setting the engineering viewpoint that sampling and periodization are equivalent (one on the time, the other on the frequency domain) has a very symmetric formulation and one can also claim:

Theorem 9.4. *For $f \in S_0(\mathbb{R}^d)$, then the sampled version of f, i.e. the sequence $(f(k))_{k \in \mathbb{Z}^d}$ is absolutely summable, i.e. belongs to the sequence space $\ell^1(\mathbb{Z}^d)$. Moreover the periodic function h arising as absolutely convergent series, namely*

$$h(x) := \sum_{k \in \mathbb{Z}^d} c_k e^{2\pi i \langle k, x \rangle}, \quad x \in \mathbb{R}^d$$

has as Fourier coefficients exactly the sampling values of the original function f.

9.8 Transferring Data Between Groups

It is one of the key observations of Fourier Analysis that one should view signals or even generalized functions as objects which *live* on some LCA group $\mathscr{G}$ (see, e.g., [11] for a good example where an applied author makes this suggestion of a *unified signal processing approach*). This author is obviously influenced by the book of L. Loomis on Abstract Harmonic Analysis ([46]).

A typical problem which is rarely treated in a mathematical satisfactory way in applied books is the transition from the continuous (and often non-periodic) setting to the finite case. Of course this transition can be realized in two steps. First we can try to approximate (in some—correctly in a distribution theoretic—sense) the non-periodic function by a periodic version with a "very large periodization constant." Afterwards the resulting function (now a function on the torus or a multi-dimensional version of the torus) has to be sampled properly in order to come up with a finite vector which can be treated in the computer, using linear algebra tools.

We also observe that these two procedures *commute* if the sampling distance (sampling grid) is a refinement of the periodization net. So for decent functions on $\mathbb{R}^d$ one can expect that it is a good idea to periodize with respect to $p \cdot \mathbb{Z}^d$ for some integer $p \in \mathbb{N}$ and to sample over the lattice $(1/q) \cdot \mathbb{Z}^d$, hence the resulting data-cube generated forms a d-dimensional matrix of size $n := p \cdot q$.

The viewpoint that a non-periodic function is the limit of its periodic versions

$$f_p := \sum_{k \in \mathbb{Z}^d} T_{pk} f,$$

for $p \to \infty$ is often used as an argument in the derivation of the formulas for the continuous Fourier transform for function in $L^1(\mathbb{R}^d)$ (not in a strict sense, but in the sense a plausibility argument). On the other hand, it is obvious that a smooth function, sampled at a high rate, is well approximated by, e.g., the piecewise linear interpolation of the sampling values. Hence overall one can expect that for a function f showing some good decay and enough smoothness a reasonable choice for the integer values p, q (large enough) will allow to retain a finite data set which captures the essence of the function f. A qualitative statement in this direction (involving the Segal algebra $S_0(\mathbb{R}^d)$) is given in ([41]).

From a group theoretical point of view it is clear that the periodization operator maps functions in $L^1(\mathbb{R}^d)$ to functions (in fact in a surjective way) on $\mathbb{R}^d/\Lambda_1$, where $\Lambda_1 = p \cdot \mathbb{Z}^d$ is the periodization lattice. On the other hand, the restriction mapping (e.g., from $S_0(\mathbb{R}^d)$ to $S_0(\Lambda_2) = \ell^1(\Lambda_2)$) is surjective as well and produces sequences over Λ_2, typically $\Lambda_2 = (1/q) \cdot \mathbb{Z}^d$.

It is also clear that the periodization step can be practically ignored if the function f has compact support, because then locally (over the fundamental domain of the lattice Λ_1) the function f is just the same as its periodized version f_p.

The combination of both operations maps function in $S_0(\mathbb{R}^d)$ to functions on the quotient group Λ_2/Λ_1, which is a finite group and thus accessible to computations.

That these transitions are not just vague limiting relations, but can be made mathematically precise using the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ has been explained in [21], and is used very much in the proof of [41]. A particular case of the results in [21] is the following statement (formulated here for the case $d = 1$):

Lemma 9.1. *Given $f \in S_0(\mathbb{R})$ let us write $\mathrm{Sp}_q(f)$ for the piecewise linear interpolation of the regular samples of f over $(1/q)\cdot\mathbb{Z}$. Then one has*

$$\|\,\mathrm{Sp}_q(f) - f\|_{S_0} \to 0 \quad as \quad q \to \infty.$$

These operations can be well described in the setting of the Banach Gelfand triple, because for any lattice $\Lambda \lhd \mathbb{R}^d$ the corresponding Dirac comb $\sqcup\!\sqcup_\Lambda := \sum_{\lambda\in\Lambda} \delta_\lambda$ belongs to $S_0'(\mathbb{R}^d)$, and sampling resp. periodization can be described as pointwise multiplication $f \mapsto \sqcup\!\sqcup_\Lambda \cdot f$ and convolution by such a Dirac comb: $\mu \mapsto \sqcup\!\sqcup_\Lambda * \mu$ for any $L^1(\mathbb{R}^d)$-function resp. bounded measure $\mu \in M_b(\mathbb{R}^d)$. If we have two such lattices, a coarse lattice Λ_1 which describes the periodization, and a refinement of this lattice, fine lattice $\Lambda_2 \lhd \Lambda_1$ (hence the quotient group is a finite Abelian group), then in fact these two operations will commute. Let us write $\sqcup\!\sqcup_s$ for the Dirac comb corresponding to the sampling lattice Λ_2, and $\sqcup\!\sqcup_p$ for the Dirac comb over the the periodization lattice Λ_1. Then we have the following commutation relation:

$$\sqcup\!\sqcup_p * (\sqcup\!\sqcup_s \cdot f) = \sqcup\!\sqcup_s \cdot (\sqcup\!\sqcup_p * f), \quad \text{for all} \quad f \in S_0(\mathbb{R}^d). \tag{9.11}$$

It is clear that for every decent function, e.g. for any smooth function with compact support, one can expect that the knowledge of these sampling values for large p and sufficiently fine sampling lattice should suffice to produce a good approximation of the function f, by cutting out the relevant local profile of f (some of the basic periods) and applying some form of (quasi-)interpolation on the data, in order to come back from a finite set of samples to the function space $S_0(\mathbb{R}^d)$. This is indeed the fact and is properly described in [41].

We will describe here only the idea with a specific prescription, restricting our attention to the case of the real line $\mathcal{G} = \mathbb{R}$ and the simple method of piecewise linear interpolation, which is also how MATLAB^{TM} presents function using finitely many sampling values.

Given $N = p^2$ we spend our budget (length of the vector to be stored) in two ways, by taking the sampling rate $1/q = 1/p$, i.e. we take p samples per unit interval of the p-periodic version of $f \in S_0(\mathbb{R})$, or alternatively, we sample the function f over the lattice $(1/p)\mathbb{Z}$ and periodize this sequence (which belongs to $\ell^1(\mathbb{Z})$) afterwards, with period $N = p^2$.

In any case, we receive a function of the cyclic group of order N (which for us is viewed as the group of unit roots of order N with the usual multiplication of complex numbers).

9.8.1 From Groups to Linear Algebra (and Back?)

Although in principle we have now an idea how to convert a function (up to some loss of information by periodizing it coarsely and sampling it at a high rate, but of course loosing some information) and to create from f a periodic and discrete signal, we still have to describe how to work with this periodic (hence infinite) signal (living on $\mathbb{Z}$) into a vector that can be treated, e.g., using MATLAB or some other mathematical software package.

It is quite clear that any basic period can be chosen to represent the periodic function, but the question is, which offset should be preferred (Fig. 9.3).

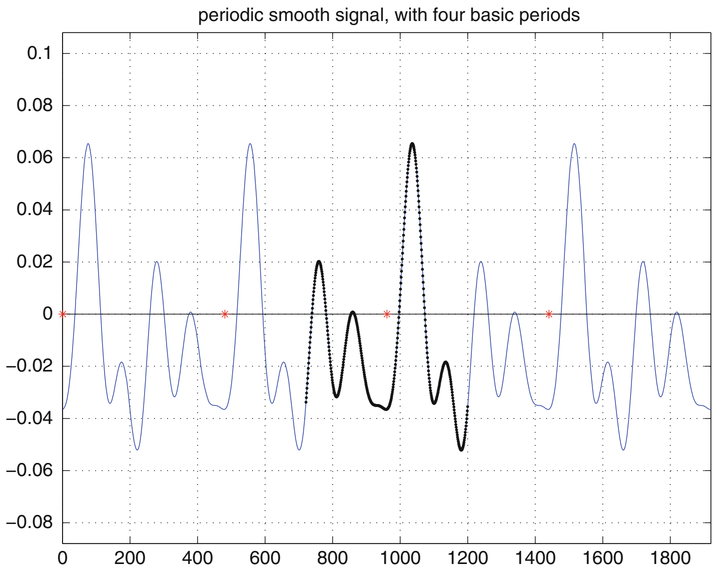

Fig. 9.3 Showing a basic period of a periodic function.

Although it may appear to choose, in the case of, e.g., a periodized Gauss function to choose the segment in such a way that the plot of that segment looks nice, i.e. with the peak centered in the middle, this is not the correct form of transfer to the finite setting. The MATLAB file `conv_example - Convolution: two Gaussian functions` provided with [65] shows the effect. It instructs the user to apply the `fftshift`-command after pointwise multiplication of the two Fourier transforms. But this is only necessary because in his transition the center is not chosen properly. The peak should be found at the *first* MATLAB component, which corresponds to the neutral element ($\omega^0 = 1$) of the group of unit roots.

9.9 Time-Frequency Analysis and Gabor Analysis

According to the current understanding, the field of *Time-Frequency Analysis* can be described as that part of mathematical analysis which is centered around the use of the so-called time-frequency shift operators (see [36] for a general reference to the field, but also [27]).

Although the natural domain for time-frequency analysis is the setting of LCA groups (one needs only a locally compact Abelian group $\mathscr{G}$ which comes with its dual group, in order to properly define TF-shifts) we recall—for simplicity—the situation for the case $\mathscr{G} = \mathbb{R}^d$. In addition to the commutative family of translation operators T_x we have (another commutative group of) pointwise multiplication operators (multiplication by characters), resp. the modulation operators $M_s : M_s(f)(z) = e^{2\pi i s \cdot z}, s, z \in \mathbb{R}^d$. When combined (first the time-shift is applied, then the modulation or frequency shift) we obtain the time-frequency shifts, i.e. the unitary operator on $L^2(\mathbb{R}^d)$, given by:

$$\pi(\lambda) = M_s T_x, \quad \text{for} \quad \lambda = (x,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{9.12}$$

Although the mapping $\lambda \mapsto \pi(\lambda)$ is not a usual group representation of the additive group $G \times \widehat{G}$, here $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, it is still a so-called *projective representation*, i.e. the composition of two operators $\pi(\lambda_1)$ and $\pi(\lambda_2)$ gives $\pi(\lambda_1 + \lambda_2)$, but up to a phase factor of absolute value one.

One of the key objects in TF-analysis is the so-called STFT (short time Fourier transform), which gives us a picture of the "energy distribution" of a signal, by localizing it to different locations (using the moving window g, which normally is assumed to be centered around the origin, smooth, non-negative, and symmetric, e.g. a Gauss-function or a B-spline), defined by

$$V_g(f)(\lambda) := \langle f, \pi(\lambda) g \rangle, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{9.13}$$

Such an approach is also well motivated by the analysis of musical sound, where the ear (and then the brain) are able to recognize a melody, find that two melodies are similar (e.g., up to transposition) used another transform, the so-called *Short-time Fourier Transform,* also called the *Sliding-Window Fourier Transform,* because it can be interpreted as follows: if one wants to analyze the change of frequency content within a given signal, one has to first *localize* the signal, and only subsequently apply Fourier Analysis methods to the localized pieces. The resulting function of two variables on the time-frequency plane is typically depicted by presenting a color-coded intensity, i.e. either $|V_g(f)|$ or $|V_g(f)|^2$. The latter can be interpreted as the energy distributions (telling us at which point in time do we have a lot of energy at which frequencies, e.g. which accord is played by which instrument), because one can interpret that the total energy equals the total energy in the signal, usually described by $\int_{\mathbb{R}^d} |f(x)|^2 dx$.

In fact, via Plancherel's theorem one can verify that one has

$$\|V_g(f)\|_{L^2(\mathbb{R}^{2d})} = \|f\|_2\|g\|_2 \quad \text{for} \quad f,g \in L^2(\mathbb{R}^d).$$

Obviously it makes a lot of sense to assume that $\|g\|_2 = 1$, because then V_g maps $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$ isometrically into $\left(L^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$, and thus the adjoint mapping V_g^* is the inverse mapping on the range, i.e. one has the (weak) recovery formula

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)\pi(\lambda)g\, d\lambda.$$

The Segal algebra $S_0(\mathbb{R}^d)$ consists exactly of those functions $f \in L^2(\mathbb{R}^d)$ with $V_g(f) \in L^1(\mathbb{R}^{2d})$, while $S_0'(\mathbb{R}^d)$ can be characterized as the set of (tempered) distributions σ with $V_g(\sigma) \in L^\infty(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

$S_0(\mathbb{R}^d)$ first appeared as a very natural space within TF-analysis. It is the smallest Banach space with the property $\|\pi(\lambda)f\|_B = \|f\|_B, \forall f \in B$, and consequently its dual is the largest such Banach space. Moreover, both of them are Fourier invariant. This was one of the original motivations to study this space [23–25].

Gabor analysis is an important branch of *time-frequency analysis*. It goes back to the idea of D. Gabor [32] to represent "every complex-valued function" on $\mathbb{R}$ as a superposition (in fact a double series) of the Gauss-function, shifted *"in time and frequency"* along the integer lattice $\mathbb{Z} \times \mathbb{Z}$. Even if it turned out that this is the "limiting case" (to use a TF-lattice at the critical density $= 1$ is not a good idea), but does not at allow to provide stable expansions of general functions $f \in L^2(\mathbb{R})$ (due to the Balian-Low theorem, see [36]) it is still a good general idea. In fact it is meanwhile well known that for any pair of lattice constants (a,b), with $0 < ab < 1$, it is true, that the Gabor family of the form $(M_{bn}T_{ak}g_0)_{(n,k)\in\mathbb{Z}^2}$ form a (stable) Gabor frame, at least for the choice g_0 equal to the Gauss function (more recently a number of other functions known explicitly). Such families have as a dual frame another Gabor family, with a so-called dual Gabor atom $\tilde{g}$, which is again a Schwartz function.

There are various properties, where good windows, meaning windows (or Gabor atoms) in $S_0(\mathbb{R}^d)$ are welcome.

First of all one can show that for any lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ the mapping $f \mapsto (V_g(f)(\lambda)_{\lambda\in\Lambda})$, i.e. the restriction of the (continuous) STFT to the lattice Λ, maps $(S_0, L^2, S_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)$, whenever $g \in S_0(\mathbb{R}^d)$ (see [25]). Hence the Gabor frame operator

$$S(f) = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

is bounded linear mapping of the Banach Gelfand triple (S_0, L^2, S_0') into itself, which by assumption is invertible at the L^2-level (cf. [13]). But as a matter of fact it is also invertible as a BGT-mapping, meaning that its inverse is also mapping $S_0(\mathbb{R}^d)$ into itself and also extends in a natural way to an inverse mapping on $S_0'(\mathbb{R}^d)$ (cf. [37]).

Consequently one has: whenever some $g \in S_0(\mathbb{R}^d)$ defines a Gabor frame (with respect to the lattice Λ then the dual Gabor atom, i.e. $\tilde{g} = S^{-1}(g)$ also belongs to $S_0(\mathbb{R}^d)$.

Another nice feature shared by all the windows $g \in S_0(\mathbb{R}^d)$ is the robustness of the lattice against jitter error (slight pointwise perturbation of the sampling points, see [18]), but more importantly also against small perturbation (e.g., small dilation or rotation) of the lattice. The main result of [21] states that for any $g \in S_0(\mathbb{R}^d)$ the set of all $2d \times 2d$ invertible matrices A, such that with $\Lambda = A(\mathbb{Z}^{2d})$ the pair (g, Λ) is an open set.

Of course also for Gabor analysis we have the problem of approximating (numerically and computationally) the dual Gabor atom (with a small error in the S_0-norm, because only this justifies to call the computed window an *approximate dual Gabor window*. Qualitatively this problem has been settled in [41].

It is a nice experiment of thoughts to pursue (at least for a little while) the following question: Was it really a natural and unavoidable step to go from the Fourier transform for periodic functions to the continuous Fourier transform, involving all the necessary tools of integration theory and complicated summability methods in order to describe the Fourier inversion results? Or was it partially (and undoubtedly an important) a historical development that could have taken another route? What if the community of people studying classical Fourier series would have first taken the modern viewpoint of time-frequency analysis and would have introduced the short-time Fourier transform before developing the Lebesgue integration theory.

In such a case it certainly would have been possible to define $S_0(\mathbb{R}^d)$, using just the Riemannian integral, and then to extend the domain of the classical Fourier transform to $S_0'(\mathbb{R}^d)$, in the way described above. Obviously such an approach would have made it necessary to discuss the question of the choice of the window, and a justification of the inverse STFT mapping.

Then one could have discussed the question of time-resolution, which can be increased by allowing very short (and well concentrated, but smooth) windows, or alternatively, guarantee a high frequency resolution by making use of a very large window size. From such a point of view the so-called sifting property of the Dirac measure would correspond to a "one-point" support of the window, i.e. the function would be viewed as a continuous superposition of its point values (of course also understood in the w^*-sense within $S_0'(\mathbb{R}^d)$!). But clearly the Dirac impulse is not a function in the ordinary sense, and not even in $L^1(\mathbb{R}^d)$, despite the generality of the Lebesgue integral. It guarantees perfect time resolution, but creates obviously some technical problems (e.g., because changing the pointwise values of a function $f \in L^1(\mathbb{R}^d)$ over a set of measure zero does not change the equivalence class of measurable functions to which f belongs). In the same way we can make the window "infinitely long," and take the resulting transform (which then has of course perfect frequency resolution over the "continuous frequency domain" $\mathbb{R}^d$) as a limiting case. But clearly the constant function (with amplitude 1) is *not an element of the Hilbert space* $L^2(\mathbb{R}^d)$, and thus we face (the usual) problem of integrability! The way out of this dilemma is of course to assume integrability of the function f whose Fourier transform should be taken, as we are used to this extra condition.

In other words, the time or frequency representations are kind of extreme cases of a TF-representations for "objects," which can be carried out without problems for $f \in S_0(\mathbb{R}^d)$, but becomes a bit "dangerous" outside of this space. We refer the reader to the classical sources in the field.

9.9.1 The Banach Gelfand Triple and w^*-Convergence

There are many advantages if one works with the Banach Gelfand Triple. Let us just pull out the advantages when working with the question of properly describing the Fourier transform. Here the space of test functions $S_0(\mathbb{R}^d)$ is well suited, because it is not so small and contains, e.g., all the classical summability kernels, but, on the other hand, allows to apply the Fourier inversion formula, and even Poisson's formula (in its strongest form). On the other hand, the dual space $S_0'(\mathbb{R}^d)$ is large to contain all the L^p-spaces, but also periodic functions of any kind, including the Dirac combs, but also pure frequencies or Dirac measures. In this enlarged setting one can describe the Fourier transform as the unique Banach Gelfand Triple automorphism of $(S_0, L^2, S_0')(\mathbb{R}^d)$ which maps pure frequencies into Dirac point measures (very much in the spirit of a musician listening to a pure tone and explaining to his friends that he is hearing a particular frequency with a given small or large amplitude).

However, when it comes to such statements one has to be careful with the type of convergence which one is taking into account. We suggest to use what is called (in courses on functional analysis) the $w^* - wst$-continuity, which means that the Fourier transform is not only norm-to-norm continuous at each level (namely, $S_0(\mathbb{R}^d), L^2(\mathbb{R}^d)$ or $S_0'(\mathbb{R}^d)$) but also preserves w^*-convergence.

Let us recall that a sequence (σ_n) of distributions in $S_0'(\mathbb{R}^d)$ is w^*-convergent to some limit $\sigma_0 \in S_0'(\mathbb{R}^d)$, if we have for every test function $f \in S_0(\mathbb{R}^d)$

$$\lim_{n\to\infty} \sigma_n(f) = \sigma_0(f).$$

Let us thus show how easy it is to argue that the generalized Fourier transform, defined for $\sigma \in S_0'(\mathbb{R}^d)$ by the rule

$$\hat{\sigma}(f) := \sigma(\hat{f}), \quad f \in S_0(\mathbb{R}^d) \tag{9.14}$$

maps w^*-convergent sequences into w^*-convergent sequences (with the correct limits). We say that $\mathscr{F}$ is a $w^* - w^*$-continuous mapping.

For a proof of this claim we only have to verify that

$$\sigma_0 = w^* - lim_{n\to\infty}\sigma_n \quad \Rightarrow \quad \hat{\sigma}_0 = w^* - lim_{n\to\infty}\hat{\sigma}_n. \tag{9.15}$$

This form of continuity is also quite important because it appears to be the correct form of approximation of distributions by either ordinary functions or by finite sums of Dirac measures (so-called finite discrete measures). It can also be used to deduce the validity of a general convolution theorem from the validity for point measures, i.e. from the fact that

$$\delta_x * \delta_y = \delta_{x+y}, \quad x, y \in \mathbb{R},$$

in conjunction with the fact that $\mathscr{F}(\delta_x) = \chi_{-x}$ the pure frequency $z \mapsto exp(2\pi i(-s)x)$, and $\chi_u \cdot \chi_v = \chi_{u+v}$ (resulting directly from the exponential law).

The mentioned continuity property can also be used to show that the "tricky method" (of defining the generalized Fourier transform as in (9.14)) is the only method which has the following property:

Proposition 9.1. *Assume that σ is the w^*-limit of a sequence of regular distributions induced by test functions (g_n) in $S_0(\mathbb{R}^d)$ (or even $L^1(\mathbb{R}^d)$), i.e. that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) g_n(x) dx = \sigma(f), \quad f \in S_0(\mathbb{R}^d).$$

Then also the sequence of their Fourier transforms has the same property, i.e. the limit

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} h(y) \hat{g}_n(y) dy$$

exists for any $h \in S_0(\mathbb{R}^d)$. Consequently it is possible to extend the Fourier transform by identifying $\hat{\sigma}$ with the functional assigning to $h \in S_0(\mathbb{R}^d)$ exactly this limit.

Even if this viewpoint suggests that the extended Fourier transform is the only natural extension of the ordinary, pointwise defined Fourier transform on the domain $L^1(\mathbb{R}^d)$ it would be cumbersome to establish this limiting relationship and then verify that the so-defined Fourier transform actually satisfies $\|\hat{\sigma}\|_{S_0'} = \|\sigma\|_{S_0'}$.

9.10 Summary

Let us shortly summarize the most important point of this pamphlet, by providing a "take-home message." The points which should be kept in mind (and giving a context in which to place our activities) are the following ones:

- First of all, we want to combine the advantages of Abstract Harmonic Analysis (AHA, which is mostly Harmonic Analysis over LCA groups) with the efficiency of Computational Harmonic Analysis, but unlike AHA we propose to keep an eye on the interaction between the different settings. So it is not just analogy which helps us to guess how to approximate (in a suitable sense) a continuous problem by a finite dimensional one, which can be performed constructively on a computer. The combination of these ideas is promoted by the idea of *Conceptual Harmonic Analysis* which we would like to see to be promoted at many different levels!
- The description of this connection between objects living on different groups is best performed by means of families of Banach spaces of functions resp. their

dual spaces, which are usually called *generalized functions*. A minimal setup suitable for a correct treatment of mathematical questions arising in engineering applications resp. in the context of AHA is the so-called *Banach Gelfand triple* (S_0, L^2, S_0'), based on the Segal algebra $\left(S_0(G), \|\cdot\|_{S_0}\right)$ and its dual. The space $S_0(G)$ contains all the "good functions" (for example, all the summability kernels), while the dual space is large enough to contain, e.g., the Dirac comb.

- While the abstract viewpoint may help us to understand connections and ideally enables us to develop efficient algorithms the opposite flow of information, i.e. from numerical experiments to theoretical insight is much less established as a research principle. According to our experience numerical simulations may provide a lot of evidence, but they also raise the question, whether what we observe is related to a corresponding "truth" in the continuous setting, or just an artifact of the involved discretization.
- In the future we should see how estimates proposed in the theoretical literature can be computationally verified, and also how one can quantify the errors made in such computations, based on smart a priori estimates.

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