Wavelets, Tiling and Spectral Sets

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Abstract

We consider functions $\varphi \in L^2(\mathbb{R}^d)$ such that $\{|\det(D)|^{\frac{1}{2}}\varphi(Dx-\lambda): \; D \in \mathcal{D}, \lambda \in \mathcal{T}\}$ forms an orthogonal basis for $L^2(\mathbb{R}^d)$, where $\mathcal{D} \subset M_d(\mathbb{R})$ and $\mathcal{T} \subset \mathbb{R}^d$. Such a function φ is called a wavelet with respect to the dilation set $\mathcal{D}$ and translation set $\mathcal{T}$. We study the following question: Under what conditions can a $\mathcal{D} \subset M_d(\mathbb{R})$ and a set $\mathcal{T} \subset \mathbb{R}^d$ can be used as respectively the dilation set and the translation set of a wavelet? When restricted to wavelets of the form $\varphi = \check{\chi}_\Omega$, this question has a surprising tie to spectral sets and their spectra.

Key words and phrases. Wavelet, wavelet set, dilation, translation, spectral set, spectrum, spectral pair, multiplicative tiling, complementing set.

1 Introduction

Let $\varphi(x) \in L^2(\mathbb{R}^d)$. We call $\varphi(x)$ a *wavelet* if there exist a set of $d \times d$ real matrices $\mathcal{D}$ and a subset $\mathcal{T}$ of $\mathbb{R}^d$ such that

$$\left\{|\det(D)|^{\frac{1}{2}}\varphi(Dx-\lambda): \; D \in \mathcal{D}, \lambda \in \mathcal{T}\right\} \tag{1.1}$$

forms an orthogonal basis for $L^2(\mathbb{R}^d)$. The sets $\mathcal{D}$ and $\mathcal{T}$ are called the *dilation set* and the *translation set* for the wavelet $\varphi(x)$, respectively.

Problem. Characterize the pairs $(\mathcal{D}, \mathcal{T})$ that are dilation sets and translation sets, respectively, of some wavelets.

Wavelets arise in many applications in both pure and applied mathematics. They play a key role in digital signal processing and scientific computations. The simplest wavelet is the Haar wavelet $\varphi(x) = \chi_{[0,1/2]}(x) - \chi_{[1/2,1)}(x)$ for the dilation set $\mathcal{D} = \{2^n : \; n \in \mathbb{Z}\}$ and the

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translation set $\mathcal{T} = \mathbb{Z}$, constructed by A. Haar in 1910. Later Daubechies [Dau] constructs a family of compactly supported wavelets for the same dilation and translation sets, which can be made arbitrarily smooth. The methods in [Dau] have been used to construct a wide variety of wavelets in $\mathbb{R}^d$. However, all such wavelets have lattice translation sets and dilation sets $\{A^n : n \in \mathbb{Z}\}$ for some expanding integer matrix A with $|\det(A)| = 2$. (For dilation matrix with $|\det A| > 2$ more than one wavelet function are needed.)

In a different direction, several authors study wavelets from their Fourier transforms, see e.g. [HW] and the references therein. Fang and Wang [FW] introduce the *minimally supported frequency wavelets* (MSF wavelet), which are studied also in Hernández, Wang and Weiss [HWW1], [HWW2] and by other authors. In particular Dai and Larson [DL] consider a special kind of MSF wavelets φ, which satisfy $\widehat{\varphi} = \chi_\Omega$ for some measurable sets Ω in $\mathbb{R}^d$. They prove that such a $\varphi(x)$ is a wavelet with dilation set $\mathcal{D} = \{2^n : n \in \mathbb{Z}\}$ and translation set $\mathcal{T} = \mathbb{Z}$ if and only if

(i) The sets $\{\Omega + \lambda : \lambda \in \mathbb{Z}\}$ is a tiling of $\mathbb{R}$.

(ii) The sets $\{2^n\Omega : n \in \mathbb{Z}\}$ is a tiling of $\mathbb{R}$.

In other words, Ω must tile $\mathbb{R}$ both translationally and multiplicatively. Here we use the term *tiling* loosely. A collection of measurable sets $\{\Omega_j\}$ is a tiling of $\mathbb{R}^d$ if it is a measure-wise disjoint partition of $\mathbb{R}^d$. The result is later extended to higher dimensions in [DLS] for $\mathcal{T} = \mathbb{Z}^d$ and $\mathcal{D} = \{A^n : n \in \mathbb{Z}\}$ where A is any expanding $d \times d$ matrix. Such an Ω is refered to as a *wavelet set* (with respect to $\mathcal{D}$ and $\mathcal{T}$).

All the studies on wavelets so far, whether from multiresolution analyses or from frequency constructions, consider wavelets whose dilation sets consist of all the powers of a single matrix and whose translation sets are lattices. These dilation sets and translation sets are rather "regular." Naturally, we may ask whether there are other dilation and translation sets. In particular, we may ask:

Question. Is it possible for a wavelet to have "irregular" dilation and translation sets $\mathcal{D}$ and $\mathcal{T}$? Can we have an aperiodic $\mathcal{T}$ and a noncommutative $\mathcal{D}$?

We will answer the above question in affirmative in this paper. To do so we consider wavelet sets in the most general setting. Let $\mathcal{D} \subseteq \mathrm{GL}(d, \mathbb{R})$, the set of all nonsingular $d \times d$ matrices and $\mathcal{T} \subseteq \mathbb{R}^d$. A measurable set $\Omega \subset \mathbb{R}^d$ with Lebesgue measure $0 < \mu(\Omega) < \infty$ is called a *wavelet set* with respect to the dilation set $\mathcal{D}$ and the translation set $\mathcal{T}$ if $\varphi(x) = \check{\chi}_\Omega(x)$ is a wavelet with respect to $\mathcal{D}$ and T. We study the following question: For which pairs of dilation sets $\mathcal{D}$ and translation sets $\mathcal{T}$ do there exist a wavelet set Ω?

This question is closely related to spectral sets and tiling. A set $\Omega \subset \mathbb{R}^d$ with $0 < \mu(\Omega) < \infty$ is called a *spectral set* if there exists a $\mathcal{T} \subseteq \mathbb{R}^d$ such that $\{e^{2\pi i\langle\lambda,\xi\rangle} : \lambda \in \mathcal{T}\}$ is an orthogonal basis for $L^2(\Omega)$. In this case we call $\mathcal{T}$ a *spectrum* of Ω and $(\Omega, \mathcal{T})$ a *spectral pair*. Spectral sets relate to tiling by the following conjecture, due to Fuglede [Fug]:

The Spectral Set Conjecture. *A set $\Omega \subset \mathbb{R}^d$ with $0 < \mu(\Omega) < \infty$ is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.*

The Spectral Set Conjecture is not resolved in either direction, even in dimension one. There are many other questions concerning spectral sets, particularly related to tiling. Some of these questions are very much related to the theme of this paper. We refer the readers to [JP99], [LW97], [LRW], [Lab2] and the references therein for results on spectral sets.

We establish the following tie between wavelet sets and spectral sets:

Theorem 1.1 *Let $\mathcal{D} \subset \mathrm{GL}(d, \mathbb{R})$ and $\mathcal{T} \subset \mathbb{R}^d$. Let Ω be a subset of $\mathbb{R}^d$ with positive and finite Lebesgue measure. If $\{D^T(\Omega) : D \in \mathcal{D}\}$ is a tiling of $\mathbb{R}^d$ and $(\Omega, \mathcal{T})$ is a spectral pair, then $\varphi = \check{\chi}_\Omega$ is a wavelet with respect to $\mathcal{D}$ and $\mathcal{T}$. Conversely, if $\varphi = \check{\chi}_\Omega$ is a wavelet with respect to $\mathcal{D}$ and $\mathcal{T}$ and $0 \in \mathcal{T}$, then $\{D^T(\Omega) : D \in \mathcal{D}\}$ is a tiling of $\mathbb{R}^d$ and $(\Omega, \mathcal{T})$ is a spectral pair.*

It was shown by Fuglede [Fug] that Ω tiles by a lattice $\mathcal{L}$ if and only if $(\Omega, \mathcal{L}^*)$ is a spectral pair, where $\mathcal{L}^*$ is the dual lattice of $\mathcal{L}$. Therefore condition (i) in the theorem of Dai and Larson [DL] stated earlier can be more appropriately stated as: (i') $(\Omega, \mathbb{Z})$ is a spectral pair. In fact, this is what the authors have shown.

The condition $0 \in \mathcal{T}$ in Theorem 1.1 cannot be dropped, as shown by Example 1 in §4.

There are two main objectives in this paper: The study of the structure of the dilation set $\mathcal{D}$ for a wavelet set, and the existence of wavelet sets for a given pair $\mathcal{D}$ and $\mathcal{T}$. We show that by allowing more general dilation sets $\mathcal{D}$ we can obtain some very elegant wavelets through simple wavelet sets. This contrasts sharply with the more restricted notion of wavelet sets where the dilation set $\mathcal{D}$ must have the form $\{A^n : n \in \mathbb{Z}\}$ for some expanding matrix A, for which non-fractal like wavelet sets are difficult to construct.

For the rest of the paper, we state our other main theorems in §2. We then prove the theorems in §3.

Throughout this paper, the Fourier transform is defined as $\widehat{\varphi}(\xi) = \int_{\mathbb{R}} e^{2\pi i\langle \xi, x\rangle}\varphi(x)\,dx$ and the inverse Fourier transform is defined as $\check{\psi}(x) = \int_{\mathbb{R}} e^{-2\pi i\langle x, \xi\rangle}\psi(\xi)\,d\xi$.

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2 Statement of Main Theorems

An important notion in the study of wavelet sets is multiplicative tiling.

Definition 2.1 Let $\mathcal{D} \subseteq \mathrm{GL}(d, \mathbb{R})$. $\mathcal{D}$ is a *multiplicative tiling set* of $\mathbb{R}^d$ if there exists a bounded $\Omega \subset \mathbb{R}^d$ of positive Lebesgue measure, with $\mathrm{dist}\,(\Omega, 0) > 0$, such that $\{D(\Omega) : D \in \mathcal{D}\}$ is a tiling of $\mathbb{R}^d$. The set Ω is called a *multiplicative $\mathcal{D}$-tile*. $\mathcal{D}$ is said to be A-invariant for some $A \in \mathrm{GL}(d, \mathbb{R})$ if $\mathcal{D}A = \mathcal{D}$.

Note that in the case of a translational tile we often require the tile be bounded. Since in some sense multiplicative tilings can be viewed as translational tiling after taking logarithm, the requirement $\operatorname{dist}(\Omega, 0) > 0$ is in fact natural and necessary for $\{\log|\xi| : \xi \in \Omega\}$ to be bounded.

We say that a multiplicative tilings set $\mathcal{D}$ satisfies the *interior condition* if there exists a multiplicative tile Ω for $\mathcal{D}$ such that Ω has nonempty interior. Similar to the interior condition for multiplicative tiling sets, we say that a spectrum $\mathcal{T}$ has *interior condition* if there exists a spectral set Ω with spectrum $\mathcal{T}$ whose interior is nonempty.

Theorem 2.1 *Let $\mathcal{D} \subset \mathrm{GL}(d, \mathbb{R})$ such that $\mathcal{D}^T := \{D^T : D \in \mathcal{D}\}$ is a multiplicative tiling set, and let $\mathcal{T} \subset \mathbb{R}^d$ be a spectrum, with both $\mathcal{D}^T$ and $\mathcal{T}$ satisfying the interior condition. Suppose that $\mathcal{D}^T$ is A-invariant for some expanding matrix A and $\mathcal{T} - \mathcal{T} \subseteq \mathcal{L}$ for some lattice $\mathcal{L}$ of $\mathbb{R}^d$. Then there exists a wavelet set Ω with respect to $\mathcal{D}$ and $\mathcal{T}$.*

The assumption $\mathcal{T} - \mathcal{T} \subseteq \mathcal{L}$ is equivalent to $\mathcal{T} \subseteq \mathcal{L} + \lambda_0$ for some $\lambda_0 \in \mathbb{R}^d$. In dimension one all known spectra have this property. Counterexamples exist in higher dimensions, the simplest of which being the spectra for the unit cube. They can be rather aperiodic, see [LRW] and [IP]. The assumption that $\mathcal{D}^T$ and $\mathcal{T}$ have the interior condition is most likely unnecessary. All known examples of multiplicative tiling sets admit a tile having nonempty interior, and likewise for spectra.

Corollary 2.2 *Let $\mathcal{D} \subset \mathrm{GL}(d, \mathbb{R})$ such that $\mathcal{D}^T$ is an A-invariant multiplicative tiling set with interior condition, where $A \in M_d(\mathbb{R})$ is expanding. Let $\mathcal{T}$ be a lattice in $\mathbb{R}^d$. Then there exists a wavelet set Ω with respect to $\mathcal{D}$ and $\mathcal{T}$.*

Corollary 2.3 ([DLS]) *Let $\mathcal{D} = \{A^j : j \in \mathbb{Z}\}$ for some expanding $A \in M_d(\mathbb{R})$. Let $\mathcal{T}$ be a lattice in $\mathbb{R}^d$. Then there exists a wavelet set Ω with respect to $\mathcal{D}$ and $\mathcal{T}$.*

In the one dimension, the structure of positive multiplicative tiling sets follows from the study by Lagarias and Wang [LW96] on translational tiles in $\mathbb{R}$. A key notion here is a special type of subsets of $\mathbb{Z}$ called complementing sets. Suppose that $\mathcal{A} \subset \mathbb{Z}$ and let $N > 1$ be an integer. The set $\mathcal{A}$ is called a *complementing set* (mod N) if there exists a set $\mathcal{B} \subset \mathbb{Z}$ such that $\mathcal{A} + \mathcal{B}$ is a direct sum and is a complete set of residues (mod N). The set $\mathcal{B}$ is called a *complement* of $\mathcal{A}$ (mod N).

Theorem 2.4 *Let $\mathcal{D} \subset \mathbb{R}$. Then*

(i) *Suppose that $\mathcal{D} \subset \mathbb{R}^+$. Then $\mathcal{D}$ is a multiplicative tiling set if and only if $\mathcal{D} = \{as^\beta : \beta \in \mathcal{E}\}$, where $a, s > 0$, $s \neq 1$ and $\mathcal{E} = \mathcal{A} + N\mathbb{Z}$ for some complementing set $\mathcal{A}$* (mod N).

(ii) *Denote $|\mathcal{D}| := \{|t| : t \in \mathcal{D}\}$ (counting multiplicity). Then $\mathcal{D}$ is a multiplicative tiling set for a centrally symmetric multiplicative tile if and only if $|\mathcal{D}|$ is a positive multiplicative tiling set, i.e. $|\mathcal{D}| = \{as^\beta : \beta \in \mathcal{E}\}$, where $a, s > 0$, $s \neq 1$ and $\mathcal{E} = \mathcal{A} + N\mathbb{Z}$ for some complementing set $\mathcal{A}$* (mod N).

Centrally symmetric wavelet sets are important, as the resulting wavelets are real. Theorem 2.4 states that if we take a positive multiplicative tiling set $\mathcal{D}$ then we may change the sign of any subset of $\mathcal{D}$, and the resulting set is still a multiplicative tiling set for a symmetric tile.

Theorem 2.5 *Let $\mathcal{D} \subset \mathbb{R}$ such that $|\mathcal{D}|$ is a multiplicative tiling set. Then*

(i) *For any spectrum $\mathcal{T}$ with the interior condition such that $\mathcal{T} - \mathcal{T} \subseteq c\mathbb{Z}$ for some $c \neq 0$ there exists a wavelet set Ω with respect to $\mathcal{D}$ and $\mathcal{T}$.*

(ii) *For any lattice $\mathcal{T}$ there exists a centrally symmetric wavelet set Ω with respect to $\mathcal{D}$ and $\mathcal{T}$.*

3 Proof of Theorems

Proof of Theorem 1.1. Let $\varphi(x) = \check{\chi}_\Omega(x)$. Then $\{\varphi_{D,\lambda} : D \in \mathcal{D}, \lambda \in \mathcal{T}\}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$ if and only if $\{\widehat{\varphi}_{D,\lambda} : D \in \mathcal{D}, \lambda \in \mathcal{T}\}$ is. Note that

$$\widehat{\varphi}_{D,\lambda}(\xi) = |\det D|^{-\frac{1}{2}} e_{D^{-1}\lambda}(\xi) \chi_{D^T(\Omega)}(\xi), \tag{3.1}$$

where $e_\omega(\xi) := e^{2\pi i \langle \omega, \xi \rangle}$.

$\Leftarrow$ Since $(\Omega, \mathcal{T})$ is a spectral pair, the set of exponentials $\{e_\lambda(\xi) : \lambda \in \mathcal{T}\}$ is an orthogonal basis for $L^2(\Omega)$. It follows that $\{e_{D^{-1}\lambda}(\xi) : \lambda \in \mathcal{T}\}$ is an orthogonal basis for $L^2(D^T(\Omega))$. Now, Ω tiles $\mathbb{R}^d$ multiplicatively by $\{D^T : D \in \mathcal{D}\}$. Hence $\{e_{D^{-1}\lambda}(\xi) : D \in \mathcal{D}, \lambda \in \mathcal{T}\}$ is an orthogonal set of functions in $L^2(\mathbb{R}^d)$. It is in fact a basis because $\{D^T(\Omega) : D \in \mathcal{D}\}$ is a partition of $\mathbb{R}^d$.

$\Rightarrow$ Let Ω be a wavelest set with respect to $\mathcal{T}$ and $\mathcal{D}$ with $0 \in \mathcal{T}$. So $\{\widehat{\varphi}_{D,\lambda} : D \in \mathcal{D}, \lambda \in \mathcal{T}\}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$. Fix $\lambda = 0 \in \mathcal{T}$. Then $\{\widehat{\varphi}_{D,0} : D \in \mathcal{D}\}$ is orthogonal. By (3.1)

$$\widehat{\varphi}_{D,0}(\xi) = |\det D|^{-\frac{1}{2}} \chi_{D^T(\Omega)}(\xi).$$

Hence $\{D^T(\Omega) : D \in \mathcal{D}\}$ must be disjoint measure-wise. Furthermore, $\operatorname{supp} \widehat{\varphi}_{D,\lambda} \subseteq D^T(\Omega)$. It follows from the fact that $\{\widehat{\varphi}_{D,\lambda} : D \in \mathcal{D}, \lambda \in \mathcal{T}\}$ is a basis that $\{D^T(\Omega) : D \in \mathcal{D}\}$ must be a tiling of $\mathbb{R}^d$.

Now fix a $D \in \mathcal{D}$. Since $\{D^T(\Omega) : D \in \mathcal{D}\}$ is a tiling, the set of exponentials $\{e_{D^{-1}\lambda}(\xi) : \lambda \in \mathcal{T}\}$ must be an orthogonal basis for $L^2(D^T(\Omega))$ by (3.1). This means that $\{e_\lambda(\xi) : \lambda \in \mathcal{T}\}$ must be an orthogonal basis for $L^2(\Omega)$. Hence $(\Omega, \mathcal{T})$ is a spectral pair. ∎

We give an example in §4 showing that the assumption $0 \in \mathcal{T}$ cannot be removed.

To prove the existence of wavelet sets we first establish several lemmas concerning spectral sets. Let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^d$. The *dual lattice* $\mathcal{L}^*$ of $\mathcal{L}$ is defined as

$$\mathcal{L}^* := \{\alpha \in \mathbb{R}^d : \langle \alpha, \beta \rangle \in \mathbb{Z} \text{ for all } \beta \in \mathcal{L}\}.$$

The next two lemmas are have been proved in [Jor] and [JP92] respectively. We include the proofs here for completeness.

Lemma 3.1 *Let $(\Omega, \mathcal{T})$ be a spectral pair such that $\mathcal{T} - \mathcal{T} \subseteq \mathcal{L}$ for some full rank lattice $\mathcal{L}$ in $\mathbb{R}^d$. Let $\alpha \in \mathcal{L}^*$. Then Ω and $\Omega + \alpha$ are measure-wise disjoint.*

Proof. Without loss of generality we may assume that $0 \in \mathcal{T}$. Therefore $\mathcal{T} \subseteq \mathcal{L}$. In particular $e_\lambda(-\alpha) = 1$ for all $\lambda \in \mathcal{T}$. Define

$$\Omega_1 = \big\{\xi \in \Omega : \ \xi + \alpha \in \Omega\big\}$$

and assume that $\mu(\Omega_1) > 0$. We derive a contradiction. Partition Ω_1 into subsets whose diameters are all less than $|\alpha|$. Let Ω' be one of the subsets that has $\mu(\Omega') > 0$. Clearly we have $\Omega' \cap (\Omega' + \alpha) = \emptyset$. Let $f = \chi_{\Omega'}$ and $g = \chi_{\Omega'+\alpha}$. Now

$$f = \sum_{\lambda\in\mathcal{T}} \langle f, e_\lambda\rangle_{L^2(\Omega)} e_\lambda, \quad g = \sum_{\lambda\in\mathcal{T}} \langle g, e_\lambda\rangle_{L^2(\Omega)} e_\lambda.$$

Observe that $e_\lambda(\xi) = e_\lambda(\xi)e_\lambda(-\alpha) = e_\lambda(\xi - \alpha)$ for all $\lambda \in \mathcal{T}$. Hence

$$\langle f, e_\lambda\rangle_{L^2(\Omega)} = \int_{\Omega'} e_\lambda(\xi)\, d\xi = \int_{\Omega'} e_\lambda(\xi - \alpha)\, d\xi = \int_{\Omega'+\alpha} e_\lambda(\xi)\, d\xi = \langle g, e_\lambda\rangle_{L^2(\Omega)}.$$

So f and g have the same Fourier series. But f and g are orthogonal. This is a contradiction. ■

We say that two measurable sets Ω and Ω^* in $\mathbb{R}^d$ are *$\mathcal{L}$-congruent* for some lattice $\mathcal{L}$ if Ω has a partition $\Omega = \bigcup_{\alpha\in\mathcal{L}} \Omega_\alpha$ such that $\Omega^* = \bigcup_{\alpha\in\mathcal{L}} (\Omega_\alpha + \alpha)$ with the union being measure-wise disjoint.

Lemma 3.2 *Let $(\Omega, \mathcal{T})$ be a spectral pair such that $\mathcal{T} - \mathcal{T} \subseteq \mathcal{L}$ for some full rank lattice $\mathcal{L}$ in $\mathbb{R}^d$. Let Ω^* be $\mathcal{L}^*$-congruent to Ω. Then $(\Omega^*, \mathcal{T})$ is also a spectral pair.*

Proof. Let $\Omega^* = \bigcup_{\alpha\in\mathcal{L}} (\Omega_\alpha + \alpha)$ where $\Omega = \bigcup_{\alpha\in\mathcal{L}} \Omega_\alpha$ is a partition. We first prove that $\{e_\lambda : \ \lambda \in \mathcal{T}\}$ is orthogonal in $L^2(\Omega^*)$. For any $\lambda_1,\, \lambda_2 \in \mathcal{T}$ we have $e_{\lambda_1-\lambda_2}(\xi - \alpha) = e_{\lambda_1-\lambda_2}(\xi)$ for all $\alpha \in \mathcal{L}^*$. Hence

$$\begin{aligned}
\int_{\Omega^*} e_{\lambda_1-\lambda_2}(\xi)d\xi &= \sum_{\alpha\in\mathcal{L}^*} \int_{\Omega_\alpha+\alpha} e_{\lambda_1-\lambda_2}(\xi)d\xi \\
&= \sum_{\alpha\in\mathcal{L}^*} \int_{\Omega_\alpha} e_{\lambda_1-\lambda_2}(\xi - \alpha)d\xi \\
&= \sum_{\alpha\in\mathcal{L}^*} \int_{\Omega_\alpha} e_{\lambda_1-\lambda_2}(\xi)d\xi \\
&= \int_{\Omega} e_{\lambda_1-\lambda_2}(\xi)d\xi \\
&= 0.
\end{aligned}$$

It remains to prove that $\{e_\lambda : \lambda \in \mathcal{T}\}$ is a basis for $L^2(\Omega^*)$. Without loss of generality we assume that $0 \in \mathcal{T}$. Then $\mathcal{T} \subseteq \mathcal{L}^*$. If not there exists a nonzero $f \in L^2(\Omega^*)$ such that $\langle f, e_\lambda \rangle_{L^2(\Omega^*)} = 0$ for all $\lambda \in \mathcal{T}$. Define $g \in L^2(\Omega^*)$ by $g(\xi) = f(\xi + \alpha)$ for $\xi \in \Omega_\alpha$. Clearly $g \neq 0$. We have for each $\lambda \in \mathcal{T}$

$$\begin{aligned}
\langle g, e_\lambda \rangle_{L^2(\Omega)} &= \sum_{\alpha \in \mathcal{L}^*} \int_{\Omega_\alpha} f(\xi + \alpha)\overline{e_\lambda(\xi)} d\xi \\
&= \sum_{\alpha \in \mathcal{L}^*} \int_{\Omega_\alpha} f(\xi + \alpha)\overline{e_\lambda(\xi + \alpha)} d\xi \\
&= \sum_{\alpha \in \mathcal{L}^*} \int_{\Omega_\alpha + \alpha} f(\xi)\overline{e_\lambda(\xi)} d\xi \\
&= \int_{\Omega^*} f(\xi)\overline{e_\lambda(\xi)}\, d\xi \\
&= 0.
\end{aligned}$$

This is a contradiction. ∎

Let Ω and $\tilde{\Omega}$ be measurable sets in $\mathbb{R}^d$, and $A \in M_d(\mathbb{R})$. We say Ω is *A-congruent* to $\tilde{\Omega}$ if there exists a partition $\Omega = \bigcup_{j\in\mathbb{Z}} \Omega_j$ such that $\bigcup_{j\in\mathbb{Z}} A^j(\Omega_j) = \tilde{\Omega}$ is a partition of $\tilde{\Omega}$. Observe that A-congruence is an equivalent relation. Furthermore if Ω tiles $\mathbb{R}^d$ multiplicatively by some A-invariant $\mathcal{D}^T$ then so does $\tilde{\Omega}$. This fact together with Lemma 3.2 allows us to prove Theorem 2.1.

Proof of Theorem 2.1. Let Ω_0^t be a multiplicative tile by $\mathcal{D}^T$ with nonempty interior and $\operatorname{dist}(\Omega_0^t, 0) > 0$. Let Ω_0^s be a spectral set with spectrum $\mathcal{T}$ and nonempty interior. Since $A^k(\Omega_0^t)$ is also a multiplicative $\mathcal{D}^T$-tile and A is expanding, we may without loss of generality assume that Ω_0^t contains a sufficiently large ball. Furthermore, since any translate of Ω_0^s is again a spectral set with spectrum $\mathcal{T}$, we may without loss of generality further assume that there exists an $\alpha^* \in \mathcal{L}^*$ such that

$$\alpha^* \in (\Omega_0^s)^o, \quad \Omega_0^s \subseteq \Omega_0^t.$$

We construct a wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$ by constructing multiplicative tiles $\{\Omega_n^t\}$ and spectral sets $\{\Omega_n^s\}$ with the property that $\Omega_n^s \subseteq \Omega_n^t$ and $\lim_{n\to\infty} \Omega_n^s = \lim_{n\to\infty} \Omega_n^t = \Omega$. The sequence of sets are constructed iteratively using congruences, a technique similar to the one used in Benedetto and Leon [BL99].

First we fix a sufficiently large $K > 0$ such that $A^{-K}(\Omega_0^t) + \alpha^* \subseteq \Omega_0^s$. Since $\alpha^* \in (\Omega_0^s)^o$ this is always possible. Let

$$\Omega_1^s = \Omega_0^s, \quad \Omega_1^t = \Omega_1^s \cup \Omega_1^e, \quad \Omega_1^e = A^{-K}(\Omega_0^t \setminus \Omega_0^s).$$

Note that $\Omega_1^s \cup (\Omega_0^t \setminus \Omega_0^s)$ is a partition of Ω_0^t, so Ω_1^t is A-congruent to Ω_0^t and hence is a $\mathcal{D}^T$-tile. (Here the letter 'e' in Ω_1^e stands for 'extra'. This is the extra piece from which Ω_1^t differs Ω_1^s.)

Observe that $\Omega_1^e + \alpha^* \subseteq \Omega_1^s$, since $\Omega_1^e \subseteq A^{-K}(\Omega_0^t)$. Define

$$\Omega_2^s = \Omega_1^e \cup \Big(\Omega_1^s \setminus (\Omega_1^e + \alpha^*)\Big), \quad \Omega_2^t = \Omega_2^s \cup \Omega_2^e, \quad \Omega_2^e = A^{-K}(\Omega_1^e + \alpha^*). \tag{3.2}$$

Note that Ω_2^s is $\mathcal{L}^*$-congruent to Ω_1^s, so by Lemma 3.2 it is a spectral set with spectrum $\mathcal{T}$. Note also that Ω_2^t is a $\mathcal{D}^T$-tile because it is A-congruent to Ω_1^t, since $\Omega_1^t = \Omega_2^s \cup (\Omega_1^e + \alpha^*)$.

It is now easy to see how we construct Ω_n^s and Ω_n^t iteratively. We let

$$\Omega_{n+1}^s = \Omega_n^e \cup \Big(\Omega_n^s \setminus (\Omega_n^e + \alpha^*)\Big), \quad \Omega_{n+1}^t = \Omega_{n+1}^s \cup \Omega_{n+1}^e, \quad \Omega_{n+1}^e = A^{-K}(\Omega_n^e + \alpha^*). \tag{3.3}$$

The set Ω_{n+1}^s is $\mathcal{L}^*$-congruent to Ω_n^s, and Ω_{n+1}^t is A-congruent to Ω_n^t. So all Ω_n^s are spectral sets with spectrum $\mathcal{T}$ and all Ω_n^t are multiplicative $\mathcal{D}^T$-tiles. Furthermore by induction the Lebesgue measure of the extra set Ω_n^e satisfies

$$\mu(\Omega_n^e) = \mu(\Omega_n^t \setminus \Omega_n^s) = |\det A|^{-K}\mu(\Omega_{n-1}^e) \le |\det A|^{-nK}\mu(\Omega_0^t). \tag{3.4}$$

Finally we show that $\lim_{n\to\infty}\Omega_n^s = \lim_{n\to\infty}\Omega_n^t = \Omega$ up to a measure zero set. This can be seen from the fact that

$$\Omega_{n+1}^s \triangle \Omega_n^s = \Omega_n^e \cup (\Omega_n^e + \alpha^*), \quad \text{and so} \;\; \mu(\Omega_{n+1}^s \triangle \Omega_n^s) = c\, r^n, \tag{3.5}$$

where $c = 2\mu(\Omega_0^t)$ and $r = |\det A|^{-K} < 1$. In other words the difference betwee successive terms decays exponentially measure-wise. Now we have

$$\Omega_n^s \triangle \textstyle\bigcap_{k\ge n}\Omega_k^s = \Omega_n^s \setminus \bigcap_{k\ge n}\Omega_k^s \subseteq \bigcup_{k\ge n}(\Omega_k^s \setminus \Omega_{k+1}^s).$$

Hence

$$\mu\big(\Omega_n^s \triangle \textstyle\bigcap_{k\ge n}\Omega_k^s\big) \le cr^n + cr^{n+1} + \cdots = c_1 r^n.$$

Similarly

$$\mu\big(\Omega_n^s \triangle \textstyle\bigcup_{k\ge n}\Omega_k^s\big) \le c_2 r^n.$$

It follows that $\limsup_n \Omega_n^s = \liminf_n \Omega_n^s$ up to a measure zero set. Let Ω be the limit. By (3.5) $\Omega = \lim_n \Omega_n^t$ up to a mesurable set. Thus Ω is a wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$. ∎

Proof of Corollaries 2.2 and 2.3. For Corollary 2.2 we only need to note that by [Fug] any lattice $\mathcal{T}$ is a spectrum of any fundamental domain of the dual lattice $\mathcal{T}^*$, which can obviously be chosen to have nonempty interior. For Corollary 2.3 we note that $\mathcal{D}^T$ is A^T-invariant, and need to show that it is a multiplicative tiling set with interior condition. Let $B_1(0)$ be the unit ball centered at 0 and let $\Omega_0 = \bigcap_{n\ge 0}(A^T)^n(B_1(0))$. The intersection is finite since $(A^T)^n(B_1(0)) \supseteq B_1(0)$ for sufficiently large n. So Ω_0 is open. Set $\Omega = A^T(\Omega_0)\setminus\Omega_0$. Then Ω is a multiplicative tile with interior, and $\operatorname{dist}(\Omega, 0) > 0$. ∎

Proof of Theorem 2.4. (i) Write $\Omega = \Omega^+ \cup \Omega^-$ where $\Omega^+ = \Omega\cap\mathbb{R}^+$ and $\Omega^- = \Omega\cap\mathbb{R}^-$. Since $\mathcal{D}\subset\mathbb{R}^+$, $\mathcal{D}(\Omega^+) := \{d\,\Omega^+ :\ d\in\mathcal{D}\}$ is a tiling og $\mathbb{R}^+$. Taking logarithm we see that $\log\mathcal{D} := \{\log d :\ d\in\mathcal{D}\}$ is a tiling set of $\mathbb{R}$, with tile $\log\Omega^+ := \{\log x :\ x\in\Omega^+\}$. Observe that $\log\Omega^+$ is bounded because $\operatorname{dist}(\Omega,0)>0$. The structure of $\log\mathcal{D}$ is classified in Lagarias and Wang [LW96]. Part (i) of the theorem follows directly from Theorem 3 of [LW96].

(ii) By central symmetry of Ω we can write Ω as $\Omega = \Omega^+ \cup (-\Omega^+)$ where $\Omega^+ = \Omega \cap \mathbb{R}^+$. Observe that $d\,\Omega = |d|\,\Omega$ for any $d \in \mathcal{D}$. So $\mathcal{D}$ is a multiplicative tiling set for Ω if and only if $|\mathcal{D}|$ is. ∎

Proof of Theorem 2.5. First we note that $\mathcal{D} = \mathcal{D}^T$. By the structure result in Theorem 2.4 $\mathcal{D}$ is s^N-invariant, where s, N are as in Theorem 2.4. It is also s^{-N}-invariant. One of s^N and s^{-N} is greater than 1. Furthermore $\mathcal{D}$ must satisfy the interior condition, see [LW96]. Theorem 2.1 now immediately implies (i).

The proof of (ii) is essentially identical to the proof of Theorem 2.1. Only minor modifications are needed.

Without loss of generality we assume that $\mathcal{T} = \mathbb{Z}$, and $|\mathcal{D}|$ is a-invariant where $a > 1$. Let Ω_0^t be a centrally symmetric multiplicative tile by $\mathcal{D}$ with nonempty interior and $\mathrm{dist}\,(\Omega_0^{\mathrm{t}}, 0) > 0$. Since $a^k(\Omega_0^t)$ is also a centrally symmetric multiplicative $\mathcal{D}$-tile we may without loss of generality assume that $[-m_0 - \frac{1}{2}, -m_0] \cup [m_0, m_0 + \frac{1}{2}] \subseteq \Omega_0^s$ for some positive integer m_0. Set $\Omega_0^s = [-m_0 - \frac{1}{2}, -m_0] \cup [m_0, m_0 + \frac{1}{2}]$. Then Ω_0^s is a spectral set with spectrum $\mathcal{T} = \mathbb{Z}$.

As in the proof of Theorem 2.1 we construct a centrally symmetric wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$ by constructing centrally symmetric multiplicative tiles $\{\Omega_n^t\}$ and centrally symmetric spectral sets $\{\Omega_n^s\}$ with the property that $\Omega_n^s \subseteq \Omega_n^t$ and $\lim_{n\to\infty} \Omega_n^s = \lim_{n\to\infty} \Omega_n^t = \Omega$.

First we fix a sufficiently large $K > 0$ such that $a^{-K}(\Omega_0^t) \subseteq (-\frac{1}{2}, \frac{1}{2})$. This is always possible because $a > 1$. Let

$$\Omega_1^s = \Omega_0^s, \quad \Omega_1^t = \Omega_1^s \cup \Omega_1^e, \quad \Omega_1^e = a^{-K}(\Omega_0^t \setminus \Omega_0^s).$$

As in the proof of Theorem 2.1, Ω_1^t is a-congruent to Ω_0^t and hence is a multiplicative $|\mathcal{D}|$-tile. Furthermore, it is centrally symmetric so it is also a multiplicative $\mathcal{D}$-tile. Note that Ω_1^s, Ω_1^t and Ω_1^e are all centrally symmetric.

For any set S in $\mathbb{R}$ we define $p_+(S) := S \cap \mathbb{R}^+$ and $p_-(S) := S \cap \mathbb{R}^-$. Now $a^{-K}(\Omega_0^t) \subseteq (-\frac{1}{2}, \frac{1}{2})$ implies that $p_+(\Omega_1^e) + m_0 \subset p_+(\Omega_1^s) = [m_0, m_0 + \frac{1}{2}]$ and $p_-(\Omega_1^e) - m_0 \subset p_-(\Omega_1^s) = [-m_0 - \frac{1}{2}, -m_0]$. Let $\tau_{m_0}(S) := (p_+(S) + m_0) \cup (p_-(S) - m_0)$.

$$\Omega_2^s = \Omega_1^e \cup \Big(\Omega_1^s \setminus \tau_{m_0}(\Omega_1^e)\Big), \quad \Omega_2^t = \Omega_1^s \cup \Omega_2^e, \quad \Omega_2^e = a^{-K}\tau_{m_0}(\Omega_1^e). \tag{3.6}$$

Note that Ω_2^s is $\mathcal{T}^*$-congruent to Ω_1^s ($\mathcal{T}^* = \mathbb{Z}$), so by Lemma 3.2 it is a spectral set with spectrum $\mathcal{T}$. Note also that Ω_2^t is a $|\mathcal{D}|$-tile because it is a-congruent to Ω_1^t. Furthermore Ω_2^s, Ω_2^t and Ω_2^e are all centrally symmetric. Thus Ω_2^t is also a multiplicative $\mathcal{D}$-tile.

It is now easy to see how we construct Ω_n^s and Ω_n^t iteratively. We let

$$\Omega_{n+1}^s = \Omega_n^e \cup \Big(\Omega_n^s \setminus \tau_{m_0}(\Omega_n^e)\Big), \quad \Omega_{n+1}^t = \Omega_{n+1}^s \cup \Omega_{n+1}^e, \quad \Omega_{n+1}^e = a^{-K}\tau_{m_0}(\Omega_n^e). \tag{3.7}$$

All these sets are centrally symmetric. The set Ω_{n+1}^s is $\mathcal{T}^*$-congruent to Ω_n^s, and Ω_{n+1}^t is a-congruent to Ω_n^t. So all Ω_n^s are spectral sets with spectrum $\mathcal{T}$ and all Ω_n^t are multiplicative $|\mathcal{D}|$-tiles. Furthermore by induction the Lebesgue measure of the extra set Ω_n^e satisfies

$$\mu(\Omega_n^e) = \mu(\Omega_n^t \setminus \Omega_n^s) = a^{-K}\mu(\Omega_{n-1}^e) \le a^{-Kn}\mu(\Omega_0^t). \tag{3.8}$$

Finally we show that $\lim_{n\to\infty}\Omega_n^s = \lim_{n\to\infty}\Omega_n^t = \Omega$ up to a measure zero set. This can be seen from the fact that

$$\Omega_{n+1}^s \triangle \Omega_n^s = \Omega_n^e \cup \tau_{m_0}(\Omega_n^e), \quad \text{and so} \;\; \mu(\Omega_{n+1}^s \triangle \Omega_n^s) = c\, a^{-Kn}, \tag{3.9}$$

where $c = 2\mu(\Omega_0^t)$. In other words the difference betwee successive terms decays exponentially measure-wise. Now we have

$$\Omega_n^s \triangle \bigcap_{k\geq n}\Omega_k^s = \Omega_n^s \setminus \bigcap_{k\geq n}\Omega_k^s \subseteq \bigcup_{k\geq n}(\Omega_k^s \setminus \Omega_{k+1}^s).$$

Hence

$$\mu\big(\Omega_n^s \triangle \bigcap_{k\geq n}\Omega_k^s\big) \leq ca^{-Kn} + ca^{-K(n+1)} + \cdots = c_1 a^{-Kn}.$$

Similarly

$$\mu\big(\Omega_n^s \triangle \bigcup_{k\geq n}\Omega_k^s\big) \leq c_2 a^{-Kn}.$$

It follows that $\limsup_n \Omega_n^s = \liminf_n \Omega_n^s$ up to a measure zero set. Let Ω be the limit. By (3.7) $\Omega = \lim_n \Omega_n^t$ up to a mesurable set. Thus Ω is a wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$. ∎

4 Examples

In this section we present several examples of wavelet sets to emphasize various aspects of the complexity of the problem.

Example 1. The condition $0 \in \mathcal{T}$ in Theorem 1.1 cannot be dropped. Let $\Omega = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. Let $\mathcal{D} = \{2^n : \; n \in \mathbb{Z}\} \cup \{-2^n : \; n \in \mathbb{Z}\}$ and $\mathcal{T} = 2\mathbb{Z} + \frac{1}{6}$. Then Ω is a wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$. But $\mathcal{D}$ is not a multiplicative tiling set of Ω, nor is $\mathcal{T}$ a spectrum of Ω.

To prove that Ω is indeed a wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$, we observe that the corresponding wavelet

$$\varphi = \check{\chi}_\Omega = \frac{\sin(\pi x) - \sin(2\pi x)}{\pi x} \tag{4.1}$$

is real and even. So $\varphi(-2^n x - \lambda) = \varphi(2^n x + \lambda)$. Therefore φ is a wavelet with respect to $\mathcal{D}$ and $\mathcal{T}$ if and only if it is a wavelet with respect to $\mathcal{D}' = \{2^n : \; n \in \mathbb{Z}\}$ and $\mathcal{T}' = \mathcal{T} \cup (-\mathcal{T})$. Now, Ω tiles multiplicatively by $\mathcal{D}'$. Furthermore, $\mathcal{T}' = 2\mathbb{Z} + \{\frac{1}{6}, -\frac{1}{6}\}$ satisfies $\mathcal{T}' - \mathcal{T}' \subset Z_\Omega \cup \{0\}$ where Z_φ denotes the set of zeros of $\widehat{\chi}_\Omega = \varphi(-x)$. Hence $(\Omega, \mathcal{T}')$ is a spectral pair (c.f. [Ped] or [LW97]). By Theorem 1.1 φ is a wavelet with respect to $\mathcal{D}'$ and $\mathcal{T}'$, so it is a wavelet with respect to $\mathcal{D}$ and $\mathcal{T}$. ∎

Example 2. A multiplicative tiling set $\mathcal{D} \subset \mathbb{R}^d$ does not have to be A-invariant for some matrix $A \neq I$, even in the one dimension. For the Ω in Example 1, the set $\mathcal{D} = \{\epsilon_n 2^n : \; n \in \mathbb{Z}\}$ where $\epsilon_n \in \{1, -1\}$ is always a multiplicative tiling set for Ω. If we choose $(\epsilon_n : \; n \in \mathbb{Z})$ to be aperiodic then $\mathcal{D}$ is not a-invariant for any $a \neq 1$. ∎

Example 3. The translation set $\mathcal{T}$ for a wavelet set Ω needs not be periodic, nor satisfy $\mathcal{T} - \mathcal{T} \subseteq \mathcal{L}$ for some lattice $\mathcal{L}$ in $\mathbb{R}^d$. Furthermore, the dilation matrices in $\mathcal{D}$ need not commute.

Let Ω be the unit square centered at $(0, \frac{3}{2})$, i.e. $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2 + (0, \frac{3}{2})$. Let

$$\mathcal{D} = \{\pm 2^n C_1, ` \pm 2^n C_2 : \ n \in \mathbb{Z}\}$$

where C_1 and C_2 are given by

$$C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}.$$

Let R denote the rectangle $[-1, 1] \times [-2, 2]$. Note that

$$C_1\Omega \cup (-C_1\Omega) \cup C_2\Omega \cup (-C_2\Omega) = R \setminus \frac{1}{2}R.$$

So Ω tiles $\mathbb{R}^2$ multiplicatively by $\mathcal{D}$.

It is well known that all cubes are spectral sets. The spectra for cubes in $\mathbb{R}^d$ have been completely classified by several authors, see [JP99] for $d = 2, 3$, and [IP] or [LRW] for general d. A set $\mathcal{T}$ is a spectrum for the unit cube Ω if and only if $\mathcal{T}$ is a tiling set for the unit cube (this unit cube should be viewed as the dual of Ω). For example, we may take

$$\mathcal{T} = \{(n, m + e^n) : \ m, n \in \mathbb{Z}\}.$$

Then $(\Omega, \mathcal{T})$ is a spectral pair. So Ω is a wavelet set with respect to $\mathcal{D}$ and $\mathcal{T}$. Observe that $\mathcal{T}$ neither is periodic, nor satisfies $\mathcal{T} - \mathcal{T} \subseteq \mathcal{L}$ for any lattice $\mathcal{L}$. Furthermore, not all matrices in $\mathcal{D}$ commute, $C_1C_2 \neq C_2C_1$.

The corresponding wavelet is

$$\varphi(x_1, x_2) = e^{-3\pi i x_2} \cdot \frac{\sin(\pi x_1)\sin(\pi x_2)}{\pi^2 x_1 x_2}.$$

∎

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