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The Spectral Function of Shift-Invariant Spaces

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1. Introduction

The shift-invariant spaces are closed subspaces of $L^2(\mathbb{R}^n)$ that are invariant under all shifts (i.e., integer translations). The theory of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ plays an important role in many areas, most notably in the theory of wavelets, spline systems, Gabor systems, and approximation theory [BMM; BDR1; BDR2; BL; Bo1; HLPS; Ji; RS1; RS2; Rz2]. The study of analogous spaces for $L^2(\mathbb{T}, \mathcal{H})$ with values in a separable Hilbert space $\mathcal{H}$ in terms of range function, often called doubly invariant spaces, is quite classical and goes back to Helson [He1].

The general structure of shift-invariant (SI) spaces was revealed in the work of de Boor, DeVore, and Ron [BDR1] with the use of fiberization techniques based on range function. In particular, conditions under which a finitely generated SI space has a generating set satisfying some desirable properties (e.g., stability, orthogonality or quasi-orthogonality) were given. This has been further developed in the work of Ron and Shen [RS1] with the introduction of the technique of Gramians and dual Gramians. The general properties of SI spaces and shift-preserving operators have also been studied by the first author [Bo1].

The contribution of this paper is a systematic study of yet another tool in SI spaces, apparently overlooked in the previous research, which we call the *spectral function*. This function was introduced by the second author in his Ph.D. thesis. It was motivated by [BDR1] and is similar to the multiplicity function studied by Baggett, Medina, and Merrill [BMM]. More precisely, to every SI subspace of $L^2(\mathbb{R}^n)$ we associate a function on $\mathbb{R}^n$ that contains much useful information about that space.

Although [BDR1] and Helson's range function [He1] is the origin of this approach, it is thanks to Weiss (see [WW]) that the spectral function has a very elementary definition. Namely, for every SI space $V \subset L^2(\mathbb{R}^n)$, there exists a countable family of functions Φ whose integer shifts form a tight frame with constant 1 for the space V, and the spectral function of V is defined as the sum of the squares of the Fourier transforms of the elements of Φ (see Lemma 2.3). It can be shown that such a function is well-defined, additive on orthogonal sums, and bounded by 1. Moreover, it behaves nicely under dilations and modulations, which makes it useful in the study of wavelet and Gabor systems. For example, it

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has already been used to show a new characterization of wavelets conjectured by Weiss (see [Rz1]), a result originally proved in [Bo2] by applying the techniques of [RS1; RS2]. The present paper is organized as follows.

In Section 2 we show several equivalent methods of defining the spectral function, and we study its basic properties such as behavior under dilations and modulations. We also show that the spectral function can be used to characterize the approximation order of SI spaces following [BDR2; BDR3]. In Section 3 we apply the spectral function to give a complete characterization of dimension functions, sometimes called multiplicity functions, associated to generalized multiresolution analyses (GMRA) and refinable spaces. This extends the results of Baggett and Medina [BM], who considered only locally integrable multiplicity functions, as well as the results of Speegle and the authors [BRS] regarding the wavelet dimension function. In Section 4 we show an analogue of the Calderón reproducing formula for GMRAs and give an explicit formula for the wavelet spectral function, whose periodization is a well-studied wavelet dimension function. Finally, in Section 5 we present an elementary proof of Rieffel's incompleteness theorem for Gabor systems utilizing the spectral function.

In order to define the spectral function we need to recall a few basic facts about shift-invariant spaces.

A closed subspace $V \subset L^2(\mathbb{R}^n)$ is called *shift-invariant* (SI) if for every function $f \in V$ we also have $T_k f \in V$ when $k \in \mathbb{Z}^n$, where $T_y f(x) = f(x-y)$ is the translation by a vector $y \in \mathbb{R}^n$. For any subset $\Phi \subset L^2(\mathbb{R}^n)$, let

$$\mathcal{S}(\Phi) = \overline{\operatorname{span}}\{T_k\varphi : \varphi \in \Phi,\ k \in \mathbb{Z}^n\}$$

be the SI space generated by Φ. A *principal shift-invariant* (PSI) space is a SI space V generated by a single function $\varphi \in L^2(\mathbb{R}^n)$, that is, $V = \mathcal{S}(\{\varphi\}) = \mathcal{S}(\varphi)$.

A *range function* is any mapping

$$J\colon \mathbb{T}^n \to \{\text{closed subspaces of } \ell^2(\mathbb{Z}^n)\},$$

where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is identified with its fundamental domain $[-1/2, 1/2)^n$. We say that J is *measurable* if the associated orthogonal projections $P_J(\xi)\colon \ell^2(\mathbb{Z}^n) \to J(\xi)$ are operator measurable; that is, $\xi \mapsto P_J(\xi)v$ is measurable for any $v \in \ell^2(\mathbb{Z}^n)$.

Given any subset $E \subset \mathbb{R}^n$, let E^P be the *periodization* of E; in other words, $E^P = \bigcup_{k\in\mathbb{Z}^n}(E+k)$. Let $\tau\colon \mathbb{R}^n \to \mathbb{T}^n$ be the *translation projection*, $\tau(\xi) = \xi + k$, where k is a unique element of $\mathbb{Z}^n$ such that $\xi + k \in \mathbb{T}^n$. Finally, let $\mathcal{T}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$ be an isometric isomorphism defined for $f \in L^2(\mathbb{R}^n)$ by

$$\mathcal{T}f\colon \mathbb{T}^n \to \ell^2(\mathbb{Z}^n), \quad \mathcal{T}f(\xi) = (\hat{f}(\xi+k))_{k\in\mathbb{Z}^n},$$

where $\hat{f}(\xi) = \int f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx$.

The following proposition, due to Helson [He1, Thm. 8], plays an important role in the theory of SI spaces in $L^2(\mathbb{R}^n)$. A proof of Proposition 1.1 can be also found in [Bo1, Prop. 1.5].

PROPOSITION 1.1. *A closed subspace $V \subset L^2(\mathbb{R}^n)$ is SI if and only if*

$$V = \{f \in L^2(\mathbb{R}^n) : \mathcal{T}f(\xi) \in J(\xi) \text{ for a.e. } \xi \in \mathbb{T}^n\}, \tag{1.1}$$

where J is a measurable range function. The correspondence between V and J is one-to-one under the convention that the range functions are identified if they are equal a.e. Furthermore, if $V = S(\Phi)$ for some countable $\Phi \subset L^2(\mathbb{R}^n)$, then

$$J(\xi) = \overline{\operatorname{span}}\{\mathcal{T}\varphi(\xi) : \varphi \in \Phi\}. \tag{1.2}$$

The *dimension function* of a SI space $V \subset L^2(\mathbb{R}^n)$ is a mapping $\dim_V : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$ given by $\dim_V(\xi) = \dim J(\xi)$, where J is the range function corresponding to V. Alternatively, the dimension function of V can be introduced as the *multiplicity function* of the projection-valued measure coming from the representation of the lattice $\mathbb{Z}^n$ on V via translations by Stone's theorem; see [BM; BMM]. The equivalence of the dimension function with the corresponding multiplicity function can then be easily deduced from [He2]. Note also that for $V = \check{L}^2(E)$, where E is a measurable subset of $\mathbb{R}^n$ and

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset E\}, \tag{1.3}$$

its dimension function is given by

$$\dim_V(\xi) = \sum_{k \in \mathbb{Z}^n} \mathbf{1}_E(\xi + k). \tag{1.4}$$

Finally, we need to recall a few facts about dual Gramian analysis of SI systems introduced by Ron and Shen [RS1]. Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is a countable set of functions such that

$$\sum_{\varphi \in \Phi} |\hat{\varphi}(\xi)|^2 < \infty \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{1.5}$$

The *dual Gramian* of a SI system $E(\Phi)$, where

$$E(\Phi) = \{T_k\varphi : k \in \mathbb{Z}^n,\ \varphi \in \Phi\}, \tag{1.6}$$

is a map $\tilde{G}$ from the fundamental domain $\mathbb{T}^n = (-1/2, 1/2]^n$ into self-adjoint infinite matrices $(g_{k,l})_{k,l \in \mathbb{Z}^n}$ defined for a.e. $\xi \in \mathbb{T}^n$ by

$$\tilde{G}(\xi)_{k,l} := \sum_{\varphi \in \Phi} \hat{\varphi}(\xi + k)\overline{\hat{\varphi}(\xi + l)} \quad \text{for } k, l \in \mathbb{Z}^n. \tag{1.7}$$

Recall that a matrix $\tilde{G}(\xi) = (\tilde{G}(\xi)_{k,l})_{k,l \in \mathbb{Z}^n}$ is bounded on $\ell^2(\mathbb{Z}^n)$ if $\tilde{G}(\xi)$ (given by $\langle \tilde{G}(\xi)e_k, e_l\rangle = \tilde{G}(\xi)_{k,l}$, where $(e_k)_{k \in \mathbb{Z}^n}$ is the standard basis of $\ell^2(\mathbb{Z}^n)$) defines a bounded operator on $\ell^2(\mathbb{Z}^n)$. It can be shown that, for any fixed $\xi \in \mathbb{T}^n$, $\{\mathcal{T}\varphi(\xi) : \varphi \in \Phi\} \subset \ell^2(\mathbb{Z}^n)$ is a Bessel family if and only if $\tilde{G}(\xi)$ is a bounded operator in $\ell^2(\mathbb{Z}^n)$. Furthermore, it follows from [RS1, Thm. 3.3.5] that (1.5) is a necessary (but not sufficient) condition for $E(\Phi)$ to be a Bessel family.

The following result due to Ron and Shen [RS1] characterizes when the system of translates of a given family of functions $E(\Phi)$ is a frame (or Bessel family if $a = 0$) in terms of the dual Gramian. See also [Bo1, Thm. 2.5(ii)].

THEOREM 1.2. *Suppose that $\Phi \subset L^2(\mathbb{R}^n)$ is countable and that Φ satisfies* (1.5). *Then the system $E(\Phi)$ is a frame for a SI space $S(\Phi)$ with frame bounds* $0 \le a \le b < \infty$, *that is,*

$$a\|f\|^2 \le \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} |\langle f, T_k\varphi\rangle|^2 \le b\|f\|^2 \quad \textit{for all } f \in S(\Phi),$$

if and only if the dual Gramian $\tilde{G}(\xi)$ satisfies

$$a\|v\|^2 \le \langle \tilde{G}(\xi)v, v\rangle \le b\|v\|^2 \quad \textit{for } v \in J(\xi) \textit{ and a.e. } \xi \in \mathbb{T}^n,$$

where $J(\xi)$ is the range function of $S(\Phi)$ given by (1.2).

2. The Spectral Function

In this section we introduce the notion of a spectral function associated to a shift-invariant space and then show its basic properties. The spectral function, which was introduced and investigated in [Rz, Sec. 1.4], contains much more information about shift-invariant spaces than the dimension function. It is a very useful tool that enables us to show many results that seem to be otherwise inaccessible by using the properties of dimension function alone.

DEFINITION 2.1. Suppose $V \subset L^2(\mathbb{R}^n)$ is SI with the range function $J(\xi)$ and the corresponding projection $P_J(\xi)$. The *spectral function* of V is a measurable mapping $\sigma_V : \mathbb{R}^n \to [0,1]$ given by

$$\sigma_V(\xi + k) = \|P_J(\xi)e_k\|^2 \quad \text{for } \xi \in \mathbb{T}^n \text{ and } k \in \mathbb{Z}^n, \tag{2.1}$$

where $\{e_k\}_{k\in\mathbb{Z}^n}$ denotes the standard basis of $\ell^2(\mathbb{Z}^n)$ and $\mathbb{T}^n = [-1/2, 1/2)^n$.

Note that $\sigma_V(\xi)$ is well-defined for a.e. $\xi \in \mathbb{R}^n$, since $\{k + \mathbb{T}^n : k \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$. Moreover, there is a simple relationship between the spectral and the dimension function:

$$\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_V(\xi + k). \tag{2.2}$$

Indeed, $\sum_{k\in\mathbb{Z}^n} \sigma_V(\xi + k) = \sum_{k\in\mathbb{Z}^n} \|P_J(\xi)e_k\|^2 = \dim \operatorname{Ran}(P_J(\xi)) = \dim_V(\xi)$.

Since our definition is rather abstract, we present a description of the spectral function of a general SI space V in terms of the spectral function of orthogonal PSI components of V.

PROPOSITION 2.2. *Let $\mathfrak{S}$ be the set of all SI subspaces of $L^2(\mathbb{R}^n)$. Then the spectral function σ_V of $V \in \mathfrak{S}$ is determined as the unique mapping*

$$\sigma : \mathfrak{S} \to L^\infty(\mathbb{R}^n)$$

that satisfies

$$\sigma_{S(\varphi)}(\xi) = \begin{cases} |\hat{\varphi}(\xi)|^2 \left(\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi + k)|^2\right)^{-1} & \textit{for } \xi \in \operatorname{supp}\hat{\varphi}, \\ 0 & \textit{otherwise,} \end{cases} \tag{2.3}$$

which is additive with respect to the orthogonal sums; *that is,*

$$V = \bigoplus_{i\in\mathbb{N}} V_i \textit{ for some } V_i \in \mathfrak{S} \implies \sigma_V = \sum_{i\in\mathbb{N}} \sigma_{V_i}. \tag{2.4}$$

Proposition 2.2 is a consequence of yet another description of the spectral function of SI space V in terms of the Fourier transform of a system of functions Φ whose shifts form a tight frame for V. Lemma 2.3 can also serve as an alternative definition of the spectral function [Rz1; Rz2]. Indeed, a direct calculation involving the standard Gabor orthonormal basis $(e^{2\pi i\langle x,j\rangle}\mathbf{1}_{\mathbb{T}^n}(x-k))_{j,k\in\mathbb{Z}^n}$ shows that formula (2.5) is well-defined and independent of the choice Φ (see [Rz2, Thm. 1.8]).

LEMMA 2.3. *Suppose a SI space $V \subset L^2(\mathbb{R}^n)$ is generated by the shifts of a countable family $\Phi \subset V$; that is, let $V = S(\Phi)$. If $E(\Phi) = \{T_k\varphi : k \in \mathbb{Z}^n, \varphi \in \Phi\}$ forms a tight frame with constant* 1 *for the space V, then*

$$\sigma_V(\xi) = \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2. \tag{2.5}$$

In particular, (2.5) *does not depend on the choice of Φ as long as $E(\Phi)$ is a tight frame with constant* 1 *for V.*

Proof. Let $J(\xi)$ be the range function of V and let $P_J(\xi)$ be the corresponding orthogonal projection onto $J(\xi)$. By [Bo1, Thm. 2.5(ii)], $E(\Phi)$ is a tight frame with constant 1 for V if and only if $\{\mathcal{T}\varphi(\xi) : \varphi \in \Phi\}$ is a tight frame with constant 1 for $J(\xi)$ for a.e. ξ. Therefore, for a.e. $\xi \in \mathbb{T}^n$,

$$\|v\|^2 = \sum_{\varphi\in\Phi} |\langle v, \mathcal{T}\varphi(\xi)\rangle|^2 \quad \text{for all } v \in J(\xi).$$

Hence

$$\|P(\xi)v\|^2 = \sum_{\varphi\in\Phi} |\langle v, \mathcal{T}\varphi(\xi)\rangle|^2 \quad \text{for all } v \in \ell^2(\mathbb{Z}^n)$$

and, in particular, for any $k \in \mathbb{Z}^n$ we have

$$\sigma_V(\xi+k) = \|P(\xi)e_k\|^2 = \sum_{\varphi\in\Phi} |\langle e_k, \mathcal{T}\varphi(\xi)\rangle|^2 = \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi+k)|^2.$$

This shows (2.5) and completes the proof of Lemma 2.3. □

REMARK. Lemma 2.3 also shows that there is a close connection between spectral functions and dual Gramians. Indeed, for a given SI space V and a family $\Phi \subset V$ such that $E(\Phi)$ is a tight frame with constant 1 for V, we can consider the corresponding dual Gramian $\tilde{G}(\xi)$ given by (1.7). By Theorem 1.2, $\tilde{G}(\xi)$ restricted to $J(\xi)$ is an identity on $J(\xi)$. Since the dual Gramian $\tilde{G}(\xi)$ is self-adjoint and since $\ker \tilde{G}(\xi) = J(\xi)^\perp$, it follows that $\tilde{G}(\xi)$ is just the orthogonal projection onto $J(\xi)$, that is, $\tilde{G}(\xi) = P_J(\xi)$. Therefore, by (1.7), the spectral function of V represents the diagonal entries of the dual Gramian of $E(\Phi)$.

Proof of Proposition 2.2. It suffices to show that the spectral function σ satisfies (2.3) and (2.4). Indeed, any mapping $\sigma : \mathfrak{S} \to L^\infty(\mathbb{R}^n)$ satisfying (2.3) and (2.4) is unique, since any SI space can be decomposed as $V = \bigoplus_{\varphi\in\Phi} \mathcal{S}(\varphi)$ for some countable family $\Phi \subset L^2(\mathbb{R}^n)$ (see e.g. [Bo1, Thm. 3.3] and [Rz2, Thm. 1.2]). To see (2.3), recall from [BL; BDR1] that if $\mathcal{S}(\varphi)$ is a PSI space then the function φ_0 given by

$$\hat{\varphi}_0(\xi) = \begin{cases} |\hat{\varphi}(\xi)|\left(\sum_{k\in\mathbb{Z}^n}|\hat{\varphi}(\xi+k)|^2\right)^{-1/2} & \text{for } \xi \in \operatorname{supp}\hat{\varphi}, \\ 0 & \text{otherwise,} \end{cases}$$

is the *quasi-orthogonal generator* for $\mathcal{S}(\varphi)$, meaning that $E(\varphi_0)$ is a tight frame with constant 1 for $\mathcal{S}(\varphi)$. Hence, by Lemma 2.3, we have (2.3).

Next, suppose $V = \bigoplus_{i\in\mathbb{N}} V_i$ for some SI spaces V_i. We can decompose each V_i as $V_i = \bigoplus_{\varphi\in\Phi_i} \mathcal{S}(\varphi)$ for some $\Phi_i \subset V_i$ such that $E(\Phi_i)$ forms a tight frame with constant 1 for V_i, $i \in \mathbb{N}$. Since $E(\Phi)$ forms a tight frame with constant 1 for V, where $\Phi = \bigcup_{i\in\mathbb{N}} \Phi_i$, by Lemma 2.3 we have

$$\sigma_V(\xi) = \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2 = \sum_{i\in\mathbb{N}} \sum_{\varphi\in\Phi_i} |\hat{\varphi}(\xi)|^2 = \sum_{i\in\mathbb{N}} \sigma_{V_i}(\xi).$$

This completes the proof of Proposition 2.2. □

Next we will show that the spectral function behaves nicely with respect to the action of modulations and dilations. This will be relevant in our study of Gabor systems and wavelets—or, more generally, framelets. Recall that the *modulation* by a vector $a \in \mathbb{R}^n$ of $f \in L^2(\mathbb{R}^n)$ is given by

$$M_a(f)(x) = e^{2\pi i\langle a,x\rangle} f(x).$$

The *dilation* by an $n \times n$ nonsingular matrix A of $f \in L^2(\mathbb{R}^n)$ is given by

$$D_A f(x) = |\det A|^{1/2} f(Ax).$$

We restrict our attention to dilations A preserving the lattice $\mathbb{Z}^n$ because this is exactly when in general we can expect that $D_A V$ is SI (with respect to the action of $\mathbb{Z}^n$) if V is SI.

THEOREM 2.4. *Let $V \subset L^2(\mathbb{R}^n)$ be SI and let A be an $n \times n$ integer matrix with* $\det A \neq 0$. *Then $D_A(V)$ is SI and*

$$\sigma_{D_A(V)}(\xi) = \sigma_V((A^*)^{-1}\xi), \tag{2.6}$$

where A^* is the transpose of A. Likewise, for any $a \in \mathbb{R}^n$, $M_a(V)$ is SI and

$$\sigma_{M_a(V)}(\xi) = \sigma_V(\xi - a). \tag{2.7}$$

Proof. First we decompose V as the orthogonal sum $V = \bigoplus_{i\in\mathbb{N}} \mathcal{S}(\varphi_i)$, where φ_i is a quasi-orthogonal generator of $\mathcal{S}(\varphi_i)$. Since D_A is an unitary operator on $L^2(\mathbb{R}^n)$ we have $D_A V = \bigoplus_{i\in\mathbb{N}} D_A\mathcal{S}(\varphi_i)$, and by Proposition 2.2 it suffices to show that

$$\sigma_{D_A\mathcal{S}(\varphi)}(\xi) = \sigma_{\mathcal{S}(\varphi)}((A^*)^{-1}\xi), \tag{2.8}$$

where φ is a quasi-orthogonal generator of $\mathcal{S}(\varphi)$.

Let $\mathcal{L}$ be a set of $|\det A|$ representatives of different cosets of $\mathbb{Z}^n/A\mathbb{Z}^n$. For $l \in \mathcal{L}$, define $\Phi_l \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$ by

$$\Phi_l(\xi) = \mathcal{T}(D_A T_l \varphi)(\xi) = (|\det A|^{-1/2} \hat{\varphi}((A^*)^{-1}(\xi + k)) e^{-2\pi i \langle (A^*)^{-1}(\xi+k), l\rangle})_{k\in\mathbb{Z}^n}.$$

Let $\mathcal{D}$ be any set of $|\det A|$ representatives of different cosets of $\mathbb{Z}^n/A^*\mathbb{Z}^n$. For $d \in \mathcal{D}$, define $\Psi_d \in L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$ by

$$\Psi_d(\xi)(k) = \begin{cases} \hat{\varphi}((A^*)^{-1}(\xi + k)) & \text{if } k \in d + A^*\mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases}$$

For $l \in \mathcal{L}$, we have

$$\Phi_l(\xi) = e^{-2\pi i \langle (A^*)^{-1}\xi, l\rangle} |\det A|^{-1/2} \sum_{d\in\mathcal{D}} e^{-2\pi i \langle (A^*)^{-1} d, l\rangle} \Psi_d(\xi).$$

Because the $|\det A| \times |\det A|$ matrix

$$(|\det A|^{-1/2} e^{-2\pi i \langle (A^*)^{-1} d, l\rangle})_{l\in\mathcal{L}, d\in\mathcal{D}}$$

is unitary, by a simple calculation we have

$$\sum_{l\in\mathcal{L}} |\langle v, \Phi_l(\xi)\rangle|^2 = \sum_{d\in\mathcal{D}} |\langle v, \Psi_d(\xi)\rangle|^2 \quad \text{for all } v \in \ell^2(\mathbb{Z}^n). \tag{2.9}$$

Since $\Psi_d(\xi) \perp \Psi_{d'}(\xi)$ for $d \neq d' \in \mathcal{D}$ and since $\|\Psi_d(\xi)\|$ is either 0 or 1, the system $\{\Psi_d(\xi) : d \in \mathcal{D}\}$ is a tight frame with constant 1 for its span. By (2.9), $\{\Phi_l(\xi) : l \in \mathcal{L}\}$ is also a tight frame with constant 1 for the space $J(\xi) = \operatorname{span}\{\Phi_l(\xi) : l \in \mathcal{L}\} = \operatorname{span}\{\Psi_d(\xi) : d \in \mathcal{D}\}$, where $J(\xi)$ is the range function of $D_A\mathcal{S}(\varphi)$. Therefore, for every $k \in \mathbb{Z}^n$,

$$\begin{aligned} \sigma_{D_A\mathcal{S}(\varphi)}(\xi + k) &= \|P_{J(\xi)} e_k\|^2 = \sum_{l\in\mathcal{L}} |\langle \Phi_l(\xi), e_k\rangle|^2 = \sum_{d\in\mathcal{D}} |\langle \Psi_d(\xi), e_k\rangle|^2 \\ &= |\hat{\varphi}((A^*)^{-1}(\xi + k))|^2 = \sigma_{\mathcal{S}(\varphi)}((A^*)^{-1}(\xi + k)). \end{aligned}$$

This shows (2.8) and therefore (2.6).

The case of modulations is much easier, since by Proposition 2.2 it suffices to show $\sigma_{M_a\mathcal{S}(\varphi)}(\xi) = \sigma_{\mathcal{S}(\varphi)}(\xi - a)$, where φ is a quasi-orthogonal generator of $\mathcal{S}(\varphi)$. Since $M_a\varphi$ is a quasi-orthogonal generator of $\mathcal{S}(M_a\varphi) = M_a\mathcal{S}(\varphi)$, we have

$$\sigma_{M_a\mathcal{S}(\varphi)}(\xi) = |\widehat{M_a\varphi}(\xi)|^2 = |\hat{\varphi}(\xi - a)|^2 = \sigma_{\mathcal{S}(\varphi)}(\xi - a),$$

which completes the proof of Theorem 2.4. □

As an immediate consequence of (2.2) and Theorem 2.4, we obtain the following corollary.

COROLLARY 2.5. *Suppose that $V \subset L^2(\mathbb{R}^n)$ is SI and that A is an $n \times n$ integer matrix with* $\det A \neq 0$. *Then*

$$\dim_{D_A V}(\xi) = \sum_{d\in\mathcal{D}} \dim_V((A^*)^{-1}(\xi + d)), \tag{2.10}$$

where $\mathcal{D}$ is the set of $|\det A|$ representatives of different cosets of $\mathbb{Z}^n/(A^*\mathbb{Z}^n)$.

EXAMPLE 1. Given a measurable set $E \subset \mathbb{R}^n$, let $V = \check{L}^2(E)$. Then $\sigma_V = \mathbf{1}_E$. Indeed, consider the family of functions $(\varphi_k)_{k\in\mathbb{Z}^n}$ given by $\hat{\varphi}_k = \mathbf{1}_{E\cap(k+\mathbb{T}^n)}$.

Clearly φ_k is a quasi-orthogonal generator of $\mathcal{S}(\varphi_k)$, and $\mathcal{S}(\varphi_k) \perp \mathcal{S}(\varphi_{k'})$ for $k \neq k' \in \mathbb{Z}^n$. Now it suffices to invoke Lemma 2.3. Conversely, if V is any SI space such that $\sigma_V = \mathbf{1}_E$ for some measurable $E \subset \mathbb{R}^n$, then necessarily $V = \check{L}^2(E)$. Indeed, let $J(\xi)$ be the range function of V. Since

$$\|P_J(\xi)e_k\|^2 = \sigma_V(\xi + k) = \mathbf{1}_E(\xi + k) = \begin{cases} 1 & \text{if } \xi + k \in E, \\ 0 & \text{if } \xi + k \notin E, \end{cases}$$

it follows that

$$P_J(\xi)e_k = \begin{cases} e_k & \text{if } \xi + k \in E, \\ 0 & \text{if } \xi + k \notin E, \end{cases}$$

and hence $J(\xi) = \overline{\text{span}}\{e_k : \xi + k \in E\}$ is the range function of $\check{L}^2(E)$. Therefore, $V = \check{L}^2(E)$.

EXAMPLE 2. There exist distinct SI spaces that have identical spectral functions. Indeed, let $\varphi_0, \varphi_1 \in L^2(\mathbb{R})$ be given by $\hat{\varphi}_0 = 2^{-1/2}(\mathbf{1}_{(0,1)} + \mathbf{1}_{(1,2)})$ and $\hat{\varphi}_1 = 2^{-1/2}(\mathbf{1}_{(0,1)} - \mathbf{1}_{(1,2)})$. Then φ_i is a quasi-orthogonal generator of $V_i = \mathcal{S}(\varphi_i)$; that is, $E(\varphi_i)$ is a tight frame of $\mathcal{S}(\varphi_i)$ with constant 1. Hence $\sigma_{V_0} = \sigma_{V_1} = 2^{-1}\mathbf{1}_{(0,2)}$, but $V_0 \perp V_1$.

We can now collect the main properties of the spectral function into a single proposition.

PROPOSITION 2.6. *Let $\mathfrak{S}$ be the set of all SI subspaces of $L^2(\mathbb{R}^n)$. Then, for $V, W \in \mathfrak{S}$, the spectral function satisfies the following properties*:

(a) $0 \le \sigma_V(\xi) \le 1$;
(b) $V = \bigoplus_{i\in\mathbb{N}} V_i \ (V_i \in \mathfrak{S}) \implies \sigma_V(\xi) = \sum_{i\in\mathbb{N}} \sigma_{V_i}(\xi)$;
(c) $V \subset W \implies \sigma_V(\xi) \le \sigma_W(\xi)$;
(d) $V \subset W \implies (V = W \iff \sigma_V(\xi) = \sigma_W(\xi))$;
(e) $\sigma_V(\xi) = \mathbf{1}_E(\xi) \iff V = \check{L}^2(E)$;
(f) $\sigma_{M_a(V)}(\xi) = \sigma(\xi - a)$, *where M_a is a modulation by $a \in \mathbb{R}^n$*;
(g) $\sigma_{D_A V}(\xi) = \sigma_V((A^*)^{-1}\xi)$, *where D_A is a dilation by nonsingular integer matrix A*;
(h) $\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_V(\xi + k)$.

We will also need the following approximation lemma.

LEMMA 2.7. *Let V be a SI space and let $(V_j)_{j\in\mathbb{N}}$ be a sequence of SI spaces. Suppose that $P_{V_j} \to P_V$ strongly as $j \to \infty$, where P_V denotes the orthogonal projection onto V; that is, for every $f \in L^2(\mathbb{R}^n)$, let $\|P_{V_j} f - P_V f\|_2 \to 0$ as $j \to \infty$. Then, for any measurable set $E \subset \mathbb{R}^n$ with finite Lebesgue measure,*

$$\int_E |\sigma_{V_j}(\xi) - \sigma_V(\xi)|\, d\xi \to 0 \text{ as } j \to \infty. \tag{2.11}$$

In particular, there exists a subsequence $(j_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sigma_{V_{j_k}}(\xi) = \sigma_V(\xi) \quad \textit{for a.e. } \xi \in \mathbb{R}^n. \tag{2.12}$$

Proof. Let J and J_j ($j \in \mathbb{N}$) be the range functions corresponding to V and V_j, respectively. Denote the corresponding projections by $P_J(\xi)$ and $P_{J_j}(\xi)$. Then, for any $f \in L^2(\mathbb{R}^n)$,

$$\mathcal{T}(P_V f)(\xi) = P_J(\xi)(\mathcal{T}f(\xi)) \quad \text{for a.e. } \xi \in \mathbb{T}^n \tag{2.13}$$

by [He1, p. 58; Bo1, Lemma 1.4]. Because

$$\begin{aligned}\|P_{V_j} f - P_V f\|^2 &= \|\mathcal{T}(P_{V_j} f - P_V f)\|^2 \\ &= \int_{\mathbb{T}^n} \|\mathcal{T}(P_{V_j} f)(\xi) - \mathcal{T}(P_V f)(\xi)\|_{\ell^2}^2 \, d\xi \\ &= \int_{\mathbb{T}^n} \|P_J(\xi)(\mathcal{T}f(\xi)) - P_{J_j}(\xi)(\mathcal{T}f(\xi))\|_{\ell^2}^2 \, d\xi,\end{aligned}$$

it follows that, for any $k \in \mathbb{Z}^n$, we obtain

$$\begin{aligned}0 &= \lim_{j\to\infty} \int_{\mathbb{T}^n} \|P_J(\xi)(e_k) - P_{J_j}(\xi)(e_k)\|_{\ell^2}^2 \, d\xi \\ &\geq \lim_{j\to\infty} \int_{k+\mathbb{T}^n} |\sigma_V(\xi)^{1/2} - \sigma_{V_j}(\xi)^{1/2}|^2 \, d\xi \\ &\geq \frac{1}{4} \lim_{j\to\infty} \int_{k+\mathbb{T}^n} |\sigma_V(\xi) - \sigma_{V_j}(\xi)|^2 \, d\xi \\ &\geq \frac{1}{4} \lim_{j\to\infty} \left(\int_{k+\mathbb{T}^n} |\sigma_V(\xi) - \sigma_{V_j}(\xi)| \, d\xi \right)^{1/2}\end{aligned}$$

by taking $f \in L^2(\mathbb{R}^n)$ such that $\mathcal{T}f(\xi) = e_k$ for all $\xi \in \mathbb{T}^n$. This shows (2.11). Finally, (2.12) is a consequence of (2.11) and the standard diagonal subsequence argument. □

The following lemma provides another way of looking at the spectral function.

LEMMA 2.8. *Suppose V is SI and $K \subset \mathbb{R}^n$ is a measurable set such that $\tau|_K$ is one-to-one. Then*

$$\|P_V(\check{\mathbf{1}}_K)\|^2 = \int_K \sigma_V(\xi) \, d\xi, \tag{2.14}$$

where P_V is an orthogonal projection onto V.

Proof. Let $f = \check{\mathbf{1}}_K$. Then, for any $\xi \in \mathbb{T}^n$,

$$\mathcal{T}f(\xi) = \begin{cases} e_l & \text{if there is an } l \in \mathbb{Z}^n \text{ such that } \xi + l \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by (2.13),

$$\begin{aligned}\|P_V(f)\|^2 &= \int_{\mathbb{T}^n} \|P_J(\xi)(\mathcal{T}f(\xi))\|^2 \, d\xi = \sum_{l\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \|P_J(\xi)(e_l)\|^2 \mathbf{1}_K(\xi + l) \, d\xi \\ &= \sum_{l\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \sigma_V(\xi + l)\mathbf{1}_K(\xi + l) \, d\xi = \int_K \sigma_V(\xi) \, d\xi,\end{aligned}$$

where J is the range function of V. □

As a consequence of Lemma 2.8 and the Lebesgue differentiation theorem, we have another formula for the spectral function:

$$\sigma_V(\xi) = \lim_{r\to 0^+} \frac{\|P_V(\check{\mathbf{1}}_{B(\xi,r)})\|^2}{|B(\xi,r)|} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.15}$$

where $B(\xi, r)$ denotes the ball with center ξ and radius r.

APPROXIMATION ORDERS OF SI SPACES. As an example of the utility of the spectral function, we will show how it can be used to characterize approximation orders of SI spaces. In order to state the main result we recall a few basic facts from [BDR1; BDR2].

Suppose $V \subset L^2(\mathbb{R}^n)$ is a closed subspace. For any $h > 0$, define the scaled space

$$V^h := \{g(\cdot/h) : g \in V\} = D_{h^{-1}}(V),$$

where $D_{h^{-1}}$ is the dilation operator by a diagonal matrix $h^{-1}\,\mathrm{Id}$. For any closed subspace $V \subset L^2(\mathbb{R}^n)$ and a function $f \in L^2(\mathbb{R}^n)$, we define the *approximation error* as

$$E(f, V) := \inf\{\|f - g\| : g \in V\}.$$

Given $k > 0$, we say that the space V provides *approximation order* k if there is a constant $C > 0$ such that

$$E(f, V^h) \le Ch^k \|f\|_{H^k(\mathbb{R}^n)} \quad \text{for all } f \in H^k(\mathbb{R}^n), \tag{2.16}$$

where $H^k(\mathbb{R}^n)$ is the *Sobolev space*

$$H^k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^k(\mathbb{R}^n)} = \|(1 + |\cdot|)^k \hat{f}\| < \infty\}.$$

We say that V provides *density order* k if, for a given $k \ge 0$, for every $f \in H^k(\mathbb{R}^n)$ we have (in addition to (2.16))

$$E(f, V^h) = o(h^k) \text{ as } h \to 0. \tag{2.17}$$

Given $\phi \in L^2(\mathbb{R}^n)$, we define the function $\Lambda_\phi : \mathbb{T}^n \to [0, 1]$ by

$$\Lambda_\phi(\xi) := \left(1 - \frac{|\hat{\phi}(\xi)|^2}{\sum_{k\in\mathbb{Z}^n} |\hat{\phi}(\xi + k)|^2}\right)^{1/2} = (1 - \sigma_{\mathcal{S}(\phi)}(\xi))^{1/2}, \tag{2.18}$$

where $0/0$ should be interpreted as 0. De Boor, DeVore, and Ron [BDR2, Thms. 1.6, 1.7] showed that the approximation and density orders of a PSI space $\mathcal{S}(\phi)$ are characterized by the behavior of the function $\xi \mapsto \Lambda_\phi(\xi)/|\xi|^k$ at the origin. Moreover, they also proved the following remarkable result showing that approximation by arbitrary (closed) SI subspaces of $L^2(\mathbb{R}^n)$ can be reduced to the case of PSI spaces.

THEOREM 2.9 [BDR2, Thm. 1.9]. *Suppose that $V \subset L^2(\mathbb{R}^n)$ is SI and that $k > 0$. Then the following statements are equivalent*:

(i) *V provides approximation order k*;
(ii) *there exists a $\phi \in V$ such that $\mathcal{S}(\phi)$ provides approximation order k*,

(iii) $\mathcal{S}(P_V(\phi_0))$ *provides approximation order* k, *where* ϕ_0 *is the sinc-function,* $\hat{\phi}_0 = \mathbf{1}_{\mathbb{T}^n}$, *and* P_V *is the orthogonal projection onto* V.

Moreover, the same is true when $k \geq 0$ *and the term "approximation order" is replaced by "density order".*

As a consequence of this result, we obtain the following characterization of approximation and density orders of SI spaces in terms of the spectral function.

THEOREM 2.10. *A SI space* $V \subset L^2(\mathbb{R}^n)$ *provides approximation order* $k > 0$ *if and only if there is a* $C > 0$ *such that*

$$\sigma_V(\xi) \geq 1 - C|\xi|^{2k} \quad \textit{for a.e. } \xi \in \mathbb{T}^n. \tag{2.19}$$

The space V *provides density order* $k \geq 0$ *if and only if* (2.19) *holds and*

$$\lim_{h \to 0} \frac{1}{h^n} \int_{h\mathbb{T}^n} |\xi|^{-2k}(1 - \sigma_V(\xi))\, d\xi = 0. \tag{2.20}$$

Proof. Let $\phi_0(x_1, \ldots, x_n) = \prod_{i=1}^n \sin(\pi x_i)/(\pi x_i) \in L^2(\mathbb{R}^n)$ be the sinc-function; that is, $\hat{\phi}_0 = \mathbf{1}_{\mathbb{T}^n}$. Let $\phi_1 = P_V(\phi_0)$ be the orthogonal projection of ϕ_0 onto V. Then, by Theorem 2.9, the approximation order and the density order provided by V are the same as the approximation order and the density order provided by its PSI subspace $\mathcal{S}(\phi_1)$.

Note that

$$\sigma_{\mathcal{S}(\phi_1)}(\xi) = \sigma_V(\xi) \quad \text{for a.e. } \xi \in \mathbb{T}^n. \tag{2.21}$$

Indeed, take any $\xi \in \mathbb{T}^n$. By (2.13),

$$\mathcal{T}(\phi_1)(\xi) = \mathcal{T}(P_V(\phi_0))(\xi) = P_J(\xi)(\mathcal{T}(\phi_0)(\xi)) = P_J(\xi)(e_0).$$

Hence, if $\mathcal{T}(\phi_1)(\xi) \neq 0$ then by (2.3) we have

$$\sigma_{\mathcal{S}(\phi_1)}(\xi) = \frac{|\hat{\phi}_1(\xi)|^2}{\|\mathcal{T}(\phi_1)(\xi)\|^2} = \frac{|\langle P_J(\xi)e_0, e_0\rangle|^2}{\|P_J(\xi)(e_0)\|^2} = \|P_J(\xi)(e_0)\|^2 = \sigma_V(\xi).$$

If $\mathcal{T}(\phi_1)(\xi) = 0$ then clearly $\sigma_{\mathcal{S}(\phi_1)}(\xi) = \sigma_V(\xi) = 0$. This shows (2.21).

By [BDR2, Thm. 1.6], $\mathcal{S}(\phi_1)$ provides approximation order $k > 0$ if $|\cdot|^{-k}\Lambda_{\phi_1}$ is in $L^\infty(\mathbb{T}^n)$. Combining this with (2.18) and (2.21) shows (2.19). Likewise, by [BDR2, Thm. 1.7], $\mathcal{S}(\phi_1)$ provides density order $k \geq 0$ if $|\cdot|^{-k}\Lambda_{\phi_1}$ is in $L^\infty(\mathbb{T}^n)$ and

$$\lim_{h \to 0} h^{-n} \int_{h\mathbb{T}^n} |\xi|^{-2k}[\Lambda_{\phi_1}(\xi)]^2\, d\xi = 0.$$

Combining this with (2.18) and (2.21) shows (2.20), which completes the proof of Theorem 2.10. □

3. The Dimension Function of a GMRA

As one of the applications of the spectral function studied in Section 2, we extend a result from [BM] by providing a characterization of dimension functions associated with an arbitrary generalized multiresolution analysis (GMRA). In other words, we characterize all multiplicity functions associated with the core subspace of a GMRA. This extends a result of Baggett and Merrill [BM], who considered only locally integrable multiplicity functions. In this work we not only allow non-integrable multiplicity functions, we also allow functions equal to ∞ on a set of nonzero measure. As an immediate consequence, we also obtain a characterization of wavelet dimension functions shown by Speegle and the authors in [BRS].

We start by recalling the notion of a GMRA, which has been studied by a number of authors [Ba2; BM; BMM; BL; Bo3; HLPS; LTW].

DEFINITION 3.1. Let A be a fixed $n \times n$ integer expansive dilation matrix (i.e., for all eigenvalues λ of A, $|\lambda| > 1$). We say that a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}} \subset L^2(\mathbb{R}^n)$ is a *generalized multiresolution analysis* (GMRA) if

$$V_j \subset V_{j+1}, \quad D_A V_j = V_{j+1}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n), \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\},$$

and V_0 is SI. The space V_0 is often called a *core space* of $(V_j)_{j\in\mathbb{Z}}$. An (orthonormal) wavelet is a collection $\Psi = \{\psi^1, \dots, \psi^L\}$ such that the system

$$\{D_{A^j} T_k \psi^l : j \in \mathbb{Z}^n,\ k \in \mathbb{Z}^n,\ l = 1, \dots, L\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$.

There is a close relationship between wavelets and GMRAs. For any orthonormal wavelet Ψ we can associate a GMRA $(V_j)_{j\in\mathbb{Z}}$ by setting

$$V_j = \overline{\operatorname{span}}\{D_{A^i} T_k \psi^l : i < j,\ k \in \mathbb{Z}^n,\ l = 1, \dots, L\}.$$

Conversely, [BMM] characterized GMRAs that can be generated by orthonormal wavelets. These are precisely GMRAs $(V_j)_{j\in\mathbb{Z}}$ such that the dimension function $\dim_{V_0}(\xi)$ of the core space V_0 is finite for a.e. ξ and satisfies the consistency equation

$$\sum_{d\in\mathcal{D}} \dim_{V_0}((A^*)^{-1}(\xi + d)) = \dim_{V_0}(\xi) + L \quad \text{for a.e. } \xi,$$

where $\mathcal{D}$ is the set of $|\det A|$ representatives of different cosets of $\mathbb{Z}^n/A^*\mathbb{Z}^n$. Another related result is the characterization in [LTW] of Riesz wavelets that generate GMRAs.

A key ingredient of a GMRA is its core space V_0, which uniquely determines the subspaces $V_j = D_{A^j} V_0$, $j \in \mathbb{Z}$. Since the core space is a refinable space, our main goal is to give a complete characterization of dimension functions of refinable spaces; see Theorem 3.2. The proof of this result will follow the ideas from [BRS]. Recall that a SI space $V \subset L^2(\mathbb{R}^n)$ is said to be *refinable* (with respect to the expansive dilation A) if $V \subset D_A V$.

THEOREM 3.2. *Suppose a SI space $V \subset L^2(\mathbb{R}^n)$ is refinable, that is, $V \subset D_A V$. Then the dimension function of V, $D(\xi) := \dim_V(\xi)$, satisfies*:

(D1) $D\colon \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$ *is a measurable* $\mathbb{Z}^n$*-periodic function*;

(D2) $\sum_{d \in \mathcal{D}} D(B^{-1}(\xi + d)) \ge D(\xi)$ *for a.e.* $\xi \in \mathbb{R}^n$, *where* $\mathcal{D}$ *is the set of representatives of different cosets of* $\mathbb{Z}^n/B\mathbb{Z}^n$ $(B = A^*)$; *and*

(D3) $\sum_{k \in \mathbb{Z}^n} \mathbf{1}_\Delta(\xi + k) \ge D(\xi)$ *for a.e.* $\xi \in \mathbb{R}^n$, *where*

$$\Delta = \{\xi \in \mathbb{R}^n : D(B^{-j}\xi) \ge 1 \text{ for } j \in \mathbb{N} \cup \{0\}\}. \tag{3.1}$$

Conversely, suppose that D satisfies (D1), (D2), *and* (D3). *Then there exists a refinable space V such that* $\dim_V(\xi) = D(\xi)$ *for a.e.* $\xi \in \mathbb{R}^n$. *Furthermore, V satisfies*

$$\bigcap_{j \in \mathbb{Z}} D_{A^j} V = \{0\}, \tag{3.2}$$

$$\overline{\bigcup_{j \in \mathbb{Z}} D_{A^j} V} = \check{L}^2\Big(\bigcup_{j \in \mathbb{Z}} B^j \Delta\Big). \tag{3.3}$$

Proof of Theorem 3.2: Necessity of (D1)–(D3). Clearly, the dimension function of any SI space V must satisfy (D1). Condition (D2) is a consequence of Corollary 2.5, since $V \subset D_A V$ implies that $\dim_V(\xi) \le \dim_{D_A V}(\xi)$. To show (D3), note that $V \subset D_A V$ implies that $\sigma_V(\xi) \le \sigma_V(B^{-1}\xi)$ by Proposition 2.6(c) and (g). By Proposition 2.6(h), $\sigma_V(\xi) \ne 0$ implies $\dim_V(\xi) \ne 0$ and hence $\dim_V(\xi) \ge 1$. Thus

$$\begin{aligned} \Delta &= \{\xi \in \mathbb{R}^n : \dim_V(B^{-j}\xi) \ge 1 \text{ for } j \in \mathbb{N} \cup \{0\}\} \\ &\supset \{\xi \in \mathbb{R}^n : \sigma_V(\xi) \ne 0\} = \operatorname{supp} \sigma_V. \end{aligned}$$

Therefore, by Proposition 2.6(a) and (h),

$$\sum_{k \in \mathbb{Z}^n} \mathbf{1}_\Delta(\xi + k) \ge \sum_{k \in \mathbb{Z}^n} \sigma_V(\xi + k) = \dim_V(\xi),$$

which shows (D3). □

The key role in the proof of the *sufficiency* of (D1)–(D3) in Theorem 3.2 is played by Lemma 3.3.

LEMMA 3.3. *Suppose that a function D satisfies* (D1), (D2), *and* (D3). *Then there exists a measurable set* $S \subset \mathbb{R}^n$ *such that*

$$D(\xi) = \sum_{k \in \mathbb{Z}^n} \mathbf{1}_S(\xi + k), \tag{3.4}$$

$$S \subset BS, \tag{3.5}$$

$$\bigcap_{j \in \mathbb{Z}} B^j S = \emptyset, \tag{3.6}$$

$$\bigcup_{j \in \mathbb{Z}} B^j S = \bigcup_{j \in \mathbb{Z}} B^j \Delta, \tag{3.7}$$

where Δ *is given by* (3.1).

The proof of Lemma 3.3 is an adaptation of the proof of the characterization of dimension functions of wavelets that follows the constructive procedure described in [BRS, Algo. 4.4]. The major difference with [BRS] is that we allow a function D to be infinite on some set with nonzero measure and that the consistency equation [BRS, (D3) in Thm. 4.2] is replaced by our consistency inequality (D2). Assuming that Lemma 3.3 holds, we can complete the proof of Theorem 3.2.

Proof of Theorem 3.2: Sufficiency of (D1)–(D3). Define a SI space $V = \check{L}^2(S)$, where S is a set guaranteed by Lemma 3.3. Clearly, $\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_S(\xi+k) = D(\xi)$ by (3.4). Hence V is refinable because $V = \check{L}^2(S) \subset \check{L}^2(BS) = D_A V$. Furthermore, V satisfies (3.2) and (3.3) by virtue of (3.6), (3.7), and the fact that $D_{A^j} V = \check{L}^2(B^j S)$ for $j \in \mathbb{Z}$. □

To prove Lemma 3.3, we will need [BRS, Lemma 4.1] as follows.

LEMMA 3.4. *Let B be a dilation and let D satisfy* (D1) *and* (D3). *Let $A_j = \{\xi \in \mathbb{T}^n : D(\xi) \geq j\}$ for $j \in \mathbb{N}$ and let $\{S_i\}_{i=1}^k$ (where $k \in \mathbb{N}$ is fixed) be a collection of measurable sets such that $\tau|_{S_i}$ is injective and onto A_i for $i = 1, \ldots, k$. Then*

(i) *there exists a measurable set $G \subset \Delta$ such that $\tau(G) = A_1$, and*
(ii) *there exists a measurable set $H \subset \Delta$ such that $\bigcup_{i=1}^k S_i \cap H = \emptyset$ and $\tau(H) = A_{k+1}$.*

Proof of Lemma 3.3 under $\{\xi \in \mathbb{T}^n : D(\xi) < \infty\}$ with nonzero measure. Let

$$\tilde{\Delta} = \bigcup_{j\in\mathbb{Z}} B^j \Delta, \tag{3.8}$$

where Δ is given by (3.1). For any $k \in \mathbb{N}$, let

$$A_k = \{\xi \in \mathbb{T}^n : D(\xi) \geq k\}. \tag{3.9}$$

The idea of the proof is to construct a sequence of sets $\{S_k\}_{k\in\mathbb{N}}$ such that $S = \bigcup_{k\in\mathbb{N}} S_k$ satisfies (3.4)–(3.7). In particular, to guarantee (3.4), we will require (among other things) that $\tau|_{S_k}$ be injective and onto A_k for each $k \in \mathbb{N}$.

Fix any measurable set $Q \subset \mathbb{R}^n$ such that $Q \subset BQ$, $\tau|_Q$ is injective, $D(\xi) \geq 1$ for $\xi \in Q$, and

$$\lim_{j\to\infty} \mathbf{1}_Q(B^{-j}\xi) = 1 \quad \text{for a.e. } \xi \in \tilde{\Delta}. \tag{3.10}$$

Since B is a dilation, $Q = \Delta \cap \bigcap_{j=0}^{\infty} B^j \mathbb{T}^n$ is an example of a set that satisfies these properties.

We are now ready to define S_1. Let $E_1 = Q$. For $m \in \mathbb{N}$, define

$$\tilde{E}_{m+1} = \left(BE_m \setminus \bigcup_{i=1}^m E_i^P \right) \cap A_1^P$$

and let $E_{m+1} \subset \tilde{E}_{m+1}$ be any measurable set such that $\tau(E_{m+1}) = \tau(\tilde{E}_{m+1})$ and $\tau|_{E_{m+1}}$ is injective. We claim that the set $S_1 = \bigcup_{m\in\mathbb{N}} E_m$ satisfies

$$Q \subset S_1, \quad S_1 \subset BS_1, \quad \tau|_{S_1} \text{ is injective and onto } A_1. \tag{3.11}$$

This can been seen by repeating verbatim the arguments in the proof of [BRS, Thm. 4.2] and then using Lemma 3.4(i). We continue defining sets $\{S_k\}_{k\in\mathbb{N}}$ by induction with the use of a technical Lemma 3.5.

LEMMA 3.5. *Suppose that there exist sets $S_1, \dots, S_k$ such that $Q \subset S_1 = P_1$ and*

$$P_i \subset BP_i \quad \text{for } i = 1, \dots, k, \quad \text{where } P_i = \bigcup_{j=1}^{i} S_j; \tag{3.12}$$

$$\tau|_{S_i} \text{ is injective and onto } A_i \quad \text{for } i = 1, \dots, k; \tag{3.13}$$

$$S_i \cap S_j = \emptyset \quad \text{for } i, j = 1, \dots, k, \ i \neq j. \tag{3.14}$$

Then there exist S_{k+1} such that $S_1, \dots, S_{k+1}$ satisfy (3.12)–(3.14).

Proof. Given $S_1, \dots, S_k$ satisfying (3.12)–(3.14), we define S_{k+1} by the inductive procedure. Let $\tilde{F}_1 = (BP_k \setminus P_k) \cap A^P_{k+1}$ and

$$\tilde{F}_{m+1} = \left(BF_m \setminus \bigcup_{i=1}^{m} F_i^P \right) \cap A^P_{k+1}, \tag{3.15}$$

where $F_m \subset \tilde{F}_m$ is any measurable set such that $\tau(F_{m+1}) = \tau(\tilde{F}_{m+1})$ and $\tau|_{F_{m+1}}$ is injective. We claim that $S_1, \dots, S_{k+1}$, where $S_{k+1} = \bigcup_{m=1}^{\infty} F_m$ satisfy (3.12)–(3.14).

To see (3.12), it suffices to show that $S_{k+1} \subset BP_{k+1}$. This follows from the fact that $F_1 \subset BP_k$ and $F_{m+1} \subset BF_m \subset BS_{k+1} \subset BP_{k+1}$ for $m \in \mathbb{N}$.

To see that $\tau|_{S_{k+1}}$ is injective, suppose that $\xi_1, \xi_2 \in S_{n+1}$ and $\tau(\xi_1) = \tau(\xi_2)$. Therefore, $\xi_1 \in F_j$ and $\xi_2 \in F_k$ for some $j, k \in \mathbb{N}$. If $j < k$ (the case $j > k$ is identical) then $\xi_2 \notin F_j^P$, which contradicts $\tau(\xi_1) = \tau(\xi_2)$. Thus $j = k$ and $\xi_1 = \xi_2$, since $\tau|_{F_k}$ is injective.

To see (3.14), it is enough to prove that $S_{k+1} \cap P_k = \emptyset$. Since $F_1 \subset BP_k \setminus P_k$ and $F_m \subset BF_{m-1}$ for $m \geq 2$, by induction we obtain $F_m \subset B^m P_k \setminus B^{m-1} P_k$ for all $m \in \mathbb{N}$. By (3.12) with $i = k$ it follows that $P_k \subset B^{m-1} P_k$ for any $m \in \mathbb{N}$. Hence $F_m \cap P_k = \emptyset$, that is, $S_{k+1} \cap P_k = \emptyset$.

The proof of the remaining part of (3.13) (i.e., $\tau(S_{k+1}) = A_{k+1}$) is much more difficult. First note that, since $\tau(F_m) \subset A_{k+1}$, we certainly have $\tau(S_{k+1}) \subset A_{k+1}$. For the reverse inclusion, we will find it useful to prove

$$A^P_{k+1} \cap BS^P_{k+1} \subset S^P_{k+1}. \tag{3.16}$$

Indeed, if $\xi \in A^P_{k+1} \cap BS^P_{k+1}$, then $B^{-1}\xi + l \in F_m$ for some $m \in \mathbb{N}$ and $l \in \mathbb{Z}^n$. Since $\xi' := \xi + Bl \in BF_m$ and $\xi \in A^P_{k+1}$, we obtain $\xi' \in BF_m \cap A^P_{k+1}$. If $\xi' \notin \bigcup_{i=1}^{m} F_i^P$, then by (3.15) we have $\xi' \in \tilde{F}_{m+1} \subset S^P_{k+1}$ and hence $\xi \in S^P_{k+1}$. However, if $\xi' \in F_i^P$ for some $i = 1, \dots, m$, then $\xi' \in S^P_{k+1}$ and hence $\xi \in S^P_{k+1}$. This shows (3.16).

Continuing with the proof of (3.13), it remains to show that $A_{k+1} \subset \tau(S_{k+1})$. By Lemma 3.4(ii) there is a set $H \subset \Delta$ such that $H \cap P_k = \emptyset$, $\tau(H) = A_{k+1}$, and $D(B^{-j}\xi) \geq 1$ for every $j \geq 0$ and $\xi \in H$. Therefore, all we have to prove is $\tau(H) \subset \tau(S_{k+1})$, that is, $H \subset S^P_{k+1}$.

We split the proof of $H \subset S^P_{k+1}$ into two cases. First, we consider all $\xi \in H$ such that, for every $j \ge 0$, $D(B^{-j}\xi) \ge k+1$; that is, we consider the set $R := H \cap \bigcap_{j=0}^{\infty} B^j A^P_{k+1}$. For a.e. $\xi \in R$, it follows from (3.10) that $B^{-j}\xi \in Q$ for some $j \ge 1$. Since $Q \subset S_1 \subset P_k$, we can consider $j_0 = \min\{j \in \mathbb{N} : B^{-j}\xi \in P_k\}$. Since $H \cap P_k = \emptyset$, it follows that $B^{-j_0+1}\xi \in BP_k \setminus P_k$. Moreover, since $B^{-j_0+1}\xi \in A^P_{k+1}$, we obtain

$$B^{-j_0+1}\xi \in (BP_k \setminus P_k) \cap A^P_{k+1} = \tilde{F}_1 \subset S^P_{k+1}.$$

Hence $R \subset \bigcup_{j=0}^{\infty} B^j S^P_{k+1}$. Therefore,

$$R = \bigcup_{j=0}^{\infty} (R \cap B^j S^P_{k+1}) \subset \bigcup_{j=0}^{\infty} \left(H \cap \bigcap_{i=0}^{j} B^i A^P_{k+1} \cap B^j S^P_{k+1} \right).$$

But (3.16) implies that, for $j \ge 0$,

$$\bigcap_{i=0}^{j} B^i A^P_{k+1} \cap B^j S^P_{k+1} \subset S^P_{k+1}, \tag{3.17}$$

so $R \subset S^P_{k+1}$.

Next we consider the set $\tilde{R} := H \setminus \bigcap_{j=0}^{\infty} B^j A^P_{k+1}$ and wish to show that $\tilde{R} \subset S^P_{k+1}$. Take any $\xi \in \tilde{R}$. Since $\xi \in H \subset A^P_{k+1}$, we can find $j_0 \ge 0$ such that $\xi \in \bigcap_{j=0}^{j_0} B^j A^P_{k+1}$ and $\xi \notin B^{j_0+1} A^P_{k+1}$. To prove that $\xi \in S^P_{k+1}$, it is enough to show that $\xi \in B^{j_0} S^P_{k+1}$ and then use (3.17) with $j = j_0$. To see why $\xi \in B^{j_0} S^P_{k+1}$, observe that $D(B^{-j_0}\xi) \ge k+1$. Hence, by the consistency inequality (D2),

$$k+1 \le D(B^{-j_0}\xi) \le \sum_{d \in \mathcal{D}} D(B^{-j_0-1}\xi + B^{-1}d), \tag{3.18}$$

where $\mathcal{D}$ is the set of representatives of different cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$. Without loss of generality we can assume that $0 \in \mathcal{D}$. For each $d \in \mathcal{D}$ we denote $k(d) = D(B^{-j_0-1}\xi + B^{-1}d)$. Then $\tau(B^{-j_0-1}\xi + B^{-1}d) \in \bigcap_{j=1}^{k(d)} A_j$. Moreover, since $d = 0 \in \mathcal{D}$, $\tau(B^{-j_0-1}\xi) \in A_{k(0)}$, and $\xi \notin B^{j_0+1} A^P_{k+1}$, we obtain $k(0) \le k$. Since $\xi \in H$, it follows that $D(B^{-j_0-1}\xi) \ge 1$, and we obtain $k(0) \ge 1$. By (3.13) we have $B^{-j_0-1}\xi \in \bigcap_{j=1}^{k(0)} S^P_j$, that is, $B^{-j_0-1}\xi + p^0_j \in S_j$, where $p^0_j \in \mathbb{Z}^n$ for $j = 1, \dots, k(0)$ are distinct by (3.14). For each $d \in \mathcal{D} \setminus \{0\}$ such that $j(d) \ne 0$, by using (3.13) again we can find distinct $p^d_j \in \mathbb{Z}^n$ such that $B^{-j_0-1}\xi + p^d_j + B^{-1}d \in S_j$, where $j = 1, \dots, \min(k(d), k)$.

Thus, for each $d \in \mathcal{D}$ such that $k(d) \ne 0$, we have

$$B^{-j_0}\xi + Bp^d_j + d \in BS_j \subset BP_k \quad \text{for } \; j = 1, \dots, \min(k(d), k). \tag{3.19}$$

We claim that this gives us at least $k+1$ distinct elements of BP_k. Indeed, if $Bp^d_j + d = Bp^{d'}_{j'} + d'$ for some $j = 1, \dots, \min(k(d), k)$ and $j' = 1, \dots, \min(k(d'), k)$, then $d - d' \in B\mathbb{Z}^n$, hence $d = d'$. Also, for fixed $d \in \mathcal{D}$, we have $p^d_j \ne p^d_{j'}$ for $j \ne j'$. What remains to check is that the number of elements in (3.19), which is equal to $\sum_{d \in \mathcal{D}} \min(k(d), k)$, is $\ge k+1$. This is indeed the case by $1 \le k(0) \le k$ and (3.18).

By the induction hypothesis (3.13), at least one of the elements in (3.19) must lie in the complement of P_k. Hence there is a $p \in \mathbb{Z}^n$, and $B^{-j_0}\xi + p \in BP_k \setminus P_k$. In addition, since $B^{-j_0}\xi \in A^P_{k+1}$, we have $B^{-j_0}\xi + p \in A^P_{k+1}$. Therefore, $B^{-j_0}\xi + p \in \tilde{F}_1 \subset S^P_{k+1}$; that is, $\xi \in B^{j_0} S^P_{k+1}$ and hence $\xi \in S^P_{k+1}$. Since both R and $\tilde{R}$ are contained in S^P_{k+1}, it follows that $H \subset S^P_{k+1}$ and so $A_{k+1} \subset S^P_{k+1}$, which completes the proof of (3.13) by the induction. We have thus shown the existence of the sets $\{S_k\}_{k\in\mathbb{N}}$ satisfying (3.12)–(3.14), which completes the proof of Lemma 3.5. □

Finally, we are ready to continue the proof of Lemma 3.3. By Lemma 3.5, define the set $S = \bigcup_{k=1}^{\infty} S_k$. We claim that S satisfies (3.4)–(3.7). To see (3.4), it suffices to show that the equality in (3.4) holds for a.e. $\xi \in \mathbb{T}^n$. By (3.14), $\mathbf{1}_S = \sum_{k=1}^{\infty} \mathbf{1}_{S_k}$, and by (3.13),

$$\sum_{l\in\mathbb{Z}^n} \mathbf{1}_S(\xi + l) = \sum_{k=1}^{\infty} \sum_{l\in\mathbb{Z}^n} \mathbf{1}_{S_k}(\xi + l)$$
$$= \sum_{k=1}^{\infty} \mathbf{1}_{\tau(S_k)}(\xi) = \sum_{k=1}^{\infty} \mathbf{1}_{A_k}(\xi) = D(\xi) \quad \text{for a.e. } \xi \in \mathbb{T}^n.$$

For (3.5) it suffices to show that $S_k \subset BS$ for every $k \in \mathbb{N}$, but this is immediate from (3.12). For (3.7) it suffices to show that $\lim_{j\to\infty} \mathbf{1}_S(B^{-j}\xi) = 1$ for a.e. $\xi \in \tilde{\Delta}$, which is an immediate consequence of $Q \subset S$ and (3.10). Finally, it remains to show (3.6). By the contradiction, suppose that $\tilde{S} = \bigcap_{j\in\mathbb{Z}} B^j S$ has a nonzero measure. Since $\tilde{S} = B\tilde{S}$, we can partition $\tilde{S}$ into a countable family of subsets $\{\tilde{S}_i\}_{i\in\mathbb{N}}$ such that $\tilde{S}_i = B\tilde{S}_i$ and $\tilde{S}_i$ has a nonzero (and hence infinite) measure for all $i \in \mathbb{N}$. Since B induces an ergodic endomorphism of $\mathbb{T}^n$ (see [BRS, proof of Thm. 5.11]), we have $\tau(\tilde{S}_i) = \mathbb{T}^n$. Therefore,

$$D(\xi) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_S(\xi + k) \geq \sum_{k\in\mathbb{Z}^n} \mathbf{1}_{\tilde{S}}(\xi + k)$$
$$= \sum_{k\in\mathbb{Z}^n} \sum_{i\in\mathbb{N}} \mathbf{1}_{\tilde{S}_i}(\xi + k) = \infty \quad \text{for a.e. } \xi,$$

which contradicts our initial assumption that $\{\xi \in \mathbb{T}^n : D(\xi) < \infty\}$ has a nonzero measure. This shows (3.6) and completes the proof of Lemma 3.3 under this assumption. □

Proof of Lemma 3.3 under $D \equiv \infty$. Finally, we need to construct a set S satisfying (3.4)–(3.7) for the case when the dimension function $D(\xi)$ is constantly ∞. Let S_0 be a set obtained from Lemma 3.3 when $D \equiv 1$. Any such set S_0 is called a *scaling set*. By (3.5)–(3.7), the family $\{(B^{j+1}S_0) \setminus (B^j S_0)\}_{j\in\mathbb{Z}}$ forms a partition of $\mathbb{R}^n$ (modulo sets of measure zero), since $\Delta = \tilde{\Delta} = \mathbb{R}^n$. Define by induction a sequence of sets $\{S_i\}_{i=0}^{\infty}$ such that, for all $i = 0, 1, \ldots,$

$$S_{i+1} \subset (BS_i) \setminus S_i, \qquad \sum_{k\in\mathbb{Z}^n} \mathbf{1}_{S_i}(\xi + k) = 1 \quad \text{for a.e. } \xi.$$

This is possible because

$$\sum_{k\in\mathbb{Z}^n} \mathbf{1}_{BS_i}(\xi+k) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_{S_i}(B^{-1}\xi + B^{-1}k)$$
$$= \sum_{k\in\mathbb{Z}^n} \sum_{d\in\mathcal{D}} \mathbf{1}_{S_i}(B^{-1}\xi + k + B^{-1}d) = |\det B|.$$

Define the set $S = \bigcup_{i=0}^{\infty} S_i$. It remains to show (3.4)–(3.7). To see (3.4), notice that $S_i \subset (B^i S_0) \setminus (B^{i-1}S_0)$ for $i \in \mathbb{N}$. Hence the sets S_i are pairwise disjoint and

$$\sum_{k\in\mathbb{Z}^n} \mathbf{1}_S(\xi+k) = \sum_{k\in\mathbb{Z}^n} \sum_{i=0}^{\infty} \mathbf{1}_{S_i}(\xi+k) = \infty.$$

Observe that (3.5) follows from

$$BS = BS_0 \cup \bigcup_{i=0}^{\infty} BS_i \supset S_0 \cup \bigcup_{i=0}^{\infty} S_{i+1} = S,$$

while (3.7) is an immediate consequence of $S_0 \subset S$ and $\bigcup_{j\in\mathbb{Z}} B^j S_0 = \mathbb{R}^n$. To see (3.6), let $c = \left|\bigcap_{j\in\mathbb{Z}} B^j S \cap ((BS_0) \setminus S_0)\right|$. For any $i \in \mathbb{N}$,

$$1 = |S_i| = |((B^i S_0) \setminus (B^{i-1}S_0) \cap S|$$
$$\geq \left|((B^i S_0) \setminus (B^{i-1}S_0)) \cap \bigcap_{j\in\mathbb{Z}} B^j S\right| = |\det B|^{i-1} c.$$

By letting $i \to \infty$, we conclude that c is zero and hence

$$\left|((B^i S_0) \setminus (B^{i-1}S_0)) \cap \bigcap_{j\in\mathbb{Z}} B^j S\right| = 0 \quad \text{for all } \ i \in \mathbb{Z}.$$

This shows (3.6) and completes the proof of Lemma 3.3. □

Finally, we are ready to show our main result.

THEOREM 3.6. *Suppose* $(V_j)_{j\in\mathbb{Z}}$ *is a GMRA. Then the dimension function of the core space* V_0, $D(\xi) := \dim_{V_0}(\xi)$, *satisfies* (D1)–(D3) *and*

(D4) $\liminf_{j\to\infty} D(B^{-j}\xi) \geq 1$ *for a.e.* $\xi \in \mathbb{R}^n$.

Conversely, if D satisfies (D1)–(D4) *then there exists a GMRA* $(V_j)_{j\in\mathbb{Z}}$ *such that* $\dim_{V_0}(\xi) = D(\xi)$ *for a.e.* $\xi \in \mathbb{R}^n$.

Proof. Suppose that $(V_j)_{j\in\mathbb{Z}}$ is a GMRA. Then, by Theorem 3.2, $D(\xi) = \dim_{V_0}(\xi)$ satisfies (D1)–(D3). To see (D4), by Proposition 2.6(g) and (h) it suffices to show that

$$\liminf_{j\to\infty} \sigma_{V_j}(\xi) = \liminf_{j\to\infty} \sigma_{V_0}(B^{-j}\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.20}$$

However, by Lemma 2.7, there exists a sequence $(j_k)_{k\in\mathbb{N}}$ such that

$$\liminf_{k\to\infty} \sigma_{V_{j_k}}(\xi) = \sigma_{L^2(\mathbb{R}^n)}(\xi) = 1 \quad \text{for a.e. } \xi,$$

because the V_j converge to $L^2(\mathbb{R}^n)$ as $j \to \infty$. Because $(\sigma_{V_j}(\xi))_{j\in\mathbb{Z}}$ is a non-decreasing sequence, by $V_j \subset V_{j+1}$ and Proposition 2.6(c) we necessarily have (3.20).

The converse is immediate by Theorem 3.2, (3.2), and (3.3), since (D4) is equivalent to $\bigcup_{j\in\mathbb{Z}} B^j\Delta = \mathbb{R}^n$. □

We conclude this section by giving an example of a GMRA whose dimension function of its core space is infinite on a set of nonzero (but not full) measure. Note that the construction of a GMRA whose dimension function constantly equals ∞ is given in the proof of Lemma 3.3. More examples of dimension functions of GMRAs associated with orthonormal wavelets can be found in [BRS, Sec. 5].

EXAMPLE 3.7. For simplicity, assume that we work in the dimension $n = 1$ and that the dilation factor $A = 2$. Given $0 < \delta < 1/2$, define a set $S \subset \mathbb{R}$ by

$$S = [-1/2, 1/2] \cup \bigcup_{j=-1}^{\infty} [2^j, 2^j + \delta).$$

An elementary calculation shows that $S \subset 2S$, $\bigcap_{j\in\mathbb{Z}} 2^j S = \emptyset$, and $\bigcup_{j\in\mathbb{Z}} 2^j S = \mathbb{R}$. Hence the sequence $(V_j)_{j\in\mathbb{Z}}$, where $V_j = \check{L}^2(2^j S)$, is a GMRA. Moreover, the dimension function of its core space satisfies

$$\dim_{V_0}(\xi) = \begin{cases} \infty & \text{for } \tau(\xi) \in (0,\delta), \\ 1 & \text{otherwise.} \end{cases}$$

4. The Spectral Function of a GMRA

In this section we investigate properties of the spectral function associated to the core space of a GMRA by showing the fundamental representation formula (4.1), which is a close analogue to the Calderón reproducing formula in the theory of wavelets. As an immediate consequence, we show an explicit formula for the wavelet spectral function, which also gives the usual well-known formula for the wavelet dimension function.

THEOREM 4.1. *Suppose $(V_j)_{j\in\mathbb{Z}}$ is a GMRA such that $\{\xi \in \mathbb{R}^n : \dim_{V_0}(\xi) < \infty\}$ has a nonzero (Lebesgue) measure. Then*

$$\sum_{j=-\infty}^{\infty} \sigma_{W_0}((A^*)^j \xi) = 1 \quad \text{for a.e. } \xi, \tag{4.1}$$

where $W_0 = V_1 \ominus V_0$. As a consequence, the spectral function of the core space V_0 can be represented as

$$\sigma_{V_0}(\xi) = \sum_{j=1}^{\infty} \sigma_{W_0}((A^*)^j \xi) \quad \text{for a.e. } \xi. \tag{4.2}$$

It is intuitively clear that (4.1) should hold. Indeed, it appears that by Proposition 2.6 we have $\sigma_{W_j}(\xi) = \sigma_{D_{A^j}W_0}(\xi) = \sigma_{W_0}((A^*)^{-j}\xi)$ and $\sum_{j\in\mathbb{Z}} \sigma_{W_j}(\xi) = 1$, since

$$\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}^n), \quad \text{where } W_j = V_{j+1} \ominus V_j. \tag{4.3}$$

However, in general the spaces W_j are not SI (with respect to the standard lattice $\mathbb{Z}^n$) for $j < 0$. Indeed, for $j < 0$ we can be sure only that W_j is SI with respect to a larger lattice $A^{-j}\mathbb{Z}^n$. Nevertheless, this idea can be transformed into a rigorous proof.

Proof of Theorem 4.1. Since $V_0 = \bigoplus_{j<0} W_j = \big(\bigoplus_{j\geq 0} W_j\big)^\perp$ is SI and since $V_0 \oplus \bigoplus_{j\geq 0} W_j = L^2(\mathbb{R}^n)$, by Proposition 2.6(b) and (g) we have

$$\begin{aligned} 1 = \sigma_{L^2(\mathbb{R}^n)}(\xi) &= \sigma_{V_0}(\xi) + \sum_{j=0}^{\infty} \sigma_{W_j}(\xi) \\ &= \sigma_{V_0}(\xi) + \sum_{j=0}^{\infty} \sigma_{W_0}((A^*)^j\xi) \quad \text{for a.e. } \xi. \end{aligned} \tag{4.4}$$

Pick any $\xi_0 \in \mathbb{R}^n$ such that (4.4) holds for any $\xi = A^i\xi_0$, where $i \in \mathbb{Z}$. By applying (4.4) for $\xi = A^i\xi_0$ and letting $i \to -\infty$, we obtain

$$\sum_{j=-\infty}^{\infty} \sigma_{W_0}((A^*)^j\xi) \leq 1 \quad \text{for a.e. } \xi. \tag{4.5}$$

Assume by way of contradiction that (4.1) fails. Then, by (4.5), there is a $\delta > 0$ such that $E = \big\{\xi \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} \sigma_{W_0}((A^*)^j\xi) < 1 - \delta\big\}$ has a nonzero measure. Since $A^*(E) = E$ and A^* is expansive, E must then have an infinite measure. Clearly, we can partition E into a countable family of subsets $\{E_i\}_{i\in\mathbb{N}}$ such that $A^*(E_i) = E_i$ and E_i has a nonzero (hence infinite) measure. Since A^* induces an ergodic endomorphism of $\mathbb{T}^n$ (see [BRS, proof of Thm. 5.11]), we have $\tau(E_i) = \mathbb{T}^n$. On the other hand, by (4.4), $\sigma_{V_0}(\xi) > \delta$ for $\xi \in E$. Therefore, by Proposition 2.6(h) for a.e. ξ,

$$\dim_{V_0}(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_{V_0}(\xi + k) \geq \sum_{i\in\mathbb{N}} \sum_{k\in\mathbb{Z}^n} \mathbf{1}_{E_i}(\xi + k)\sigma_{V_0}(\xi + k) \geq \sum_{i\in\mathbb{N}} \delta = \infty.$$

This contradicts our initial hypothesis and so proves (4.1). Equation (4.2) follows from (4.1) and (4.4). □

THEOREM 4.2. *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is a semi-orthogonal wavelet (i.e., that the affine system $\{D_{A^j}T_k\psi : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi\}$ is a tight frame with constant 1) and let $W_i \perp W_j$ for $i \neq j$, where*

$$W_i = \overline{\operatorname{span}}\{D_{A^i}T_k\psi : k \in \mathbb{Z}^n, \ \psi \in \Psi\} = D_{A^i}(\mathcal{S}(\Psi)).$$

Then the spectral function of the core space V_0 *of the GMRA* $(V_j)_{j\in\mathbb{Z}}$, $V_j = \bigoplus_{i<j} W_i$, *is given by*

$$\sigma_{V_0}(\xi) = \sum_{\psi\in\Psi}\sum_{j=1}^{\infty}|\hat{\psi}((A^*)^j\xi)|^2. \tag{4.6}$$

Proof. By the semi-orthogonality condition, $E(\Psi)$ forms a tight frame with constant 1 for $W_0 = \mathcal{S}(\Psi)$ and thus $\sigma_{W_0}(\xi) = \sum_{\psi\in\Psi}|\hat{\psi}(\xi)|^2$ by Lemma 2.3. By (4.4),

$$\begin{aligned}\sigma_{V_0}(\xi) &= 1 - \sum_{i=0}^{\infty}\sigma_{W_0}((A^*)^j\xi)\\ &= 1 - \sum_{j=0}^{\infty}\sum_{\psi\in\Psi}|\hat{\psi}((A^*)^j\xi)|^2 = \sum_{\psi\in\Psi}\sum_{j=1}^{\infty}|\hat{\psi}((A^*)^j\xi)|^2,\end{aligned}$$

where we used the Calderón formula $\sum_{\psi\in\Psi}\sum_{j=-\infty}^{\infty}|\hat{\psi}((A^*)^j\xi)|^2 = 1$ for a.e. ξ (see e.g. [Bo2; HW]). This completes the proof of Theorem 4.2. □

As a corollary to Proposition 2.6(h) and Theorem 4.2 we obtain the usual formula for the wavelet dimension function [BRS; LTW; RS4; We].

COROLLARY 4.3. *Suppose* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ *is a semi-orthogonal wavelet. Then the dimension function of the core space* V_0 *of the GMRA* $(V_j)_{j\in\mathbb{Z}}$ *associated with* Ψ *is given by*

$$\dim_{V_0}(\xi) = \sum_{k\in\mathbb{Z}^n}\sigma_{V_0}(\xi+k) = \sum_{\psi\in\Psi}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}|\hat{\psi}((A^*)^j(\xi+k))|^2. \tag{4.7}$$

REMARK. It should be noted that neither Theorem 4.2 nor Corollary 4.3 are true for general affine tight frame systems with constant 1. Indeed, Theorem 3.1 of [PSWX] shows that a (dyadic) normalized tight frame wavelet $\psi \in L^2(\mathbb{R})$ is semi-orthogonal if and only if $\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}|\hat{\psi}(2^j(\xi+k))|^2$ is integer-valued almost everywhere. Therefore, one can expect that the right-hand side of (4.7) is a dimension function of V_0, and thus integer-valued, only if Ψ is a semi-orthogonal wavelet. As a result, Theorem 4.2 and Corollary 4.3 are not valid for more general tight affine frame systems.

5. Rieffel's Incompleteness Theorem for Gabor Systems

In this section we present an elementary proof of Rieffel's incompleteness theorem for Gabor systems that utilizes the spectral function introduced in Section 2. We will start by giving some historical background information.

In 1986, Daubechies, Grossmann, and Meyer [DGM] constructed a Gabor system

$$\{g_{bl,am}(x) = e^{2\pi iblx}g(x-am)\}_{l,m\in\mathbb{Z}}$$

forming a tight frame for $L^2(\mathbb{R})$ with g a compactly supported function in the Schwartz class for every positive a and b such that $ab < 1$. Lyubarskii [Ly] and independently Seip and Wallstén [SW] proved that the original choice of Gabor [Ga] (i.e., g a Gaussian) gives rise to a frame of $L^2(\mathbb{R})$ if $ab < 1$ (this was conjectured by Daubechies and Grossmann in 1988).

Both of these results utilize the situation $ab < 1$, since for $ab > 1$ the Gabor system $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ can never be complete in $L^2(\mathbb{R})$ for any choice of $g \in L^2(\mathbb{R})$. This simple fact was communicated by Baggett [Ba1] and Daubechies [D], and it follows from the work of Rieffel [Ri] on von Neumann algebras (see also [DLL]). Even though the argument in [DLL] is rather simple, it is based completely on [Ri], which requires a reasonable knowledge of the theory of C^*-algebras. This, in turn, has motivated several researchers to find an "elementary" proof of the incompleteness result.

Daubechies [D] was able to find a very elegant argument for the case when $ab > 1$ is rational by constructing a function orthogonal to every element of $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$. The problem of (explicitly) constructing such a function in the general case $ab > 1$ is still open. In 1993 Landau [La] proved, under certain assumptions on the decay of g and $\hat{g}$, that $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ cannot be a frame for $L^2(\mathbb{R})$ if $ab > 1$. Janssen [Ja] showed the same without any decay assumptions for a general $g \in L^2(\mathbb{R}^n)$; see also [CDH; RS3]. Furthermore, Ramanathan and Steger [RSt] proved Landau's result for irregular Gabor systems without any decay assumptions. Their methods allowed them to recover Rieffel's incompleteness theorem. They also conjectured that the incompleteness result can be extended to the case of irregular sampling sets with a uniform density smaller than 1, which was later disproved in [BHW] by exhibiting a counterexample based on Landau's result from 1960s. Another proof of Rieffel's incompleteness theorem (again based on von Neumann algebras) was recently given by Gabardo and Han [GH] (see also [HWa, Thm. 3.3]).

We shall now show Rieffel's incompleteness theorem in its full generality—that is, for Gabor multi-systems in $L^2(\mathbb{R}^n)$. The advantage of our approach is that we do not use any results about von Neumann algebras; instead, our proof is based on the spectral function introduced in Section 2. Moreover, we are not aware of any other proofs of Theorem 5.1 in its full generality that do not use the machinery of von Neumann algebras. For example, [RS3, Cor. 4.7] contains a proof of this result for compressible Gabor systems, which correspond in one dimension to the case when ab is rational.

THEOREM 5.1. *If* $\{g^1, \dots, g^L\} \subset L^2(\mathbb{R}^n)$ *and if* A, B *are* $n \times n$ *nonsingular matrices, then the Gabor system*

$$\{g^i_{Bl,Am} : l, m \in \mathbb{Z}^n,\ i = 1, \dots, L\},$$

where

$$g_{Bl,Am}(x) = M_{Bl}T_{Am}g = e^{2\pi i\langle Bl,x\rangle}g(x - Am), \quad g \in L^2(\mathbb{R}^n), \tag{5.1}$$

is incomplete in $L^2(\mathbb{R}^n)$ *if* $|\det A| \cdot |\det B| > L$.

Proof. The standard dilation argument allows us to reduce the general case to $A = \mathrm{Id}$. Indeed, this follows immediately from $D_A(g_{Bl,Am}) = (D_A g)_{A^T Bl,m}$, where $l, m \in \mathbb{Z}^n$. We present a proof by contradiction; that is, we shall assume that the system $\{g^i_{Bl,m}\}^{i=1,\dots,L}_{l,m\in\mathbb{Z}^n}$ is complete and then prove that $|\det B| \le L$ follows.

We start by constructing a multiresolution scheme for Gabor systems. Suppose that $\{g^1, \dots, g^L\} \subset L^2(\mathbb{R}^n)$ and that $\{g^i_{Bl,m}\}^{i=1,\dots,L}_{l,m\in\mathbb{Z}^n}$ is the corresponding Gabor system. For every $l \in \mathbb{Z}^n$, define

$$G_l = \overline{\mathrm{span}}\{g^i_{Bl,m} : m \in \mathbb{Z}^n,\ i = 1, \dots, L\} = \mathcal{S}(\{M_{Bl}(g^1), \dots, M_{Bl}(g^L)\}). \quad (5.2)$$

We will use the lexicographic order on $(\mathbb{Z}^n, <)$ defined by $(k_1, k_2, \dots, k_n) < (l_1, l_2, \dots, l_n)$ if there exist $r = 1, \dots, n$ such that $k_j = l_j$ for all $j < r$ and $k_r < l_r$. For any $l \in \mathbb{Z}^n$, let

$$V_l = \overline{\mathrm{span}}\{g_{Bk,m} : k < l,\ k, m \in \mathbb{Z}^n\} = \overline{\biguplus_{k<l} G_k}. \quad (5.3)$$

Let S be the closed subspace of $L^2(\mathbb{R}^n)$ spanned by $\{g_{Bl,m}\}_{l,m\in\mathbb{Z}^n}$. Our discussion is general and we do not yet assume that the system is complete. We would expect to have the following properties of the "multiresolution analysis" $(V_l)_{l\in\mathbb{Z}^n}$:

(G1) $V_k \subset V_l$ for any $k < l \in \mathbb{Z}^n$;
(G2) $M_{Bk}(V_l) = V_{l+k}$ for any $k, l \in \mathbb{Z}^n$;
(G3) $\bigcup_{l\in\mathbb{Z}^n} V_l$ is dense in S;
(G4) $\bigcap_{l\in\mathbb{Z}^n} V_l = \{0\}$;
(G5) $V_{\mathbf{0}}$ is shift-invariant, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^n$.

Indeed, all of these conditions except (G4) follow immediately from (5.3). However, a serious difficulty arises because (G4) does not hold in general (see Example 5.2). Nevertheless, if we assume in addition that the functions $g^1, \dots, g^L$ are band-limited, then this easily implies that (G4) holds. Indeed, if $\mathrm{supp}\,\hat{g}^1, \dots, \mathrm{supp}\,\hat{g}^L \subset K$ for some bounded set $K \subset \mathbb{R}^n$, then $G_l \subset \check{L}^2(Bl+K)$ for any $l \in \mathbb{Z}^n$. Therefore,

$$V_k = \overline{\biguplus_{l<k} G_l} \subset \check{L}^2\Big(\bigcup_{l<k}(Bl + K)\Big)$$

and

$$\bigcap_{k\in\mathbb{Z}^n} V_k \subset \bigcap_{k\in\mathbb{Z}^n} \check{L}^2\Big(\bigcup_{l<k}(Bl + K)\Big) = \check{L}^2\Big(B\Big(\bigcap_{k\in\mathbb{Z}^n}\Big(\bigcup_{l<k}(l + B^{-1}K)\Big)\Big)\Big) = \{0\},$$

and (G4) follows.

Finally, we will need an additional condition that, in general, is guaranteed to hold only in the band-limited case. Recall that $I \subset \mathbb{Z}^n$ is a *beginning interval* if, for every $k, l \in \mathbb{Z}^n$, we have that $k < l$ and $l \in I$ together imply $k \in I$. It is not hard to show that all nonempty beginning intervals of $(\mathbb{Z}^n, <)$ can be associated with its ending, which consists of either

(i) elements of $\mathbb{Z}^n$ or
(ii) elements of the form $(k_1, \dots, k_r, \infty)$, where $0 \le r < n$ and $k_1, \dots, k_r \in \mathbb{Z}$.

That is, for any beginning interval $I \subset \mathbb{Z}^n$, there exist $r = 0, 1, \dots, n$ and $k \in \mathbb{Z}^r$ such that $I = \{l \in \mathbb{Z}^n : (l_1, \dots, l_r) \le (k_1, \dots, k_r)\}$.

We claim that, for any beginning interval $I \subset \mathbb{Z}^n$ of type (ii),

(G6) $\bigcap_{l\in\mathbb{Z}^n\setminus I} V_l = \overline{\bigcup_{l\in I} V_l}$.

The inclusion "$\supset$" in (G6) is obvious. To show "$\subset$", suppose that a beginning interval I is associated with $(k_1, \dots, k_r, \infty)$ and that $\operatorname{supp} \hat{g}^1, \dots, \operatorname{supp} \hat{g}^L \subset K$ for some bounded set $K \subset \mathbb{R}^n$. For any $r > 0$, consider an orthogonal projection P_r of $L^2(\mathbb{R}^n)$ onto $\check{L}^2((-r,r)^n)$ defined by $\widehat{P_r f} = \mathbf{1}_{(-r,r)^n} \hat{f}$. For any $j \in \mathbb{Z}$, consider an index set I_j,

$$I_j = \{(k_1, \dots, k_r + 1, m_1, \dots, m_{n-r}) \in \mathbb{Z}^n : m_1 \le j,\ (m_1, \dots, m_{n-r}) \in \mathbb{Z}^{n-r}\}.$$

We now claim that, for sufficiently small j, $P_r\big(\overline{\biguplus_{l\in I_j} G_l}\big) = \{0\}$. Indeed, if $f \in \overline{\biguplus_{l\in I_j} G_l}$ then

$$\begin{aligned} \operatorname{supp} \hat{f} \cap (-r,r)^n &\subset \bigcup_{l\in I_j} (Bl + K) \cap (-r,r)^n \\ &= B\bigg(\bigcup_{l\in I_j} (l + B^{-1}K) \cap B^{-1}(-r,r)^n\bigg) = \emptyset. \end{aligned}$$

Thus, for sufficiently small j,

$$\begin{aligned} P_r\bigg(\bigcap_{l\in\mathbb{Z}^n\setminus I} V_l\bigg) &\subset P_r(V_{(k_1,\dots,k_r+1,j,0,\dots,0)}) \\ &\subset P_r\bigg(\overline{\biguplus_{l\in I} G_l \uplus \biguplus_{l\in I_j} G_l}\bigg) = P_r\bigg(\overline{\biguplus_{l\in I} G_l}\bigg) = P_r\bigg(\overline{\bigcup_{l\in I} V_l}\bigg). \end{aligned}$$

Since $r > 0$ is arbitrary, the preceding formula yields (G6).

We shall use (G1)–(G6) in the same way one uses the multiresolution analysis scheme to construct wavelets. This is exactly true in dimension $n = 1$, but an additional argument based on (G6) is needed in dimensions $n > 1$. In our case, for every $l \in \mathbb{Z}^n$ we define a space

$$W_l = V_{l+\mathbf{1}} \ominus V_l, \tag{5.4}$$

where $\mathbf{1} = (0, \dots, 0, 1) \in \mathbb{Z}^n$. By (G2) we have

$$M_{Bk}(W_l) = W_{l+k} \quad \text{for } k \in \mathbb{Z}^n. \tag{5.5}$$

We claim that from (G1)–(G6) it follows that

$$S = \bigoplus_{l\in\mathbb{Z}^n} W_l. \tag{5.6}$$

Indeed, by (G1), (G3), and (G4),

$$S = \bigoplus_{I} \bigg(\bigcap_{l\in\mathbb{Z}^n\setminus I} V_l \ominus \overline{\bigcup_{l\in I} V_l}\bigg), \tag{5.7}$$

where the orthogonal sum is taken over all nonempty beginning intervals $I \subset \mathbb{Z}^n$. By (G6), the orthogonal sum in (5.7) is effectively taken only over the beginning intervals of type (i), which clearly implies (5.6).

The decomposition (5.6) allows us to see that the spectral function of S is given by a simple formula. In fact, we have

$$W_{\mathbf{0}} = V_{\mathbf{1}} \ominus V_{\mathbf{0}} = \overline{(V_{\mathbf{0}} \uplus G_{\mathbf{0}})} \ominus V_{\mathbf{0}}.$$

By (5.2), $G_{\mathbf{0}}$ is a SI space with the dimension function $\dim_{G_{\mathbf{0}}}(\xi) \le L$ for a.e. $\xi \in \mathbb{R}^n$. Hence, by (G5), $W_{\mathbf{0}}$ is also a SI space with the dimension function that satisfies $\dim_{W_{\mathbf{0}}}(\xi) \le \dim_{G_{\mathbf{0}}}(\xi) \le L$ for a.e. $\xi \in \mathbb{R}^n$. Therefore, $W_{\mathbf{0}}$ has a *quasi-orthogonal basis* $\Psi \subset W_{\mathbf{0}}$ with cardinality at most L; that is, $E(\Psi)$ is a tight frame with constant 1 for $W_{\mathbf{0}}$ (see [BDR1]). The spectral function of $W_{\mathbf{0}}$ is thus $\sigma_{W_{\mathbf{0}}} = \sum_{\psi\in\Psi} |\hat{\psi}|^2$ by Lemma 2.3. Therefore, by Proposition 2.6, (5.5), and (5.6), we have

$$\begin{aligned} \sigma_S(\xi) &= \sum_{l\in\mathbb{Z}^n} \sigma_{W_l}(\xi) = \sum_{l\in\mathbb{Z}^n} \sigma_{M_{Bl}(W_{\mathbf{0}})}(\xi) \\ &= \sum_{l\in\mathbb{Z}^n} \sigma_{W_{\mathbf{0}}}(\xi - Bl) = \sum_{l\in\mathbb{Z}^n} \sum_{\psi\in\Psi} |\hat{\psi}(\xi - Bl)|^2 \end{aligned} \tag{5.8}$$

for a.e. $\xi \in \mathbb{R}^n$.

The system $\{g_{Bl,m}\}_{l,m\in\mathbb{Z}^n}$ is complete in $L^2(\mathbb{R}^n)$ if and only if $S = L^2(\mathbb{R}^n)$, which by Proposition 2.6 is equivalent to having $\sigma_S = 1$ a.e. If we integrate this equality over the cube $B([0,1]^n)$, then by (5.8) we have

$$\begin{aligned} |\det B| &= \int_{B([0,1]^n)} \sum_{l\in\mathbb{Z}^n} \sum_{\psi\in\Psi} |\hat{\psi}(\xi - Bl)|^2 \, d\xi \\ &= \sum_{\psi\in\Psi} \sum_{l\in\mathbb{Z}^n} \int_{B(l+[0,1]^n)} |\hat{\psi}(\xi)|^2 \, d\xi = \sum_{\psi\in\Psi} \|\psi\|_2^2. \end{aligned}$$

Now Ψ is a quasi-orthogonal basis of $W_{\mathbf{0}}$ and so we must have $\|\psi\|_2 \le 1$ for every $\psi \in \Psi$, which proves that $|\det B| \le L$.

To drop the assumption that $g^1, \dots, g^L$ are band-limited, we approximate each g^i by a sequence of band-limited functions $\{g^{i,r}\}_{r\in\mathbb{N}} \subset L^2(\mathbb{R}^n)$ in the $L^2(\mathbb{R}^n)$-norm. For example, we can define such a sequence by $\widehat{g^{i,r}} = \mathbf{1}_{(-r,r)^n} \widehat{g^i}$ for $r \in \mathbb{N}$, and then

$$\lim_{r\to\infty} \|g^i - g^{i,r}\|_2 = 0. \tag{5.9}$$

Let S^r be the closure of the space spanned by the Gabor system $\{g^{i,r}_{Bl,m}\}^{i=1,\dots,L}_{l,m\in\mathbb{Z}^n}$ and let P_{S^r} be the orthogonal projection onto S^r for $r \in \mathbb{N}$. For every $l, m \in \mathbb{Z}^n$, we have

$$\|g^i_{Bl,m} - g^{i,r}_{Bl,m}\|_2 = \|g^i - g^{i,r}\|_2.$$

Therefore, the completeness of the system $\{g^i_{Bl,m}\}^{i=1,\dots,L}_{l,m\in\mathbb{Z}^n}$ and (5.9) imply that, for every $f \in L^2(\mathbb{R}^n)$,

$$\lim_{r\to\infty} \|P_{S^r} f - f\|_2 = 0.$$

Thus, by Lemma 2.7 it follows that

$$\lim_{j\to\infty} \sigma_{S^{r_j}}(\xi) = 1 \quad \text{for a.e. } \xi \tag{5.10}$$

for some subsequence $\{r_j\}_{j\in\mathbb{N}}$. For every $j \in \mathbb{N}$, the functions $g^{1,r_j}, \dots, g^{L,r_j}$ are band-limited and so by (5.8) we have

$$\sigma_{S^{r_j}}(\xi) = \sum_{l\in\mathbb{Z}^n} \sum_{\psi\in\Psi_j} |\hat{\psi}(\xi - Bl)|^2,$$

where $\Psi_j \subset L^2(\mathbb{R}^n)$ has cardinality $\le L$ and $\|\psi\|_2 \le 1$ for $\psi \in \Psi_j$. This equality together with (5.10) gives us

$$\lim_{j\to\infty} \sum_{l\in\mathbb{Z}^n} \sum_{\psi\in\Psi_j} |\hat{\psi}(\xi - Bl)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Integrating this limit equality over the cube $B([0,1]^n)$ yields $|\det B| \le L$. Indeed, by Fatou's lemma we have

$$\begin{aligned} |\det B| &= \int_{B([0,1]^n)} \lim_{j\to\infty} \sum_{l\in\mathbb{Z}^n} \sum_{\psi\in\Psi_j} |\hat{\psi}(\xi - Bl)|^2 \, d\xi \\ &\le \liminf_{j\to\infty} \int_{B([0,1]^n)} \sum_{l\in\mathbb{Z}^n} \sum_{\psi\in\Psi_j} |\hat{\psi}(\xi - Bl)|^2 \, d\xi = \liminf_{j\to\infty} \sum_{\psi\in\Psi_j} \|\psi\|_2^2 \le L, \end{aligned}$$

which concludes the proof of Theorem 5.1. □

The following example shows that condition (G4) does not hold in general.

EXAMPLE 5.2. We will construct a function $g \in L^2(\mathbb{R})$ such that the corresponding Gabor system $\{g_{l,m}(x) = e^{2\pi lx} g(x-m)\}_{l,m\in\mathbb{Z}}$ fails condition (G4)—that is, such that

$$\bigcap_{l\in\mathbb{Z}} V_l \neq \{0\}, \quad \text{where } V_l = \overline{\operatorname{span}}\{g_{k,m} : k < l,\ k, m \in \mathbb{Z}\}. \tag{5.11}$$

Let $g \in L^2(\mathbb{R})$ be such that $g(x) \neq 0$ if and only if $x \in (0,1)$, and also let $\int_0^1 \log|g(x)|\, dx = -\infty$. Then, by a classical result of Helson [He1, pp. 13, 21], the system of functions $\{e^{2\pi kx} g(x) : k \in \mathbb{N}\}$ is complete in $L^2(0,1) := \{f \in L^2(\mathbb{R}) : \operatorname{supp} f \subset (0,1)\}$. As an immediate consequence,

$$\overline{\operatorname{span}}\{e^{2\pi kx} g(x) : k < l\} = L^2(0,1) \quad \text{for any } l \in \mathbb{Z}.$$

Therefore, $V_l = L^2(\mathbb{R})$ for any $l \in \mathbb{Z}$, and (5.11) holds. This example shows that we cannot expect (G4) to hold unless g satisfies some additional hypotheses (e.g., that g be band-limited).

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