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A Characterization of Dimension Functions of Wavelets

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This paper is devoted to the study of the dimension functions of (multi)wavelets, which was introduced and investigated by P. Auscher in 1995 (*J. Geom. Anal.* **5**, 181–236). Our main result provides a characterization of functions which are dimension functions of a (multi)wavelet. As a corollary, we obtain that for every function D that is the dimension function of a (multi)wavelet, there is a minimally supported frequency (multi)wavelet whose dimension function is D. In addition, we show that if a dimension function of a wavelet not associated with a multiresolution analysis (MRA) attains the value K, then it attains all integer values from 0 to K. Moreover, we prove that every expansive matrix which preserves $\mathbb{Z}^N$ admits an MRA structure with an analytic (multi)wavelet.

Key Words: orthonormal (multi)wavelet; MSF (multi)wavelet; dimension function.

1. INTRODUCTION

The *dimension function* of an orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is defined as

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + k))|^2.$$

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The importance of the dimension function was discovered by Lemarié, who used it to prove that certain wavelets are associated with a multiresolution analysis (MRA); see [15, 16]. After that Gripenberg [11] and Wang [22] independently characterized all wavelets associated with an MRA. The well-known characterization establishes that a wavelet is an MRA wavelet if and only if $D_\psi(\xi) = 1$ for almost every $\xi \in \mathbb{R}$. However, Auscher [1] proved something more. His deep theorem shows that the dimension function of a wavelet describes dimensions of certain subspaces of $\ell^2(\mathbb{Z})$; in particular, it is integer valued. Recently, Baggett, Medina, and Merrill [2, 3] observed that the dimension function has one more property, i.e., it satisfies the following *consistency equation*:

$$D_\psi(\xi) + D_\psi(\xi + 1/2) = D_\psi(2\xi) + 1 \qquad \text{a.e.} \tag{1.1}$$

We shall prove that a third condition is always satisfied, namely,

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + k) \geq D_\psi(\xi) \qquad \text{a.e.,} \tag{1.2}$$

where $\Delta = \{\xi \in \mathbb{R} : D_\psi(2^{-j}\xi) \geq 1 \text{ for } j \in \mathbb{N} \cup \{0\}\}$ and $\mathbf{1}_\Delta$ denotes the characteristic function of Δ. It turns out that the three properties together with three obvious ones — that D_ψ is 1-periodic, $\liminf_{n\to\infty} D_\psi(2^{-n}\xi) \geq 1$, and $\int_{-1/2}^{1/2} D_\psi(\xi)\, d\xi = 1$ — fully characterize dimension functions. Baggett and Merrill [4] used the condition (1.2) found in the preliminary version of this paper, which only dealt with the one-dimensional case, to prove a similar characterization for the multiplicity function.

In Section 5 we give a brief study of the collection of all dimension functions. Included are examples of a construction of MSF (minimally supported frequency) wavelets using the MRA dimension function and the Journé dimension function. In Example 5.8 we use an idea due to Madych [17] to prove that for any expansive matrix A which preserves $\mathbb{Z}^N$ there exists an analytic (multi)wavelet associated with an MRA. In Theorem 5.11 we prove that there are no skips in the range of a non-MRA dimension function; that is, if a dimension function not associated with an MRA attains the value K, then it attains all of the integer values from zero to K.

2. PRELIMINARIES

Since our main result holds in greater generality than described in the Introduction, let us review the necessary terminology. For this paper, a *dilation* matrix A will be an expansive matrix which preserves $\mathbb{Z}^N$, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$ and $A\mathbb{Z}^N \subset \mathbb{Z}^N$. The transpose of A is denoted by $B = A^{\mathrm{T}}$. A finite set $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is called an *orthonormal multiwavelet* if the system $\{\psi^l_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^N,\ l = 1, \ldots, L\}$ is an orthonormal basis for $L^2(\mathbb{R}^N)$, where for $\psi \in L^2(\mathbb{R}^N)$ we use the convention

$$\psi_{j,k} = |\det A|^{j/2}\psi(A^j x - k) \qquad \text{for all } j \in \mathbb{Z}, k \in \mathbb{Z}^N.$$

If a multiwavelet Ψ consists of a single element ψ then we say that ψ is a *wavelet*. The following result establishes a characterization of orthonormal multiwavelets (see [6, 7, 10, 11]).

THEOREM 2.1. *A subset* $\Psi = \{\psi^1, \ldots, \psi^L\}$ *is an orthonormal multiwavelet if and only if*

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^l(B^j \xi)|^2 = 1 \qquad a.e.\ \xi \in \mathbb{R}^N, \tag{2.1}$$

$$t_s(\xi) \equiv \sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\psi}^l(B^j \xi) \overline{\hat{\psi}^l(B^j(\xi + s))} = 0 \qquad a.e.\ \xi \in \mathbb{R}^N,\ s \in \mathbb{Z}^N \backslash B\mathbb{Z}^N, \tag{2.2}$$

and $\|\psi^l\| = 1$ *for* $l = 1, \ldots, L$, *where* $B = A^{\mathrm{T}}$.

In the theorem above and throughout the paper the Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx.$$

Denote $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$, which we identify with the set $(-1/2, 1/2]^N$. The $\mathbb{Z}^N$-periodization of a set $E \subset \mathbb{R}^N$ is defined by $E^P = \bigcup_{k \in \mathbb{Z}^N}(E + k)$. The *translation projection* τ is defined on $\mathbb{R}^N$ by $\tau(\xi) = \xi'$, where $\xi' \in \mathbb{T}^N$ and $\xi' - \xi = k$ for some $k \in \mathbb{Z}^N$. In our convention the set of natural numbers $\mathbb{N}$ does not contain zero. The Lebesgue measure of a set $E \in \mathbb{R}^N$ is denoted by $|E|$.

An easy to justify property of the mapping τ is the following.

LEMMA 2.2. *Let* $\tilde{E}$ *be a measurable subset of* $\mathbb{R}^N$. *Then there exists a measurable set* $E \subset \tilde{E}$ *such that* $\tau(E) = \tau(\tilde{E})$ *and* $\tau|_E$ *is injective.*

DEFINITION 2.3. An *MSF (minimally supported frequency) multiwavelet* (of order L) is an orthonormal multiwavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ such that $|\hat{\psi}^l| = \mathbf{1}_{W_l}$ for some measurable sets $W_l \subset \mathbb{R}^N$, $l = 1, \ldots, L$. An MSF multiwavelet of order 1 is simply referred to as an *MSF wavelet.*

The following theorem characterizes all MSF multiwavelets (see [9] for a similar characterization of MSF wavelets). Note that Theorems 2.4 and 2.6 hold without the assumption that the dilation A preserves the lattice $\mathbb{Z}^N$.

THEOREM 2.4. *A set* $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^N)$ *such that* $|\hat{\psi}^l| = \mathbf{1}_{W_l}$ *for* $l = 1, \ldots, L$ *is an orthonormal multiwavelet associated with the dilation* A *if and only if*

$$\sum_{k \in \mathbb{Z}^N} \mathbf{1}_{W_l}(\xi + k) \mathbf{1}_{W_{l'}}(\xi + k) = \delta_{l,l'} \qquad a.e.\ \xi \in \mathbb{R}^N,\ l, l' = 1, \ldots, L, \tag{2.3}$$

$$\sum_{j \in \mathbb{Z}} \sum_{l=1}^{L} \mathbf{1}_{W_l}(B^j \xi) = 1 \qquad a.e.\ \xi \in \mathbb{R}^N, \tag{2.4}$$

where $B = A^{\mathrm{T}}$.

Proof. Suppose $\psi \in L^2(\mathbb{R}^N)$ with $|\hat{\psi}| = \mathbf{1}_W$, for some measurable set W. The set $\{\psi_{0,k} : k \in \mathbb{Z}^N\}$ is an orthonormal family if and only if $\sum_{k \in \mathbb{Z}^N} |\hat{\psi}(\xi + k)|^2 = \sum_{k \in \mathbb{Z}^N} \mathbf{1}_W(\xi + k) = 1$ for a.e. ξ. In this case

$$\overline{\mathrm{span}}\{\psi_{0,k} : k \in \mathbb{Z}^N\} = \{f \in L^2(\mathbb{R}^N) : \mathrm{supp}\, \hat{f} \subset W\},$$

where $\operatorname{supp}\hat{f} := \{\xi \in \mathbb{R}^N : \hat{f}(\xi) \neq 0\}$. Therefore $\{\psi^l_{0,k} : k \in \mathbb{Z}^N, l = 1, \dots, L\}$ is an orthonormal family if and only if

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_{W_l}(\xi + k) = 1 \qquad \text{for } l = 1, \dots, L, \text{ a.e. } \xi \in \mathbb{R}^N,$$

and the W_l's are pairwise disjoint (modulo sets of measure zero), i.e., (2.3). In this case

$$W_0 := \overline{\operatorname{span}}\{\psi^l_{0,k} : k \in \mathbb{Z}^N,\ l = 1, \dots, L\} = \left\{ f \in L^2(\mathbb{R}^N) : \operatorname{supp}\hat{f} \subset \bigcup_{l=1}^{L} W_l \right\},$$

and by scaling for any $j \in \mathbb{Z}$,

$$W_j := \overline{\operatorname{span}}\{\psi^l_{j,k} : k \in \mathbb{Z}^N,\ l = 1, \dots, L\} = \left\{ f \in L^2(\mathbb{R}^N) : \operatorname{supp}\hat{f} \subset B^j \left(\bigcup_{l=1}^{L} W_l \right) \right\}.$$

Therefore, Ψ is a multiwavelet if and only if (2.3) is true and $\bigoplus_{j\in\mathbb{Z}} Wj = L^2(\mathbb{R}^N)$ if and only if (2.3) is true and $\{B^j(\bigcup_{l=1}^L W_l) : j \in \mathbb{Z}\}$ partitions $\mathbb{R}^N$ (modulo sets of measure zero), i.e., (2.4) holds. ∎

DEFINITION 2.5. A set $W \subset \mathbb{R}^N$ is a *multiwavelet set* (of order L) if $W = \bigcup_{l=1}^L W_l$ for some $W_1, \dots, W_L$ satisfying (2.3) and (2.4). A multiwavelet set of order 1 is called a *wavelet set*.

The following theorem characterizes all multiwavelet sets.

THEOREM 2.6. *A measurable set $W \subset \mathbb{R}^N$ is a multiwavelet set of order L if and only if*

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_W(\xi + k) = L \qquad \text{a.e. } \xi \in \mathbb{R}^N, \tag{2.5}$$

$$\sum_{j\in\mathbb{Z}} \mathbf{1}_W(B^j\xi) = 1 \qquad \text{a.e. } \xi \in \mathbb{R}^N. \tag{2.6}$$

Proof. Suppose W is a multiwavelet set of order L. Since $\mathbf{1}_W = \sum_{l=1}^L \mathbf{1}_{W_l}$, Theorem 2.4 implies (2.5) and (2.6). Conversely, suppose W satisfies (2.5) and (2.6). We proceed to define $\{W_l\}_{l=1}^L$ inductively. Let W_1 be a subset of W such that $\tau(W_1) = \mathbb{T}^N$ (modulo sets of measure zero) and $\tau|_{W_1}$ is injective; the existence of W_1 is guaranteed by Lemma 2.2. Suppose that for some $1 \le n < L$ we have defined disjoint sets $W_1, \dots, W_n \subset W$ such that $\tau(W_i) = \mathbb{T}^N$ and $\tau|_{W_i}$ is injective for $i = 1, \dots, n$. Therefore $\sum_{k\in\mathbb{Z}^N} \mathbf{1}_R(\xi + k) = L - n$ a.e. ξ, where $R = W \setminus \bigcup_{i=1}^n W_i$, and by Lemma 2.2 we can find $W_{n+1} \subset R$, such that $\tau(W_{n+1}) = \mathbb{T}^N$ and $\tau|_{W_{n+1}}$ is injective. Therefore, we have a disjoint partition of $W = \bigcup_{l=1}^L W_l$ such that

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_{W_l}(\xi + k) = 1 \qquad \text{a.e. } \xi, \text{ for } l = 1, \dots, L. \tag{2.7}$$

Thus, (2.3) holds. Equation (2.4) is an immediate consequence of (2.6) and the pairwise disjointness of the W_l's. ∎

For a finite subset subset $F = \{f^1, \ldots, f^L\} \subset L^2(\mathbb{R}^N)$ and a dilation A, define the $\mathbb{Z}^N$-periodic function D_F by

$$D_F(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} |\hat{f}^l(B^j(\xi + k))|^2, \tag{2.8}$$

where $B = A^{\mathrm{T}}$. The following fact implies that D_F is finite for a.e. $\xi \in \mathbb{R}^N$.

PROPOSITION 2.7. *If $F \subset L^2(\mathbb{R}^N)$ is finite, then*

$$\|D_F\|_{L^1(\mathbb{T}^N)} = \frac{1}{q-1} \sum_{l=1}^{L} \|f^l\|^2_{L^2(\mathbb{R}^N)},$$

where $q = |\det B|$.

Proof. We have

$$\int_{\mathbb{T}^N} D_F(\xi)\, d\xi = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{T}^N} |\hat{f}^l(B^j(\xi + k))|^2\, d\xi = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \int_{\mathbb{R}^N} |\hat{f}^l(B^j \xi)|^2\, d\xi$$

$$= \sum_{l=1}^{L} \|\hat{f}^l\|_2^2 \sum_{j=1}^{\infty} |\det B|^{-j} = \frac{1}{q-1} \sum_{l=1}^{L} \|\hat{f}^l\|_2^2 = \frac{1}{q-1} \sum_{l=1}^{L} \|f^l\|_2^2. \qquad \blacksquare$$

DEFINITION 2.8. The *dimension function* of a multiwavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ associated with a dilation A is the function D_Ψ given by (2.8); that is,

$$D_\Psi(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} |\hat{\psi}^l(B^j(\xi + k))|^2,$$

where $B = A^{\mathrm{T}}$.

A priori, it is not obvious from the definition that D_Ψ has integer values. It is also not immediate why D_Ψ is referred to as a dimension function.

THEOREM 2.9. *Suppose $\Psi = \{\psi^1, \ldots, \psi^L\}$ is a multiwavelet. Then $D_\Psi(\xi)$ is a nonnegative integer for a.e. $\xi \in \mathbb{R}^N$.*

This result was proved by Auscher [1] for wavelets in $L^2(\mathbb{R})$. It is not a surprise that it holds in a much more general setting. In the following argument it also becomes clear why D_Ψ is called a dimension function.

Proof. For $l = 1, \ldots, L$, $j \geq 1$, and a.e. $\xi \in \mathbb{T}^N$ we define a vector

$$\Psi_{l,j}(\xi) = (\hat{\psi}^l(B^j(\xi + k)))_{k \in \mathbb{Z}^N}$$

which belongs to $l^2(\mathbb{Z}^N)$. The key observation is that the vectors $\Psi_{k,p}(\xi)$ satisfy the reproducing formula (see [7, Lemma 4.2] or [13, Chap. 7, Eq. (3.5)] for the case of wavelets in $L^2(\mathbb{R})$)

$$\Psi_{k,p}(\xi) = \sum_{l=1}^{L} \sum_{j\geq 1} \langle \Psi_{k,p}(\xi), \Psi_{l,j}(\xi)\rangle_{l^2(\mathbb{Z}^N)} \Psi_{l,j}(\xi) \qquad \text{a.e. } \xi \in \mathbb{T}^N,$$

for $k = 1, \ldots, L$, $p \geq 1$. As we pointed out before, Proposition 2.7 implies that $D_\Psi(\xi)$ is finite a.e. Moreover, a simple calculation shows that $D_\Psi(\xi) = \sum_{l=1}^{L} \sum_{j\geq 1} \|\Psi_{l,j}(\xi)\|^2_{l^2(\mathbb{Z}^N)}$, which allows us to apply Auscher's geometrical lemma [13, Chap. 7, Lemma 3.7] to get

$$D_\Psi(\xi) = \sum_{l=1}^{L} \sum_{j\geq 1} \|\Psi_{l,j}(\xi)\|^2 = \dim \overline{\operatorname{span}}\{\Psi_{l,j}(\xi) : j \geq 1,\ l = 1, \ldots, L\}$$

for a.e. $\xi \in \mathbb{T}^N$. ■

In [23] Weber proved that the multiplicity function introduced in [3] is equal to the dimension function in the case of single wavelets on $\mathbb{R}$. It is reasonable to suspect that this result can be extended to the case of the dimension function of multiwavelets on $\mathbb{R}^N$.

Before we present our main result, let us develop a notion of a generalized scaling set which we use to characterize dimension functions.

3. GENERALIZED SCALING SETS

One of the main features in the theory of (multi)wavelets is the idea of multiresolution analysis introduced by Mallat [18]. We say that a multiwavelet $\Psi \subset L^2(\mathbb{R}^N)$ is an *MRA multiwavelet* if there exists a function $\varphi \in L^2(\mathbb{R}^N)$ such that its integer translations form an orthonormal basis of the space $V_0 := \bigoplus_{j<0} W_j$, where $W_j := \overline{\operatorname{span}}\{\psi^l_{j,k} : k \in \mathbb{Z}^N,\ l = 1, \ldots, L\}$. The function φ is called a *scaling function*. It is easy to check that if Ψ is an MSF multiwavelet which comes from a multiresolution analysis, then its scaling function φ satisfies $|\hat{\varphi}| = \mathbf{1}_S$ for some measurable set $S \subset \mathbb{R}^N$. Conversely, if φ is a scaling function of some MRA, such that $|\hat{\varphi}| = \mathbf{1}_S$, then there is an MSF multiwavelet associated with this MRA. Such a set S is called a *scaling set*.

In [7] Calogero extended Gripenberg's result by showing that a multiwavelet Ψ comes from an MRA if and only if $D_\Psi(\xi) = 1$ a.e. This implies that the multiwavelets with nontrivial dimension function cannot be constructed by means of multiresolution analysis. One importance of a multiresolution structure is based on the fact that a MRA multiwavelet can be easily recovered from its scaling function. It turns out that a similar property is true for all MSF multiwavelets. In fact, in this case $W_0 = \{f \in L^2(\mathbb{R}^N) : \operatorname{supp} \hat{f} \subset W\}$, where W is a multiwavelet set, and $V_0 = \{f \in L^2(\mathbb{R}^N) : \operatorname{supp} \hat{f} \subset \bigcup_{j=1}^{\infty} B^{-j} W\}$. Therefore the role of a scaling function can be played by $\mathbf{1}_S \in V_0$, where $S = \bigcup_{j=1}^{\infty} B^{-j} W$, because then the set W can be easily obtained from S, that is, $W = BS \setminus S$. These ideas can be formulated precisely as follows.

DEFINITION 3.1. For fixed $L \in \mathbb{N}$, a set $S \subset \mathbb{R}^N$ is called a *generalized scaling set* (of order L) associated with a dilation A if $|S| = L/(q-1)$ and $BS \setminus S$ is a multiwavelet set (of order L) associated with the dilation A, where $B = A^{\mathrm{T}}$ and $q = |\det A|$.

An equivalent definition can be stated as follows.

PROPOSITION 3.2. *A set $S \subset \mathbb{R}^N$ is a generalized scaling set (of order L) if and only if $S = \bigcup_{j=1}^{\infty} B^{-j} W$ for some multiwavelet set W (of order L).*

Proof. Let us assume that $S = \bigcup_{j=1}^{\infty} B^{-j} W$ for some multiwavelet set W. From (2.6) it follows that the union is disjoint; therefore, $BS \setminus S = W$. Moreover, (2.5) implies that $|W| = L$; hence, $|S| = L/(q-1)$.

To prove the other implication, denote $BS \setminus S$ by W and observe that since $|W| = L$ we must have $S \subset BS$. From this it easily follows that $\bigcup_{j=1}^{\infty} B^{-j} W \subset S$, but since both these sets have the same measure they must be equal. ■

In [19] a simple characterization of scaling sets in $\mathbb{R}$ is given. Theorem 2.1 of [3] can be viewed as a similar characterization of generalized scaling sets of order 1 in $\mathbb{R}^N$. The proof presented there uses methods of abstract harmonic analysis (in particular, the multiplicity function of projection valued measures). Let us present an elementary proof of this theorem extended to the case of generalized scaling sets of order L.

THEOREM 3.3. *A measurable set $S \subset \mathbb{R}^N$ is a* generalized scaling set *(of order L) if and only if*

(i) $|S| = L/(q-1)$,
(ii) $S \subset BS$,
(iii) $\lim_{n\to\infty} \mathbf{1}_S(B^{-n}\xi) = 1$ *for a.e.* $\xi \in \mathbb{R}^N$,
(iv) $\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = D(B\xi) + L$ *a.e., where* $D(\xi) = \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k)$ *and* $\mathcal{D}$ *is the set of q representatives of distinct cosets of* $\mathbb{Z}^N / B\mathbb{Z}^N$.

Proof. Let us prove that conditions (i)–(iv) are necessary. The first one is guaranteed by definition; the second one follows easily from Proposition 3.2. Moreover, since $S = \bigcup_{j=1}^{\infty} B^{-j} W$ for some multiwavelet set W and the union is disjoint, we can write $\mathbf{1}_S(B^{-n}\xi) = \sum_{j=-n+1}^{\infty} \mathbf{1}_W(B^j\xi)$ and use (2.6) to obtain that the limit as $n \to \infty$ is 1, which establishes (iii). A simple calculation shows that

$$\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = \sum_{d\in\mathcal{D}} \sum_{k\in\mathbb{Z}^N} \sum_{j=1}^{\infty} \mathbf{1}_W(B^j(\xi + B^{-1}d + k))$$
$$= \sum_{k\in\mathbb{Z}^N} \sum_{j=0}^{\infty} \mathbf{1}_W(B^j(B\xi + k)) = D(B\xi) + \sum_{k\in\mathbb{Z}^N} \mathbf{1}_W(B\xi + k), \quad (3.1)$$

therefore (iv) follows from (2.5).

To prove sufficiency define W as $BS \setminus S$. It is easy to see that W and $B^j W$ are disjoint for every $j \in \mathbb{N}$. In fact, $B^j W = B^{j+1} S \setminus B^j S$ and, by (ii), $W \subset B^j S$, which together imply $W \cap B^j W = \emptyset$. In this way we obtain that $B^j W \cap B^k W = \emptyset$ for $j, k \in \mathbb{Z}$, $j \neq k$. This allows us to check that $S = \bigcup_{j=1}^{\infty} B^{-j} W$. Indeed, by (ii) we have $\bigcup_{j=1}^{\infty} B^{-j} W \subset S$, but both these sets have the same measure; therefore, they are equal. This shows that $\mathbf{1}_S(\xi) = \sum_{j=1}^{\infty} \mathbf{1}_W(B^j\xi)$, so (iii) implies that W satisfies (2.6). Using (iv) together with (3.1), we obtain that (2.5) is fulfilled as well; therefore, W is a multiwavelet set. ■

Remark. Suppose that a measurable set S satisfies conditions (i)–(iv) of Theorem 3.3. Then, as we have shown, $W = BS \setminus S$ is a multiwavelet set and S decomposes into a disjoint sum $S = \bigcup_{j=1}^{\infty} B^{-j} W$. By Definition 2.5 we can find disjoint sets W_l, $l = 1, \ldots, L$, such that $W = \bigcup_{l=1}^{L} W_l$ and $\Psi = \{\mathbf{1}_{W_1}, \ldots, \mathbf{1}_{W_L}\}$ is an MSF multiwavelet. Hence we conclude that

$$D_\Psi(\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N} \mathbf{1}_{W_l}(B^j(\xi+k)) = \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi+k) \qquad \text{for a.e. } \xi\in\mathbb{R}^N. \tag{3.2}$$

As we have mentioned before, a multiwavelet Ψ can be associated with an MRA if and only if $D_\Psi(\xi)=1$ a.e. Therefore, it follows that if Ψ is an MSF multiwavelet, then it comes from an MRA if and only if $\sum_{k\in\mathbb{Z}^N}\mathbf{1}_S(\xi+k)=1$ a.e. In this way we obtain the following.

COROLLARY 3.4. *A measurable set $S\subset\mathbb{R}^N$ is a scaling set if and only if it is a generalized scaling set of order $q-1$ and $\sum_{k\in\mathbb{Z}^N}\mathbf{1}_S(\xi+k)=1$ a.e.*

4. MAIN RESULT

Our key contribution to the study of the dimension function of a multiwavelet Ψ associated with a dilation A was noticing that its support must be big enough in the sense of condition (1.2) (that is,

$$\sum_{k\in\mathbb{Z}^N}\mathbf{1}_\Delta(\xi+k) \ge D_\Psi(\xi) \qquad \text{for a.e. } \xi\in\mathbb{R}^N,$$

where $\Delta=\{\xi\in\mathbb{R}^N : D_\Psi(B^{-j}\xi)\ge 1 \text{ for } j\in\mathbb{N}\cup\{0\}\}$, $B=A^{\mathrm{T}}$) and proving that, together with already known conditions, this condition characterizes dimension functions. Before we proceed further let us prove a technical lemma which converts this condition into a statement which is more convenient for the proof of our main result.

LEMMA 4.1. *Let B be a dilation and $D\colon \mathbb{R}^N\to\mathbb{N}\cup\{0\}$ be a measurable $\mathbb{Z}^N$-periodic function which satisfies*

$$\sum_{k\in\mathbb{Z}^N}\mathbf{1}_\Delta(\xi+k) \ge D(\xi) \qquad \text{for a.e. } \xi\in\mathbb{R}^N, \tag{4.1}$$

where $\Delta=\{\xi\in\mathbb{R}^N : D(B^{-j}\xi)\ge 1$ for $j\in\mathbb{N}\cup\{0\}\}$. Let $A_j=\{\xi\in\mathbb{T}^N : D(\xi)\ge j\}$ for $j\in\mathbb{N}$, and let $\{S_i\}_{i=1}^n$, where $n\in\mathbb{N}$ is fixed, be a collection of measurable sets such that $\tau(S_i)=A_i$ and $\tau|_{S_i}$ is injective for $i=1,\dots,n$. Then

(a) *there exists a measurable set G such that $\tau(G)=A_1$ and $D(B^{-j}\xi)\ge 1$ for $\xi\in G$ and $j\ge 0$,*

(b) *there exists a measurable set H disjoint from $\bigcup_{i=1}^n S_i$ such that $\tau(H)=A_{n+1}$ and $D(B^{-j}\xi)\ge 1$ for $\xi\in H$ and $j\ge 0$,*

Proof. (a) Recall that $\Delta=\{\xi\in\mathbb{R}^N : D(B^{-j}\xi)\ge 1 \text{ for all } j\ge 0\}$. Define $G=\Delta$. All we have to prove is that $\tau(G)=A_1$. Since the inclusion $\tau(G)\subset A_1$ is obvious, it is enough to check that the opposite inclusion holds. By condition (4.1) we have $\sum_{k\in\mathbb{Z}^N}\mathbf{1}_G(\xi+k)\ge D(\xi)\ge \mathbf{1}_{A_1}(\xi)$, which implies that $A_1\subset G^P$, i.e., $A_1\subset\tau(G)$.

(b) Define $H=(\Delta\setminus\bigcup_{i=1}^n S_i)\cap A_{n+1}^P$. Since $H\cap\bigcup_{i=1}^n S_i=\emptyset$ and $\tau(H)\subset A_{n+1}$, we can see that all we have to do is prove that $A_{n+1}\subset H^P$. By condition (4.1) and our assumption about $\{S_i\}_{i=1}^n$, we have

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_\Delta(\xi+k) \geq D(\xi) \geq \sum_{k\in\mathbb{Z}^N} \sum_{i=1}^{n+1} \mathbf{1}_{A_i}(\xi+k)$$
$$= \sum_{k\in\mathbb{Z}^N} \sum_{i=1}^{n} \mathbf{1}_{S_i}(\xi+k) + \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{A_{n+1}}(\xi+k)$$
$$\geq \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{\bigcup_{i=1}^n S_i}(\xi+k) + \mathbf{1}_{A_{n+1}}(\xi).$$

In this way we obtain

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_{\Delta\setminus\bigcup_{i=1}^n S_i}(\xi+k) \geq \sum_{k\in\mathbb{Z}^N} \mathbf{1}_\Delta(\xi+k) - \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{\bigcup_{i=1}^n S_i}(\xi+k) \geq \mathbf{1}_{A_{n+1}}(\xi),$$

which implies that $A_{n+1} \subset (\Delta\setminus\bigcup_{i=1}^n S_i)^P$, i.e., $A_{n+1} \subset H^P$. ■

As before A denotes some fixed dilation, $B = A^{\mathrm{T}}$, and $q = |\det A| = |\det B|$ is the order of the quotient group $\mathbb{Z}^N/B\mathbb{Z}^N$. Let $\mathcal{D} = \{d_1, \ldots, d_q\}$, where $d_1 = 0$, be representatives of different cosets of $\mathbb{Z}^N/B\mathbb{Z}^N$. The following theorem gives a full characterization of the dimension functions of a multiwavelet.

THEOREM 4.2. *Let D: $\mathbb{R}^N \to \mathbb{N}\cup\{0\}$ be a measurable $\mathbb{Z}^N$-periodic function. Then, D is the dimension function of some multiwavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ associated with a dilation A if and only if the following conditions are satisfied*:

(D1) $\int_{\mathbb{T}^N} D(\xi)\, d\xi = L/(q-1)$,
(D2) $\liminf_{n\to\infty} D(B^{-n}\xi) \geq 1$,
(D3) $\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = D(B\xi) + L$ *a.e.*,
(D4) $\sum_{k\in\mathbb{Z}^N} \mathbf{1}_\Delta(\xi+k) \geq D(\xi)$ *a.e., where* $\Delta = \{\xi\in\mathbb{R}^N : D(B^{-j}\xi) \geq 1$ *for* $j \in \mathbb{N}\cup\{0\}\}$ *and* $B = A^{\mathrm{T}}$.

Proof. Let us begin by proving that the conditions (D1)–(D4) are necessary. To do so we assume that D is the dimension function of some multiwavelet Ψ, i.e.,

$$D(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}^N} |\hat{\psi}^l(B^j(\xi+k))|^2.$$

Then, by Proposition 2.7 we obtain $\int_{\mathbb{T}^N} D(\xi)\, d\xi = 1/(q-1)\sum_{l=1}^L \|\psi^l\|^2 = L/(q-1)$, so (D1) is proven. To see that (D2) is true, let us observe that $D(\xi) \geq \sum_{l=1}^L \sum_{j=1}^\infty |\hat{\psi}^l(B^j\xi)|^2$. Thus, (D2) follows from (2.1). An easy computation similar to (3.1) shows that

$$\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = D(B\xi) + \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^N} |\hat{\psi}^l(B\xi+k)|^2,$$

so since $\sum_{k\in\mathbb{Z}^N} |\hat{\psi}^l(\xi+k)|^2 = 1$ a.e. we obtain (D3). To prove the last condition, let us denote $s(\xi) = \sum_{l=1}^L \sum_{j=1}^\infty |\hat{\psi}^l(B^j\xi)|^2$. By (2.1) it is clear that $s(\xi) \leq 1$; moreover, we have $s(\xi) \geq s(B\xi)$ and $D(\xi) = \sum_{k\in\mathbb{Z}^N} s(\xi+k)$. Therefore all we have to prove is $s(\xi) \leq \mathbf{1}_\Delta(\xi)$ a.e. This is true for $\xi \in \Delta$. On the other hand, if $\xi \notin \Delta$ then $D(B^{-j}\xi) = 0$

for some $j \geq 0$, therefore $s(B^{-j}\xi) = 0$, but $s(B^{-j}\xi) \geq s(\xi)$, so we obtain that $s(\xi) = 0$; i.e., $s \leq \mathbf{1}_\Delta$ holds almost everywhere.

We turn now to proving the sufficiency of the conditions (D1)–(D4). By Theorem 3.3 and the remark following it, it suffices to find a measurable set S such that

(a) $|S| = L/(q-1)$,
(b) $S \subset BS$,
(c) $\lim_{n\to\infty} \mathbf{1}_S(B^{-n}\xi) = 1$ for a.e. $\xi \in \mathbb{R}^N$, and
(d) $D(\xi) = \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k)$.

Denote $A_k = \{\xi \in \mathbb{T}^N : D(\xi) \geq k\}$ and fix any measurable set $Q \subset \mathbb{R}^N$ such that $Q \subset BQ$, $\tau|_Q$ is injective, $\lim_{n\to\infty} \mathbf{1}_Q(B^{-n}\xi) = 1$ a.e, and $D(\xi) \geq 1$ for $\xi \in Q$. Condition (D2) and the fact that B is a dilation imply that $Q = \Delta \cap \bigcap_{j=0}^{\infty} B^j \mathbb{T}^N$ is an example of a set which satisfies these properties.

CLAIM 1. *There is a measurable set $S_1 \subset \mathbb{R}^N$ such that*

(i) $Q \subset S_1$,
(ii) $S_1 \subset BS_1$,
(iii) $\tau|_{S_1}$ *is injective, and*
(iv) $\tau(S_1) = A_1$.

Proof of Claim 1. Let us denote $E_1 = Q$. For $n \in \mathbb{N}$, define

$$\tilde{E}_{n+1} = \left(BE_n \setminus \bigcup_{i=1}^{n} E_i^P \right) \cap A_1^P$$

and let $E_{n+1} \subset \tilde{E}_{n+1}$ be the set guaranteed to exist by Lemma 2.2.

Define $S_1 = \bigcup_{i=1}^{\infty} E_i$. We submit that S_1 satisfies the conditions (i)–(iv) above. Indeed, (i) is obvious from the definition. To see (ii), note that by the construction $E_1 \subset BE_1$ and $E_n \subset BE_{n-1}$ for all $n \geq 2$. Therefore, $BS_1 = \bigcup_{i=1}^{\infty} BE_i \supset \bigcup_{i=1}^{\infty} E_i = S_1$.

To see (iii), if $\xi_1, \xi_2 \in S_1$ and $\tau(\xi_1) = \tau(\xi_2)$, then there exist $j, k \in \mathbb{N}$ such that $\xi_1 \in E_j$ and $\xi_2 \in E_k$. Without loss of generality, $j \leq k$. If $j < k$, then $\xi_2 \notin E_j^P$, which contradicts $\tau(\xi_1) = \tau(\xi_2)$. Therefore, $j = k$; however, $\tau|_{E_j}$ is injective, so $\xi_1 = \xi_2$.

We now turn to proving (iv). By the definition of S_1 we have $\tau(S_1) \subset A_1$. It remains to show that $A_1 \subset \tau(S_1)$. By the condition (D4) and Lemma 4.1 there is a set $G \subset \mathbb{R}^N$ such that $\tau(G) = A_1$, and $D(B^{-j}\xi) \geq 1$ for every $j \geq 0$ and every $\xi \in G$.

SUBCLAIM. *For $j \geq 1$ we have $G \cap B^j S_1^P \subset B^{j-1} S_1^P$.*

Proof of Subclaim. Let $\xi \in G \cap B^j S_1^P$. Since $B^{-j}\xi + l \in S_1$ for some $l \in \mathbb{Z}^N$, it follows that $B^{-j}\xi + l \in E_m$ for some $m \in \mathbb{N}$. Then $\xi' := B_{-j+1}\xi + Bl \in BE_m$, and by the definition of G, $D(\xi') \geq 1$, so $\xi' \in BE_m \cap A_1^P$. Recall that $\tilde{E}_{m+1} = (BE_m \setminus \bigcup_{i=1}^m E_i^P) \cup A_1^P$. Therefore, if $\xi' \notin \bigcup_{i=1}^m E_i^P$, then $\xi' \in \tilde{E}_{m+1} \subset S_1^P$, i.e., $\xi \in B^{j-1} S_1^P$. On the other hand, if $\xi' \in E_i^P$ for some $i = 1, \ldots, m$, then again $\xi' \in S_1^P$, so $\xi \in B^{j-1} S_1^P$ and the proof of the subclaim is completed.

To finish the proof of Claim 1, note that from the assumptions we made about Q it follows that $\bigcup_{j=1}^{\infty} B^j Q = \mathbb{R}^N$. Therefore, $G = \bigcup_{j=1}^{\infty} (G \cap B^j Q) \subset \bigcup_{j=1}^{\infty} (G \cap B^j S_1^P)$. By iterating the subclaim, we obtain that $G \cap B^j S_1^P \subset S_1^P$ for $j \geq 1$. So, $G \subset S_1^P$, i.e., $\tau(G) \subset \tau(S_1)$, and since $\tau(G) = A_1$ the proof of Claim 1 is finished.

We continue defining the set S. Suppose that there exist sets $S_1, \dots, S_n$ such that if we define $P_i = \bigcup_{j=1}^{i} S_j$ for $i \le n$, then the collections $\{S_i\}_{i=1}^n$ and $\{P_i\}_{i=1}^n$ satisfy $Q \subset P_1$, and

(1) $P_i \subset BP_i$ for $i = 1, \dots, n$,
(2) $\tau|_{S_i}$ is injective for $i = 1, \dots, n$,
(3) $S_i \cap S_j = \emptyset$ for all $i, j = 1, \dots, n$, where $i \ne j$, and
(4) $\tau(S_i) = A_i$ for $i = 1, \dots, n$.

Then, we will construct S_{n+1} such that $S_1, \dots, S_{n+1}$ and $P_1, \dots, P_{n+1}$ satisfy (1)–(4). Recall that $A_{n+1} = \{\xi \in \mathbb{T}^N : D(\xi) \ge n+1\}$.

Define $\tilde{F}_1 = (BP_n \backslash P_n) \cap A_{n+1}^P$. By Lemma 2.2, there is a measurable subset $F_1 \subset \tilde{F}_1$ such that $\tau|_{F_1}$ is injective and $\tau(F_1) = \tau(\tilde{F}_1)$.

For $m \in \mathbb{N}$ define

$$\tilde{F}_{m+1} = \left(BF_m \Big\backslash \bigcup_{i=1}^{m} F_i^P \right) \cap A_{n+1}^P,$$

and let F_{m+1} be the subset of $\tilde{F}_{m+1}$ guaranteed to exist by Lemma 2.2.

Let $S_{n+1} = \bigcup_{m=1}^{\infty} F_m$. We claim that the collections $\{S_i\}_{i=1}^{n+1}$ and $\{P_i\}_{i=1}^{n+1}$ satisfy the four conditions above. For the condition (1) it suffices to show that $S_{n+1} \subset BP_{n+1}$. This follows from the fact that $F_1 \subset BP_n$, while $F_m \subset BF_{m-1} \subset BP_{n+1}$ for $m \ge 2$.

For Condition (2) we must show that $\tau|_{S_{n+1}}$ is injective. Suppose that $\xi_1, \xi_2 \in S_{n+1}$ and $\tau(\xi_1) = \tau(\xi_2)$; then, without loss of generality, for some $j \le k$, $\xi_1 \in F_j$ and $\xi_2 \in F_k$. If $j < k$, then $\xi_2 \notin F_j^P$, which contradicts $\tau(\xi_1) = \tau(\xi_2)$. Therefore, $j = k$; however, $\tau|_{F_k}$ is injective, so $\xi_1 = \xi_2$.

To see (3) it is enough to prove that $S_{n+1} \cap P_n = \emptyset$. As we noted before, for $m \ge 2$ we have $F_m \subset BF_{m-1}$. But since $F_1 \subset BP_n \backslash P_n$, by induction we obtain $F_m \subset B^m P_n \backslash B^{m-1} P_n$ for $m \ge 1$. From (1) with $i = n$, it follows that $P_n \subset B^{m-1} P_n$ for $m \ge 1$. Therefore, for such m we obtain $F_m \cap P_n = \emptyset$, i.e., $S_{n+1} \cap P_n = \emptyset$.

The proof of (4) is more difficult. First note that since $\tau(F_m) \subset A_{n+1}$, we have $\tau(S_{n+1}) \subset A_{n+1}$. For the reverse inclusion, we will find it useful to prove the following.

CLAIM 2. *We have* $A_{n+1}^P \cap BS_{n+1}^P \subset S_{n+1}^P$.

Proof of Claim 2. If $\xi \in A_{n+1}^P \cap BS_{n+1}^P$, then $B^{-1}\xi + k \in F_m$ for some $m \in \mathbb{N}$ and $k \in \mathbb{Z}^N$. Therefore, $\xi' := \xi + Bk \in BF_m$; moreover, since $\xi \in A_{n+1}^P$, we obtain $\xi' \in BF_m \cap A_{n+1}^P$. Now, if $\xi' \notin \bigcup_{i=1}^m F_i^P$ then $\xi' \in \tilde{F}_{m+1} \subset S_{n+1}^P$, i.e., $\xi \in S_{n+1}^P$. On the contrary, if $\xi' \in F_i^P$ for some $i = 1, \dots, m$, then $\xi' \in S_{n+1}^P$ as well, so $\xi \in S_{n+1}^P$, which ends the proof of Claim 2.

Continuing with the proof of Condition (4), we need to show that $A_{n+1} \subset \tau(S_{n+1})$. By Condition (D4) and Lemma 4.1, there is a set $H \subset \mathbb{R}^N$ such that $H \cap P_n = \emptyset$, $\tau(H) = A_{n+1}$, and $D(B^{-j}\xi) \ge 1$ for every $j \ge 0$ and $\xi \in H$. Therefore all we have to prove is that $\tau(H) \subset \tau(S_{n+1})$, i.e., $H \subset S_{n+1}^P$.

We split into two cases; first, we consider all $\xi \in H$ such that, for every $j \ge 0$, $D(B^{-j}\xi) \ge n+1$, i.e., the set $R := H \cap \bigcap_{j=1}^{\infty} B^j A_{n+1}^P$. We will show that $R \subset S_{n+1}^P$. Let $\xi \in R$. As we mentioned before, $\bigcup_{j=1}^{\infty} B^j Q = \mathbb{R}^N$; therefore, $B^{-j}\xi \in Q$ for some $j \ge 1$. Condition (1) implies that $Q \subset P_n$; hence, we can consider $j_0 = \min\{j \in \mathbb{N} : B^{-j}\xi \in P_n\}$. Since $H \cap P_n = \emptyset$, it follows that $B^{-j_0+1}\xi \in BP_n \backslash P_n$. Moreover, since $B^{-j_0+1}\xi \in A_{n+1}^P$, we obtain $B^{-j_0+1}\xi \in (BP_n \backslash P_n) \cap A^P_{n+1} = \tilde{F}_1 \subset S^P_{n+1}$. In this way we prove that $R \subset \bigcup_{k=0}^\infty B^k S^P_{n+1}$. Therefore,

$$R = \bigcup_{k=0}^{\infty} (R \cap B^k S^P_{n+1}) \subset \bigcup_{k=0}^{\infty} \left(H \cap \bigcap_{j=0}^{k} B^j A^P_{n+1} \cap B^k S^P_{n+1} \right).$$

But Claim 2 implies that for $k \geq 0$,

$$\bigcap_{j=0}^{k} B^j A^P_{n+1} \cap B^k S^P_{n+1} \subset S^P_{n+1}, \tag{4.2}$$

hence $R \subset S^P_{n+1}$.

The second case deals with the set $C := H \backslash \bigcap_{j=0}^\infty B^j A^P_{n+1}$. We still wish to show that $C \subset S^P_{n+1}$. If $\xi \in C$ then $\xi \in H \subset A^P_{n+1}$, so we can find $j_0 \geq 0$ such that $\xi \in \bigcap_{j=0}^{j_0} B^j A^P_{n+1}$ and $\xi \notin B^{j_0+1} A^P_{n+1}$. To prove that $\xi \in S^P_{n+1}$, it is enough to show that $\xi \in B^{j_0} S^P_{n+1}$ and then to use the formula (4.2) with $k = j_0$. To see why $\xi \in B^{j_0} S^P_{n+1}$ observe that $D(B^{-j_0}\xi) \geq n+1$. Therefore, by the consistency equation, i.e., Condition (D3), we obtain

$$n + L + 1 \leq L + D(B^{-j_0}\xi) = \sum_{d \in \mathcal{D}} D(B^{-j_0-1}\xi + B^{-1}d). \tag{4.3}$$

For each $d \in \mathcal{D}$ set $K(d) = D(B^{-j_0-1}\xi + B^{-1}d)$, then $\tau(B^{-j_0-1}\xi + B^{-1}d) \in \bigcap_{k=1}^{K(d)} A_k$. Moreover, since $d = 0 \in \mathcal{D}$, $\tau(B^{-j_0-1}\xi) \in A_{K(0)}$, and $\xi \notin B^{j_0+1} A^P_{n+1}$, we obtain $K(0) \leq n$. Since $\xi \in H$, it follows that $D(B^{-j_0-1}\xi) \geq 1$, and we obtain $K(0) \geq 1$. By (4) we have $B^{-j_0-1}\xi \in \bigcap_{k=1}^{K(0)} S^P_k$, i.e., $B^{-j_0-1}\xi + p^0_k \in S_k$, where $p^0_k \in \mathbb{Z}^N$ for $k = 1, \ldots, K(0)$ are distinct by (3). For each $d \in \mathcal{D} \backslash \{0\}$ such that $K(d) \neq 0$, by using (4) again we can find distinct $p^d_k \in \mathbb{Z}^N$ such that $B^{-j_0-1}\xi + p^d_k + B^{-1}d \in S_k$, where $k = 1, \ldots, \min(K(d), n)$.

In this way for each $d \in \mathcal{D}$ such that $K(d) \neq 0$ we obtain

$$B^{-j_0}\xi + Bp^d_k + d \in BS_k \subset BP_n \qquad \text{for } k = 1, \ldots, \min(K(d), n). \tag{4.4}$$

We claim that this gives us at least $n+1$ distinct elements of BP_n. If $Bp^d_k + d = Bp^{d'}_{k'} + d'$ for some $k = 1, \ldots, \min(K(d), n)$, $k' = 1, \ldots, \min(K(d'), n)$, then $d - d' \in B\mathbb{Z}^N$, hence $d = d'$. Also, for fixed $d \in \mathcal{D}$, $p^d_k \neq p^d_{k'}$, for $k \neq k'$. What remains to check is that $\sum_{d \in \mathcal{D}} \min(K(d), n) \geq n+1$. Since $1 \leq K(0) \leq n$, Formula (4.3) yields $K(0) + \sum_{d \in \mathcal{D} \backslash \{0\}} \min(K(d), n) \geq n + 1$.

By Property (2) of the induction hypothesis, at least one of the elements given in (4.4) must lie in the complement of P_n. So, for some $k \in \mathbb{Z}^N$, $B^{-j_0}\xi + k \in BP_n \backslash P_n$. In addition, since $B^{-j_0}\xi \in A^P_{n+1}$, we have $B^{-j_0}\xi + k \in A^P_{n+1}$. Therefore, $B^{-j_0}\xi + k \in \tilde{F}_1 \subset S^P_{n+1}$, i.e., $\xi \in B^{-j_0} S^P_{n+1}$, which completes the proof of condition (4).

To recap, we have defined sets $\{S_i\}_{i=1}^\infty$ such that if we set $P_i = \bigcup_{j=1}^i S_j$ then the following conditions hold:

(0) $Q \subset P_1$,
(1) $P_i \subset BP_i$ for $i \in \mathbb{N}$,
(2) $\tau|_{S_i}$ is injective for $i \in \mathbb{N}$,

(3) $S_i \cap S_j = \emptyset$ whenever $i \neq j$, and
(4) $\tau(S_i) = A_i$ for $i \in \mathbb{N}$.

Define $S = \bigcup_{i=1}^{\infty} S_i$. We claim that S satisfies properties (a)–(d) listed above. To show that $S \subset BS$, it suffices to show that $S_i \subset BS$ for every $i \in \mathbb{N}$, but this is immediate from Condition (1).

To show that $\lim_{n\to\infty} \mathbf{1}_S(B^{-n}\xi) = 1$ almost everywhere, we note that $Q \subset S$, and since $\lim_{n\to\infty} \mathbf{1}_Q(B^{-n}\xi) = 1$ we obtain $\lim_{n\to\infty} \mathbf{1}_S(B^{-n}\xi) = 1$.

To prove that $D(\xi) = \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k)$ it suffices to show that the equality holds for all $\xi \in \mathbb{T}^N$. By Condition (3) we have $\mathbf{1}_S = \sum_{i=1}^{\infty} \mathbf{1}_{S_i}$, therefore

$$D'(\xi) := \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k) = \sum_{k\in\mathbb{Z}^N} \sum_{i=1}^{\infty} \mathbf{1}_{S_i}(\xi + k).$$

It follows from (2) that $\sum_{k\in\mathbb{Z}^N} \mathbf{1}_{S_i}(\xi + k) = \mathbf{1}_{\tau(S_i)}(\xi)$ for $\xi \in \mathbb{T}^N$. Therefore, using (4) we obtain (for $\xi \in \mathbb{T}^N$)

$$D'(\xi) = \sum_{i=1}^{\infty} \mathbf{1}_{\tau(S_i)}(\xi) = \sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\xi) = D(\xi).$$

Finally, we show that $|S| = L/(q-1)$. By Condition (D1) we have

$$|S| = \int_{\mathbb{R}^N} \mathbf{1}_S(\xi)\, d\xi = \int_{\mathbb{T}^N} \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k)\, d\xi = \int_{\mathbb{T}^N} D(\xi)\, d\xi = L/(q-1),$$

which completes the proof of the theorem. ■

An immediate implication of the proof of Theorem 4.2 is the following,

COROLLARY 4.3. *If D is a dimension function of some multiwavelet then there exists an MSF multiwavelet Ψ of the same order such that $D_\Psi = D$.*

Theorem 4.2 gives us an algorithm for constructing multiwavelet sets if the dimension function is given.

ALGORITHM 4.4. Assume that D is a function given which satisfies the assumptions of Theorem 4.2 for some $L \in \mathbb{N}$. Denote $A_k = \{\xi \in \mathbb{T}^N : D(\xi) \geq k\}$.

1. Fix a measurable set $Q \subset \mathbb{R}^N$ such that $Q \subset BQ$, $\lim_{n\to\infty} \mathbf{1}_Q(B^{-n}\xi) = 1$ a.e., $\tau|_Q$ is injective, and $D(\xi) \geq 1$ for $\xi \in Q$.
2. Let $E_1 = Q$. For $m \in \mathbb{N}$ define $\tilde{E}_{m+1} = (BE_m \backslash \bigcup_{i=1}^{m} E_i^P) \cap A_1^P$ and choose any measurable $E_{m+1} \subset \tilde{E}_{m+1}$ such that $\tau(E_{m+1}) = \tau(\tilde{E}_{m+1})$ and $\tau|_{E_{m+1}}$ is injective. Let $S_1 = \bigcup_{m=1}^{\infty} E_m$.
3. If S_i are constructed for $1 \leq i \leq n$, let $P_n = \bigcup_{i=1}^{n} S_i$. Define $\tilde{F}_1 = (BP_n \backslash P_n) \cap A_{n+1}^P$ and $\tilde{F}_{m+1} = (BF_m \backslash \bigcup_{i=1}^{m} F_i^P) \cap A_{n+1}^P$, where again $F_{m+1} \subset \tilde{F}_{m+1}$ is such that $\tau(F_{m+1}) = \tau(\tilde{F}_{m+1})$ and $\tau|_{F_{m+1}}$ is injective. Let $S_{n+1} = \bigcup_{m=1}^{\infty} F_m$.
4. Let $S = \bigcup_{n=1}^{\infty} S_n$, then $W = BS \backslash S$ is a multiwavelet set (of order L) with dimension function equal to D.

If the function D is bounded by n then $S_i = \emptyset$ for $i > n$. In particular, if we wish to construct MRA multiwavelet sets (i.e., $D \equiv 1$) the algorithm is much simpler.

5. EXAMPLES AND REMARKS

In the first part of this section we will restrict our attention to dimension functions of a single wavelet in $\mathbb{R}$. Let us denote by $\mathbb{D}$ the set of all functions which are dimension functions of some wavelet associated with dilation $A = 2$ in dimension $N = 1$. Theorem 4.2 provides some information about $\mathbb{D}$, but we are not aware of a general constructive procedure which would allow us to produce all dimension functions. As a result, further investigation of $\mathbb{D}$ relies on other techniques and in studying examples. Below, we present several dimension functions and provide examples which illustrate the construction of wavelet sets via Algorithm 4.4.

EXAMPLE 5.1. Wavelet Sets with Dimension Function Equal to 1. In the MRA case, the dimension function is identically 1. We illustrate how to construct MSF wavelets with this dimension function using Algorithm 4.4. If we choose $Q = [-1/2, 1/2]$, then $Q = S_1$, and the MSF wavelet we obtain is the Shannon wavelet, $\check{\mathbf{1}}_{[-1,-1/2)\cup[1/2,1)}$ ($\check{\mathbf{1}}_S$ is denotes the inverse Fourier transform of $\mathbf{1}_S$). If we choose $Q = [-a, 1-a]$ (for $0 < a < 1$), then we obtain $S_1 = Q$ and the MSF wavelet $\check{\mathbf{1}}_{[-2a,-a)\cup[1-a,2-2a)}$.

A nontrivial example of our construction is obtained by taking $Q = [-\frac{1}{8}, \frac{1}{4}] \cup [\frac{3}{8}, \frac{1}{2}]$. Then, we can choose $S_1 = Q \cup [\frac{1}{4}, \frac{3}{8}] \cup [\frac{3}{4}, \frac{7}{8}] \cup [\frac{3}{2}, \frac{7}{4}]$. The MSF wavelet we obtain has support $W = [-\frac{1}{4}, -\frac{1}{8}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{7}{8}, 1] \cup [3, \frac{7}{2}]$.

EXAMPLE 5.2. Wavelet Sets with the Journé Dimension Function. A nontrivial example of a dimension function is given by the Journé wavelet

$$D_\psi(\xi) = \begin{cases} 2 & \text{for } \xi \in [-\frac{1}{7}, \frac{1}{7}] \\ 1 & \text{for } \xi \in [-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{1}{7}, -\frac{2}{7}] \cup [\frac{1}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}] \\ 0 & \text{for } \xi \in [-\frac{2}{7}, -\frac{3}{7}] \cup [\frac{2}{7}, \frac{3}{7}], \end{cases}$$

where $\hat{\psi} = \mathbf{1}_{[-16/7,-2]\cup[-1/2,-2/7]\cup[2/7,1/2]\cup[2,16/7]}$ (see Fig. 1).

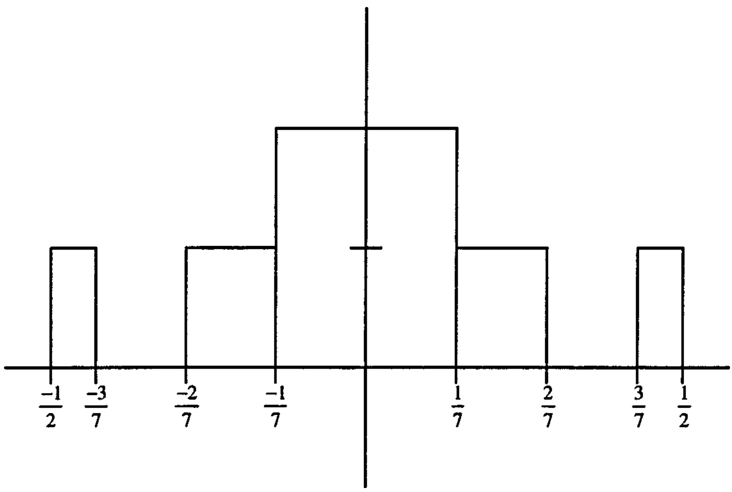

FIG. 1. Journé dimension function.

We proceed to construct an MSF wavelet with the dimension function above via Algorithm 4.4. We can choose $Q = [-\frac{2}{7}, \frac{2}{7}]$, and S_1 can be chosen to be $Q \cup [-\frac{1}{2}, -\frac{3}{7}] \cup [\frac{3}{7}, \frac{1}{2}]$. Then S_2 can be chosen to be $[-1, -\frac{6}{7}] \cup [\frac{6}{7}, 1]$. These choices lead to the MSF wavelet with support

$$W_1 = \left[-2, -\tfrac{12}{7}\right] \cup \left[-\tfrac{4}{7}, -\tfrac{1}{2}\right] \cup \left[-\tfrac{3}{7}, -\tfrac{2}{7}\right] \cup \left[\tfrac{2}{7}, \tfrac{3}{7}\right] \cup \left[\tfrac{1}{2}, \tfrac{4}{7}\right] \cup \left[\tfrac{12}{7}, 2\right].$$

We also could have chosen $S_1 = Q \cup [-\frac{4}{7}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{4}{7}]$ and $S_2 = 2(S_1 \backslash Q)$, which would have yielded the usual Journé wavelet.

For the choice of $Q = [-\frac{1}{7}, \frac{2}{7}]$ we can get $S_1 = Q \cup [-\frac{2}{7}, \frac{1}{7}] \cup [\frac{3}{7}, \frac{4}{7}]$ and $S_2 = [\frac{6}{7}, \frac{8}{7}]$. In this case, we obtain the wavelet set

$$W_2 = \left[-\tfrac{4}{7}, -\tfrac{2}{7}\right] \cup \left[\tfrac{2}{7}, \tfrac{3}{7}\right] \cup \left[\tfrac{12}{7}, \tfrac{16}{7}\right].$$

EXAMPLE 5.3. A Nonsymmetric Dimension Function. Another nontrivial example of a dimension function which is bounded by two can be obtained by computing the dimension function of an MSF wavelet with support $W = [-\frac{4}{3}, -1] \cup [-\frac{1}{2}, -\frac{1}{3}] \cup [\frac{1}{5}, \frac{1}{3}] \cup [\frac{4}{3}, \frac{3}{2}] \cup [3, \frac{16}{5}]$ (this wavelet set is considered by Dai and Larson in [8]). Then for $\hat{\psi} = \mathbf{1}_W$ we have

$$D_\psi(\xi) = \begin{cases} 2 & \text{for } \xi \in [-\frac{1}{3}, -\frac{1}{5}] \cup [\frac{1}{3}, \frac{2}{5}] \\ 1 & \text{for } \xi \in [-\frac{1}{2}, -\frac{2}{5}] \cup [-\frac{1}{5}, \frac{1}{5}] \cup [\frac{2}{5}, \frac{1}{2}] \\ 0 & \text{for } \xi \in [-\frac{2}{5}, -\frac{1}{3}] \cup [\frac{1}{5}, \frac{1}{3}], \end{cases}$$

which yields a nonsymmetric dimension function.

Before proceeding to the next example, we mention a fact that simplifies checking whether a given function satisfies the consistency equation.

FACT 5.4. *Let D be a one-periodic function such that $D(\xi) + D(\xi + 1/2) = D(2\xi) + 1$ for all $\xi \in [-\frac{1}{4}, \frac{1}{4}]$. Then D satisfies the consistency equation for all $\xi \in \mathbb{R}$. Moreover, if D is symmetric, then it suffices to check the consistency equation on* $[0, \frac{1}{4}]$.

Proof. It suffices to show that whenever ξ satisfies $D(\xi) + D(\xi + 1/2) = D(2\xi) + 1$ so does $\xi + 1/2$ and $\xi - 1/2$. However, this is obvious since $D(\xi + 1/2) + D((\xi + 1/2) + 1/2) = D(\xi + 1/2) + D(\xi) = D(2\xi) + 1 = D(2\xi + 1) + 1 = D(2(\xi + 1/2)) + 1$, and analogous reasoning works for $\xi - 1/2$.

To see the moreover statement, if D is symmetric and the consistency equation is satisfied on $[0, \frac{1}{4}]$, then for $\xi \in [-\frac{1}{4}, 0]$ we have $D(\xi) + D(\xi + 1/2) = D(\xi) + D(\xi - 1/2) = D(-\xi) + D(-\xi + 1/2) = D(2(-\xi)) + 1 = D(2\xi) + 1$. ■

In the following example we present an interesting family of dimension functions which appears also in [5].

EXAMPLE 5.5. Dimension Functions with Arbitrarily Large Supremums. Let n be a positive integer. We show that there is a dimension function D_n with $\|D_n\|_\infty = n$. Let $\epsilon = 1/(2^{n+2} - 2)$. Define $T_1 = [-2\epsilon, 2\epsilon]$, and for $i = 2, \ldots, n$ define $T_i = [-2^i\epsilon, 2^{i-1}\epsilon] \cup [2^{i-1}\epsilon, 2^i\epsilon]$. Finally, define $U = [-1/2 + \epsilon, -2^n\epsilon] \cup [2^n\epsilon, 1/2 - \epsilon]$ and

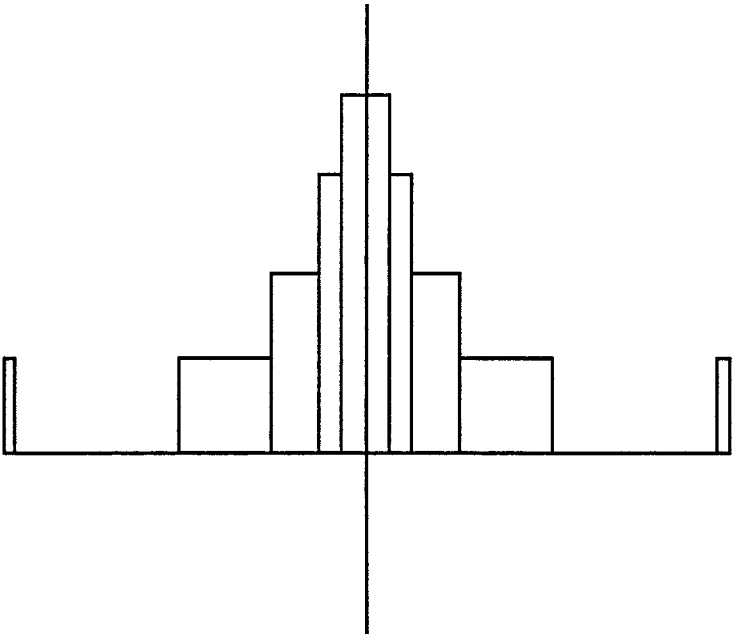

FIG. 2. The function D_4.

$V = [-1/2, -1/2+\epsilon] \cup [1/2-\epsilon, 1/2]$. Then, we define (see Fig. 2)

$$D_n(\xi) = \begin{cases} n-i+1 & \text{for } \xi \in T_i,\ 1 \le i \le n \\ 0 & \text{for } \xi \in U \\ 1 & \text{for } \xi \in V, \end{cases}$$

and extend 1-periodically.

Note that when $n = 1$ we obtain the MRA dimension function, and when $n = 2$ we obtain the Journé dimension function. We turn to showing that D_n satisfies the consistency equation. For $\xi \in [0, \epsilon]$, we have that $D_n(\xi) + D_n(\xi + 1/2) = n + 1$, while $D_n(2\xi) = n$, so the consistency equation is satisfied for those ξ. For $\xi \in [\epsilon, (2^n - 1)\epsilon] \cap T_i$, we have that $-1/2 + \epsilon \le \xi - 1/2 \le -1/2 + (2^n - 1)\epsilon = -2^n\epsilon$ and that $2\xi \in T_{i+1}$. Therefore, $D_n(\xi) + D_n(\xi + 1/2) = D_n(\xi) = n - i + 1 = D_n(2\xi) + 1$. It remains to check that the consistency equation is satisfied for $\xi \in [(2^n - 1)\epsilon, 1/4]$. In this case, $2\xi \in V$ and $\xi - 1/2 \in T_n$. Therefore, $D_n(\xi) + D_n(\xi + 1/2) = 2 = D_n(2\xi) + 1$, as desired.

We now show that the function D_n satisfies the condition (D4) of Theorem 4.2. Since $A_1 \subset \Delta$ we only need to check for $\xi \in A_2$, i.e., $\xi \in T_i$, $i \le n - 1$. (Here, as before, $A_j = \{\xi \in \mathbb{T}^N : D_n(\xi) \ge j\}$.) For such ξ, we have that $\xi + 2^{j-1} \in \Delta$ for all $j \ge n - 1$. Indeed, $\xi + 2^{j-1} \in [2^{j-1}, 2^{j-1} + 2^{n-1}\epsilon] \subset \Delta$. Hence, $\sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + k) = \infty$ for all $\xi \in A_2$, as desired.

We note here that for $n = 3$, an MSF wavelet, the construction in Theorem 4.2 yields $Q = [-\frac{4}{15}, \frac{4}{15}]$, $S_1 = Q \cup [-\frac{1}{2}, -\frac{7}{15}] \cup [\frac{7}{15}, \frac{1}{2}]$, $S_2 = [-\frac{29}{15}, -\frac{28}{15}] \cup [-1, -\frac{14}{15}] \cup [\frac{14}{15}, 1] \cup [\frac{28}{15}, \frac{29}{15}]$, $S_3 = [-2, -\frac{29}{15}] \cup [\frac{29}{15}, 2]$, and the MSF wavelet has support

$$W_3 = \left[-4, -\tfrac{56}{15}\right] \cup \left[-\tfrac{8}{15}, -\tfrac{1}{2}\right] \cup \left[-\tfrac{7}{15}, -\tfrac{4}{15}\right] \cup \left[\tfrac{4}{15}, \tfrac{7}{15}\right] \cup \left[\tfrac{1}{2}, \tfrac{8}{15}\right] \cup \left[\tfrac{56}{15}, 4\right].$$

One might think that letting n go to ∞ in the above example would yield an unbounded dimension function. However, this is not the case. Indeed, the limit function would be $D_\infty(\xi) = i$ for all $\xi \in [-1/2^{i+1}, -1/2^{i+2}] \cup [1/2^{i+2}, 1/2^{i+1}]$. In this limit, the zero set of D_∞ contains $[-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$, so $\Delta = [-\frac{1}{4}, \frac{1}{4}]$, and Condition (D4) of Theorem 4.2 is not satisfied. The function D_∞ also is an example of a function which satisfies Conditions (D1)–(D3) of Theorem 4.2, but not Condition (D4). In particular, Condition (D4) is independent of the other conditions, at least for unbounded dimension functions.

The following example shows that, in spite of the setback above, unbounded dimension functions do exist.

EXAMPLE 5.6. An MSF Wavelet with Unbounded Dimension Function. Recall that for a function $f \in L^2(\mathbb{R})$ we defined $D_f(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{f}(2^j(\xi + k))|^2$. If f is a wavelet, then this is just the usual dimension function. It is clear that whenever $\hat{f} \le \hat{g}$, then $D_f(\xi) \le D_g(\xi)$ almost everywhere.

We will proceed by constructing a set S such that $D_{\check{\mathbf{1}}_S}$ is unbounded and then by showing that S is a subset of some wavelet set W. Since we will have $\mathbf{1}_S \le \mathbf{1}_W$ this will be enough to conclude that $D_{\check{\mathbf{1}}_W}$ is unbounded. Set $S_k = [2^{k-1} + 1/2^{k+2}, 2^{k-1} + 1/2^{k+1})$ and $S = \bigcup_{k=1}^{\infty} S_k$. We show that $D_{\check{\mathbf{1}}_S}$ is unbounded.

CLAIM. *$D_{\check{\mathbf{1}}_S}(\xi) \ge m/2 - 1$ for even $m \in \mathbb{N}$ and $\xi \in [1/2^{m+2}, 1/2^{m+1})$.*

Proof of Claim. If $1/2^{m+2} \le \xi < 1/2^{m+1}$, then for all $l \in \mathbb{N}$,

$$\frac{1}{2^{m+2}} + 2^{l-1} \le \xi + 2^{l-1} < \frac{1}{2^{m+1}} + 2^{l-1},$$

from which it follows that

$$\frac{1}{2^{(m+l)/2+2}} + 2^{(m+l)/2-1} \le 2^{(m-l)/2}(\xi + 2^{l-1}) < \frac{1}{2^{(m+l)/2+1}} + 2^{(m+l)/2-1}.$$

That is, $2^{(m-l)/2}(\xi + 2^{l-1}) \in S_{(m+l)/2}$ whenever $(m+l)/2 \in \mathbb{N}$, i.e., whenever l is even.

We conclude that for even $m \ge 4$ and $\xi \in [1/2^{m+2}, 1/2^{m+1})$,

$$D_{\check{\mathbf{1}}_S}(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\mathbf{1}_S(2^j(\xi + k))| \ge \sum_{j=1}^{m/2-1} |\mathbf{1}_S(2^{(m-2j)/2}(\xi + 2^{2j-1}))| = m/2 - 1.$$

From the claim, it immediately follows that $D_{\check{\mathbf{1}}_S}$ is unbounded. We now turn to showing that there is a wavelet set $W \supset S$. The *dilation projection* d is defined on $\mathbb{R}\backslash\{0\}$ by $d(\xi) = \xi'$, where $\xi' \in [-\frac{1}{2}, \frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2})$ and $\xi'/\xi = 2^k$ for some $k \in \mathbb{Z}$. It is easy to check that the translation projection τ and the dilation projection d satisfy the properties that $\tau|_S$ and $d|_S$ are injections, $\tau(S) \subset [0, \frac{1}{4}]$, and $d(S) \subset [\frac{1}{4}, \frac{1}{2})$.

Set $T = [\frac{1}{4}, \frac{1}{2})\backslash d(S)$ and note that $\tau|_{S\cup T}$ and $d|_{S\cup T}$ are injections and that $d(S \cup T) = [\frac{1}{4}, \frac{1}{2})$, while $\tau(S \cup T) = [0, \frac{1}{2})$. We will use the following theorem, due to Ionascu and Pearcy, with $S \cup T = U$.

THEOREM [14]. *A measurable set $U \subset \mathbb{R}$ is a subset of a wavelet set if and only if the following two conditions hold:*

(i) *There is a set* $A \supset U$ *such that* $\tau|_A$ *and* $d|_A$ *are injective and* $d(A) = [-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2})$.

(ii) *There is a set* $B \supset U$ *such that* $\tau|_B$ *and* $d|_B$ *are injective and* $\tau(B) = [-\frac{1}{2}, \frac{1}{2})$. *Moreover, in this case, there is a wavelet set* W *satisfying* $U \subset W \subset A \cup B$.

Indeed, it is clear that $A = [-\frac{1}{2}, -\frac{1}{4}) \cup U$ satisfies Condition (i). Also, if we set $V = [-\frac{1}{2}, \frac{1}{2})\backslash\tau(U)$, we will show that $B = V - 3$ satisfies Condition (ii). First, note that $V - 3 \subset [-\frac{7}{2}, -\frac{5}{2})$, so $d(V - 3) \subset [-\frac{1}{2}, -\frac{1}{4})$ and $d|_V$ is injective. Therefore, $d|_{(V-3)\cup U}$ is also injective. Finally, note that $\tau(V - 3) = V$, so $\tau|_{(V-3)\cup U}$ is injective and $\tau((V-3)\cup U) = [-\frac{1}{2}, \frac{1}{2})$. Therefore, by Ionascu and Pearcy's result mentioned above, U (and hence S) is contained in a wavelet set, as desired.

The last fact concerning single wavelets in $\mathbb{R}$ which we are going to present implies that there are "many" dimension functions, enough to connect all of the examples we have included so far.

FACT 5.7. *The set* D *is arcwise connected in the* $L^1(\mathbb{T})$ *topology.*

Proof. Let $D, D' \in \mathbb{D}$. By Corollary 4.3 we can find MSF wavelets ψ, ψ' with corresponding wavelet sets W, W' such that $D_\psi = D$ and $D_{\psi'} = D'$. It is easy to justify that $\|D - D'\|_{L^1(\mathbb{T})} \le \|\mathbf{1}_W - \mathbf{1}_{W'}\|^2_{L^2(\mathbb{R})}$. In fact,

$$\begin{aligned}\|D - D'\|_{L^1(\mathbb{T})} &= \int_{-1/2}^{1/2} \left| \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \big(\mathbf{1}_W(2^j(\xi+k)) - \mathbf{1}_{W'}(2^j(\xi+k))\big) \right| d\xi \\ &\le \int_{-1/2}^{1/2} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \big|\mathbf{1}_W(2^j(\xi+k)) - \mathbf{1}_{W'}(2^j(\xi+k))\big| d\xi \\ &= \|\mathbf{1}_W - \mathbf{1}_{W'}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

The arcwise connectivity of $\mathbb{D}$ follows from [20], where it was shown that the set of all characteristic functions of wavelet sets is arcwise connected in the $L^2(\mathbb{R})$ norm. ■

We can also consider $\sqrt{\mathbb{D}} = \{\sqrt{D} : D \in \mathbb{D}\}$ with the $L^2(\mathbb{T})$ norm. The advantage of this comes from the fact that for $f, g \in L^2(\mathbb{R})$ we have $\|\sqrt{D_f} - \sqrt{D_g}\|_{L^2(\mathbb{T})} \le \|f - g\|_{L^2(\mathbb{R})}$, where D_f and D_g are defined as in formula (2.8). In particular, if $\psi_n \to \psi$ in $L^2(\mathbb{R})$, where ψ_n and ψ are orthonormal wavelets, then $\sqrt{D_{\psi_n}} \to \sqrt{D_\psi}$ in $L^2(\mathbb{T})$. The connectivity of $\sqrt{\mathbb{D}}$ in the $L^2(\mathbb{T})$ norm follows again from [20, Corollary 4.3].

In the final part of our paper we consider the multidimensional case. Of course, a detailed study of dimension functions requires fixing a dilation and an order number; this, however, goes beyond the scope of our paper. We offer instead some examples and facts which hold for any dilation.

The following example is devoted to the construction of an analytic MRA multiwavelet for any dilation A. In [17] Madych defined a set Ω, which can be thought of as a scaling set, and showed how to use it to obtain an analytic scaling function. However, the problem of constructing such a set was not considered in his paper. We present here a detailed treatment of this problem using Algorithm 4.4.

EXAMPLE 5.8. An Analytic Multiwavelet. Consider the function $D \equiv 1$, which clearly satisfies (D1)–(D4) of Theorem 4.2 with $L = q - 1$, where $q = |\det A|$, A is some dilation, and $B = A^{\mathrm{T}}$. Algorithm 4.4 yields then a generalized scaling set S of order $q - 1$ satisfying

$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k) = 1$ for a.e. $\xi \in \mathbb{R}^N$; therefore, by Corollary 3.4, S is a scaling set. To start Algorithm 4.4 we must choose a set Q. Note that $Q = \mathbb{T}^N$ will not work for every dilation, since in general we do not have the inclusion $\mathbb{T}^N \subset B\mathbb{T}^N$. Nevertheless, choose as Q the set $\bigcap_{j=0}^{\infty} B^j \mathbb{T}^N$ which is bounded and contains some neighborhood of the origin. We claim that such a choice of Q yields a bounded scaling set S.

Indeed, the sets E_m from Algorithm 4.4 satisfy $E_1 = Q$ and, for $m \in \mathbb{N}$,

$$E_{m+1}^P = (BE_m)^P \Big\backslash \bigcup_{i=1}^{m} E_i^P. \tag{5.1}$$

Hence

$$E_{m+1}^P \subset (BE_m)^P \subset E_1^P \cup \cdots \cup E_{m+1}^P. \tag{5.2}$$

We claim that for any $m \in \mathbb{N}$,

$$(B^m E_1)^P \subset E_1^P \cup \cdots \cup E_{m+1}^P. \tag{5.3}$$

Indeed, (5.3) is true for $m = 1$ by (5.2). Assume, by induction, that (5.3) holds for some m, then

$$\begin{aligned}(B^{m+1}E_1)^P &= (B(B^m E_1))^P \subset (B(E_1^P \cup \cdots \cup E_{m+1}^P))^P = (B(E_1 \cup \cdots \cup E_{m+1}))^P \\ &= (BE_1)^P \cup \cdots \cup (BE_{m+1})^P \subset E_1^P \cup \cdots \cup E_{m+2}^P,\end{aligned}$$

therefore (5.3) is true for $m + 1$, and hence for all $m \in \mathbb{N}$. The set $E_1 = Q$ contains some neighborhood of the origin, therefore $(B^M E_1)^P = \mathbb{R}^N$ for sufficiently large $M \in \mathbb{N}$. Thus by (5.3) $E_1^P \cup \cdots \cup E_{M+1}^P = \mathbb{R}^N$, and by (5.1) $E_n = \emptyset$ for $n \geq M + 2$. Since each E_m is bounded so is $S = \bigcup_{m=1}^{M+1} E_m$.

By the remark following Theorem 3.3 we conclude that the bounded set $W = BS\backslash S$ is a multiwavelet set of order $q - 1$. Definition 2.5 allows us to partition W into sets $W_1, \ldots, W_{q-1}$ so that $\Psi = \{\psi^1, \ldots, \psi^{q-1}\}$ is an MSF multiwavelet, where $\hat{\psi}^l = \mathbf{1}_{W_l}$. The functions ψ^l are C^∞ (even analytic) in the space domain and are associated with an MRA with a smooth scaling function φ, where $\hat{\varphi} = \mathbf{1}_S$. Naturally, φ and ψ^l do not have good decay properties.

In the above example we can consider a special case of a dilation matrix A such that $|\det A| = 2$. This leads to a multiwavelet of order 1; that is, we obtain a single wavelet. The fact that the condition $|\det A| = 2$ is sufficient for obtaining MRA wavelets was already noted by Gu and Han in [12]. It is not clear whether the techniques in [12] can be used to obtain analytic wavelets.

The following example shows the existence of nontrivial dimension functions on $\mathbb{R}^N$.

EXAMPLE 5.9. The "Stairway to Heaven" Dimension Function. Suppose A is a dilation, $B = A^{\mathrm{T}}$, and $q = |\det A| = |\det B|$. Consider a scaling set S associated with the dilation A; that is, a set satisfying the conditions of Theorem 3.3 with $L = q - 1$ and

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(\xi + k) = 1 \qquad \text{a.e. } \xi \in \mathbb{R}^N. \tag{5.4}$$

Example 5.8 guarantees the existence of such a set S. Fix $J \in \mathbb{N}$ and define the function D by

$$D(\xi) = 1 + jJ \qquad \text{if } \xi \in (B^{-j}S \backslash B^{-j-1}S)^P \text{ for some } j \geq 0. \tag{5.5}$$

D is defined for a.e. $\xi \in \mathbb{R}^N$ because the sets $\{B^{-j}S \backslash B^{-j-1}S\}_{j\geq 0}$ form a partition of S, and $\{S + k\}_{k\in\mathbb{Z}^N}$ partitions $\mathbb{R}^N$ modulo sets of measure zero. We will show that D satisfies Conditions (D1)–(D4) of Theorem 4.2 with $L = J + (q-1)$.

The first condition is fulfilled because

$$\begin{aligned}\int_{\mathbb{T}^N} D(\xi)\,d\xi &= \int_S D(\xi)\,d\xi = \sum_{j=0}^{\infty} \int_{B^{-j}S\backslash B^{-j-1}S} D(\xi)\,d\xi \\ &= \sum_{j=0}^{\infty} (1 + jJ)(q^{-j} - q^{-j-1}) = 1 + \sum_{j=1}^{\infty} q^{-j}J = 1 + J/(q-1).\end{aligned}$$

Since $D(\xi) \geq 1$ a.e., Conditions (D2) and (D4) are automatically satisfied. To check the consistency equation we shall show that $\{(B^{-1}S)^P + B^{-1}d\}_{d\in\mathcal{D}}$ partitions $\mathbb{R}^N$, where $\mathcal{D}$ denotes the set of q representatives of different cosets of $\mathbb{Z}^N/B\mathbb{Z}^N$. In fact, by (5.4) we have

$$\begin{aligned}\sum_{d\in\mathcal{D}} \mathbf{1}_{(B^{-1}S)^P}(\xi + B^{-1}d) &= \sum_{d\in\mathcal{D}} \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{B^{-1}S}(\xi + B^{-1}d + k) \\ &= \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{B^{-1}S}(\xi + B^{-1}k) = \sum_{k\in\mathbb{Z}^N} \mathbf{1}_S(B\xi + k) = 1.\end{aligned}$$

Since $\mathbb{R}^N = \bigcup_{j=0}^{\infty} (B^{-j}S \backslash B^{-j-1}S)^P$ it is enough to prove that (D3) holds on $(B^{-j}S \backslash B^{-j-1}S)^P$ for every $j \geq 0$. First, let us consider $\xi \in (B^{-j}S \backslash B^{-j-1}S)^P$, where $j \geq 1$. It is easy to see that then we must have $B\xi \in (B^{-j+1}S \backslash B^{-j}S)^P$. Moreover, since $S \subset BS$, we have $\xi \in (B^{-1}S)^P$ which together with the fact that $\{(B^{-1}S)^P + B^{-1}d\}_{d\in\mathcal{D}}$ partitions $\mathbb{R}^N$ and $S^P = \mathbb{R}^N$, allows us to conclude that $\xi + B^{-1}d \in (S \backslash B^{-1}S)^P$ for $d \in \mathcal{D}\backslash\{0\}$. Therefore, by (5.5),

$$\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = 1 + jJ + (q-1) = 1 + (j-1)J + J + (q-1) = D(B\xi) + L.$$

If $j = 0$ and $\xi \in S \backslash B^{-1}S$, we can choose $d \in \mathcal{D}$ such that $\xi + B^{-1}d \in (B^{-1}S)^P$, since $\{(B^{-1}S)^P + B^{-1}d\}_{d\in\mathcal{D}}$ partitions $\mathbb{R}^N$. Replacing ξ by $\xi + B^{-1}d$ does not affect values of the expressions in (D3), so we can assume that $\xi \in (B^{-1}S)^P$, which corresponds exactly to the case $j \geq 1$. Thus the consistency equation is satisfied a.e., and D is a dimension function of a multiwavelet of order $L = J + (q-1)$.

The dimension function we constructed above is clearly unbounded (note, however, that for $q = 2$ the order L must be at least 2, therefore we cannot use this construction to produce anything similar to Example 5.6). The following fact shows that band limited wavelets have a bounded dimension function.

FACT 5.10. *Suppose $\Psi = \{\psi^1, \ldots, \psi^L\}$ is a multiwavelet and* $\operatorname{supp} \hat{\psi}^l$ *is bounded for each $l = 1, \ldots, L$. Then there is an $n \in \mathbb{N}$ such that $D_\Psi \leq n$.*

Proof. Let $s(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} |\hat{\psi}^l(B^j\xi)|^2$. By the support condition on $\hat{\psi}^l$ we see that $\operatorname{supp} s$ is bounded. Therefore (2.1) implies that $s \le \mathbf{1}_{\operatorname{supp} s}$, and we obtain

$$D_\Psi(\xi) = \sum_{k\in\mathbb{Z}^N} s(\xi+k) \le \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{\operatorname{supp} s}(\xi+k) = \sum_{k\in\mathbb{Z}^N} \mathbf{1}_{k+\operatorname{supp} s}(\xi) \le n,$$

where n denotes the number of distinct k's such that $\mathbb{T}^N \cap (k + \operatorname{supp} s) \ne \emptyset$. ■

As we saw, every dimension function presented in Example 5.5 attains all integer values between zero and its supremum. The following fact shows that the skips in the range of a dimension function of a multiwavelet of order L are at most of the size L. In particular, if a dimension function of a single wavelet not associated with an MRA attains the value M, then it attains all values from zero to M.

THEOREM 5.11. *Let Ψ be an orthonormal multiwavelet (of order L) with dimension function D_Ψ, and let M be an integer $M > L/(q-1)$. If there is a set $C \subset \mathbb{R}^N$ such that its Lebesgue measure $|C| > 0$ and $D_\Psi(\xi) \ge M$ for all $\xi \in C$, then there is a set $R \subset \mathbb{R}^N$ such that $|R| > 0$ and $M > D_\Psi(\xi) \ge M - L$ for all $\xi \in R$.*

Proof. By the consistency equation (see Theorem 4.2(D3))

$$D_\Psi(B\xi) = \sum_{d\in\mathcal{D}} D_\Psi(\xi + B^{-1}d) - L,$$

it follows that $D_\Psi(B\xi) \ge D_\Psi(\xi) - L$ a.e. Suppose that $|\{\xi \in \mathbb{R}^N : M > D_\Psi(\xi) \ge M - L\}| = 0$. Then, the inequality implies that for almost every $\xi \in BC$, $D_\Psi(\xi) \ge M$. Repeating this argument, we obtain $D_\Psi(\xi) \ge M$ for all $\xi \in B^jC$ and all $j \ge 1$. So, by the $\mathbb{Z}^N$-periodicity of D_Ψ, we have that for every $j \ge 1$ and $\xi \in \tau(B^jC)$, $D_\Psi(\xi) \ge M$; that is, $D_\Psi(\xi) \ge M$ for $\xi \in \bigcup_{j=1}^{\infty} \tau(B^jC)$. From the claim below, it follows that $|\bigcup_{j=1}^{\infty} \tau(B^jC)| = 1$, hence

$$\int_{\mathbb{T}^N} D_\Psi(\xi)\,d\xi \ge M \left| \bigcup_{j=1}^{\infty} \tau(B^jC) \right| = M > L/(q-1),$$

which contradicts the fact that $\int_{\mathbb{T}^N} D_\Psi(\xi)\,d\xi = L/(q-1)$ (see Theorem 4.2(D1)). ■

CLAIM. *If $C \subset \mathbb{R}^N$ is a set of positive Lebesgue measure, then $|\bigcup_{j=1}^{\infty} \tau(B^jC)| = 1$.*

Proof. Since the matrix $B\colon \mathbb{R}^N \to \mathbb{R}^N$ preserves the lattice $\mathbb{Z}^N$, it induces a measure preserving endomorphism $\tilde{B}\colon \mathbb{T}^N \to \mathbb{T}^N$. Moreover, $\tilde{B}$ is ergodic by [21, Corollary 1.10.1] because B is expansive. Since $Z = \bigcup_{j=1}^{\infty} \tau(B^jC) = \bigcup_{j=1}^{\infty} \tilde{B}^j \tau(C)$, it follows that $\tilde{B}Z \subset Z$. We have $Z \subset \tilde{B}^{-1}\tilde{B}Z \subset \tilde{B}^{-1}Z$, and since $\tilde{B}$ is measure preserving $Z = \tilde{B}^{-1}Z$ (modulo sets of measure zero). By the ergodicity of $\tilde{B}$ we have that either $|Z| = 0$ or $|Z| = 1$. Since C is of positive measure we must have that $|\tau(Z)| > 0$, hence $|Z| = 1$. ■

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