# MINIMAL GENERATOR SETS FOR FINITELY GENERATED SHIFT-INVARIANT SUBSPACES OF $\mathbf{L^2}(\mathbb{R}^\mathbf{n})$

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ABSTRACT. Let $S$ be a shift-invariant subspace of $L^2(\mathbb{R}^n)$ defined by $N$ generators and suppose that its length $L$, the minimal number of generators of $S$, is smaller than $N$. Then we show that at least one reduced family of generators can always be obtained by a linear combination of the original generators, without using translations. In fact, we prove that almost every such combination yields a new generator set. On the other hand, we construct an example where any rational linear combination fails.

## 1. Introduction and Main Result

Given a family of functions $\phi_1, \ldots, \phi_N \in L^2(\mathbb{R}^n)$, let $S = S(\phi_1, \ldots, \phi_N)$ denote the closed subspace of $L^2(\mathbb{R}^n)$ generated by their integer translates. That is, $S$ is the closure of the set of all functions $f$ of the form

$$f(t) = \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}^n} c_{j,k}\ \phi_j(t-k), \qquad t \in \mathbb{R}^n, \tag{1}$$

where finitely many $c_{j,k} \in \mathbb{C}$ are nonzero. By construction, these spaces $S \subset L^2(\mathbb{R}^n)$ are invariant under shifts, i.e., integer translations and they are called finitely generated shift-invariant spaces. Shift-invariant spaces play an important role in analysis, most notably in the areas of spline approximation, wavelets, Gabor (Weyl-Heisenberg) systems, subdivision schemes and uniform sampling. The structure of this type of spaces is analyzed in [1], see also [2, 3, 4, 9]. Only implicitly we are concerned with the dependence properties of sets of generators, for details on this topic we refer to [7, 8].

The minimal number $L \le N$ of generators for the space $S$ is called the length of $S$. Although we include the case $L = N$, our results are motivated by the case $L < N$. In this latter case, there exists a smaller family of generators $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R}^n)$ such that

$$S(\phi_1, \ldots, \phi_N) = S(\psi_1, \ldots, \psi_L), \qquad \text{with } L < N.$$

Since the new generators $\psi_1, \ldots, \psi_L$ belong to $S$, they can be approximated in the $L^2$-norm by functions of the form (1), i.e., by finite sums of shifts of the original generators. However, we prove that at least one reduced set of generators can be obtained from a linear combination of the original

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generators without translations. In particular, no limit or infinite summation is required. In fact, we show that almost every such linear combination yields a valid family of generators. On the other hand, we show that those combinations which fail to produce a generator set can be dense. That is, combining generators can be a sensitive procedure.

Let $M_{N,L}(\mathbb{C})$ denote the space of complex $N \times L$ matrices endowed with the product Lebesgue measure of $\mathbb{C}^{NL} \cong \mathbb{R}^{2NL}$.

**Theorem 1.** *Given $\phi_1, \ldots, \phi_N \in L^2(\mathbb{R}^n)$, let $S = S(\phi_1, \ldots, \phi_N)$ and let $L \leq N$ be the length of $S$. Let $\mathscr{R} \subset M_{N,L}(\mathbb{C})$ denote the set of those matrices $\Lambda = (\lambda_{j,k})_{1\leq j\leq N,\, 1\leq k\leq L}$ such that the linear combinations $\psi_k = \sum_{j=1}^{N} \lambda_{j,k}\, \phi_j$, for $k = 1, \ldots, L$, yield $S = S(\psi_1, \ldots, \psi_L)$.*

(i) *Then $\mathscr{R} = M_{N,L}(\mathbb{C}) \setminus \mathscr{N}$, where $\mathscr{N}$ is a null-set in $M_{N,L}(\mathbb{C})$.*

(ii) *The set $\mathscr{N}$ in* (i) *can be dense in $M_{N,L}(\mathbb{C})$.*

*Remark* 1. The conclusions of Thm. 1 also hold when the complex matrices $M_{N,L}(\mathbb{C})$ are replaced by real matrices $M_{L,N}(\mathbb{R})$.

We note that our results are not restricted to the case of compactly supported generators. We also mention that the shift-invariant space $S$ in Thm. 1 is not required to be regular nor quasi-regular, see [1] for these notions.

## 2. Examples

The main result, Thm. 1, has been formulated for finitely generated shift-invariant (FSI) spaces. We will illustrate this result by examples in the special case of principal shift-invariant (PSI) spaces. A shift-invariant subspace, initially possibly defined by more than one generator, is called principal if it can be defined by a single generator, i.e., if its length is $L = 1$. As an immediate consequence of Thm. 1 in the case of PSI spaces we have the following corollary.

**Corollary 1.** *Given $\phi_1, \ldots, \phi_N \in L^2(\mathbb{R}^n)$, let $S = S(\phi_1, \ldots, \phi_N)$ and suppose that $S$ is principal, i.e., the length of $S$ is $L = 1$. Let $\mathscr{R} \subset \mathbb{C}^N$ denote the set of those vectors $\lambda = (\lambda_1, \ldots, \lambda_N)$ such that the linear combination $\psi = \sum_{j=1}^{N} \lambda_j\, \phi_j$ yields $S = S(\psi)$.*

(i) *Then $\mathscr{R} = \mathbb{C}^N \setminus \mathscr{N}$, where $\mathscr{N}$ is a null-set in $\mathbb{C}^N$.*

(ii) *The set $\mathscr{N}$ in* (i) *can be dense in $\mathbb{C}^N$.*

The subsequent examples illustrate Corollary 1 and show that the set $\mathscr{N}$ can be a singleton (Example 1) or indeed be dense in $\mathbb{C}^N$ (Example 2). Example 3 demonstrates how the exceptional set $\mathscr{N}$ can be computed. We use the following normalization for the Fourier transform

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i \langle t,x\rangle} dt, \qquad x \in \mathbb{R}^n.$$

The support of a function $f$ is defined without closure, $\operatorname{supp} f = \{t \in \mathbb{R}^n \colon f(t) \neq 0\}$. Let $|F|$ denote the Lebesgue measure of a subset $F \subset \mathbb{R}^n$. We will need the following elementary lemma, which can be easily deduced from [1, Thm. 1.7].

**Lemma 1.** *Suppose that $\Sigma \subset \mathbb{R}^n$ is measurable and*

$$|\Sigma \cap (k + \Sigma)| = 0, \qquad \text{for all } k \in \mathbb{Z}^n \setminus \{0\}.$$

*Let $\phi \in L^2(\mathbb{R}^n)$ be given by $\widehat{\phi} = \chi_\Sigma$. Then $S = S(\phi)$ is a PSI space of the form*

$$S = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \widehat{f} \subseteq \Sigma\}$$

*Moreover, for any $\psi \in S$, we have*

$$S(\psi) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \widehat{f} \subseteq \operatorname{supp} \widehat{\psi}\}.$$

**Example 1.** The function sinc $\in L^2(\mathbb{R})$ is defined by

$$\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}, \qquad t \in \mathbb{R}.$$

Given $N \geq 1$, let $\phi_1, \ldots, \phi_N \in L^2(\mathbb{R})$ be a collection of distinct translations of the sinc-function. That is,

$$\phi_j(t) = \operatorname{sinc}(t - t_j), \qquad t \in \mathbb{R}, \qquad j = 1, \ldots, N,$$

where $t_1, \ldots, t_N \in \mathbb{R}$ satisfy $t_j \neq t_k$, for $j \neq k$. Let $S = S(\phi_1, \ldots, \phi_N)$. Then $S$ is the Paley-Wiener space of functions in $L^2(\mathbb{R})$ which are band-limited to $[-\frac{1}{2}, \frac{1}{2}]$, so $S$ is principal. For example, the sinc-function itself or any of its shifts individually generate $S$. Indeed any linear combination of the original generators yields a single generator for $S$ as well, unless all coefficients are zero. Hence, in this example the set $\mathscr{N}$ of Thm. 1 is

$$\mathscr{N} = \{0\}, \qquad 0 = (0, \ldots, 0) \in \mathbb{C}^N,$$

consisting of the zero vector only.

*Proof.* The Fourier transform of any nonzero linear combination $\psi = \sum_{j=1}^{N} \lambda_j \phi_j$ of the given generators is a nonzero trigonometric polynomial restricted to the interval $[-\frac{1}{2}, \frac{1}{2}]$. Such a trigonometric polynomial cannot vanish on a subset of $[-\frac{1}{2}, \frac{1}{2}]$ with positive measure. Hence, we have $\operatorname{supp} \widehat{\psi} = [-\frac{1}{2}, \frac{1}{2}]$, modulo null sets. Therefore, by Lemma 1 we obtain $S = S(\psi)$, independently of the choice of $(\lambda_1, \ldots, \lambda_N) \neq 0$, so $S = S(\psi)$ holds for any such linear combination. □

**Example 2.** For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer less or equal $x$. We define a discretized version of the Archimedean spiral by $\gamma \colon [0, 1) \to \mathbb{Z}^2$,

$$\gamma(x) = (\lfloor u \cos 2\pi u \rfloor, \lfloor u \sin 2\pi u \rfloor), \qquad u = \tan \tfrac{\pi}{2} x, \quad x \in [0, 1).$$

Next, let

$$\gamma^\circ(x) = \begin{cases} \gamma(x)/|\gamma(x)|, & \text{if } \gamma(x) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad x \in [0, 1).$$

Now define $\phi_1, \phi_2 \in L^2(\mathbb{R})$ by their Fourier transforms, obtained from $\gamma^\circ = (\gamma_1^\circ, \gamma_2^\circ)$ by

$$\widehat{\phi}_j(x) = \begin{cases} \gamma_j^\circ(x), & x \in [0, 1), \\ 0, & x \in \mathbb{R} \setminus [0, 1), \end{cases} \qquad j = 1, 2.$$

Let $S = S(\phi_1, \phi_2)$. Then $S$ is principal. In fact, the function $\psi = \lambda_1 \phi_1 + \lambda_2 \phi_2$ is a single generator, $S = S(\psi)$, if and only if $\lambda_1$ and $\lambda_2$ are rationally linearly independent. So here the set $\mathscr{N}$ of Thm. 1 is

$$\mathscr{N} = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \colon \lambda_1 \text{ and } \lambda_2 \text{ rationally linear dependent}\}.$$

In particular, any rational linear combination of $\phi_1$, $\phi_2$ fails to generate $S$. This example illustrates Corollary 1 for the case of real coefficients, cf. Remark 1. Namely, $\mathscr{N} \cap \mathbb{R}^2$ is a null-set in $\mathbb{R}^2$ yet it contains $\mathbb{Q}^2$, so it is dense in $\mathbb{R}^2$. For the extension to the case of complex coefficients, see the proof of Thm. 1(ii).

*Proof.* The Archimedean spiral is sufficiently close to $\mathbb{Z}^2$ such that its discretization $\gamma$ contains all of $\mathbb{Z}^2$. In fact, for each $z \in \mathbb{Z}^2$, the pre-image

$$I_z := \gamma^{-1}(z) \subset \mathbb{R}$$

has positive measure, it contains at least one interval of positive length. Now suppose that $\lambda_1, \lambda_2 \in \mathbb{C}$ are linearly dependent over the rationals or, equivalently, over the integers. That is,

$$\lambda_1 z_1 + \lambda_2 z_2 = 0, \qquad \text{for some } z = (z_1, z_2) \in \mathbb{Z}^2 \setminus \{0\}.$$

Then, for $\psi = \lambda_1\phi_1 + \lambda_2\phi_2$ we obtain

$$\widehat{\psi}(x) = \lambda_1\widehat{\phi}_1(x) + \lambda_2\widehat{\phi}_2(x) = |z|^{-1}(\lambda_1 z_1 + \lambda_2 z_2) = 0 \qquad \text{for all } x \in I_z. \tag{2}$$

Since $z \neq 0$, we have

$$I_z \subset \operatorname{supp}\widehat{\phi}_1 \cup \operatorname{supp}\widehat{\phi}_2,$$

while (2) implies that

$$I_z \cap \operatorname{supp}\widehat{\psi} = \varnothing,$$

so we conclude that

$$|\,(\operatorname{supp}\widehat{\phi}_1 \cup \operatorname{supp}\widehat{\phi}_2) \setminus \operatorname{supp}\widehat{\psi}\,| \geq |I_z| > 0.$$

Thus, using Lemma 1 with $\Sigma = \operatorname{supp}\widehat{\phi}_1 \cup \operatorname{supp}\widehat{\phi}_2 \subset [0,1]$ we obtain $S(\psi) \neq S(\phi_1, \phi_2)$, for any rationally dependent $\lambda_1, \lambda_2$.

With complementary arguments we obtain that if $\lambda_1, \lambda_2$ are rationally independent, then

$$\widehat{\psi}(x) \neq 0, \qquad \text{for a.e. } x \in \bigcup_{z\neq 0} I_z = \operatorname{supp}\widehat{\phi}_1 \cup \operatorname{supp}\widehat{\phi}_2,$$

and consequently $S(\psi) = S(\phi_1, \phi_2)$. $\square$

**Example 3.** Let $\rho\colon [0,1] \to [0,1]$ be an arbitrary measurable function. Define $\phi_1, \phi_2 \in L^2(\mathbb{R})$ by their Fourier transform

$$\widehat{\phi}_1(x) = \chi_{[0,1]}(x)\,\cos 2\pi\rho(x), \qquad \widehat{\phi}_2(x) = \chi_{[0,1]}(x)\,\sin 2\pi\rho(x), \qquad x \in \mathbb{R},$$

and let $S = S(\phi_1, \phi_2)$. Then

$$S = \{f \in L^2(\mathbb{R})\colon \operatorname{supp}\widehat{f} \subseteq [0,1]\}.$$

In this example, the set $\mathscr{N}$ of Thm. 1 is determined by

$$\mathscr{N} = \{0\} \cup \bigcup_{\theta\in\Theta} (u_\theta)^\perp,$$

where $u_\theta = (\cos 2\pi\theta, \sin 2\pi\theta) \in \mathbb{C}^2$, for given $\theta \in [0,1]$, and

$$\Theta = \{\theta \in [0,1]\colon |\rho^{-1}(\theta)| > 0\}.$$

*Proof.* First, we observe that

$$\operatorname{supp}\widehat{\phi}_1 \cup \operatorname{supp}\widehat{\phi}_2 = [0,1].$$

Hence, for $\psi = \lambda_1\phi_1 + \lambda_2\phi_2$ with $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, by using Lemma 1 with $\Sigma = [0,1]$ we have the equivalence

$$S = S(\psi) \;\Leftrightarrow\; \operatorname{supp}\widehat{\psi} = [0,1] \quad \text{(modulo null sets)}$$

$$\begin{aligned}
&\Leftrightarrow\ \lambda_1\widehat{\phi}_1(x)+\lambda_2\widehat{\phi}_2(x)\neq 0, \quad \text{for a.e. } x\in[0,1]\\
&\Leftrightarrow\ (\lambda_1,\lambda_2)\not\perp(\cos 2\pi\rho(x),\sin 2\pi\rho(x)), \quad \text{for a.e. } x\in[0,1]\\
&\Leftrightarrow\ \lambda\not\perp u_{\rho(x)}, \quad \text{for a.e. } x\in[0,1].
\end{aligned}$$

Therefore, we obtain the complementary characterization

$$\begin{aligned}
S\neq S(\psi)\ &\Leftrightarrow\ \lambda\perp u_{\rho(x)}, \quad \text{for all } x \text{ from a set of positive measure,}\\
&\Leftrightarrow\ \begin{cases}\lambda=0 & \text{or}\\ \lambda\in(u_\theta)^\perp, & \text{for some } \theta\in\Theta,\end{cases}
\end{aligned}$$

since $(u_{\theta_1})^\perp\cap(u_{\theta_2})^\perp=\{0\}$, for $\theta_1\neq\theta_2$. □

## 3. Proof of Thm. 1

We generally identify an operator $A\colon\mathbb{C}^N\to\ell^2(\mathbb{Z}^n), v\mapsto Av$, with its matrix representation,

$$A=(a_1,\dots,a_N),\qquad a_1,\dots,a_N\in\ell^2(\mathbb{Z}^n), \tag{3}$$

that is,

$$Av(k)=\sum_{j=1}^N v_j\,a_j(k),\qquad k\in\mathbb{Z}^n. \tag{4}$$

The composition of $A$ with $\Lambda\in M_{N,L}(\mathbb{C})$ of the form $A\circ\Lambda$ can thus be viewed as a matrix multiplication. Recall the notion $\operatorname{rank}(A)=\dim(\operatorname{range}(A))$.

**Lemma 2.** *Let $L\leq N$ and suppose that $A\colon\mathbb{C}^N\to\ell^2(\mathbb{Z}^n)$ satisfies $\operatorname{rank}(A)\leq L$. Define*

$$\mathscr{N}_A=\{\Lambda\in M_{N,L}(\mathbb{C})\colon \operatorname{rank}(A\circ\Lambda)\neq\operatorname{rank}(A)\}.$$

*Then $\mathscr{N}_A$ is a null-set in $M_{N,L}(\mathbb{C})$.*

*Remark* 2. (i) Since $A$ and $\Lambda$ in Lemma 2 are finite rank operators, we notice that the identity $\operatorname{rank}(A\circ\Lambda)=\operatorname{rank}(A)$ holds if and only if $\operatorname{range}(A\circ\Lambda)=\operatorname{range}(A)$, for both of these range spaces are finite-dimensional hence they are closed.

(ii) If $\operatorname{rank}(A)=L=N$, then

$$\mathscr{N}_A=\{\Lambda\in M_{N,N}(\mathbb{C})\colon \Lambda \text{ singular}\}.$$

If $\operatorname{rank}(A)=L=1$, then $A=s\otimes a$ for some $s\in\ell^2(\mathbb{Z}^n)$, $a\in\mathbb{C}^N$, and

$$\mathscr{N}_A=a^\perp=\{\lambda\in\mathbb{C}^N\colon \lambda\perp a\}.$$

*Proof.* Let $K=\operatorname{rank}(A)$ and note that

$$K=\dim(\operatorname{range}(A))=\dim(\ker(A)^\perp).$$

Let $\{q_1,\dots,q_K\}\subset\mathbb{C}^N$ denote an orthonormal basis for $\ker(A)^\perp$ and let $\{q_{K+1},\dots,q_N\}\subset\mathbb{C}^N$ denote an orthonormal basis for $\ker(A)$. Define $Q\in M_{N,N}(\mathbb{C})$ by its column vectors

$$Q=(q_1,\dots,q_N)$$

and notice that $Q$ is unitary. Then the operator $R=A\circ Q\colon\mathbb{C}^N\to\ell^2(\mathbb{Z}^n)$ has the matrix representation

$$R=A\circ Q=(Aq_1,\dots,Aq_N)=(r_1,\dots,r_K,0,\dots,0),$$

where $r_1, \dots, r_K \in \ell^2(\mathbb{Z}^n)$ are linearly independent. The special structure of $R$ allows us to determine

$$\begin{aligned} \mathscr{N}_R &= \{\Lambda \in M_{N,L}(\mathbb{C})\colon\ \operatorname{rank}(R \circ \Lambda) \neq \operatorname{rank}(R)\} \\ &= \left\{\Lambda = \left(\begin{smallmatrix}\Lambda_1 \\ \Lambda_2\end{smallmatrix}\right) \in M_{K,L}(\mathbb{C}) \times M_{N-K,L}(\mathbb{C})\colon\ \operatorname{rank}(\Lambda_1) \leq K-1\right\}. \end{aligned} \tag{5}$$

Since the rank-deficient matrices in $M_{K,L}(\mathbb{C})$ are a null-set in $M_{K,L}(\mathbb{C})$, we conclude from (5) that $\mathscr{N}_R$ is a null-set in $M_{N,L}(\mathbb{C})$, i.e.,

$$|\mathscr{N}_R| = 0. \tag{6}$$

Now we observe that

$$\begin{aligned} \mathscr{N}_A &= \{\Lambda \in M_{N,L}(\mathbb{C})\colon\ \operatorname{rank}(A \circ \Lambda) \neq \operatorname{rank}(A)\} \\ &= \{\Lambda \in M_{N,L}(\mathbb{C})\colon\ \operatorname{rank}(R \circ Q^{-1} \circ \Lambda) \neq \operatorname{rank}(R \circ Q^{-1})\} \\ &= \{Q \circ \Lambda \in M_{N,L}(\mathbb{C})\colon\ \operatorname{rank}(R \circ \Lambda) \neq \operatorname{rank}(R)\} = Q\mathscr{N}_R. \end{aligned} \tag{7}$$

Since $Q$ is unitary, we notice that the bijection $\Lambda \leftrightarrow Q\Lambda$ is measure preserving in $M_{N,L}(\mathbb{C})$. Therefore, from (6) and (7) we obtain

$$|\mathscr{N}_A| = |Q\,\mathscr{N}_R| = |\mathscr{N}_R| = 0. \qquad \square$$

The next lemma is a variant of the Fubini theorem without integrals, it is sometimes called Fubini's theorem for null sets. In some cases it is an implicit preliminary result for proving the Fubini-Tonelli theorem. Conversely, it also follows immediately from the Fubini-Tonelli theorem, so we omit a reference.

**Lemma 3.** *Given $X \subseteq \mathbb{R}^{n_1}$ and $Y \subseteq \mathbb{R}^{n_2}$, suppose $F \subseteq X \times Y$. For $x \in X$, let $F_x = \{y \in Y\colon (x,y) \in F\}$ and for $y \in Y$, let $F_y = \{x \in X\colon (x,y) \in F\}$. If $F$ is measurable, then the following are equivalent.*
(i) $|F| = 0$,
(ii) $|F_x| = 0$, *for a.e.* $x \in X$,
(iii) $|F_y| = 0$, *for a.e.* $y \in Y$.

*Remark* 3. It is known from a paradoxical set of Sierpiński that the assumption that $F$ is measurable in Lemma 3 cannot be removed.

*Proof of Thm. 1.* (i) Given $f \in L^2(\mathbb{R}^n)$, let $Tf\colon \mathbb{T}^n \to \ell^2(\mathbb{Z}^n)$ denote the fibration mapping for shift-invariant spaces [1, 3], defined by

$$Tf(x) = \big(\widehat{f}(x+k)\big)_{k\in\mathbb{Z}^n}, \qquad x \in \mathbb{T}^n,$$

where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is identified with the fundamental domain $[-\frac{1}{2}, \frac{1}{2})^n$. For $x \in \mathbb{T}^n$, define $A(x)\colon \mathbb{C}^N \to \ell^2(\mathbb{Z}^n)$, $v \mapsto A(x)\,v$, by its matrix representation, cf. (3) and (4),

$$A(x) = (T\phi_1(x), \dots, T\phi_N(x)), \qquad x \in \mathbb{T}^n.$$

We note that each column of $A(x)$ is an element of $\ell^2(\mathbb{Z}^n)$. Then for the composed operator $A(x)\circ\Lambda\colon \mathbb{C}^L \to \ell^2(\mathbb{Z}^n)$ we have

$$\begin{aligned} A(x)\circ\Lambda &= (T\phi_1(x),\dots,T\phi_N(x))\begin{pmatrix}\lambda_{1,1} & \dots & \lambda_{1,L}\\ \vdots & & \vdots\\ \lambda_{N,1} & \dots & \lambda_{N,L}\end{pmatrix}\\ &= (T\psi_1(x),\dots,T\psi_L(x)), \qquad\qquad x\in\mathbb{T}^n. \end{aligned}$$

Define the subset $F\subset \mathbb{T}^n\times M_{N,L}(\mathbb{C})$ by

$$F=\{(x,\Lambda)\in\mathbb{T}^n\times\mathbb{C}^N\colon\ \operatorname{rank}(A(x)\circ\Lambda)\neq\operatorname{rank}\big(A(x)\big)\}.$$

Since $x\mapsto T\phi_j(x)$ is measurable, for $j=1,\dots,N$, we have that $x\mapsto A(x)$ is measurable so the function $h\colon\mathbb{T}^n\times M_{N,L}(\mathbb{C})\to\mathbb{N}$ defined by

$$h(x,\Lambda)=\operatorname{rank}(A(x))-\operatorname{rank}(A(x)\circ\Lambda)$$

is measurable. Now the set $F$ is the pre-image under $h$ of $\{1,\dots,L\}$, i.e.,

$$F=h^{-1}(\{1,\dots,L\}),$$

so we conclude that $F$ is measurable. Denote

$$\begin{aligned} F_x&=\{\Lambda\in M_{N,L}(\mathbb{C})\colon\ \operatorname{rank}(A(x)\circ\Lambda)\neq\operatorname{rank}(A(x))\},\qquad x\in\mathbb{T}^n,\quad\text{and}\\ F_\Lambda&=\{x\in\mathbb{T}^n\colon\ \operatorname{rank}(A(x)\circ\Lambda)\neq\operatorname{rank}(A(x))\},\qquad \Lambda\in M_{N,L}(\mathbb{C}). \end{aligned}$$

By Lemma 2 we have that

$$|F_x|=0,\qquad\text{for a.e. } x\in\mathbb{T}^n.$$

So Lemma 3 implies that

$$|F_\Lambda|=0,\qquad\text{for a.e. } \Lambda\in M_{N,L}(\mathbb{C}). \tag{8}$$

Next, by Remark 2 and using the notion of the range function $J_S$ of a shift-invariant space $S$ [1, 3, 4], we have for $x\in\mathbb{T}^n$,

$$\begin{aligned} &\operatorname{rank}(A(x)\circ\Lambda)=\operatorname{rank}(A(x)),\\ \Leftrightarrow\ &\operatorname{range}(A(x)\circ\Lambda)=\operatorname{range}(A(x)),\\ \Leftrightarrow\ &\operatorname{span}\big(T\psi_1(x),\dots T\psi_L(x)\big)=\operatorname{span}\big(T\phi_1(x),\dots T\phi_N(x)\big),\\ \Leftrightarrow\ &J_{S(\psi_1,\dots,\psi_L)}(x)=J_{S(\phi_1,\dots,\phi_N)}(x). \end{aligned}\tag{9}$$

Thus, from the characterization of FSI spaces in terms of the range function [1, 3], by using (9) we obtain the following equivalence,

$$\begin{aligned} &S(\psi_1,\dots,\psi_L)=S(\phi_1,\dots,\phi_N)\\ \Leftrightarrow\ &J_{S(\psi_1,\dots,\psi_L)}(x)=J_{S(\phi_1,\dots,\phi_N)}(x),\qquad\text{for a.e. } x\in\mathbb{T}^n,\\ \Leftrightarrow\ &\operatorname{rank}(A(x)\circ\Lambda)=\operatorname{rank}(A(x)),\qquad\text{for a.e. } x\in\mathbb{T}^n. \end{aligned}\tag{10}$$

Then using (10) the set $\mathscr{N}$ of Thm. 1 can be expressed as follows,

$$\begin{aligned} \mathscr{N}&=\{\Lambda\in M_{N,L}(\mathbb{C})\colon S(\psi_1,\dots,\psi_L)\neq S(\phi_1,\dots,\phi_N)\}\\ &=\{\Lambda\in M_{N,L}(\mathbb{C})\colon\ \operatorname{rank}(A(x)\circ\Lambda)\neq\operatorname{rank}\big(A(x)\big)\text{ for all } x \text{ from a set} \end{aligned}$$

$$\text{of positive measure}\}$$
$$= \{\Lambda \in M_{N,L}(\mathbb{C})\colon |F_\Lambda| > 0\}.$$

Therefore, (8) implies that $\mathscr{N}$ is a null set.

(ii) For principal shift-invariant spaces ($L = 1$) and in dimension $n = 1$, the statement is verified by Example 2 for the case of $N = 2$ generators and real coefficients. The construction can be extended to general $N$ and complex coefficients. Namely, replace the Archimedean spiral in $\mathbb{R}^2$, which comes close to each point of the lattice $\mathbb{Z}^2$, with a continuous curve $\gamma$ in $\mathbb{C}^N \cong \mathbb{R}^{2N}$ which comes close to the lattice $\mathbb{Z}^{2N}$. More precisely, we require that $\gamma$ intersects with every open cube $k + (0,1)^{2N}$, for $k \in \mathbb{Z}^{2N}$. The case of general $L = 1, \dots, N$ is obtained by extending the pair of generators $\phi_1, \phi_2$ to the set of $2L$ generators $\{\phi_1^{(k)}, \phi_2^{(k)}\}_{1\le k\le L}$, where

$$\phi_j^{(k)}(x) = e^{2\pi ikx}\phi_j(x), \qquad k = 1, \dots, L, \quad j = 1, 2, \quad x \in \mathbb{R}.$$

The extension of this construction to arbitrary dimension $n = 1, 2, \dots$ yields no additional difficulty, and hence we omit the details. □

## 4. Final Remarks

We have shown that some minimal generator sets for finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^n)$ can always be obtained as linear combinations of the original generators without using translations. It is interesting to ask whether the same holds for finitely generated shift-invariant subspaces of $L^p(\mathbb{R}^n)$, where $1 \le p \le \infty$ and $p \ne 2$. For a few properties of these spaces we refer to [5, 6]. Since the proof of Thm. 1 relies heavily on fiberization techniques for $p = 2$ and on the characterization of shift-invariant spaces in terms of range functions, this question remains open for $p \ne 2$.

## References

[1] C. de Boor, R. A. DeVore, and A. Ron, *The structure of finitely generated shift-invariant spaces in $L_2(\mathbf{R}^d)$*, J. Funct. Anal. **119** (1994), no. 1, 37–78.

[2] ———, *Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$*, Trans. Amer. Math. Soc. **341** (1994), no. 2, 787–806.

[3] M. Bownik, *The structure of shift-invariant subspaces of $L^2(\mathbf{R}^n)$*, J. Funct. Anal. **177** (2000), no. 2, 282–309.

[4] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[5] R.-Q. Jia, *Shift-invariant spaces on the real line*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 785–793.

[6] ———, *Stability of the shifts of a finite number of functions*, J. Approx. Theory **95** (1998), no. 2, 194–202.

[7] R. Q. Jia and C. A. Micchelli, *On linear independence for integer translates of a finite number of functions*, Proc. Edinburgh Math. Soc. **36** (1993), no. 1, 69–85.

[8] A. Ron, *A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution*, Constr. Approx. **5** (1989), no. 3, 297–308.

[9] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of $L_2(\mathbf{R}^d)$*, Canad. J. Math. **47** (1995), no. 5, 1051–1094.

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