Contemporary Mathematics

A Fractal Set Constructed from a Class of Wavelet Sets

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1. Introduction

1.1. Background. A function ψ in $L^2(\mathbb{R}^d)$ is called a *single dyadic orthonormal wavelet* if the family of functions $\left\{\psi_{j,k}(\cdot) \equiv 2^{jd/2}\psi(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\right\}$, is an orthonormal basis for $L^2(\mathbb{R}^d)$. More generally, a collection of functions $\psi^1, \psi^2, \ldots, \psi^M$ in $L^2(\mathbb{R}^d)$ is called a *wavelet collection* if the dyadic dilations and integer translations of the ψ^is,

$$\left\{\psi^i_{j,k}(\cdot) \equiv 2^{jd/2}\psi^i(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d, i = 1, \ldots, M\right\},$$

form an orthonormal basis for $L^2(\mathbb{R}^d)$.

The first wavelet in L^2 is the Haar wavelet $\psi = \mathbf{1}_{[0,\frac{1}{2})} - \mathbf{1}_{[\frac{1}{2},1)}$ discovered by Haar [**Haa10**] in 1909. The existence of smooth wavelets had not been established until March 1987 when Daubechies [**Dau88**] showed that for any $n \in \mathbb{N}$ there exists a compactly supported orthonormal wavelet which is n-times continuously differentiable. This remarkable achievement was proved at about the same time as Mallat and Meyer's discovery of a method for constructing orthonormal wavelets from a family of subspaces of $L^2(\mathbb{R}^d)$ satisfying certain properties, e.g., see [**Mal85**], [**Mal89**], [**Mey86**], [**Mey87**], [**Mey89**], [**Mey90**]. This family is called a *multiresolution analysis*(MRA); and Meyer [**Mey87**] first observed that Daubechies' theorem can be formulated in this context . It is now known, e.g., see [**Aus95**], [**Gri95**], [**Wan**], that if $\left\{\psi^1, \psi^2, \ldots, \psi^M\right\}$ is a wavelet collection associated with an MRA, then M must be $2^d - 1$. Further, it was proved by Auscher [**Aus95**] that every wavelet collection $\left\{\psi^1, \psi^2, \ldots, \psi^M\right\}$ for $L^2(\mathbb{R}^d)$, whose members satisfy a weak smoothness and decay condition on the Fourier transform side, must come from an MRA. Because of these results, and notwithstanding Journé's celebrated example of a non-MRA wavelet basis for $L^2(\mathbb{R})$, e.g., [**Dau92**], there was some question during the mid-1990s about the existence of single dyadic orthonormal wavelets in $L^2(\mathbb{R}^d)$.

The existence of single dyadic orthonormal wavelets in $L^2(\mathbb{R}^d)$, $d > 1$, was proved using operator algebra methods by Dai and Larson, e.g., see [**DLS97**]. The

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wavelets constructed in [**DLS97**] are inverse Fourier transforms of characteristic functions of certain measurable sets in $\widehat{\mathbb{R}}^d$. These sets are called wavelet sets. A *wavelet set* in $\widehat{\mathbb{R}}^d$ is defined as a measurable set K such that the characteristic function $\mathbf{1}_K$ is the Fourier transform $\widehat{\psi}$ of some dyadic orthonormal wavelet ψ in $L^2(\mathbb{R}^d)$. Soardi and Weiland [**SW98**] later constructed several wavelet sets in $\widehat{\mathbb{R}}^2$; there are also examples from the same period due to Zakharov [**Zak96**]. A necessary and sufficient condition for the existence of a wavelet set K in $\widehat{\mathbb{R}}^d$ is given by the tiling properties:

$$\{K + k : k \in \mathbb{Z}^d\} \text{ is a partition for } \widehat{\mathbb{R}}^d \text{ a.e.,} \tag{1 a}$$

$$\{2^j K : j \in \mathbb{Z}\} \text{ is a partition for } \widehat{\mathbb{R}}^d \text{ a.e.,} \tag{1 b}$$

e.g., see [**BL99**], [**BL01**].

For comparison, it should be pointed out that a complete characterization of single orthonormal wavelets ψ with $\|\psi\|_2 = 1$ is given by the equations,

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j\xi)\right|^2 = 1 \qquad \text{a.e. } \xi \in \widehat{\mathbb{R}}^d, \tag{2 a}$$

$$\sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi + k))} = 0 \qquad \text{a.e. } \xi \in \widehat{\mathbb{R}}^d,\ k \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d, \tag{2 b}$$

see, e.g., section 7.1 in [**HW96**].

1.2. Contents. In section 2, we review a general construction of wavelet sets in $\widehat{\mathbb{R}}^d$, introduced in [**BL01**], which we call the *neighborhood-mapping method.* We then give a reformulation of this construction when such wavelet sets are contained in $[-1, 1]^d$, and point out a self-similarity property of these sets. This is the content of section 3. Section 4 explains how a fractal set $\mathcal{B}$ arises from this class of wavelet sets, and there is a characterization of $\mathcal{B}$ in equations (9), (11), and Theorem 8. This will be used to prove a result concerning the generality of the neighborhood-mapping method in section 5. A brief analysis of another construction method of wavelet sets in $\widehat{\mathbb{R}}^d$, due to Baggett, Medina, and Merrill [**BMM99**], is given in section 6.

1.3. Notation and definitions. We shall use the standard notation and results from harmonic analysis and fractal theory as found in [**Ben97**], [**SW71**], and [**Fal90**]. For completeness we provide the following definitions which are used in the sequel.

Definition 1. A map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is called a *contraction* if there is a positive number $c < 1$ such that

$$\|\varphi(x) - \varphi(y)\| \leq c\,\|x - y\| \quad \forall x, y \in \mathbb{R}^d$$

We say that φ is a *similarity* with *ratio* $c < 1$ if $\|\varphi(x) - \varphi(y)\| = c\,\|x - y\|$ for all $x, y \in \mathbb{R}^d$

A set F that is invariant under a finite collection of similarities $\varphi_1, \ldots, \varphi_n$, that is

$$F = \bigcup_{i=1}^{n} \varphi_i(F),$$

is called a *self-similar set.*

Definition 2. a. For any non-empty set U in $\mathbb{R}^d$, the *diameter* of U is defined as

$$|U| = \sup\{|x-y| : x, y \in U\}.$$

Let F be a subset of $\mathbb{R}^d$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, i.e., $F \subseteq \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of F.

b. For a subset $F \subseteq \mathbb{R}^d$, a non-negative number s, and any $\delta > 0$, we define

$$\mathcal{H}_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } F\right\}.$$

Note that $\mathcal{H}_\delta^s(F)$ is increasing with δ, but decreasing with s, when the other two variables are fixed. This gives rise to a well-defined measure,

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F),$$

taking values in $[0, \infty]$, called the *s-dimensional Hausdorff measure of F* .

Moreover if $s < t$ and $\{U_i\}_{i=1}^{\infty}$ is a δ-cover of F, we have $\mathcal{H}_\delta^t(F) \leq \delta^{t-s}\mathcal{H}_\delta^s(F)$. Therefore,

- if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ for all $t > s$, and
- if $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$ for all $s < t$.

c. The *Hausdorff dimension* of F is

$$\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

A set F whose Hausdorff dimension is not an integer is said to be a *fractal set*.

Throughout the paper, especially in section 3, we denote the unit cube $[-\frac{1}{2}, \frac{1}{2}]$ by Q. The set $\{-1, 0, 1\}^d \setminus \{(0, \ldots, 0)\}$ will be denoted by Λ.

2. A general construction of wavelet sets

A general construction of wavelet sets in $\mathbb{R}^d$ was introduced by Benedetto and Leon [**BL01**] in 1997. They begin with a fixed number N and a neighborhood of the origin $K_0 \subseteq [-N, N]^d$, which is τ-congruent to the unit cube $Q \equiv [-\frac{1}{2}, \frac{1}{2}]^d$. This means that there exist countable measurable partitions $\{P_m : m \in \mathbb{Z}\}$ and $\{Q_m : m \in \mathbb{Z}\}$ of K_0 and Q, respectively, and a sequence $\{n_m : m \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that $P_m = Q_m + n_m$ for all $m \in \mathbb{Z}$. K_0 is then successively transformed by a given measurable injective integer-translated map,

$$\begin{aligned} T : K_0 &\to \overline{[-2N, 2N]^d \setminus [-N, N]^d} \\ x &\mapsto x + n_x \quad \text{for some } n_x \in \mathbb{Z}^d, \end{aligned}$$

in the following way.

We first define

$$A_0 = K_0 \cap \left[\bigcup_{j \geq 1} 2^{-j} K_0\right] \quad \text{and} \quad K_1 = [K_0 \setminus A_0] \cup T(A_0).$$

Note that

$$K_1^- \equiv K_0 \setminus A_0 \subseteq K_0 \subseteq [-N, N]^d,$$

and

$$K_1^+ \equiv T(A_0) \subseteq \overline{[-2N,2N]^d \setminus [-N,N]^d}.$$

Generally, for a given $K_n = K_n^- \cup K_n^+$, where K_n^- is contained in $[-N,N]^d$ and K_n^+ is contained in the "annulus" $\overline{[-2N,2N]^d \setminus [-N,N]^d}$, $n \geq 1$, we define

$$A_n = K_n \cap \left[\bigcup_{j\geq 1} 2^{-j} K_n\right]$$

and

$$K_{n+1} = [K_n^- \setminus A_n] \cup [K_n^+ \cup T(A_n)] \equiv K_{n+1}^- \cup K_{n+1}^+.$$

Thus,

$$K_{n+1} = \left[K_0 \setminus \bigcup_{k=0}^{n} A_k\right] \cup \left[\bigcup_{k=0}^{n} T(A_k)\right].$$

Finally, we define

$$K = \left[K_0 \setminus \bigcup_{k=0}^{\infty} A_k\right] \cup \left[\bigcup_{k=0}^{\infty} T(A_k)\right]. \tag{3}$$

It is not difficult to show that K satisfies tiling conditions in (1 a) and (1 b), and hence K is a wavelet set. This is the construction in [**BL01**].

Remark 3. Even though the definition of A_n is intuitively clear, there are some unnecessary set computations involved. For the sake of computational efficiency, we shall show that

$$A_n = \left[K_0 \setminus \bigcup_{k=0}^{n-1} A_k\right] \cap \left[\bigcup_{j\geq 1} 2^{-j} T(A_{n-1})\right]. \tag{4}$$

Indeed, from the decomposition of K_n into K_n^- inside the cube $[-N,N]^d$ and K_n^+ outside the open cube $(-N,N)^d$, but still inside $[-2N,2N]^d$, it is clear that

$$A_n = K_n^- \cap \left[\bigcup_{j\geq 1} 2^{-j} K_n\right] \quad \text{a.e.}$$

Moreover, $K_n^- \cap \left[\bigcup_{j\geq 1} 2^{-j} K_n^-\right]$ is empty. In fact, if $x \in K_n^- \subseteq K_0 \setminus A_0$, then $x \notin 2^{-j} K_0$ for all $j \geq 1$, and so x cannot be in the union $\bigcup_{j\geq 1} 2^{-j} K_n^-$. Therefore, for each $n \geq 1$, we have

$$\begin{aligned} A_n &= K_n^- \cap \left[\bigcup_{j\geq 1} 2^{-j} K_n^+\right] \\ &= \left[K_0 \setminus \bigcup_{k=0}^{n-1} A_k\right] \cap \left[\bigcup_{j\geq 1} 2^{-j} \left(\bigcup_{k=0}^{n-1} T(A_k)\right)\right]. \end{aligned} \tag{5}$$

Note that the difference between (5) and the right side of (4) can be rewritten as

$$\left[K_0 \setminus \bigcup_{k=0}^{n-2} A_k\right] \cap \left[\bigcup_{j\geq 1} 2^{-j}\left(\bigcup_{k=0}^{n-2} T(A_k)\right)\right] \setminus A_{n-1},$$

which, because of (5), is empty.

3. Wavelet sets in $[-1,1]^d$

It is a consequence of the geometrical characterization in (1) that a wavelet set can not be contained in a closed proper sub-cube of $I \equiv [-1,1]^d$. In fact, we have the following result.

PROPOSITION 4. *For any $\alpha < 1$, a wavelet set K can not be contained in $[-\alpha,\alpha]^d$.*

PROOF. Suppose that $K \subseteq [-\alpha,\alpha]^d$ is a wavelet set. By (1), the integral translates of K will cover $\widehat{\mathbb{R}}^d$, i.e.,

$$\dot{\bigcup}_{k\in\mathbb{Z}^d}(K+k) = \widehat{\mathbb{R}}^d,$$

where $\dot{\bigcup}$ designates disjoint union.

Observe that, for each $i = 1,\ldots,d$, the union of all translates $K + (k_1,\ldots,k_d)$ with $k_i \neq 0$ leaves out the band $B_i \equiv \{(x_1,x_2,\ldots,x_d) \in \mathbb{R}^d : \alpha - 1 < x_i < 1-\alpha\}$. That is, for each $i = 1,2,\ldots,d$,

$$\dot{\bigcup}_{k_i\neq 0}(K+k) \cap B_i = \emptyset.$$

Therefore,

$$\dot{\bigcup}_{k\neq(0,\ldots,0)}(K+k) \cap B = \emptyset,$$

where $B \equiv \bigcap_{i=1}^d B_i = (\alpha-1, 1-\alpha)^d$. This clearly implies that $B \subseteq K$ and that $\emptyset \neq B \subseteq K \cap 2K$, a contradiction to the second condition in (1). $\square$

To construct wavelet sets K by the neighborhood-mapping method in section 2, we note that N must always be at least $\frac{1}{2}$ since K_0 has to be τ-congruent to $[-\frac{1}{2},\frac{1}{2}]^d$ and contained in $[-N,N]^d$. Further, if $K \subseteq [-1,1]^d$, then it is also necessary that $N = \frac{1}{2}$. In fact, it is impossible to map a point in $[-N,N]^d \setminus [-\frac{1}{2},\frac{1}{2}]^d$ into $\overline{[-1,1]^d \setminus [-N,N]^d}$ by $\mathbb{Z}^d$-translation; and this is a requirement for the map T. Hence, the only possible K_0 in this case is $Q \equiv [-\frac{1}{2},\frac{1}{2}]^d$, and the function T is a mapping from Q to $\overline{2Q \setminus Q}$ defined by $T(x) = x + n_x$ for some $n_x \in \Lambda \equiv \{-1,0,1\}^d \setminus \{(0,\ldots,0)\}$.

In this case, the A_is can be obtained by successively computing $\frac{1}{2}T$ of $\frac{1}{2}Q$, i.e.,

$$\begin{aligned}A_0 &= \frac{1}{2}Q,\\ A_1 &= \frac{1}{2}T\left(\frac{1}{2}Q\right),\\ A_2 &= \frac{1}{2}T\left(\frac{1}{2}T\left(\frac{1}{2}Q\right)\right),\\ &\vdots\\ A_k &= \frac{1}{2}S^k(Q),\end{aligned} \tag{6}$$

where $S(x) \equiv T\left(\frac{1}{2}x\right)$. First, it is clear by definition that $A_0 = \frac{1}{2}Q = \left[-\frac{1}{4}, \frac{1}{4}\right]^d$. Since $2^{-j}T(E) \subseteq \left[-\frac{1}{4}, \frac{1}{4}\right]^d$ for all $j \geq 2$, $E \subseteq Q$, we also have

$$A_n = \left[K_0 \setminus \bigcup_{k=0}^{n-1} A_k\right] \cap \left[\frac{1}{2}T(A_{n-1})\right].$$

It is then left to show that $\frac{1}{2}T(A_{n-1})$ does not intersect $\bigcup_{k=0}^{n-1} A_k$, so that $A_n = \frac{1}{2}T(A_{n-1})$. This in turn follows from the limitation of the map T, that it can only translate by at most 1 both horizontally and vertically, and an induction argument showing that each A_n is contained in a "square annulus" $B_n = \overline{[-b_n, b_n]^2 \setminus [-a_n, a_n]^2}$, where $b_n - a_n = 2^{-n-2}$ and the B_ns are mutually disjoint.

Therefore, combining (3) and (6), we obtain

$$K = \left[Q \setminus \bigcup_{k=0}^{\infty} \frac{1}{2}S^k(Q)\right] \cup \left[\bigcup_{k=0}^{\infty} T\left(\frac{1}{2}S^k(Q)\right)\right]. \tag{7}$$

When T is regular enough, e.g., $|\cup_{k\in\Lambda}\partial\Omega_k| = 0$ where $\Omega_k \equiv \{x \in Q : T(x) = x + n_x\}$, we also have

$$K = \left[Q \setminus \bigcup_{k=0}^{\infty} \frac{1}{2}\overline{S^k(Q)}\right] \cup \left[\bigcup_{k=1}^{\infty} \overline{S^k(Q)}\right] \quad \text{a.e.}$$

In light of this reformulation of the wavelet set K in the case $K_0 = Q$, and because of our approach in section 4, we set

$$M \equiv \bigcup_{k=1}^{\infty} \overline{S^k(Q)} \subseteq \overline{2Q \setminus Q}, \tag{8}$$

where, of course, S and M both depend on the map T.

Example 5. Consider the map $T_1 : \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \to [-1,1]^2 \setminus \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ defined by

$$T_1(x, y) = (x, y) - (\operatorname{sgn} x, \operatorname{sgn} y);$$

and write $S_1 \equiv S_{T_1}$ and $M_1 \equiv M_{T_1}$.

Note that using this map T_1, the wavelet set constructed by the above method, shown in Figure 1, is a generalization to $\mathbb{R}^2$ of the wavelet set $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ associated with the Shannon or Littlewood-Paley wavelet. Observe also that the map $S_1 \equiv T_1\left(\frac{1}{2}\cdot\right)$ is a similarity map from one quadrant to the opposite with ratio $\frac{1}{2}$ when restricted to each quadrant. The resulting set M_1 in Figure 2 is "self-similar" at the four limit points $F_1 \equiv \left\{\left(\pm\frac{2}{3}, \pm\frac{2}{3}\right)\right\}$ in the sense that there is a compact

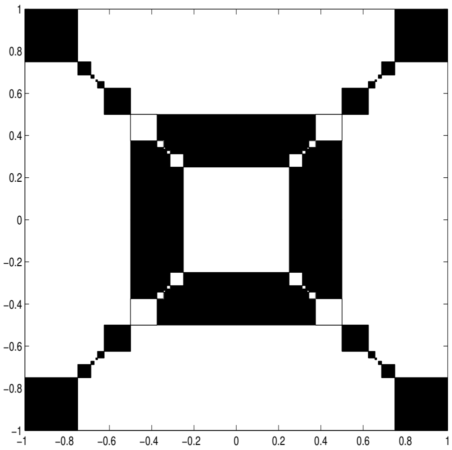


FIGURE 1. The set K_7

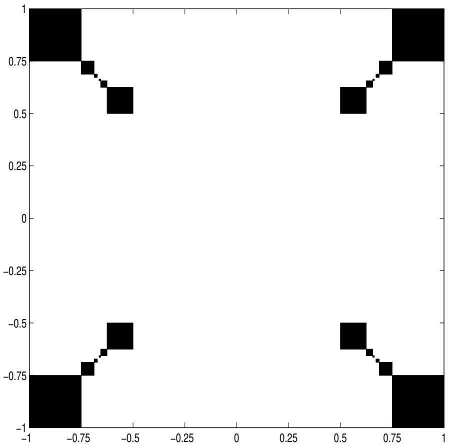


FIGURE 2. The set $\bigcup_{k=1}^{6} \overline{S_1^k(Q)}$

subset P_1, viz., $P_1 = \left([-1,-\frac{3}{4}] \cup [\frac{3}{4},1]\right)^2$ consisting of the four largest squares in the outer corners of M_1, such that

$$M_1 = S_1(M_1) \cup P_1 \quad \text{and} \quad F_1 = S_1(F_1).$$

4. A fractal set arising from the neighborhood-mapping construction

We define a set function $\mathcal{T}$ mapping a given set $E \subseteq Q$ to the union

$$\mathcal{T}(E) \equiv \bigcup \{T(E) | T : Q \to 2Q \setminus Q \text{ is a measurable 1-1 } \mathbb{Z}\text{-translated map}\},$$

and define

$$\mathcal{S}(E) \equiv \mathcal{T}\left(\frac{1}{2}E\right),$$

$$\mathcal{M} \equiv \bigcup_{k=1}^{\infty} \mathcal{S}^k(Q).$$

Observe that since T maps $\left[-\frac{1}{2},\frac{1}{2}\right]^2$ into $[-1,1]^2 \setminus \left[-\frac{1}{2},\frac{1}{2}\right]^2$ by integer translation, then $T(x,y) = (x - \varepsilon_1 \operatorname{sgn} x, y - \varepsilon_2 \operatorname{sgn} y)$ for some $(\varepsilon_1,\varepsilon_2) \in \{0,1\}^2 \setminus \{(0,0)\}$. Thus,

$$\mathcal{T}(E) = \left\{(x - \varepsilon_1 \operatorname{sgn} x, y - \varepsilon_2 \operatorname{sgn} y) : (x,y) \in E, (\varepsilon_1,\varepsilon_2) \in \{0,1\}^2 \setminus \{(0,0)\}\right\}.$$

Clearly, for any such map T, its associated set $M = M_T$ defined in (8), and any set $E \subseteq Q$, we have the inclusions $T(E) \subseteq \mathcal{T}(E)$ and $M_T \subseteq \mathcal{M}$. Figure 3 shows an approximation of $\mathcal{M}$.

Moreover, the limit points of M lie outside the set $\mathcal{M}$, i.e., inside the white jagged line $\mathcal{B} \equiv (\overline{2Q \setminus Q}) \setminus \mathcal{M}$ in Figure 3. It can be shown that $\mathcal{B}$ is a fractal set with Hausdorff dimension $\log 3/\log 2$. In fact, the set $\mathcal{B}$ can be viewed as twelve self-similar sets of the same dimension. Consider, for instance, the set $\mathcal{B}' = \mathcal{B} \cap [\frac{1}{2},1]^2 - (\frac{1}{2},\frac{1}{2})$. We shall illustrate, see Figures 4 and 5, and also prove that

$$\mathcal{B}' = \bigcup_{i=1}^{3} \varphi_i(\mathcal{B}'), \tag{9}$$

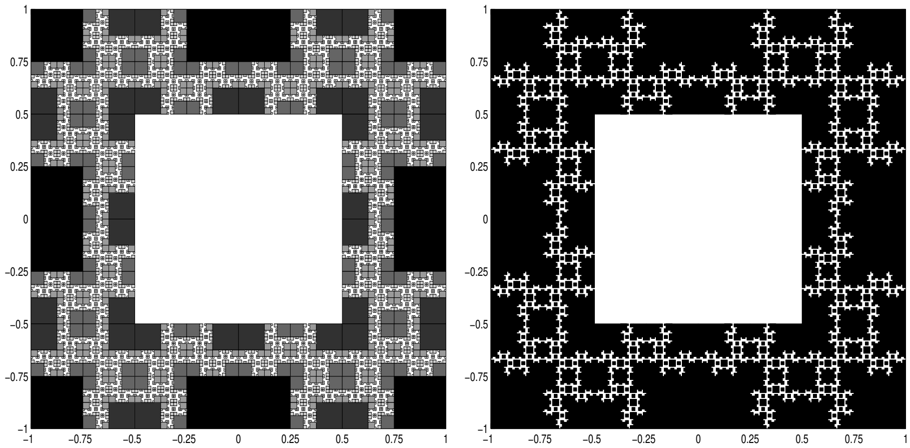


FIGURE 3. The set $\bigcup_{k=1}^{6} \mathcal{S}^k(Q) \subseteq \mathcal{M}$

where

$$\begin{aligned}
\varphi_1(x,y) &= \frac{1}{2}\left(\frac{1}{2}-y, \frac{1}{2}-x\right) = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\varphi_2(x,y) &= \frac{1}{2}\left(\frac{1}{2}+y, \frac{1}{2}-x\right) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\varphi_3(x,y) &= \frac{1}{2}\left(\frac{1}{2}-y, \frac{1}{2}+x\right) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Since φ_1, φ_2 and φ_3 are similarity maps with ratio $1/2$ and the disjoint union $\dot{\bigcup}_{i=1}^{3} \varphi_i\left(\left(0,\frac{1}{2}\right)^2\right)$ is contained in $\left(0,\frac{1}{2}\right)^2$, we conclude by the Hutchinson Theorem(Theorem 6) that the Hausdorff dimension of $\mathcal{B}'$ is $\log 3/\log 2$. Therefore $\dim_H(\mathcal{B}) = \log 3/\log 2$ since $\mathcal{B}$ is the union of twelve reflections of $\mathcal{B} \cap [\frac{1}{2}, 1]^2$.

THEOREM 6. *Let φ_i be contractions on $\mathbb{R}^d$ with ratios $c_i < 1$ for each $i = 1, \ldots, m$. Then there exists a unique non-empty compact set F that satisfies*

$$F = \bigcup_{i=1}^{m} \varphi_i(F).$$

Furthermore, suppose each φ_i is a similarity with ratio $c_i < 1$, and the φ_is satisfy the open set condition, which says that there exists a non-empty bounded open set V such that

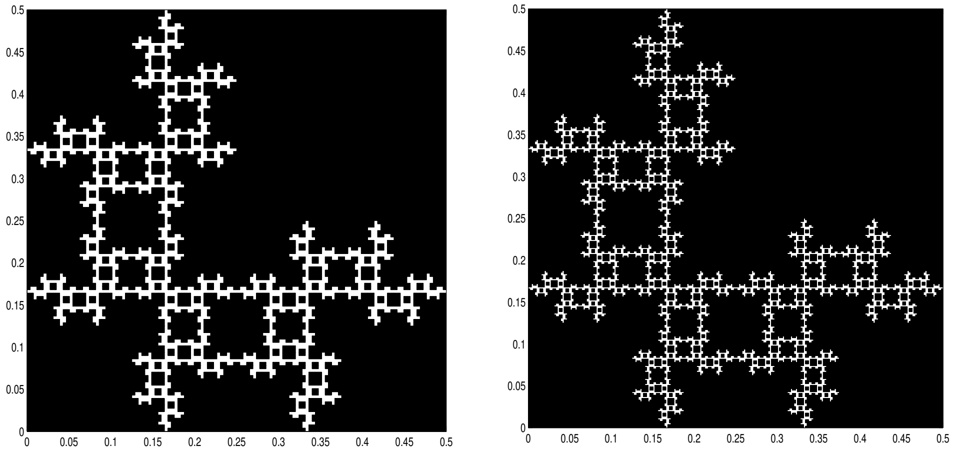
$$\forall i = 1, \ldots, m, \quad \varphi_i(V) \subseteq V$$

and

$$\forall i \neq j, \quad \varphi_i(V) \cap \varphi_j(V) = \emptyset.$$

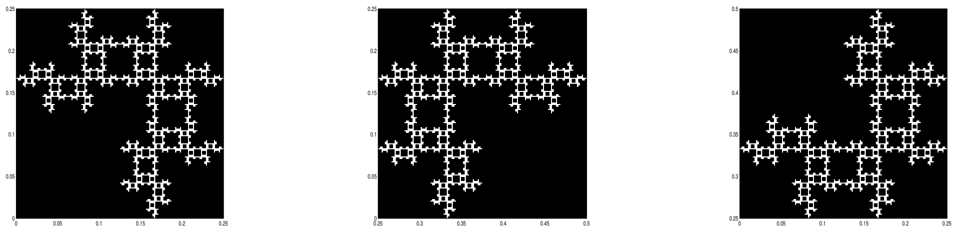
Then the Hausdorff dimension of the invariant set F is s, where s is given by

$$\sum_{i=1}^{m} c_i^s = 1.$$



(a) $\mathcal{M}_6'$ (b) $\mathcal{M}_7'$

FIGURE 4. $\mathcal{M}_n' = \bigcup_{k=1}^{n} \mathcal{S}^k(Q) \cap \left[\frac{1}{2}, 1\right]^2 - \left(\frac{1}{2}, \frac{1}{2}\right)$



(a) $\varphi_1(\mathcal{M}_6')$ (b) $\varphi_2(\mathcal{M}_6')$ (c) $\varphi_3(\mathcal{M}_6')$

FIGURE 5. The images of $\mathcal{M}_6'$ under each map $\varphi_i, i = 1, 2, 3$

In order to prove (9) and to prove that $\mathcal{B}$ is the union of certain reflections of $\mathcal{B} \cap [\frac{1}{2}, 1]^2$, a characterization of $\mathcal{M}$, and hence $\mathcal{B}$, is required. Since this involves binary representations, we set up the following conventions. For a given point $x \in [-1, 1]$, let $\pm.x^{(1)}x^{(2)}\ldots$ denote its binary representation. When ambiguity arises, we pick the representation with infinitely many 1's. For example, $\frac{1}{4}$ is written as $.00111\ldots$ instead of $.01$. The following lemma characterizes all points in $\mathcal{M} \equiv \bigcup_{k=1}^{\infty} \mathcal{S}^k(Q)$.

LEMMA 7. *Let (x, y) be a point in the "square annulus" $\overline{2Q \setminus Q}$. Then (x, y) belongs to $\mathcal{M}$ if and only if there exists $k \in \mathbb{N}$ such that $x^{(k)} = x^{(k+1)}$ and $y^{(k)} = y^{(k+1)}$.*

PROOF. By induction on $k \in \mathbb{N}$, we prove the stronger statement that

$$(x, y) \in \mathcal{S}^k(Q) \quad \text{if and only if} \quad x^{(k)} = x^{(k+1)} \text{ and } y^{(k)} = y^{(k+1)}. \tag{10}$$

First, let us observe that if $x^{(k)} = x^{(k+1)}$, then $(x+n)^{(k)} = (x+n)^{(k+1)}$ for any $n \in \mathbb{N}$. If $(x,y) \in \mathcal{S}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^2\right) = \mathcal{T}\left(\left[-\frac{1}{4},\frac{1}{4}\right]^2\right)$, then $x^{(1)} = x^{(2)}$ and $y^{(1)} = y^{(2)}$ because the first two digits of both x and y are zeros when $(x,y) \in \left[-\frac{1}{4},\frac{1}{4}\right]^2$ and $\mathcal{T}$ translates only by integers. Conversely, if $x^{(1)} = x^{(2)}$ and $y^{(1)} = y^{(2)}$, then $(x,y) - (x^{(1)} \operatorname{sgn} x, y^{(1)} \operatorname{sgn} y)$ lies in $\left[-\frac{1}{4},\frac{1}{4}\right]^2$ and so $(x,y) \in \mathcal{T}\left(\frac{1}{2}Q\right)$.

Now let us assume (10) and let $(u,v) \in \mathcal{S}^{k+1}(Q)$. So $(u,v) = \frac{1}{2}(x,y) + (a,b)$ for some $(a,b) \in \Lambda$ and $(x,y) \in \mathcal{S}^k(Q)$. Hence by the "only if" part of (10), $\left(\frac{x}{2}\right)^{(k+1)} = x^{(k)} = x^{(k+1)} = \left(\frac{x}{2}\right)^{(k+2)}$ and similarly for y. Therefore, $u^{(k+1)} = u^{(k+2)}$ and $v^{(k+1)} = v^{(k+2)}$.

To prove the "if" part of (P_{k+1}), assume $u^{(k+1)} = u^{(k+2)}$ and $v^{(k+1)} = v^{(k+2)}$. Since $(u,v) \in [-1,1]^2$, there is $(a,b) \in \Lambda$ such that $(u,v) - (a,b)$ is in $\left[-\frac{1}{2},\frac{1}{2}\right]^2$. It is then clear that $(2(u-a))^{(k)} = (u-a)^{(k+1)} = (u-a)^{(k+2)} = (2(u-a))^{(k+1)}$ and the similar statement holds for v, hence $2\left[(u,v)-(a,b)\right] \in \mathcal{S}^k(Q)$. Therefore $(u,v) \in \frac{1}{2}\mathcal{S}^k(Q) + (a,b) \subseteq \mathcal{S}^{k+1}(Q)$. The proof is complete. $\square$

To sum up, we have proved that

$$\mathcal{M} \equiv \bigcup_{k=1}^{\infty} \mathcal{S}^k(Q) = \bigcup_{k=1}^{\infty} \left\{(x,y) \in \overline{2Q \setminus Q} : x^{(k)} = x^{(k+1)} \text{ and } y^{(k)} = y^{(k+1)}\right\} \tag{11}$$

We are now ready to prove (9). It follows from Lemma 7 that $(x,y) \in \mathcal{B}'$ if and only if $x^{(1)} = y^{(1)} = 0$, $x^{(2)} \neq 1$ or $y^{(2)} \neq 1$ and for all $k \geq 2$, $x^{(k)} \neq x^{(k+1)}$ or $y^{(k)} \neq y^{(k+1)}$. For brevity, let us consider only the set equation

$$\mathcal{B}' \cap \left[0, \frac{1}{4}\right]^2 = \varphi_1(\mathcal{B}'). \tag{12}$$

When $(x,y) \in \mathcal{B}'$ and $\varphi_1(x,y) \equiv (x_1,y_1)$, we have $x_1^{(k)} = y_1^{(k)} = 0$ for $k = 1,2$ and $x_1^{(k)} = 1 - x^{(k-1)}$, $y_1^{(k)} = 1 - y^{(k-1)}$ for all $k \geq 3$. Therefore, for $k \geq 2$ $x_1^{(k)} \neq x_1^{(k+1)}$ or $y_1^{(k)} \neq y_1^{(k+1)}$ with the first two digits of both x_1 and y_1 vanish and hence $(x_1,y_1) \in \mathcal{B}' \cap \left[0,\frac{1}{4}\right]^2$. The converse arguments are also valid and the equality (12) is proved. The other two cases, e.g., $\mathcal{B}' \cap \left(\left[\frac{1}{4},\frac{1}{2}\right] \times \left[0,\frac{1}{4}\right]\right) = \varphi_2(\mathcal{B}')$, are similar and the proof will be omitted.

Theorem 8. *Let $T : Q \to 2Q \setminus Q$ be measurable injective integer-translated map and $S = T(\frac{1}{2}\cdot)$. The limit points of $M = \bigcup_{k=1}^{\infty} \overline{S^k(Q)}$ are contained in the set $\overline{2Q \setminus Q} \setminus \mathcal{M}$.*

Proof. Let $(x,y) \in Q$. Then for each $k \in \mathbb{N}_0$ there exists $\varepsilon_k = (\zeta_k, \eta_k)$ in Λ such that

$$\begin{aligned}
S(x,y) &= \frac{1}{2}(x,y) + \varepsilon_0 \\
S^2(x,y) &= \frac{1}{2^2}(x,y) + \frac{\varepsilon_0}{2} + \varepsilon_1 \\
&\vdots \\
S^{n+1}(x,y) &= \frac{(x,y)}{2^{n+1}} + \frac{\varepsilon_0}{2^n} + \frac{\varepsilon_1}{2^{n-1}} + \cdots + \varepsilon_n.
\end{aligned}$$

Since S maps $[-1,1]^2$ into $[-1,1]^2 \setminus [-\frac{1}{2},\frac{1}{2}]^2$, there must exist strictly increasing sequences of integers

$$\{p : \zeta_p \neq 0\} = \{p_1 < p_2 < \dots\}\,, \quad \{q : \eta_q \neq 0\} = \{q_1 < q_2 < \dots\}$$

and, for a given $n \in \mathbb{N}$, unique indices $i = i(n)$ and $j = j(n)$ for which p_i is the largest p less than or equal to n such that $\zeta_p \neq 0$ and q_j is the largest q less than or equal to n such that $\eta_q \neq 0$. That is, $p_i \leq n < p_{i+1}$ and $q_j \leq n < q_{j+1}$. So

$$\begin{aligned}(\tilde{a}_n, \tilde{b}_n) &\equiv S^{n+1}(x,y)\\ &= \left(\frac{x}{2^{n+1}} + \frac{\zeta_{p_1}}{2^{n-p_1}} + \cdots + \frac{\zeta_{p_i}}{2^{n-p_i}}, \frac{y}{2^{n+1}} + \frac{\eta_{q_1}}{2^{n-q_1}} + \cdots + \frac{\eta_{q_j}}{2^{n-q_j}}\right).\end{aligned}$$

Furthermore, $\zeta_{p_k} = (-1)^k \operatorname{sgn} x$, $\eta_{q_k} = (-1)^k \operatorname{sgn} y$ and $\{p : \zeta_p \neq 0\} \cup \{q : \eta_q \neq 0\} = \{0,1,2,\dots\} \equiv \mathbb{N}_0$ because (ζ_k, η_k) can never be $(0,0)$. For brevity, we suppose $x, y \geq 0$. So

$$a_n \equiv \tilde{a}_n - \frac{x}{2^{n+1}} = -\frac{1}{2^{n-p_1}} + \frac{1}{2^{n-p_2}} + \cdots + \frac{(-1)^i}{2^{n-p_i}}$$

and

$$b_n \equiv \tilde{b}_n - \frac{y}{2^{n+1}} = -\frac{1}{2^{n-q_1}} + \frac{1}{2^{n-q_2}} + \cdots + \frac{(-1)^j}{2^{n-q_j}}.$$

In binary representation,

$$a_n \equiv \tilde{a}_n - \frac{x_0}{2^{n+1}} = \begin{cases} +.\underbrace{0\dots0}_{n-p_i}\underbrace{1\dots1}_{p_i-p_{i-1}}\dots\dots\underbrace{0\dots0}_{p_3-p_2}\underbrace{1\dots1}_{p_2-p_1} & \text{if } i \text{ is even}\\ -.\overbrace{0\dots0}\overbrace{1\dots1}\dots\dots\overbrace{1\dots1}\overbrace{0\dots0}\dot{1} & \text{if } i \text{ is odd}\end{cases}$$

and

$$b_n \equiv \tilde{b}_n - \frac{y_0}{2^{n+1}} = \begin{cases} +.\underbrace{0\dots0}_{n-q_j}\underbrace{1\dots1}_{q_j-q_{j-1}}\dots\dots\underbrace{0\dots0}_{q_3-q_2}\underbrace{1\dots1}_{q_2-q_1} & \text{if } j \text{ is even}\\ -.\overbrace{0\dots0}\overbrace{1\dots1}\dots\dots\overbrace{1\dots1}\overbrace{0\dots0}\dot{1} & \text{if } j \text{ is odd}\end{cases}$$

Let (a,b) be a limit point of $\left\{S^{n+1}(x,y)\right\}_{n\in\mathbb{N}}$ and hence of $\{(a_n,b_n)\}_{n\in\mathbb{N}}$. Clearly, $(a,b) \in \overline{2Q \setminus Q}$. We will prove that $(a,b) \notin \mathcal{M}$ by showing that for each $k \in \mathcal{N}$, either $a^{(k)} \neq a^{(k+1)}$ or $b^{(k)} \neq b^{(k+1)}$.

For any $N \in \mathbb{N}$, there exists an integer $n > N$ for which

$$|a_n - a| \leq \frac{1}{2^N} \text{ and } |b_n - b| \leq \frac{1}{2^N}.$$

This means exactly that the first N digits of a_n coincide with those of a. Of course, the same statement also holds for b_n and b. This gives rise to a sequence $\{n_l\}_{l=1}^{\infty}$ of strictly increasing integers such that $\left|a_{n_{l+1}} - a\right| \leq \frac{1}{2^{n_l}}$ for all l.

It then suffices to prove that for each integer $n = 2, 3, \dots,$

$$(M_n) \qquad \forall k = 1,2,\dots,n-1, \text{ either } a_n^{(k)} \neq a_n^{(k+1)} \text{ or } b_n^{(k)} \neq b_n^{(k+1)}.$$

When $n = 2$, we only have to show that $a_2^{(1)} \neq a_2^{(2)}$ or $b_2^{(1)} \neq b_2^{(2)}$. In fact, it can be checked that if $p_m = 1$ for some m, then $a_2^{(1)} \neq a_2^{(2)}$. Since either p_m or q_m must be

1 for some m, the case $n=2$ is done. Now assume (M_n) with $p_{i(n)} \leq n < p_{i(n)+1}$ and $q_{j(n)} \leq n < q_{j(n)+1}$. When $p_{i(n)+1} = n+1$, one has $i(n+1) = i(n)+1$ and

$$\begin{aligned} n+1-p_{i(n+1)} &= 0, \\ p_{i(n+1)} - p_{i(n+1)-1} &= n+1-p_{i(n)}, \\ p_{i(n+1)-1} - p_{i(n+1)-2} &= p_{i(n)} - p_{i(n)-1}, \\ &\vdots \end{aligned}$$

Thus a_{n+1} equals the "reverse"$(0 \to 1$ and $1 \to 0)$ of a_n shifted one place to the right with 1 as the first digit provided that $p_{i(n)+1} = n+1$. If, in addition, $p_{i(n)} = n$ then the second digit will be 0. Since either $p_{i(n)+1} = n+1$ or $q_{j(n)+1} = n+1$, (M_{n+1}) clearly holds for $k = 2, \ldots, n$. (M_{n+1}) is also the case for $k=1$ because either $p_{i(n)}$ or $q_{j(n)}$ has to be n. $\square$

5. Generality of the neighborhood-mapping construction

It is natural to investigate on how general the neighborhood-mapping construction of wavelet sets is. In fact, the algorithm can construct every wavelet set in the annulus $[-1,1]^2 \setminus [-\frac{1}{4}, \frac{1}{4}]^2$. We prove the following theorem.

THEOREM 9. *Every wavelet set in* $[-1,1]^2 \setminus [-\frac{1}{4}, \frac{1}{4}]^2$ *can be constructed by means of the neighborhood-mapping method.*

PROOF. Let K be a wavelet set contained in $[-1,1]^2 \setminus [-\frac{1}{4}, \frac{1}{4}]^2$. Define the integer-translated map $T : [-\frac{1}{2}, \frac{1}{2}]^2 \to [-1,1]^2 \setminus [-\frac{1}{2}, \frac{1}{2}]^2$ by

$$T(x) = \begin{cases} x - n & \text{if } x \in K + n, \\ x - \operatorname{sgn}(x) & \text{if } x \in K. \end{cases}$$

T is well-defined almost everywhere because $\{K + n : n \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$ a.e. Since the domain of T is a unit cube, it is clearly injective. Observe that for almost every $x \in Q = [-\frac{1}{2}, \frac{1}{2}]^2$,

$$x \in K \text{ if and only if } T(x) \notin K$$

and for almost every $x \in 2Q \setminus Q$,

$$x \in K \text{ if and only if } \frac{1}{2}x \notin K.$$

From the neighborhood mapping construction with $S = T(\frac{1}{2}\cdot)$

$$\tilde{K} \equiv \left[Q \setminus \bigcup_{k=0}^{\infty} \frac{1}{2} S^k(Q) \right] \dot{\cup} \left[\bigcup_{k=0}^{\infty} T\left(\frac{1}{2} S^k(Q) \right) \right]$$

is a wavelet set. In order to prove that $K = \tilde{K}$ a.e., it suffices to show that

$$K \cap Q \subseteq Q \setminus \bigcup_{k=0}^{\infty} \frac{1}{2} S^k(Q) \quad \text{a.e.} \tag{13}$$

and

$$K \setminus Q \subseteq \bigcup_{k=1}^{\infty} S^k(Q) \quad \text{a.e.} \tag{14}$$

In fact, if $K \subseteq \tilde{K}$ and $|K| = |\tilde{K}| = 1$ then $K = \tilde{K}$ a.e.

Since $\frac{1}{2}S^k(Q) \subseteq Q$, it is easy to check that (13) is equivalent to saying that $\bigcup_{k=0}^{\infty} \left(K \cap \frac{1}{2}S^k(Q)\right) = \emptyset$ a.e. We show that $K \cap \frac{1}{2}S^k(Q) = \emptyset$ a.e. by induction on k. The case $k = 0$ is clear by the assumption. If $\frac{1}{2}S^k(Q) \cap K = \emptyset$ a.e., then $S^{k+1}(Q) = T\left(\frac{1}{2}S^k(Q)\right) \subseteq K$ a.e. and so $\frac{1}{2}S^{k+1}(Q) \cap K = \emptyset$ a.e.

In order to prove (14), it is sufficient to show, for almost everywhere, that $\bigcup_{k=1}^{\infty} (K \setminus Q) \cap \mathcal{S}^k(Q) = (K \setminus Q) \cap \left(\bigcup_{k=1}^{\infty} \mathcal{S}^k(Q)\right)$ is a subset of $\bigcup_{k=1}^{\infty} S^k(Q)$ because $(K \setminus Q) \setminus \left(\bigcup_{k=1}^{\infty} \mathcal{S}^k(Q)\right) \subseteq 2Q \setminus Q \setminus \mathcal{M} = \emptyset$. We first make $\bigcup_{k=1}^{\infty} \mathcal{S}^k(Q)$ a disjoint union by letting

$$
\begin{aligned}
Q_1 &= \mathcal{S}(Q), \\
Q_2 &= \mathcal{S}^2(Q) \setminus \mathcal{S}(Q), \\
&\vdots \\
Q_k &= \mathcal{S}^k(Q) \setminus \bigcup_{l<k} \mathcal{S}^l(Q),
\end{aligned}
$$

and then show by induction on k that $(K \setminus Q) \cap Q_k \subseteq S^k(Q)$ a.e.

Let $(x, y) \in (K \setminus Q) \cap \mathcal{S}(Q)$. Then $x^{(1)} = x^{(2)}$ and $y^{(1)} = y^{(2)}$. From the surjectivity of T onto $(2Q \setminus Q) \cap K = K \setminus Q$, $(x, y) = T\left(\frac{1}{2}(u, v)\right)$ for some $(u, v) \in 2Q$. Since the first two binary digits of $u/2$ and $v/2$ are equal, i.e., $(u/2)^{(1)} = (u/2)^{(2)}$ and $(v/2)^{(1)} = (v/2)^{(2)}$, and $-\frac{1}{2} \leq \frac{u}{2}, \frac{v}{2} \leq \frac{1}{2}$, the first two digits of both $u/2$ and $v/2$ must be zero. Therefore $-\frac{1}{4} \leq \frac{u}{2}, \frac{v}{2} \leq \frac{1}{4}$ and $(x, y) \in T\left(\frac{1}{2}Q\right) = S(Q)$.

Assume that $(K \setminus Q) \cap Q_k \subseteq S^k(Q)$ a.e. and let $(x, y) \in (K \setminus Q) \cap Q_{k+1}$ be given. By the definition of Q_{k+1}, $k + 1$ is the first (smallest) i such that $x^{(i)} = x^{(i+1)}$, $y^{(i)} = y^{(i+1)}$, and there exists $(u, v) \in 2Q$ for which $(x, y) = T(\frac{1}{2}(u, v))$. In fact $(u, v) \in K$ since K tiles $\mathbb{R}^2$ by both integer translation and dyadic dilation and $K \subseteq [-1, 1]^2 \setminus [-\frac{1}{4}, \frac{1}{4}]^2$. Besides $(u, v) \in Q_k$. To verify that $(u, v) \notin Q = [-\frac{1}{2}, \frac{1}{2}]^2$, one observes that (x, y) not being in $Q_1 = \mathcal{S}(Q)$ implies that $(u, v) = 2T^{-1}(x, y) \notin Q$. Thus $(u, v) \in (K \setminus Q) \cap Q_k$ which is contained in $S^k(Q)$ by the assumption. Consequently $(x, y) = T\left(\frac{1}{2}(u, v)\right) \in S^{k+1}(Q)$ as desired. □

Corollary 10. *For every wavelet set K in $[-1, 1]^2$,*

$$\left|K \cap [-c, c]^2\right| > 0 \quad \textit{for all } c > \frac{1}{4}.$$

Proof. If $\left|K \cap \left[-\frac{1}{4}, \frac{1}{4}\right]^2\right| > 0$, then we are done. Otherwise, by Theorem 9, the wavelet set K can be constructed by the neighborhood-mapping method. Due to the property of the map T, it can be shown that A_2 is the only A_n that intersects the "square annulus" $\overline{\left[-\frac{5}{16}, \frac{5}{16}\right] \setminus \left[-\frac{1}{4}, \frac{1}{4}\right]}$. Again, since T is $\mathbb{Z}^2$-translated map, it is not possible for A_2 to cover $[-c, c]^2 \setminus \left[-\frac{1}{4}, \frac{1}{4}\right]^2$ for any $c > \frac{1}{4}$. This completes the proof. □

6. Analysis on Baggett, Medina, and Merrill's wavelet set construction

In [**BMM99**], Baggett, Medina, and Merrill introduced a general construction of wavelet sets and subspace wavelet sets. For brevity, they wrote $x \equiv y$ if $x - y = n$ for some $n \in \mathbb{Z}^d$ and introduced the notion of *complementary pair*.

DEFINITION 11. Let $E \subseteq \widehat{\mathbb{R}}^d$ be invariant under multiplication by $1/2$, i.e., $\frac{1}{2}E \subseteq E$. By a complementary pair for E, we mean a pair (R, R') of measurable injective maps $R : Q \to \widehat{\mathbb{R}}^d$ and $R' : E \to E$ satisfying

(1) $R(Q) \subseteq E$ and $R'(E) \subseteq E \setminus R(Q)$
(2) $2R(x) \equiv x$ and $2R'(x) \equiv x$ a.e.
(3) $E = \bigcup_{j \geq 0} R'^j(R(Q))$.

In their paper [**BMM99**], Baggett et al. have proven the following results.

(A) If (R, R') is a complementary pair for a set $E \subseteq \widehat{\mathbb{R}}^d$, then $W = 2E \setminus E$ is a *subspace* wavelet set. Moreover, if, in addition, $\bigcup_{j \in \mathbb{Z}} 2^j(E)$ is a neighborhood of the origin, then W is a wavelet set for $L^2(\mathbb{R}^d)$. (Note that $\bigcup_{j \in \mathbb{Z}} 2^j(E) = \bigcup_{j \in \mathbb{Z}} 2^j(W)$.)
(B) If W is a *subspace* wavelet set, then there exists a complementary pair (R, R') for the set $E = \bigcup_{j<0} 2^j(W)$. In fact, R' can always be taken equal to $1/2$.
(C) If a map $R : Q \to \widehat{\mathbb{R}}^d$ satisfies $2R(x) \equiv x$ a.e., then there exists a set E and a map $R' : E \to E \setminus R(Q)$ such that (R, R') is a complementary pair for E. (The set E and the map R' are by no means uniquely determined by R. Therefore the resulting wavelet set $W = 2E \setminus E$ is not unique.)

Since $E \subseteq 2E$, we see in (A) and (B) that $W = 2E \setminus E$ if and only if $E = \bigcup_{j<0} 2^j(W)$.

In part (C), not every such a map R will result in a wavelet set for L^2 of the whole of $\mathbb{R}^d$. For instance, let us take

$$R(x) = \begin{cases} (x+1)/2 & \text{if } x \geq 0 \\ (x-1)/2 & \text{if } x < 0. \end{cases}$$

Then $R'(x)$ can be defined as $\frac{1}{2}x$, and

$$E = \bigcup_{j \geq 0} \frac{1}{2^j} R(Q) = \bigcup_{j \geq 0} \frac{1}{2^j} \left(\left[-\frac{3}{4}, -\frac{1}{2} \right] \cup \left[\frac{1}{2}, \frac{3}{4} \right] \right).$$

The set $W = 2E \setminus E = \left[-\frac{3}{2}, -1\right] \cup \left[1, \frac{3}{2}\right]$ is only a proper subspace wavelet set.

Let us first consider how the neighborhood-mapping construction of wavelet sets in $[-1,1]^d$ fits into this construction. In this case, we have a measurable injective $\mathbb{Z}^d$-translated map $T : Q \to 2Q \setminus Q$ and $S(x) = T\left(\frac{1}{2}x\right)$. Then the set

$$K = \left[Q \setminus \bigcup_{k=0}^{\infty} \frac{1}{2} S^k(Q) \right] \cup \left[\bigcup_{k=1}^{\infty} S^k(Q) \right]$$

was shown to be a wavelet set. If we let $E = \bigcup_{k \geq 0} \frac{1}{2} S^k(Q)$, it is clear that

$$2E \setminus E = \left[Q \cup \bigcup_{k \geq 1} S^k(Q) \right] \setminus \left[\bigcup_{k \geq 0} \frac{1}{2} S^k(Q) \right] = K.$$

Moreover E can be written as the union $\bigcup_{j \geq 0} R'^j(R(Q))$ where $R'(x) = \frac{1}{2}T(x)$ and $R(x) = \frac{1}{2}x$. Since, in addition, $R(Q) = \frac{1}{2}Q \subseteq E$ and $R'(E) = E \setminus \frac{1}{2}Q$ the pair $(R, R') = (\frac{1}{2}, \frac{1}{2}T)$ is a complementary pair for E. But E has yet another complementary pair. Define $\widetilde{R}: Q \to E$ by letting

$$\widetilde{R}(x) = \begin{cases} T(x)/2 & \text{if } x \in \bigcup_{k=0}^{\infty} \frac{1}{2} S^k(Q) \\ x/2 & \text{otherwise.} \end{cases}$$

Then it is easy to check that $(\widetilde{R}, \frac{1}{2})$ is a complementary pair for E.

For more general wavelet sets K constructed by the neighborhood-mapping method, we let T_0 be the injective $\mathbb{Z}^d$-translated function that maps the unit cube Q onto the neighborhood K_0 of the origin used in the set up of the construction. The existence of T_0 follows from the condition that K_0 must be τ-congruent to Q. We then define

$$E = \bigcup_{j \geq 1} 2^{-j} K = \bigcup_{j \geq 1} 2^{-j} \left[\left(K_0 \setminus \bigcup_{k=0}^{\infty} A_k \right) \cup T \left(\bigcup_{k=0}^{\infty} A_k \right) \right]$$

and $\breve{R}: K_0 \to E$ as

$$\breve{R}(x) = \begin{cases} T(x)/2 & \text{if } x \in \bigcup_{k=0}^{\infty} A_k \\ x/2 & \text{otherwise.} \end{cases} \tag{15}$$

It can then be shown that $\left(\breve{R} \circ T_0, \frac{1}{2}\right)$ is a complementary pair for E. In fact

$$\breve{R} \circ T_0(Q) = \breve{R}(K_0) = \frac{1}{2} \left[\left(K_0 \setminus \bigcup_{k=0}^{\infty} A_k \right) \cup T \left(\bigcup_{k=0}^{\infty} A_k \right) \right] \subseteq E,$$

$$\frac{1}{2} E = E \setminus R(Q),$$

and

$$\bigcup_{j \geq 0} 2^{-j} R(Q) = \bigcup_{j \geq 1} 2^{-j} K = E.$$

It should be mentioned that the above complementary pair corresponding to the wavelet set K constructed by neighborhood-mapping method comprises of the simple map $1/2$ and the somewhat complicated map $\breve{R}$. The definition of $\breve{R}$ in (15) essentially depends on the very wavelet set K we would like to construct. It is clear that the Baggett et al. wavelet set construction is theoretically more general than the neighborhood-mapping method. One way to see this is by considering the boundedness of wavelet sets. There are examples of unbounded wavelet sets, e.g., [**DLS98**], but the neighborhood-mapping wavelet sets are always in the cube $[-2N, 2N]^d$. In practice while a neighborhood-mapping wavelet set K is constructed by a couple of simple objects, i.e. a neighborhood K_0 of 0 and a map $T : K_0 \to \overline{[-2N, 2N]^d \setminus [-N, N]^d}$ satisfying some properties, we are not able to find a simpler complementary pair that depends on some simple entities.

Finally, it is worth noting that if we set

$$E_n = \bigcup_{j \geq 1} 2^{-j} \left[\left(K_0 \setminus \bigcup_{k=0}^{n} A_k \right) \cup T \left(\bigcup_{k=0}^{n} A_k \right) \right],$$

then

$$2E_n \setminus E_n = \left(K_0 \setminus \bigcup_{k=0}^{n+1} A_k \right) \cup T \left(\bigcup_{k=0}^{n} A_k \right).$$

References

[Aus95] Pascal Auscher. Solution of two problems on wavelets. *J. Geom. Anal.*, 5(2):181–236, 1995.

[Ben97] John J. Benedetto. *Harmonic Analysis and Applications.* CRC Press, Inc., Boca Raton, Florida, 1997.

[BL99] John J. Benedetto and Manuel T. Leon. The construction of multiple dyadic minimally supported frequency wavelets on $\mathbf{R}^d$. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, pages 43–74. Amer. Math. Soc., Providence, RI, 1999.

[BL01] John J. Benedetto and Manuel T. Leon. The construction of single wavelets in d-dimensions. *J. Geom. Anal.*, 11(1):1–15, 2001.

[BMM99] Lawrence W. Baggett, Herbert A. Medina, and Kathy D. Merrill. Generalized multiresolution analyses and a construction procedure for all wavelet sets in $\mathbf{R}^n$. *J. Fourier Anal. Appl.*, 5(6):563–573, 1999.

[Dau88] Ingrid Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41(7):909–996, 1988.

[Dau92] Ingrid Daubechies. *Ten lectures on wavelets.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[DLS97] Xingde Dai, David R. Larson, and Darrin M. Speegle. Wavelet sets in $\mathbf{R}^n$. *J. Fourier Anal. Appl.*, 3(4):451–456, 1997.

[DLS98] Xingde Dai, David R. Larson, and Darrin M. Speegle. Wavelet sets in $\mathbf{R}^n$. II. In *Wavelets, multiwavelets, and their applications (San Diego, CA, 1997)*, pages 15–40. Amer. Math. Soc., Providence, RI, 1998.

[Fal90] Kenneth Falconer. *Fractal geometry.* John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.

[Gri95] Gustaf Gripenberg. A necessary and sufficient condition for the existence of a father wavelet. *Studia Math.*, 114(3):207–226, 1995.

[Haa10] Alfred Haar. Zur Theorie der orthogonalen Funktionen-Systeme. *Math. Ann.*, 69:331–371, 1910.

[HW96] Eugenio Hernández and Guido Weiss. *A first course on wavelets.* CRC Press, Boca Raton, FL, 1996.

[Mal85] Stéphane G. Mallat. An efficient image representation for multiscale analysis. Technical report, GRASP Laboratory, Dept. of Computer and Information Science, U. of Pennsylvania, Philadelphia, (about 1985).

[Mal89] Stéphane G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. *Trans. Amer. Math. Soc.*, 315(1):69–87, September 1989.

[Mey86] Yves Meyer. Ondelettes, fonctions splines et analyses graduées. Lectures given at the University of Torino, Italy, 1986.

[Mey87] Yves Meyer. Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs. *Astérisque*, (145-146):4, 209–223, 1987. Séminaire Bourbaki, Vol. 1985/86.

[Mey89] Yves Meyer. Ondelettes, filtres miroirs en quadrature et traitement numérique de l'image. *Gaz. Math.*, (40):31–42, 1989.

[Mey90] Yves Meyer. *Ondelettes et Opérateurs.* Hermann, Paris, 1990.

[SW71] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[SW98] Paolo M. Soardi and David Weiland. Single wavelets in n-dimensions. *J. Fourier Anal. Appl.*, 4(3):299–315, 1998.

[Wan] X. Wang. *The study of wavelets from the properties of their Fourier transforms.* PhD thesis, Washington University, St. Louis, MO.

[Zak96] V. Zakharov. Nonseparable multidimensional Littlewood-Paley like wavelet bases. Preprint, 1996.

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