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Nonseparable multidimensional Littlewood-Paley like wavelet bases

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Abstract

This paper presents a method for the construction of multidimensional orthonormal wavelet bases of L^2, formed by using only one type of functions, regardless of the dimension of space. Unlike other nonseparable wavelet bases, our bases do not rely on the matrix dilation approach.

Key-Words: Wavelets, Nonseparable bases.

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Classical multidimensional wavelet bases are based on the multiresolution approach and variable separation in each spatial direction. In particular, N-dimensional wavelet basis can be obtained by tensor products of two one-dimensional functions (the basic wavelet and scale function)[6]. Such wavelet bases require $2^N - 1$ types of functions and are intrinsically anisotropic. This is inconvenient from different points of view and is a stimulus to construct multidimensional wavelet bases formed by one type of functions.

In this paper we shall extend to the multidimensional case one of the simplest wavelet bases – Littlewood-Paley's basis [5]. It looks especially simple in Fourier space

$$\hat{\psi}(\xi) = (2\pi)^{-1/2}\chi_I(\xi), \tag{1}$$

where χ_I is indicator function of the set $I = [-2\pi, -\pi] \cup [\pi, 2\pi]$. The proof, that functions $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)$, $j, k \in \mathbb{Z}$ form complete orthonormal basis of $L^2(\mathbb{R})$ is based on geometrical properties of the support I of $\hat{\psi}$ (see, for example, [1]). Namely

- $|I| = 2\pi$;
- $\overline{\bigcup_{j\in\mathbb{Z}} I^j} = \mathbb{R}, \quad I^j \cap I^{j'} = \{\emptyset\},\ j \neq j'$, where I^j is obtained from I by dilation by 2^j (for $j < 0$ it is contraction). In other words, the sets I^j span the whole line $\mathbb{R}$ without overlappings and gaps;
- It is possible to tile the interval $[-\pi, \pi]$ by 2π-translation of the pieces of I.

For any set I satisfying the above-mentioned conditions, the function $\hat{\psi}(\xi) = (2\pi)^{-1/2}\chi_I(\xi)$ will supply a wavelet basis of $L^2(\mathbb{R})$.

These conditions easily can be extended to case of N dimensions: let $I \subset \mathbb{R}^N$, assume that

i) the measure of I is $(2\pi)^N$;

ii) $\overline{\bigcup_{j\in\mathbb{Z}} I^j} = \mathbb{R}^N, \quad I^j \cap I^{j'} = \{\emptyset\},\ j \neq j'$;

iii) it is possible to tile the cube $[-\pi, \pi]^N$ by 2π-translations along coordinate axes of the parts of I.

Let us prove that

For any set $I \in \mathbb{R}^N$, satisfying conditions i), ii) and iii), the set of functions $\psi_{m,k}(x) = 2^{-mN/2}\psi(2^{-m}x - k)$, $x \in \mathbb{R}^N$, $k \in \mathbb{Z}^N$, where $\hat{\psi}(\xi) = (2\pi)^{-N/2}\chi_I(\xi)$, $\xi \in \mathbb{R}^N$, constitutes an orthonormal basis of $L^2(\mathbb{R}^N)$.

Proof. Using the first condition on I it follows that $\|\psi_{m,k}\| = 1$ for all $m \in \mathbb{Z}$, $k \in \mathbb{Z}^N$. For any $f \in L^2(\mathbb{R}^N)$

$$\sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} |\langle f, \psi_{m,k}\rangle|^2 = \sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} 2^{mN} \left| \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\hat{\psi}(2^m\xi)} e^{i2^m k\cdot\xi} d\xi \right|^2$$

(where $\langle \cdot, \cdot \rangle$ is scalar product and overbar stands for complex conjugate).

$$= \sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} (2\pi)^{-N} 2^{mN} \left| \int_{I^{-m}} \hat{f}(\xi) e^{i2^m k\cdot\xi} d\xi \right|^2 = \sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} (2\pi)^{-N} 2^{-mN} \left| \int_{I^0} \hat{f}(2^{-m}\zeta) e^{ik\cdot\zeta} d\zeta \right|^2$$

(use a change of variables in last equality).

In accordance with condition iii) there is a partition $\{I^0_\lambda\}$ of the set I^0, such that we can tile the cube $[-\pi,\pi]^N$ by 2π-shifts of the sets I^0_λ along some coordinate axes. Denote by h_λ the vector of shift, corresponding to the subset I^0_λ. (Obviously the vectors $h_\lambda \in \mathbb{R}^N$ will have components $\{\pm 2\pi, 0\}$.) Define $\hat{f}_1(2^{-m}\zeta) = \sum_\lambda \hat{f}(2^{-m}(\zeta - h_\lambda))\chi_{I^0_\lambda}(\zeta - h_\lambda)$. Then using 2π-periodicity of the functions $e^{ik\cdot\zeta}$, we can write

$$\sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} (2\pi)^{-N} 2^{-mN} \left| \int_{I^0} \hat{f}(2^{-m}\zeta) e^{ik\cdot\zeta} d\zeta \right|^2$$
$$= \sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} (2\pi)^{-N} 2^{-mN} \left| \int_{[-\pi,\pi]^N} \hat{f}_1(2^{-m}\zeta) e^{ik\cdot\zeta} d\zeta \right|^2 .$$

Now Parseval's equality can be applied to the system of the functions $e^{ik\cdot\zeta}$, $k \in \mathbb{Z}^N$:

$$\sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} (2\pi)^{-N} 2^{-mN} \left| \int_{[-\pi,\pi]^N} \hat{f}_1(2^{-m}\zeta) e^{ik\cdot\zeta} d\zeta \right|^2 = \sum_{m\in\mathbb{Z}} 2^{-mN} \int_{[-\pi,\pi]^N} \left| \hat{f}_1(2^{-m}\zeta) \right|^2 d\zeta.$$

As the sets I^0_λ do not overlap, we can return to the set I^0

$$\sum_{m\in\mathbb{Z}} 2^{-mN} \int_{[-\pi,\pi]^N} \left| \hat{f}_1(2^{-m}\zeta) \right|^2 d\zeta = \sum_{m\in\mathbb{Z}} 2^{-mN} \int_{I^0} \left| \hat{f}(2^{-m}\zeta) \right|^2 d\zeta.$$

Finally, since the sets $I^m, m \in \mathbb{Z}$ cover all space $\mathbb{R}^N$ without overlappings and gaps, we have

$$\sum_{m\in\mathbb{Z}} 2^{-mN} \int_{I^0} \left| \hat{f}(2^{-m}\zeta) \right|^2 d\zeta = \sum_{m\in\mathbb{Z}} \int_{I^{-m}} \left| \hat{f}(\xi) \right|^2 d\xi = \int_{\mathbb{R}^N} \left| \hat{f}(\xi) \right|^2 d\xi = \|f\|^2.$$

Therefore, for $\forall f \in L^2(\mathbb{R}^N)$, we obtain Parseval's equality

$$\sum_{m\in\mathbb{Z},k\in\mathbb{Z}^N} |\langle f, \psi_{m,k}\rangle|^2 = \|f\|^2,$$

which proves the completeness and orthogonality of our wavelet basis. ■

The conditions i), ii) and iii) seem rather contradictory and hard to satisfy. We can offer a method to construct such sets. Take the "initial" set I_0 as cube $[-\pi,\pi]^N$, contract it by 2 (which will be denoted as I_0^{-1}) and consider the intersection $I_0 \cap I_0^{-1}$. Then make some partition of this intersection, such that it is possible to move parts of $I_0 \cap I_0^{-1}$ out of the cube $[-\pi,\pi]^N$ by 2π-shifts along coordinate axes so that these parts will remain in $[-2\pi,2\pi]^N$. Denote obtained set as $I_1 = T_0 I_0$ (where T_0 stands for the map $I_0 \to I_1$). Repeating this procedure, i.e. contracting I_1 by 2 and removing intersection $I_1 \cap I_1^{-1}$ out of the cube $[-\pi,\pi]^N$, we obtain set $I_2 = T_1 I_1$, etc. If sequence of the maps $T_j I_j \to I_{j+1}$ has fixed point, the set $I = \lim_{j\to\infty} I_j$ will satisfy all conditions i), ii) and iii). We do not derive conditions which guarantee that the sequence of maps T_j has a fixed point, but for concrete reasonable partitions, the existence of a fixed point is obvious. Fig.1 represents examples of the construction of I for 2-dimensional case.

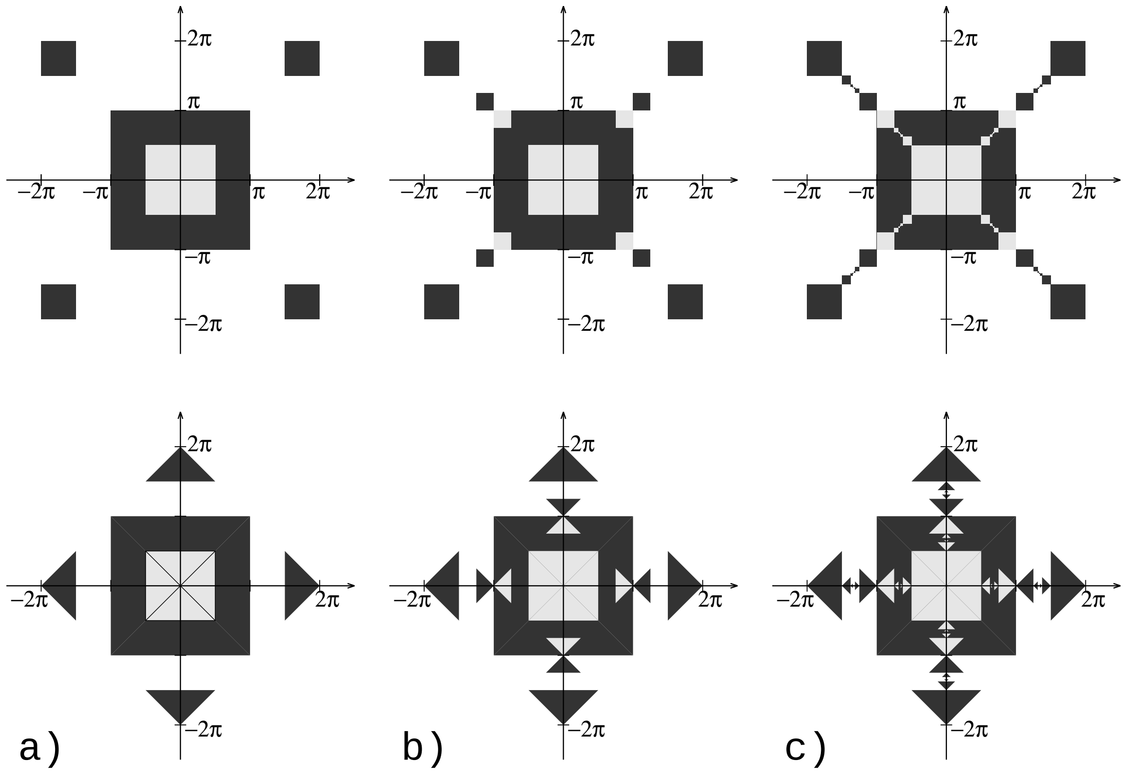

Figure 1: Two examples of set I construction. a) first iteration: I_1; b) second iteration: I_2; c) the limit set I and set J (light gray).

We want to return to separable case of N-dimensional wavelet bases, in order to explain why it needs several wavelet spaces in higher dimensions and why it is sufficient to have one wavelet in our case.

In two dimensions, for example, the approximation space V_0 is generated by the translation of one function $\phi(x,y)$ over $\mathbb{Z}^2$; the space V_{-1} is generated by the translation of $\phi(2x,2y)$ over $\frac{1}{2}\mathbb{Z}^2$, or equivalently by the $\mathbb{Z}^2$-translates of the four functions $\phi(2x,2y)$, $\phi(2x-1,2y)$, $\phi(2x,2y-1)$, $\phi(2x-1,2y-1)$. V_{-1} is therefore "four times as big" as V_0, on the other hand each of the W_0^j-space is generated by the $\mathbb{Z}^2$-translations of a single function $\psi^j(x,y)$ and is therefore "of the same size" as V_0. It follows that one needs three (four minus one) spaces W_0^j (hence three wavelets ψ^j) to make up the complement of V_0 in V_{-1}. It can be rephrased in more mathematical terms: the number of wavelets is equal to the number of different cosets of the subgroup $\mathbb{Z}^2$ in the group $\frac{1}{2}\mathbb{Z}^2$.

In our case it is also possible to introduce the multiresolution analysis. Scaling function belonging to the space V_0 can be defined naturally as indicator function of the set J, removed from the square $[-\pi,\pi]^N$, i.e. $\hat{\phi}(\xi) = |J|^{-1/2}\chi_J(\xi)$ and $J = \bigcup_{j=-1}^{-\infty} I^j$, fig.1 c), where $|J| = (2\pi)^N/(2^N-1)$. These scaling functions satisfy all conditions of the multiresolution analysis, excepting one that these functions form tight frame with frame constant $A = 2^N - 1$ (see, for example, review of frames [3]):

$$\forall f \in V_0 \subset L^2(\mathbb{R}^N) \qquad \sum_{k\in\mathbb{Z}^N} |\langle f, \phi(\cdot - k)\rangle|^2 = A\|f\|^2.$$

If $\|\phi\| = 1$, then the frame constant can serve as measure of the redundancy of the set of functions. Thus the space V_0 has "A times less functions" as in the case of the orthogonal basis and it needs only one wavelet space to make up the complement of V_0 in V_{-1}.

Certainly, all bases built by our approach will have a shortcoming inherited from Littlewood-Paley's basis. Since these functions are not continuous in Fourier space, they decay slowly in the real space (like $1/x$), leading to bad space localization. Unfortunately, the attempts to construct more smooth functions within the framework of our approach are prevented by following statement.

If the set of functions $\{\phi(x-k),\ x \in \mathbb{R}^N,\ k \in \mathbb{Z}^N\}$, $\|\phi\| = 1$ *forms tight frame with frame constant* A, *then* $\sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2$ *is the indicator function of some set* $S \subset [0, 2\pi]^N$, *i.e.*

$$\sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 = A(2\pi)^{-N} \chi_S(\xi),$$

and the measure of S *is* $(2\pi)^N/A$.

Proof. The $\sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2$ can be expanded in a Fourier series:

$$\sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 = \sum_{k\in\mathbb{Z}^N} c_k e^{-ik\cdot\xi}, \tag{2}$$

where the Fourier coefficients are

$$\begin{aligned} c_k = (2\pi)^{-N} \int_{[0,2\pi]^N} e^{ik\cdot\xi} \sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 d\xi &= (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ik\cdot\xi} |\hat{\phi}(\xi)|^2 d\xi \\ &= (2\pi)^{-N} \int_{\mathbb{R}^N} \phi(x)\overline{\phi(x-k)} dx. \end{aligned} \tag{3}$$

Since the functions $\phi(\cdot - k)$ form tight frame, and using (3), ϕ can be expressed as follows

$$\phi(x) = A^{-1} \sum_{k\in\mathbb{Z}^N} \langle \phi, \phi(\cdot - k)\rangle \phi(x-k) = A^{-1}(2\pi)^N \sum_{k\in\mathbb{Z}^N} c_k \phi(x-k). \tag{4}$$

Now using (2) and (4), we can write

$$\begin{aligned} &\left\| \sum_{k\in\mathbb{Z}^N} c_k \phi(\cdot - k) - A(2\pi)^{-N}\phi \right\|^2 \\ &\quad = \int_{\mathbb{R}^N} \left| \sum_{k\in\mathbb{Z}^N} c_k e^{-ik\cdot\xi} - A(2\pi)^{-N} \right|^2 |\hat{\phi}(\xi)|^2 d\xi \\ &\quad = \int_{[0,2\pi]^N} \left| \sum_{k\in\mathbb{Z}^N} c_k e^{-ik\cdot\xi} - A(2\pi)^{-N} \right|^2 \sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 d\xi \\ &\quad = \int_{[0,2\pi]^N} \left| \sum_{m\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi m)|^2 - A(2\pi)^{-N} \right|^2 \sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 d\xi = 0. \end{aligned} \tag{5}$$

Obviously the integral (5) can be equal to zero if and only if

$$\sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 = A(2\pi)^{-N} \chi_S(\xi),$$

where S is some subset of cube $[0, 2\pi]^N$. Consequently

$$\int_{[0,2\pi]^N} \left| \sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 \right|^2 d\xi = A^2 (2\pi)^{-2N} |S|. \tag{6}$$

On the other side we have

$$\int_{[0,2\pi]^N} \left| \sum_{l\in\mathbb{Z}^N} |\hat{\phi}(\xi + 2\pi l)|^2 \right|^2 d\xi = \int_{[0,2\pi]^N} \left| \sum_{k\in\mathbb{Z}^N} c_k e^{-ik\cdot\xi} \right|^2 d\xi = (2\pi)^N \sum_{k\in\mathbb{Z}^N} |c_k|^2$$
$$= (2\pi)^{-N} \sum_{k\in\mathbb{Z}^N} |\langle \phi, \phi(\cdot - k)\rangle|^2 = (2\pi)^{-N} A \|\phi\|^2 = (2\pi)^{-N} A. \tag{7}$$

Comparing (6) and (7), it follows that the measure of S is $(2\pi)^N/A$. ■

It is difficult to imagine a good smooth function ϕ forming a tight frame of the space $V_0 = \text{span}\{\phi(\cdot - k),\ k \in \mathbb{Z}^N\}$.

In conclusion, we can also mention the approach based on so called matrix dilation [2, 4]: the multiresolution spaces are subspaces of $L^2(\mathbb{R}^N)$ and the dilation is determined by a matrix D with integer entries so that $D\mathbb{Z}^N \subset \mathbb{Z}^N$. The number of wavelets is again determined by the number of cosets of $D\mathbb{Z}^N$. A particularly interesting case is given by the "quincunx lattice", i.e. the two-dimensional case where $D\mathbb{Z}^2 = \{(m, n); m + n \in 2\mathbb{Z}\}$. In this case there is only one other coset, and therefore only one wavelet is needed. The sufficiency of only one wavelet space can be understood from another point of view: a single dilation can be regarded as a dilation by $\sqrt{2}$ (combining with a rotation and/or a reflection) and thus the space V_{-1} is only "two times as big" as V_0.

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