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Wavelet Sets in $\mathbb{R}^n$

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ABSTRACT. A congruency theorem is proven for an ordered pair of groups of homeomorphisms of a metric space satisfying an abstract dilation-translation relationship. A corollary is the existence of wavelet sets, and hence of single-function wavelets, for arbitrary expansive matrix dilations on $L^2(\mathbb{R}^n)$. Moreover, for any expansive matrix dilation, it is proven that there are sufficiently many wavelet sets to generate the Borel structure of $\mathbb{R}^n$.

A dyadic orthonormal (or orthogonal) wavelet is a function $\psi \in L^2(\mathbb{R})$, (Lebesgue measure), with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - l) \colon n, l \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$ (see [1, 2]). For certain measurable sets, E, the normalized characteristic function $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of such a wavelet. There are several characterizations of such sets (see [3] chapt. 4, and independently [5]). In [3] they are called wavelet sets. In [5, 6, 7] they are the support sets of MSF (minimally supported frequency) wavelets.

Dilation factors on $\mathbb{R}$ other than 2 have been studied in the literature, and analogous wavelet sets corresponding to all dilations > 1 are known to exist ([3], Example 4.5, part 10). Matrix dilations (for real expansive matrices) on $\mathbb{R}^n$ have also been considered in the literature, usually for a "multi-" notion of wavelet. The translations involved are those along the coordinate axes. The purpose of this article is to prove a general-principle type of result that shows, as a corollary, that analogous wavelet sets exist (and are plentiful) for all such dilations. In particular, "single-function" wavelets always exist. This appears to be new. Theorem 1 seems to belong to the mathematics behind wavelet theory. For this reason we prove it in a more abstract setting than needed for our wavelet results. Essentially, it is a dual-dynamical system congruency principle. The general proof is no more difficult than that for $\mathbb{R}^n$.

We point out that the wavelets we obtain, which are analogs of Shannon's wavelet, need not satisfy the regularity properties often desired (see [8]) in applications.

Let X be a metric space, and let m be a σ-finite nonatomic Borel measure on X for which the measure of every open set is positive and for which bounded sets have finite measure. Let $\mathcal{T}$ and $\mathcal{D}$ be countable groups of homeomorphisms of X which map bounded sets to bounded sets and which are absolutely continuous in the sense that they map m-null sets to m-null sets. A countable group $\mathcal{G}$ of absolutely continuous Borel isomorphisms of X determines an equivalence relation on the family $\mathcal{B}$ of Borel sets of X in a natural way: E and F are *$\mathcal{G}$-congruent* (written $E \sim_{\mathcal{G}} F$) if

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there are measurable partitions $\{E_g\colon g \in \mathcal{G}\}$ and $\{F_g\colon g \in \mathcal{G}\}$ of E and F, respectively, such that $F_g = g(E_g)$ for each $g \in \mathcal{G}$, modulo m-null sets.

If $r > 0$ and $y \in X$, we write $B_r(y) := \{x \in X\colon \|x - y\| < r\}$, and abbreviate $B_r := B_r(0)$.

We will say that $(\mathcal{D}, \mathcal{T})$ is an *abstract dilation-translation pair* if (1) for each bounded set E and each open set F there are elements $\delta \in \mathcal{D}$ and $\tau \in \mathcal{T}$ such that $\tau(E) \subseteq \delta(F)$, and (2) there is a fixed point θ for $\mathcal{D}$ in X which has the property that if N is any nhood of θ and E is any bounded set, there is an element $\delta \in \mathcal{D}$ such that $\delta(E) \subseteq N$.

Theorem 1.

Let $X, \mathcal{B}, m, \mathcal{D}, \mathcal{T}$ be as above, with $(\mathcal{D}, \mathcal{T})$ an abstract dilation-translation pair, and with θ the $\mathcal{D}$-fixed point as above. Let E and F be bounded measurable sets in X such that E contains a nhood of θ, and F has nonempty interior and is bounded away from θ. Then there is a measurable set $G \subseteq X$, contained in $\bigcup_{\delta\in\mathcal{D}} \delta(F)$, which is both $\mathcal{D}$-congruent to F and $\mathcal{T}$-congruent to E.

Proof. We will use the term "$\mathcal{D}$-dilate" to denote the image of a set Ω under an element of $\mathcal{D}$, and "$\mathcal{T}$-translate" for the image of Ω under an element of $\mathcal{T}$.

We will construct a disjoint family $\{G_{ij}\colon i \in \mathbb{N}, j \in \{1, 2\}\}$ of measurable sets whose $\mathcal{D}$-dilates form a partition $\{F_{ij}\}$ of F and whose $\mathcal{T}$-translates form a partition $\{E_{ij}\}$ of E, modulo m-null sets. Then $G = \bigcup_{i,j} G_{ij}$ will clearly satisfy our requirements. The i^{th} induction step will consist of constructing G_{i1} and G_{i2}.

Let $\{\alpha_i\}$ and $\{\beta_i\}$ be sequences of positive constants decreasing to 0. Let $N_1 \subset E$ be a ball centered at θ with radius $< \alpha_1$ such that $m(E\backslash N_1) > 0$. Let $E_{11} = E\backslash N_1$.

Observe that we may choose $\delta_1 \in \mathcal{D}$, $\tau_1 \in \mathcal{T}$, so that $(\delta_1^{-1} \circ \tau_1)(E_{11})$ is a subset of F whose relative complement in F has a nonempty interior. This is possible because since the interior of F is nonempty, there is a δ_1-dilate of F which contains a ball large enough to contain some τ_1-translate of E with ample room left over. Now set $F_{11} := (\delta_1^{-1} \circ \tau_1)(E_{11})$. (In this context, clearly we may choose δ_1 and τ_1 such that, in addition, the τ_1-translate of E is disjoint from any prescribed bounded set — a fact that will be useful in the second and subsequent steps.)

Let $G_{11} := \tau_1(E_{11}) = \delta_1(F_{11})$. Since δ_1 is a homeomorphism of X which fixes θ, G_{11} is bounded away from θ since F_{11} is. Let F_{12} be a measurable subset of F of positive measure, disjoint from F_{11}, such that the difference $F\backslash(F_{11} \cup F_{12})$ has a nonempty interior and measure $< \beta_1$. Choose $\gamma_1 \in \mathcal{D}$ such that $\gamma_1(F_{12})$ is contained in N_1 and is disjoint from G_{11}. Set $E_{12} := \gamma_1(F_{12})$, and set $G_{12} := E_{12}$. The first step is complete.

For the second step, note that since F is bounded away from θ, $N_1\backslash E_{12}$ contains a ball N_2 centered at θ with radius $< \alpha_2$ such that $N_1\backslash(E_{12} \cup N_2)$ has positive measure. Let

$$E_{21} := N_1\backslash(E_{12} \cup N_2) = E\backslash(E_{11} \cup E_{12} \cup N_2).$$

Choose $\delta_2 \in \mathcal{D}$, $\tau_2 \in \mathcal{T}$, using similar reasoning to that used above, such that $(\delta_2^{-1} \circ \tau_2)(E_{21})$ is a subset of $F\backslash(F_{11} \cup F_{12})$ whose relative complement in $F\backslash(F_{11} \cup F_{12})$ has a nonempty interior, and for which $\tau_2(E_{21})$ is disjoint from G_{11} and G_{12}. Let $F_{21} := (\delta_2^{-1} \circ \tau_2)(E_{21})$, and let $G_{21} := \tau_2(E_{21})$.

Choose a measurable subset $F_{22} \subset F$ of positive measure disjoint from F_{11}, F_{12}, F_{21} such that $F\backslash(F_{11} \cup F_{12} \cup F_{21} \cup F_{22})$ has a nonempty interior and measure $< \beta_2$. Noting that G_{11}, G_{12}, G_{21} are bounded away from θ, choose $\gamma_2 \in \mathcal{D}$ such that $\gamma_2(F_{22})$ is contained in N_2 and is disjoint from G_{11}, G_{12}, G_{21}. Set $E_{22} := \gamma_2(F_{22})$, and let $G_{22} := E_{22}$.

Now proceed inductively, obtaining disjointed families of sets of positive measure $\{E_{ij}\}$ in E, $\{F_{ij}\}$ in F, and $\{G_{ij}\}$, such that

$$\begin{aligned} \tau_i^{-1}(G_{i1}) &= E_{i1}, G_{i2} = E_{i2}, \delta_i^{-1}(G_{i1}) = F_{i1}, \\ \gamma_i^{-1}(G_{i2}) &= F_{i2}, \text{ for } i = 1, 2, \ldots \text{ and } j = 1, 2. \end{aligned}$$

We have $E\backslash(\cup E_{ij}) = \{\theta\}$, a null set, since $\alpha_i \to 0$, and $F\backslash(\cup F_{ij})$ is a null set since $\beta_i \to 0$. Let $G = \cup G_{ij}$. Since $\delta_i, \gamma_i \in \mathcal{D}$ we have $G_{ij} \in F$ for all i, j. So $G \subseteq \bigcup_{\delta\in\mathcal{D}} \delta(F)$. The proof is complete. □

Remark. Suppose K is any bounded set that is bounded away from θ (i.e., K is contained in an annulus centered at θ). Then the set G in Theorem 1 can be taken *disjoint* from K. This follows immediately from the way the sets G_{ij} in the proof are constructed. Moreover, for each n a disjoint n-tuple can be constructed, all of which satisfy the properties of G and K above. To see this, mimic the proof of Theorem 1, at each step constructing $G_{ij}^1, \ldots, G_{ij}^n$ simultaneously, making sure that they are disjoint from each other and also from all of the previous G_{lk}^h that have been constructed to that point. This construction can easily be modified to yield an infinite pairwise disjoint family $\{G^k\}_{k=1}^{\infty}$. □

We will now relate Theorem 1 to wavelets.

Let $1 \le m < \infty$, and let A be an $n \times n$ real matrix which is *expansive* (equivalently, all eigenvalues have modulus > 1 (see [9])). By a dilation-A orthonormal wavelet we mean a function $\psi \in L^2(\mathbb{R}^n)$ such that

$$(*) \qquad \{|\det(A)|^{\frac{n}{2}} \psi(A^n t - (l_1, l_2, \ldots, l_n)^t \colon n, l \in \mathbb{Z}\},$$

where $t = (t_1, \ldots, t_n)^t$, is an orthonormal basis for $L^2(\mathbb{R}^n; m)$. (Here m is product Lebesgue measure, and the superscript "t" means transpose.)

It is useful to introduce dilation and translation unitary operators. If $A \in M_n(\mathbb{R})$ is invertible (so in particular if A is expansive), then the operator defined by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At),$$

$f \in L^2(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, is unitary. For $1 \le i \le n$, let T_i be the unitary operator determined by translation by 1 in the i^{th} coordinate direction. The set $(*)$ is then

$$\{D_A^k T_1^{l_1} \ldots T_n^{l_n} \psi \colon k, l_i \in \mathbb{Z}\}.$$

The term orthogonal wavelet has been extended in the literature to include a "multi" notion, which is an orthonormal p-tuple $(f_1, \ldots, f_p)$ of functions in $L^2(\mathbb{R}^n)$, each of which separately generates an incomplete orthonormal set under the system of unitaries, and which together form an o.n. basis.

Let $\mathcal{F}$ be the Fourier-Plancherel transform on $L^2(\mathbb{R})$, normalized so it is a unitary transformation. For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\mathcal{F}(f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t) dt$$

and

$$\mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s) ds.$$

On $L^2(\mathbb{R}^n)$ the Fourier transform is

$$(\mathcal{F} f)(s) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(s \circ t)} f(t) dm,$$

where $s \circ t$ denotes the real inner product. Write $\hat{f} = \mathcal{F} f$, and for $A \in B(\mathbb{R}^n)$ write $\hat{A} := \mathcal{F} A \mathcal{F}^{-1}$. We have $\widehat{D}_A = D_{(A^t)^{-1}} (= D_{A^t}^{-1} = D_{A^t}^*)$, where A^t is the transpose of A, and $\widehat{T_j} = M_{e^{-is_j}}$, the multiplication operator on $\mathbb{R}^n$ with symbol $f(s_1, \ldots, s_n) = e^{-is_j}$.

By a *dilation-A wavelet set* we will mean a measurable subset of $\mathbb{R}^n$ (necessarily of finite measure) for which the inverse Fourier transform of $(m(E))^{-\frac{1}{2}}\chi_E$ is a dilation-A orthonormal wavelet.

We will say that measurable subsets H and K of $\mathbb{R}^n$ are *A-dilation congruent* if there exist measurable partitions $\{H_l\}$ of H and $\{K_l\}$ of K such that $K_l = A^l H_l$, $l \in \mathbb{Z}$, modulo Lebesgue null-sets. Write $H \sim_{\delta_A} K$. We will also say that E, F are *2π-translation congruent* (write this $E \sim_{\tau_{2\pi}} F$) if there exist measurable partitions $\{E_l\colon l = (l_1, \ldots, l_n) \in \mathbb{Z}^n\}$ of E and $\{F_l\colon l \in \mathbb{Z}^n\}$ of F such that $F_l = E_l + 2\pi l$, $l \in \mathbb{Z}^n$, modulo null sets. If W is a measurable subset of $\mathbb{R}^n$ which is 2π-translation congruent to the n-cube $E = [-\pi, \pi) \times \cdots \times [-\pi, \pi)$, it is clear from the exponential form of $\widehat{T}_j$ that $\{\widehat{T}_1^{l_1}\widehat{T}_2^{l_2}\ldots\widehat{T}_n^{l_n} \cdot (m(W))^{-\frac{1}{2}}\chi_W\colon (l_1, \ldots, l_n) \in \mathbb{Z}^n\}$ is an o.n. basis for $L^2(W)$.

If A is a strict dilation, so $\|A^{-1}\| < 1$, then $AB_1 \supseteq B_{\|A^{-1}\|^{-1}}$. It follows that if $F = AB_1 \backslash B_1$, then $\{A^k F\colon k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^n \backslash \{0\}$. If A is expansive, then A is *similar* to a strict dilation. Therefore, $A = TCT^{-1}$ for T a real invertible $n \times n$ matrix, and with $\|C^{-1}\| < 1$. If $F_A = T(CB_1 \backslash B_1)$, then $\{A^k F_A\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}^n\backslash\{0\}$. Therefore, an expansive matrix has a measurable complete *wandering set* $F_A \subset \mathbb{R}^n$. It follows that $L^2(F_A)$, considered as a subspace of $L^2(\mathbb{R}^n)$, is a complete wandering subspace for D_A. That is, $L^2(\mathbb{R}^n)$ is the direct sum decomposition of the subspaces $\{D_A^k L^2(F_A)\}_{k\in\mathbb{Z}}$. Moreover, it is clear that any measurable set F' with $F' \sim_{\delta_A} F_A$ has this same property.

Corollary 1.
Let $1 \le n < \infty$ and let $A \in M_n(\mathbb{R})$ be expansive. There exist dilation-A wavelet sets.

Proof. Let $\mathcal{A}$ be the group of homeomorphisms of $\mathbb{R}^n$ generated by the map $x \to A^t x$. Let $\mathcal{T}$ be the group of homeomorphisms of $\mathbb{R}^n$ generated by the translations in each of the coordinate directions by the integral multiples of 2π. Then A-dilation-congruency means $\mathcal{A}$-congruency and 2π-translation-congruency means $\mathcal{T}$-congruency. Moreover, it is clear that $(\mathcal{A}, \mathcal{T})$ is an abstract dilation-translation pair on $\mathbb{R}^n$ in the sense of Theorem 1, with $\theta = 0$.

By Theorem 1, a measurable set W exists with $W \sim_{\delta_{A^t}} F_{A^t}$ and $W \sim_{\tau_{2\pi}} E$, where $E = [-\pi, \pi)^n$. As remarked above, the set $\{\widehat{T}_1^{l_1}\widehat{T}_2^{l_2}\ldots\widehat{T}_n^{l_n}\widehat{\psi}_W\colon l_j \in \mathbb{Z}, 1 \le j \le n\}$ is an o.n. basis for $L^2(W)$, (with $\widehat{\psi}_W = (m(W))^{-\frac{1}{2}}\chi_W$). Therefore, since $L^2(W)$, regarded as a subspace of $L^2(\mathbb{R}^n)$, is wandering for $\widehat{D} := \widehat{D}_A = D_{A^t}^{-1}$, the set

$$\{\widehat{D}^k\widehat{T}_1^{l_1}\ldots\widehat{T}_n^{l_n}\widehat{\psi}\colon k \in \mathbb{Z}, l_j \in \mathbb{Z}, 1 \le j \le n\}$$

is an o.n. basis for $L^2(\mathbb{R}^n)$, and W is a wavelet set for A. (Moreover, by Remark 5 it follows that there is a countably infinite pairwise disjoint family of such sets.) □

A *Hardy* dyadic orthonormal wavelet is a function $\psi \in L^2(\mathbb{R})$ for which $\{2^{\frac{n}{2}}\psi(2^n t - \ell)\colon n, l \in \mathbb{Z}\}$ is an o.n. basis for the Hardy space of L^2-functions f whose Fourier transform $\hat{f}$ has support contained in $[0, \infty)$. An example is $\widehat{\psi} = (2\pi)^{-\frac{1}{2}}\chi_{[2\pi,4\pi)}$. Therefore, $[2\pi, 4\pi)$ is a Hardy wavelet set. This idea can be generalized.

Corollary 2.
Let $A \in M_n(\mathbb{R})$ be expansive, and let $M \subseteq \mathbb{R}^n$ be a measurable set of positive measures that is stable under A^t in the sense that $A^t M = M$. Suppose $M \cap F_{A^t}$ has a nonempty interior. Then there exist measurable sets $W \subset M$ with the property that, if $\widehat{\psi}_W := (2\pi)^{-\frac{n}{2}}\chi_W$, then

$$\{D_A^k T_1^{l_1}\ldots T_n^{l_n}\psi_W\colon k, l_i \in \mathbb{Z}\}$$

is an orthonormal basis for $\mathcal{F}^{-1}(L^2(M))$.

(Wavelets of this type were studied in [4] for the dyadic, $n = 1$ case, where they were called *subspace wavelets*. The concept is that they are wavelets for proper subspaces of $L^2(\mathbb{R})$.)

Proof. Apply Theorem 1, with $F = M \cap F_{A^t}$ and $E = [-\pi, \pi) \times \cdots \times [-\pi, \pi)$, obtaining W with $W \sim_{\tau_{2\pi}} E$ and $W \sim_{\delta_{A^t}} F$. Since M is A^t-stable, $W \subset M$. Also, $\{(A^t)^k W\colon k \in \mathbb{Z}\}$ is a measurable partition of M. Therefore, an argument similar to that before shows that $\{\widehat{D}_A^k \widehat{T}_1^{l_1} \ldots \widehat{T}_n^{l_n} \widehat{\psi}_W\colon k, l_i \in \mathbb{Z}\}$ is an o.n. basis for $L^2(M)$. □

The following result points out that the set of wavelet sets for any dilation is large. We will call an orthonormal wavelet for a dilation-factor $a > 1$, $a \in \mathbb{R}$, an *a-adic* orthonormal wavelet.

Corollary 3.

Let $A \in M_n(\mathbb{R})$ be expansive. Every measurable subset of $\mathbb{R}^n$ is a countable union of intersections of pairs of dilation-A wavelet sets. The family of Borel dilation-A wavelet sets generates the Borel structure of $\mathbb{R}^n$.

Proof. We first prove the a-adic case. Let $a > 1$ be arbitrary. Let $d(\cdot)$ denote the projection map from $\mathbb{R}\backslash\{0\}$ onto $F = [-a, -1) \cup [1, a)$ determined by a-dilation, and let $t(\cdot)$ denote the projection map from $\mathbb{R}$ onto $E = [-\pi, \pi)$ determined by 2π-translation. That is, for $x \in \mathbb{R}\backslash\{0\}$, $d(x)$ is the unique a-dilate of x contained in F, and for $x \in \mathbb{R}$, $t(x)$ is the unique 2π-translate of x contained in E. Note that $E \sim_{\tau_{2\pi}} [0, 2\pi) \sim_{\tau_{2\pi}} ([-2\pi, \pi) \cup [\pi, 2\pi)$. Suppose K is a measurable set in $\mathbb{R}\backslash\{0\}$ for which the restrictions $d|_K$ and $t|_K$ are one-to-one. Let $E_0 = E\backslash t(K)$ and $F_0 = F\backslash d(K)$. If E_0 contains a nhood of 0 and F_0 has a nonempty interior then by Theorem 1 and Remark 5 there are disjoint measurable sets G_1, G_2 with $G_i \sim_{\tau_{2\pi}} E_0$ and $G_i \sim_{\delta_a} F_0$, $i = 1, 2$. (By the construction in the proof of Theorem 1 (and Remark 5) if K is Borel, then these can be taken Borel.) Let $W_i = K \cup G_i$. Then $W_i \sim_{\tau_{2\pi}} E$ and $W_i \sim_{\delta_a} F$. So each W_i is an a-adic wavelet set. We have $K = W_1 \cap W_2$. We will show that each measurable set $G \subseteq \mathbb{R}$ has a measurable partition $\{G_j\}_j$ where each G_j has the property of K.

Observe that if K has the property in the above paragraph, i.e., $d(\cdot)$ and $t(\cdot)$ are 1-1, E_0 contains a nhood of 0 and F_0 has nonempty interior then every subset of K also has the property.

Suppose $0 < \alpha < \beta$, and let $J = [\alpha, \beta]$. If $\beta - \alpha < 2\pi$ then $t|_J$ is 1-1, and if $\beta < a\alpha$ then $d|_J$ is 1-1. If, in addition, J contains no integral multiple of 2π, then J satisfies the property of K above. Let $\mathcal{J}_+$ be the set of all intervals $[\alpha, \beta]$ with $0 < \alpha < \beta$, $\beta < \min\{a\alpha, \alpha + 2\pi\}$, $[\alpha, \beta] \cap 2\pi\mathbb{Z} = \phi$, α and β rational. Observe that $\cup\{J\colon J \in \mathcal{J}_+\} = (0, \infty)\backslash 2\pi\mathbb{Z}$. Let $\mathcal{J}_- = \{[-\beta, -\alpha]\colon [\alpha, \beta] \in \mathcal{J}_+\}$, and $\mathcal{J} = \mathcal{J}_+ \cup \mathcal{J}_-$. Then $\bigcup_{J \in \mathcal{J}} J = \mathbb{R}\backslash 2\pi\mathbb{Z}$. Let $J_1, J_2, \ldots$ be an enumeration of $\mathcal{J}$, and let $L_1 = J_1$, and

$$L_{j+1} = J_{j+1}\backslash(J_1 \cup \cdots \cup J_j) \quad \text{for} \quad j \geq 1.$$

Then $\{L_j\colon j \in \mathbb{N}\}$ is a measurable partition of $\mathbb{R}\backslash 2\pi\mathbb{Z}$.

Let $G \subseteq \mathbb{R}$ be a measurable set. Clearly we may assume $G \cap 2\pi\mathbb{Z} = \phi$. Let $G_j = G \cap L_j$. Then $\{G_j\}$ is a measurable partition of G satisfying our requirements. If G is Borel, then each G_j is Borel.

We adapt the previous proof to the general case. Replace F with F_{A^t}, E with the n-cube $[-\pi, \pi) \times \cdots \times [-\pi, \pi)$, and $d(\cdot)$ and $t(\cdot)$ with the corresponding projections from $\mathbb{R}^n\backslash\{0\}$ to F_{A^t} and from $\mathbb{R}^n$ to E, respectively. If $K \subset \mathbb{R}^n$ has the property in paragraph one relative to these, the same argument shows that K is the intersection of two dilation-A wavelet sets. The boundary ∂C of the n-cube $C = [0, 2\pi) \times \cdots \times [0, 2\pi)$ is an m-null set. Let $Q = \cup\{(\partial C) + 2\pi\ell\colon \ell \in \mathbb{Z}^{(n)}\}$. By construction ∂F_{A^t} is also an m-null set. If $J = B_r(y)$ is a ball in $\mathbb{R}^n$ contained in one of the annuli $(A^t)^\ell F_{A^t}$ and which is also bounded away from Q, then J satisfies the property of K. Let $\mathcal{J}$ be the set of all such balls that have rational center and radius. Enumerate $\mathcal{J}$, define L_j as above, and observe that $\{L_j\colon j \in \mathbb{N}\}$ is a partition of $\mathbb{R}^n$ modulo a null set. As above, if $G \subseteq \mathbb{R}^n$ is a measurable, the partition $\{G \cap L_j\colon j \in \mathbb{N}\}$ satisfies our requirements. □

Remark. Theorem 1 can be improved in several further ways.

- 1. It is not necessary that m be nonatomic in Theorem 1. All that is needed is that $\{\theta\}$ is not an atom for m.
- 2. The hypothesis that E contains a nhood of θ in Theorem 1 can be replaced with the hypothesis that for each $\epsilon > 0$ there exists $\delta \in \mathcal{D}$ such that $\delta(F) \subseteq E \cap B_\epsilon(\theta)$. If we let $\widetilde{F} = \cup\{\delta(F)\colon\ \delta \in \mathcal{D}\} \cup \{\theta\}$, then this is equivalent to the requirement that E contain a subset of $\widetilde{F}$ which is a nhood of θ in the relative topology of $\widetilde{F}$ in X. Remark 5 generalizes as well.
- 3. Theorem 1 remains true, in the general form of Remark 5 and 1, 2 above, if we drop the hypotheses that E and F are bounded and F is bounded away from θ. To adapt the proof, write $E = \bigcup\limits_{i=0}^{\infty} E_i$, $F = \bigcup\limits_{i=0}^{\infty} F_i$, $\{E_i\}$, $\{F_i\}$ disjoint, bounded, F_i bounded away from θ, and such that E_0 and F_0 play the role of E, F in the proof of Theorem 1; so E_0 contains a nhood of θ and F_0 has a nonempty interior. Then, for $k \geq 1$, in the k^{th} induction step (in which G_{k1} and G_{k2} are constructed), replace E with $E_0 \cup E_1 \cup \cdots \cup E_k$ and F with $F_0 \cup F_1 \cup \cdots \cup F_k$. The proof, thus modified, is easily seen to be valid.

□

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