The Structure of Shift-Invariant Subspaces of $L^2(\mathbb{R}^n)$

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Using the range function approach to shift invariant spaces in $L^2(\mathbb{R}^n)$ we give a simple characterization of frames and Riesz families generated by shifts of a countable set of generators in terms of their behavior on subspaces of $l^2(\mathbb{Z}^n)$. This in turn gives a simplified approach to the analysis of frames and Riesz families done by Gramians and dual Gramians. We prove a decomposition of a shift invariant space into the orthogonal sum of spaces each of which is generated by a quasi orthogonal generator. As an application of this fact we characterize shift preserving operators in terms of range operators and prove some facts about the dimension function. © 2000 Academic Press

Key Words: shift invariant space; range function; shift-preserving operator; range operator; dimension function; Bessel family; frame; Riesz family; dual frame.

1. INTRODUCTION

The aim of this paper is to investigate the structure of shift invariant spaces in $L^2(\mathbb{R}^n)$ under the action of some lattice $\Gamma = P\mathbb{Z}^n$, where P is a nonsingular n by n real matrix. Without loss of generality, and to simplify the proofs, we will work only with the standard lattice $\mathbb{Z}^n$. The results about general lattices follow from the corresponding $\mathbb{Z}^n$ results by standard arguments.

In the introduction we present the necessary definitions and the proof of the characterization of shift invariant spaces in terms of the range function. This result is perhaps not widely known even though it plays the central role in the L^2 theory of shift invariant spaces. The proof follows an idea from Helson's book [H] adapted to our setting. In the next section we show that shifts of a given set of functions form a frame (a Riesz family) in $L^2(\mathbb{R}^n)$ precisely when these functions form a frame (a Riesz family) with uniform constants on the fibers over the base space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ in the

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Fourier domain. This allows us to reproduce some results of Ron and Shen [RS1] involving the Gramian and the dual Gramian matrices. In the third section we show that every (even infinitely generated) shift invariant space can be decomposed as an orthogonal sum of spaces, each of which is generated by a single function whose shifts form a tight frame with constant 1. This result enables us to prove the Representation Theorem 4.5 for shift preserving operators in terms of range operators. These operators are defined on subspaces determined by a range function, they are uniformly bounded, and when glued together they satisfy a measurability condition. We also prove some properties of the dimension function. Among them the fact that two shift invariant spaces can be mapped onto each other with an iso-morphism commuting with shifts precisely when they have identical dimension functions almost everywhere. In the last section we derive a result about dual frames using a range operator approach.

DEFINITION 1.1. A closed subspace $V \subset L^2(\mathbb{R}^n)$ is *shift invariant* if $f \in V$ implies $T_k f \in V$ for any $k \in \mathbb{Z}^n$. Here $T_y f(x) = f(x - y)$ is the translation by the vector $y \in \mathbb{R}^n$. For any subset $\mathscr{A} \subset L^2(\mathbb{R}^n)$ let

$$S(\mathscr{A}) = \overline{\operatorname{span}}\{T_k f : f \in \mathscr{A}, k \in \mathbb{Z}^n\}, \tag{1.1}$$

be the shift invariant space generated by $\mathscr{A}$. If $\mathscr{A} = \{\varphi\}$ we will also write $S(\varphi) = S(\{\varphi\})$. If $V = S(\varphi)$ for some function φ we say V is *principal shift invariant* (PSI). If $V = S(\mathscr{A})$ for some finite $\mathscr{A}$ we say V is *finitely generated shift invariant* (FSI).

Convention. We will identify $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with its fundamental domain; that is, $\mathbb{T}^n = [-1/2, 1/2)^n$. The Fourier transform is given by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) \, e^{-2\pi i \langle x, y \rangle} \, dx.$$

The Hilbert space of square integrable vector functions $L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$, consists of all vector valued measurable functions $\Phi \colon \mathbb{T}^n \to l^2(\mathbb{Z}^n)$ with the norm

$$\|\Phi\| = \left(\int_{\mathbb{T}^n} \|\Phi(x)\|_{l^2}^2 \, dx\right)^{1/2} < \infty.$$

We will denote by $\{e_k : k \in \mathbb{Z}^n\}$ the standard basis of $l^2(\mathbb{Z}^n)$, and we will frequently abbreviate $l^2 = l^2(\mathbb{Z}^n)$.

PROPOSITION 1.2. $\mathscr{T}: L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ *defined for* $f \in L^2(\mathbb{R}^n)$ *by*

$$\mathscr{T}f: \mathbb{T}^n \to l^2(\mathbb{Z}^n), \qquad \mathscr{T}f(x) = (\hat{f}(x+k))_{k \in \mathbb{Z}^n}, \tag{1.2}$$

is an isometric isomorphism between $L^2(\mathbb{R}^n)$ *and* $L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$.

DEFINITION 1.3. A *range function* is a mapping

$$J: \mathbb{T}^n \to \{\text{closed subspaces of } l^2(\mathbb{Z}^n)\}.$$

J is *measurable* if the associated orthogonal projections $P(x): l^2(\mathbb{Z}^n) \to J(x)$ are weakly operator measurable; i.e., $x \mapsto \langle P(x) a, b \rangle$ is a measurable scalar function for each $a, b \in l^2(\mathbb{Z}^n)$. For a given range function J, we define the space

$$M_J = \{\Phi \in L^2(\mathbb{T}^n, l^2)): \Phi(x) \in J(x) \text{ for a.e. } x \in \mathbb{T}^n\}. \tag{1.3}$$

Remark (i). Note that in the separable Hilbert space measurability is equivalent to weak measurability. Therefore, the condition on P is equivalent to $x \mapsto P(x) a$ being vector measurable for each $a \in l^2$, or $x \mapsto P(x)(\Phi(x))$ being vector measurable for each vector measurable $\Phi: \mathbb{T}^n \to l^2$.

Remark (ii). Suppose J is a range function which is not necessarily measurable. Then M_J given by (1.3) is a closed subspace of $L^2(\mathbb{T}^n, l^2)$. Indeed, take any sequence $(\Phi_i) \subset M_J$ converging to Φ in $L^2(\mathbb{T}^n, l^2)$. We can find a subsequence (Φ_{i_j}) converging pointwise a.e. to Φ in l^2; that is,

$$\Phi_{i_j}(x) \to \Phi(x) \qquad \text{as} \quad i_j \to \infty, \quad \text{for a.e. } x \in \mathbb{T}^n.$$

Since the space $J(x)$ is closed, $\Phi \in M_J$. Hence M_J is closed. This observation will play an important role subsequently.

Remark (iii). Suppose $M_J = M_K$ for some measurable range functions J and K with associated projections P and Q, respectively. Then $J(x) = K(x)$ for a.e. $x \in \mathbb{T}^n$. Indeed, if we apply Lemma 1.4 to the constant function $\Phi(x) = e_k$, where e_k is a standard vector in l^2, then we have

$$P(x) e_k = Q(x) e_k \qquad \text{for all} \quad k \in \mathbb{Z}^n, \quad \text{for a.e. } x \in \mathbb{T}^n.$$

Therefore $P(x) = Q(x)$ for a.e. $x \in \mathbb{T}^n$

The following lemma is due to Helson in [H]. We present its proof for the sake of completeness.

LEMMA 1.4. *Let J be a measurable range function with associated projections P. Let $\mathscr{P}$ denote the orthogonal projection of $L^2(\mathbb{T}^n, l^2)$ onto M_J. Then for any $\mathscr{L}^2(\mathbb{T}^n, l^2)$*

$$(\mathscr{P}\Phi)(x) = P(x)(\Phi(x)) \qquad \text{for a.e.} \quad x \in \mathbb{T}^n. \tag{1.4}$$

Proof. Define $\mathscr{P}': L^2(\mathbb{T}^n, l^2) \to L^2(\mathbb{T}^n, l^2)$, by

$$\mathscr{P}'\Phi(x) = P(x)(\Phi(x)).$$

Note that since $\|P(x)\| \leqslant 1$ the right hand side is a measurable vector function which belongs to $L^2(\mathbb{T}^n, l^2)$. Clearly, $(\mathscr{P}')^2 = \mathscr{P}'$, and $(\mathscr{P}')^* = \mathscr{P}'$, since $P(x)$ has these properties for a.e. $x \in \mathbb{T}^n$. Therefore $\mathscr{P}'$ is an orthogonal projection with range M'. We automatically have the inclusion $M' \subset M_J$. To end the proof we must show that $M' = M_J$. Suppose, by contradiction, that there exists $0 \neq \Psi \in M_J$, which is orthogonal to M_J. Then

$$0 = \int_{\mathbb{T}^n} \langle P(x)\,\Phi(x), \Psi(x) \rangle_{l^2}\, dx$$

$$= \int_{\mathbb{T}^n} \langle \Phi(x), P(x)\,\Psi(x) \rangle_{l^2} \qquad \text{for all} \quad \Phi \in L^2(\mathbb{T}^n, l^2).$$

Since $\Psi(x) \in J(x)$, we have $\Psi(x) = P(x)\,\Psi(x) = 0$ for a.e. $x \in \mathbb{T}^n$, which is a contradiction. $\blacksquare$

The next proposition, due to Helson in [H], plays the central role in the theory of shift invariant spaces in $L^2(\mathbb{R}^n)$. Since Proposition 1.5 is a minor modification of the original theorem in [H] we present its proof.

PROPOSITION 1.5. *A closed subspace $V \subset L^2(\mathbb{R}^n)$ is shift invariant if and only if*

$$V = \{ f \in L^2(\mathbb{R}^n) : \mathscr{T}f(x) \in J(x) \text{ for a.e. } x \in \mathbb{T}^n \}, \tag{1.5}$$

where J is a measurable range function. The correspondence between V and J is one-to-one under the convention that the range functions are identified if they are equal a.e. Furthermore, if $V = S(\mathscr{A})$ for some countable $\mathscr{A} \subset L^2(\mathbb{R}^n)$, then

$$J(x) = \overline{\operatorname{span}}\{ \mathscr{T}\varphi(x) : \varphi \in \mathscr{A} \}. \tag{1.6}$$

Note that for any $f \in L^2(\mathbb{R}^n)$, $k \in \mathbb{Z}^n$

$$\mathscr{T}T_k f(x) = e^{-2\pi i \langle x, k \rangle} \mathscr{T}f(x) \qquad \text{for} \quad x \in \mathbb{T}^n. \tag{1.7}$$

Therefore $V \subset L^2(\mathbb{R}^n)$ is a shift invariant space if and only if $M \subset L^2(\mathbb{T}^n, l^2)$ is a closed subspace closed under multiplication by exponentials; i.e.,

$$\Phi(\cdot) \in M \Rightarrow e^{-2\pi i \langle \cdot, k \rangle} \Phi(\cdot) \in M \qquad \text{for all} \quad k \in \mathbb{Z}^n,$$

where $\cdot$ represents the generic variable. The correspondence is given by $V = \mathcal{T}M$.

Proof of Proposition 1.5. Suppose $V = S(\mathcal{A})$ is a shift invariant space, $M = \mathcal{T}V$, and $J(x)$ is given by (1.6). For any $\Phi \in M$ we can find a sequence (Φ_i) converging to Φ, such that

$$\mathcal{T}^{-1}\Phi_i \in \text{span}\{T_k \varphi \colon \varphi \in \mathcal{A}, k \in \mathbb{Z}^n\};$$

hence by (1.7) $\Phi_i(x) \in J(x)$. As in Remark (ii) we can conclude that $\Phi(x) \in J(x)$, and therefore $M \subset M_J$. Take any $0 \neq \Psi \in L^2(\mathbb{T}^n, l^2)$ which is orthogonal to M. For any $\Phi \in \mathcal{T}\mathcal{A}$ and $k \in \mathbb{Z}^n$, we have $e^{-2\pi i \langle \cdot, k \rangle} \Phi(\cdot) \in \mathcal{T}V$; hence

$$0 = \int_{\mathbb{T}^n} \langle e^{-2\pi i \langle x, k \rangle} \Phi(x), \Psi(x) \rangle \, dx = \int_{\mathbb{T}^n} e^{-2\pi i \langle x, k \rangle} \langle \Phi(x), \Psi(x) \rangle \, dx.$$

Therefore all Fourier coefficients of the scalar function $x \mapsto \langle \Phi(x), \Psi(x) \rangle$ vanish. Hence

$$\langle \Phi(x), \Psi(x) \rangle = 0 \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad \Phi \in \mathcal{T}\mathcal{A};$$

that is, $\Psi(x) \in J(x)^\perp$ for a.e. x. Thus there is no $0 \neq \Psi \in M_J$ which is orthogonal to M, and therefore $M = M_J$.

Finally, we need to show that J given by (1.6) is measurable. Let $\mathscr{P}$ denote the orthogonal projection of $L^2(\mathbb{T}^n, l^2)$ onto M, and let $P(x)$ be the projection onto $J(x)$. Take any $\Psi \in L^2(\mathbb{R}^n, l^2)$; then $(I - \mathscr{P})\Psi$ is orthogonal to M. By the above argument $\Psi(x) - \mathscr{P}\Psi(x) \in J(x)^\perp$ for a.e. x. Therefore,

$$P(x)(\Psi(x)) = P(x)(\mathscr{P}\Psi(x)) = \mathscr{P}\Psi(x) \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \qquad (1.8)$$

because $\mathscr{P}\Psi(x) \in J(x)$ a.e. since $M = M_J$. Take a constant vector function $\Psi(x) = a \in l^2$. Since $\mathscr{P}\Psi(x)$ is a measurable vector function, by (1.8) so is $x \mapsto P(x)a$. Therefore J is measurable.

Conversely, if we start with a measurable range function J then by Remark (ii) $V = \mathcal{T}^{-1}M_J$ is a closed shift invariant space. By Lemma 1.4 the space V clearly satisfies (1.5). The correspondence between V and J is one-to-one by Remark (iii). $\blacksquare$

COROLLARY 1.6.   *Suppose J is a range function* (*not necessarily measurable*). *Then there exists a unique measurable range function K such that $K(x) \subset J(x)$ for a.e. $x \in \mathbb{T}^n$, and $M_J = M_K$.*

DEFINITION 1.7.   The *dimension function* of V is a mapping $\dim_V \colon \mathbb{T}^n \to \mathbb{N} \cup \{0, \infty\}$ given by $\dim_V(x) = \dim J(x)$, where J is the range function associated with V. The *spectrum* of V is defined by $\sigma(V) = \{x \in \mathbb{T}^n : J(x) \neq \{0\}\}$.

Note that this spectrum has nothing to do with a spectrum of an operator even though we use the symbol σ. We use this terminology following [BDR1, BDR2].

2. FRAMES AND RIESZ FAMILIES

For $\mathscr{A} \subset L^2(\mathbb{R}^n)$ we define the family of shifts of $\mathscr{A}$ by

$$E(\mathscr{A}) = \{T_k \varphi \colon k \in \mathbb{Z}^n, \varphi \in \mathscr{A}\}.$$

DEFINITION 2.1.   Suppose $\mathscr{H}$ is a Hilbert space. $X \subset \mathscr{H}$ is a *Bessel family* with constant $B > 0$, if

$$\sum_{\eta \in X} |\langle h, \eta \rangle|^2 \leqslant B \|h\|^2 \qquad \text{for} \quad h \in \operatorname{span}(X).$$

If in addition there exist $0 < A \leqslant B$ so that

$$A \|h\|^2 \leqslant \sum_{\eta \in X} |\langle h, \eta \rangle|^2 \leqslant B \|h\|^2 \qquad \text{for} \quad h \in \operatorname{span}(X),$$

then X is a *frame* with constants A, B. It is *tight* if A, B can be chosen so that $A = B$. X is a *fundamental frame* if X is *complete* in $\mathscr{H}$, i.e. $\operatorname{span}(X)$ is dense in $\mathscr{H}$.

DEFINITION 2.2. $X \subset \mathscr{H}$ is a *Riesz family* with constants A, B, if

$$A \sum_{\eta \in X} |a_\eta|^2 \leqslant \left\| \sum_{\eta \in X} a_\eta \eta \right\|_{\mathscr{H}}^2 \leqslant B \sum_{\eta \in X} |a_\eta|^2,$$

for all (finitely supported) sequences $(a_\eta)_{\eta \in X}$. If a Riesz family X is complete in $\mathscr{H}$, we say X forms a *Riesz basis*. If we can choose $A = B = 1$, then X is an *orthogonal family*. It is an *orthogonal basis* if X is complete in $\mathscr{H}$.

Remark.   In Definition 2.1 we could replace $\operatorname{span}(X)$ by its closure and obtain an equivalent condition. In Definitions 2.1 and 2.2 we think of

$X \subset \mathscr{H}$ as a set with multiplicity; i.e., some elements may be repeated. Naturally, if X is a Riesz family, then no element of X can be repeated. Nevertheless, we must follow this convention to state correctly the main result of this section.

Theorem 2.3, which appears implicitly in the work of Ron and Shen [RS1], characterizes the system of translates $E(\mathscr{A})$ as being a Bessel family, a frame, or a Riesz family in terms of fibers.

THEOREM 2.3. *Suppose $\mathscr{A} \subset L^2(\mathbb{R}^n)$ is countable.*

(i) *$E(\mathscr{A})$ is a frame with constants A, B (or a Bessel family with constant B) if and only if $\{\mathscr{T}\varphi(x) : \varphi \in \mathscr{A}\} \subset l^2$ is a frame with constants A, B (or a Bessel family with constant B) for a.e. $x \in \mathbb{T}^n$. Moreover, $E(\mathscr{A})$ is a fundamental frame if and only if $\{\mathscr{T}\varphi(x) : \varphi \in \mathscr{A}\} \subset l^2$ is a fundamental frame for a.e. $x \in \mathbb{T}^n$.*

(ii) *$E(\mathscr{A})$ is a Riesz family with constants A, B if and only if $\{\mathscr{T}\varphi(x) : \varphi \in \mathscr{A}\} \subset l^2$ is a Riesz family with constants A, B for a.e. $x \in \mathbb{T}^n$. Moreover, $E(\mathscr{A})$ is a Riesz basis if and only if $\{\mathscr{T}\varphi(x) : \varphi \in \mathscr{A}\} \subset l^2$ is a Riesz basis of l^2 for a.e. $x \in \mathbb{T}^n$.*

Proof of (i). The key lies in the following computation:

$$
\begin{aligned}
\sum_{\varphi \in \mathscr{A}} \sum_{k \in \mathbb{Z}^n} |\langle T_k \varphi, f \rangle|^2 &= \sum_{\varphi \in \mathscr{A}} \sum_{k \in \mathbb{Z}^n} |\langle \mathscr{T} T_k \varphi, \mathscr{T} f \rangle|^2 \\
&= \sum_{\varphi \in \mathscr{A}} \sum_{k \in \mathbb{Z}^n} \left| \int_{\mathbb{T}^n} e^{-2\pi i \langle x, k \rangle} \langle \mathscr{T}\varphi(x), \mathscr{T} f(x) \rangle \, dx \right|^2 \\
&= \sum_{\varphi \in \mathscr{A}} \int_{\mathbb{T}^n} |\langle \mathscr{T}\varphi(x), \mathscr{T} f(x) \rangle|^2 \, dx. \qquad (2.1)
\end{aligned}
$$

Let J denotes the range function associated with $S(\mathscr{A})$, i.e., J is given by (1.6). Suppose $\{\mathscr{T}\varphi(x) : \varphi \in \mathscr{A}\} \subset l^2$ is a frame with constants A, B (or a Bessel family with constant B) for a.e. $x \in \mathbb{T}^n$; that is,

$$
A \|a\|^2 \leqslant \sum_{\varphi \in \mathscr{A}} |\langle \mathscr{T}\varphi(x), a \rangle|^2 \leqslant B \|a\|^2 \qquad \text{for} \quad a \in J(x). \qquad (2.2)
$$

If $f \in S(\mathscr{A})$ then $a = \mathscr{T} f(x) \in J(x)$ for a.e. $x \in \mathbb{T}^n$, and by (2.1) and (2.2), $E(\mathscr{A})$ is a frame with constants A, B (or a Bessel family with constant B).

The converse requires more work. Suppose $E(\mathscr{A})$ is a frame (or a Bessel family). Let $\{d_1, d_2, ...\}$ be a dense subset of l^2. Our aim is to show that

$$
\begin{aligned}
A \|P(x) d_i\|^2 &\leqslant \sum_{\varphi \in \mathscr{A}} |\langle \mathscr{T}\varphi(x), P(x) d_i \rangle|^2 \\
&\leqslant B \|P(x) d_i\|^2 \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad i \in \mathbb{N}. \qquad (2.3)
\end{aligned}
$$

If (2.3) fails then there exists a measurable set $D \subset \mathbb{T}^n$, with $|D| > 0$, $i_0 \in \mathbb{N}$, and $\varepsilon > 0$ such that at least one of the following two happens (or, in the Bessel case, only (2.4)):

$$\sum_{\varphi \in \mathscr{A}} |\langle \mathscr{T}\varphi(x), P(x) \, d_{i_0} \rangle|^2 > (B + \varepsilon) \, \|P(x) \, d_{i_0}\| \qquad \text{for} \quad x \in D, \qquad (2.4)$$

$$\sum_{\varphi \in \mathscr{A}} |\langle \mathscr{T}\varphi(x), P(x) \, d_{i_0} \rangle|^2 < (A - \varepsilon) \, \|P(x) \, d_{i_0}\| \qquad \text{for} \quad x \in D. \qquad (2.5)$$

Indeed, if (2.3) fails for $i = i_0$, then set $F(x) = \sum_{\varphi \in \mathscr{A}} |\langle \mathscr{T}\varphi(x), P(x) \, d_{i_0} \rangle|^2$. F is a measurable function on $\mathbb{T}^n$ and we can assume $F(x)$ is finite for a.e. x; otherwise $D = \{x \in \mathbb{T}^n : F(x) = \infty\}$ works in (2.4). At least one of the two sets below on the left hand side has nonzero measure,

$$\{x \in \mathbb{T}^n : F(x) - B \, \|P(x) \, d_{i_0}\| > 0\}$$
$$= \bigcup_{j=1}^{\infty} \{x \in \mathbb{T}^n : F(x) - (B + 1/j) \, \|P(x) \, d_{i_0}\| > 0\},$$
$$\{x \in \mathbb{T}^n : F(x) - A \, \|P(x) \, d_{i_0}\| < 0\}$$
$$= \bigcup_{j=1}^{\infty} \{x \in \mathbb{T}^n : F(x) - (A - 1/j) \, \|P(x) \, d_{i_0}\| < 0\},$$

and thus at least one of the sets in the unions has nonzero measure. Therefore either (2.4) or (2.5) holds. Suppose first that (2.4) happens. Let $f \in S(\mathscr{A})$ be given by $\mathscr{T}f(x) = \mathbf{1}_D(x) \, P(x) \, d_{i_0}$. Then by (2.1)

$$\sum_{\varphi \in \mathscr{A}} \sum_{k \in \mathbb{Z}^n} |\langle T_k \varphi, f \rangle|^2 = \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |\langle \mathscr{T}\varphi(x), \mathbf{1}_D(x) \, P(x) \, d_{i_0} \rangle|^2 \, dx$$

$$\geqslant (B + \varepsilon) \int_{\mathbb{T}^n} \mathbf{1}_D(x) \, \|P(x) \, d_{i_0}\|^2 \, dx$$

$$= (B + \varepsilon) \int_{\mathbb{T}^n} \|\mathscr{T}f(x)\|^2 = (B + \varepsilon) \, \|f\|^2,$$

which is a contradiction. A similar argument (in the frame case) shows that (2.5) cannot hold. Therefore (2.3) is true; hence (2.2) holds for a.e. $x \in \mathbb{T}^n$. The statement about a fundamental frame is an immediate consequence of Proposition 1.5. ∎

Proof of (ii). Let $(a_{\varphi, k})_{(\varphi, k) \in \mathscr{A} \times \mathbb{Z}^n}$ be any sequence with finitely many nonzero terms. Define polynomials $p_\varphi(x) = \sum_{k \in \mathbb{Z}^n} a_{\varphi, k} e^{-2\pi i \langle x, k \rangle}$. Only a

finite number of the p_φ's are non zero. By Proposition 1.2 and (1.7) we have

$$\left\| \sum_{(\varphi, k) \in \mathscr{A} \times \mathbb{Z}^n} a_{\varphi, k} T_k \varphi \right\|^2 = \left\| \sum_{(\varphi, k) \in \mathscr{A} \times \mathbb{Z}^n} a_{\varphi, k} \mathscr{T} T_k \varphi \right\|^2$$

$$= \left\| \sum_{(\varphi, k) \in \mathscr{A} \times \mathbb{Z}^n} a_{\varphi, k} e^{-2\pi i \langle \cdot, k \rangle} \mathscr{T} \varphi \right\|^2$$

$$= \int_{\mathbb{T}^n} \left\| \sum_{\varphi \in \mathscr{A}} p_\varphi(x) \, \mathscr{T} \varphi(x) \right\|^2_{l^2} dx, \qquad (2.6)$$

and by Plancherel formula

$$\sum_{(\varphi, k) \in \mathscr{A} \times \mathbb{Z}^n} |a_{\varphi, k}|^2 = \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |p_\varphi(x)|^2 \, dx. \qquad (2.7)$$

Suppose that for a.e. $x \in \mathbb{T}^n$, $\{\mathscr{T}\varphi(x) : \varphi \in \mathscr{A}\} \subset l^2$ is a Riesz family with constants A, B. Then

$$A \sum_{\varphi \in \mathscr{A}} |p_\varphi(x)|^2 \leqslant \left\| \sum_{\varphi \in \mathscr{A}} p_\varphi(x) \, \mathscr{T} \varphi(x) \right\|^2 \leqslant B \sum_{\varphi \in \mathscr{A}} |p_\varphi(x)|^2. \qquad (2.8)$$

Integrating (2.8) over $\mathbb{T}^n$ and using (2.6) and (2.7) we obtain $E(\mathscr{A})$ is also a Riesz family with the same constants.

Conversely, suppose $E(\mathscr{A})$ is a Riesz family, then by (2.6) and (2.7)

$$A \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |p_\varphi(x)|^2 \, dx \leqslant \int_{\mathbb{T}^n} \left\| \sum_{\varphi \in \mathscr{A}} p_\varphi(x) \, \mathscr{T} \varphi(x) \right\|^2_{l^2} dx$$

$$\leqslant B \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |p_\varphi(x)|^2 \, dx, \qquad (2.9)$$

where only finite number of the polynomials p_φ is nonzero.

Take any family $\{m_\varphi \in L^\infty(\mathbb{T}^n) : \varphi \in \mathscr{A}\}$ of functions with finitely nonzero m_φ's. By Luzin's Theorem, for every $\varphi \in \mathscr{A}$ we can find a sequence of polynomials $(p_\varphi^i)_{i \in \mathbb{N}}$, so that

$$\|p_\varphi^i\|_\infty \leqslant \|m_\varphi\|_\infty \qquad \text{for all} \quad i \in \mathbb{N}, \qquad \varphi \in \mathscr{A},$$

$$p_\varphi^i(x) \to m_\varphi(x) \quad \text{as } i \to \infty \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad \varphi \in \mathscr{A}.$$

By the Lebesgue Dominated Convergence Theorem we can strengthen (2.9) to

$$A \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |m_\varphi(x)|^2 \, dx \leqslant \int_{\mathbb{T}^n} \left\| \sum_{\varphi \in \mathscr{A}} m_\varphi(x) \, \mathscr{T}\varphi(x) \right\|_{l^2}^2 dx$$

$$\leqslant B \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |m_\varphi(x)|^2 \, dx. \tag{2.10}$$

Let $(d^i)_{i \in \mathbb{N}}$, where $d^i = (d_\varphi^i)_{\varphi \in \mathscr{A}} \in l^2(\mathscr{A})$, be a dense sequence in $l^2(\mathscr{A})$, with each d^i having finitely many non zero coordinates. Our goal is to show that

$$A \sum_{\varphi \in \mathscr{A}} |d_\varphi^i|^2 \leqslant \left\| \sum_{\varphi \in \mathscr{A}} d_\varphi^i \, \mathscr{T}\varphi(x) \right\|_{l^2}$$

$$\leqslant B \sum_{\varphi \in \mathscr{A}} |d_\varphi^i|^2 \qquad \text{for all} \quad i \in \mathbb{N}, \quad \text{for a.e.} \quad x \in \mathbb{T}^n. \tag{2.11}$$

If (2.11) fails then, as in the proof of (2.3), we could find a measurable set D, $|D| > 0$, $i_0 \in \mathbb{N}$, and $\varepsilon > 0$ such that at least one of the following happens:

$$\left\| \sum_{\varphi \in \mathscr{A}} d_\varphi^{i_0} \, \mathscr{T}\varphi(x) \right\|_{l^2} > (B + \varepsilon) \sum_{\varphi \in \mathscr{A}} |d_\varphi^{i_0}|^2 \qquad \text{for} \quad x \in D, \tag{2.12}$$

$$\left\| \sum_{\varphi \in \mathscr{A}} d_\varphi^{i_0} \, \mathscr{T}\varphi(x) \right\|_{l^2} < (A - \varepsilon) \sum_{\varphi \in \mathscr{A}} |d_\varphi^{i_0}|^2 \qquad \text{for} \quad x \in D. \tag{2.13}$$

Consider the family of functions $m_\varphi = d_\varphi^{i_0} \mathbf{1}_D$. If, for example, (2.12) happens then

$$\int_{\mathbb{T}^n} \left\| \sum_{\varphi \in \mathscr{A}} m_\varphi(x) \, \mathscr{T}\varphi(x) \right\|_{l^2}^2 dx = \int_D \left\| \sum_{\varphi \in \mathscr{A}} d_\varphi^{i_0} \, \mathscr{T}\varphi(x) \right\|_{l^2}^2 dx$$

$$\geqslant (B + \varepsilon) \, |D| \sum_{\varphi \in \mathscr{A}} |d_\varphi^{i_0}|^2$$

$$= (B + \varepsilon) \int_{\mathbb{T}^n} \sum_{\varphi \in \mathscr{A}} |m_\varphi(x)|^2 \, dx,$$

which contradicts (2.10). Therefore (2.11) holds.

The statement about a Riesz basis is an immediate consequence of Proposition 1.5. ∎

Theorem 2.3 allows us to reduce the problem of checking whether $E(\mathscr{A})$ is a frame, or a Riesz family, in a "big" subspace of $L^2(\mathbb{R}^n)$ to analyzing the fibers in "smaller" subspaces of $l^2(\mathbb{Z}^n)$ parameterized by the base space $\mathbb{T}^n$.

To simplify the notation enumerate the functions in $\mathscr{A} = \{\varphi_i : i \in I\}$, where $I = \{1, ..., L\}$, or $I = \mathbb{N}$. For a fixed $x \in \mathbb{T}^n$, set $t_i = \mathscr{T}\varphi_i(x) \in l^2$.

For a given family of vectors $\{t_i : i \in I\} \subset l^2$ define operator $K: l^2(I) \to l^2(\mathbb{Z}^n)$, initially on sequences $(c_i)_{i \in I}$ with compact support, by

$$K(c) = \sum_{i \in I} c_i t_i. \tag{2.14}$$

If K extends to a bounded operator then its adjoint, $K^*: l^2(\mathbb{Z}^n) \to l^2(I)$, is given by

$$K^*(a) = (\langle a, t_i \rangle)_{i \in I} \qquad \text{for} \quad a = (a_k) \in l^2(\mathbb{Z}^n). \tag{2.15}$$

Note that $\{t_i : i \in I\}$ is a Bessel family with constant B precisely when K^* is bounded and $\|K^*\|^2 \leqslant B$. Therefore, the operator K given by (2.14) is bounded if and only if K^* given by (2.15) is bounded and hence if and only if $\{t_i : i \in I\}$ is a Bessel family.

DEFINITION 2.4.   Suppose $\{t_i : i \in I\} \subset l^2(\mathbb{Z}^n)$, and K is defined by (2.14). The *Gramian* of the system $\{t_i : i \in I\}$ is $G: l^2(I) \to l^2(I)$ defined by $G = K^*K$. The *dual Gramian* of the system $\{t_i : i \in I\}$ is $\tilde{G}: l^2(\mathbb{Z}^n) \to l^2(\mathbb{Z}^n)$ defined by $\tilde{G} = KK^*$. In the case where K is unbounded, we say $\|G\| = \|\tilde{G}\| = \infty$.

Consider the Gramian and dual Gramian of $\{t_i = \mathscr{T}\varphi_i(x) : i \in I\}$ for some fixed $x \in \mathbb{T}^n$. In the standard basis $(e_i)_{i \in I}$ of $l^2(I)$ the Gramian G acts by $\langle Ge_i, e_j \rangle = \langle t_i, t_j \rangle$, for $i, j \in I$. Thus G can be associated with a matrix function

$$G = G(x) = (\langle \mathscr{T}\varphi_i(x), \mathscr{T}\varphi_j(x) \rangle)_{i, j \in I} = \left(\sum_{k \in \mathbb{Z}^n} \hat{\varphi}_i(x+k) \overline{\hat{\varphi}_j(x+k)} \right)_{i, j \in I}. \tag{2.16}$$

Similarly, in the basis $(e_k)_{k \in \mathbb{Z}^n}$ the dual Gramian $\tilde{G}$ acts by $\langle \tilde{G}e_k, e_l \rangle = \sum_{i \in I} t_i(k) \overline{t_i(l)}$, so

$$\tilde{G} = \tilde{G}(x) = \left(\sum_{i \in I} \hat{\varphi}_i(x+k) \overline{\hat{\varphi}_i(x+l)} \right)_{k, l \in \mathbb{Z}^n}. \tag{2.17}$$

Remark (i).   Note that the entries of the matrix G are always well defined. If the matrix G represents a bounded operator on $l^2(I)$, then G is

non-negative definite (self-adjoint). In this case, K is also bounded, and $\|G\| = \|K\|^2$.

Remark (ii).   However, the entries of matrix $\tilde{G}$ are meaningfully defined if, at least,

$$\sum_{i \in I} |\hat{\varphi}_i(x+k)|^2 < \infty \qquad \text{for} \quad k \in \mathbb{Z}^n. \tag{2.18}$$

If $\tilde{G}$ represents a bounded operator on $l^2(\mathbb{Z}^n)$ then $\tilde{G}$ is also non-negative definite (self-adjoint). In this case, K^* is also bounded, and $\|\tilde{G}\| = \|K^*\|^2$.

Remark (iii).   We can conveniently summarize by saying that the (un-)boundedness of one of the operators K, $K^\star$, G, $\tilde{G}$ implies the (un-)boundedness of the others. In any case $\|K\|^2 = \|K^*\|^2 = \|G\| = \|\tilde{G}\|$.

Now we can easily recover the characterization of frames and Riesz families of shift invariant systems obtained by Ron and Shen in [RS1]. For the applications of this result to wavelets, or more appropriately affine and quasi-affine systems, and Weyl–Heisenberg systems we refer the reader to [RS2, RS3].

THEOREM 2.5.   *Let* $\mathcal{A} = \{\varphi_i : i \in I\} \subset L^2(\mathbb{R}^n)$. *For* $x \in \mathbb{T}^n$ *let* $G(x)$ *and* $\tilde{G}(x)$ *denote the Gramian and dual Gramian of* $\{\mathcal{T}\varphi_i(x): i \in I\}$.

(i) $E(\mathcal{A})$ *is a Bessel family with a constant* $B \Leftrightarrow$

$$\sup_{x \in \mathbb{T}^n} \mathrm{ess} \ \|G(x)\| \leqslant B$$

$\Leftrightarrow$

$$\sup_{x \in \mathbb{T}^n} \mathrm{ess} \ \|\tilde{G}(x)\| \leqslant B.$$

(ii) $E(\mathcal{A})$ *is a frame with constants* $A, B \Leftrightarrow$

$$A \|a\|^2 \leqslant \langle \tilde{G}(x) a, a \rangle$$
$$\leqslant B \|a\|^2 \quad \text{for} \quad a \in \mathrm{span}\{\mathcal{T}\varphi_i(x): i \in I\}, \ \text{for a.e.} \ x \in \mathbb{T}^n, \tag{2.19}$$

$\Leftrightarrow$

$$\sigma(\tilde{G}(x)) \subset \{0\} \cup [A, B] \qquad \text{for a.e.} \ x \in \mathbb{T}^n. \tag{2.20}$$

Moreover, $E(\mathscr{A})$ is a fundamental frame with constants A, $B \Leftrightarrow \sigma(\tilde{G}(x)) \subset [A, B]$ for a.e. $x \in \mathbb{T}^n$.

(iii) $E(\mathscr{A})$ *is a Riesz family with constants A, $B \Leftrightarrow$*

$$A \|c\|^2 \leqslant \langle G(x)\, c, c \rangle \leqslant B \|c\|^2 \qquad for \quad c \in l^2(I), \quad for\ a.e.\ x \in \mathbb{T}^n, \qquad (2.21)$$

$\Leftrightarrow$

$$\sigma(G(x)) \subset [A, B] \qquad for\ a.e. \quad x \in \mathbb{T}^n. \qquad (2.22)$$

Moreover, $E(\mathscr{A})$ is a Riesz basis $\Leftrightarrow$ (2.21), (2.22) and $0 \notin \sigma(\tilde{G}(x))$ for a.e. $x \in \mathbb{T}^n$.

The conditions (2.19)–(2.22) are understood in the sense that any of them can possibly hold only if $G(x)$ or $\tilde{G}(x)$ are bounded for a.e. $x \in \mathbb{T}^n$.

Proof. (i) follows from Theorem 2.3(i) and the observation made before Definition 2.4.

The equivalence between $E(\mathscr{A})$ being a frame and (2.19) is a consequence of

$$\langle \tilde{G}a, a \rangle = \langle K^*a, K^*a \rangle = \sum_{i \in I} |\langle a, t_i \rangle|^2 \qquad for \quad a \in l^2(\mathbb{Z}^n),$$

and Theorem 2.3(i). To justify $(2.19) \Leftrightarrow (2.20)$, note that $\ker \tilde{G}(x) \oplus \overline{\operatorname{ran} \tilde{G}(x)} = l^2(\mathbb{Z}^n)$, because $\tilde{G}(x)$ is self-adjoint. Moreover, $\ker \tilde{G}(x) = \ker K^* = J(x)^\perp$, where $J(x)$ is the range function of $S(\mathscr{A})$. Therefore $\overline{\operatorname{ran} \tilde{G}(x)} = J(x)$, and it suffices to consider the non-negative (self-adjoint) operator $\tilde{G}(x)$ restricted to $J(x)$ to see this equivalence. Moreover, $E(\mathscr{A})$ is a fundamental frame if additionally $\ker \tilde{G}(x) = J(x)^\perp = \{0\}$ for a.e. $x \in \mathbb{T}^n$.

The equivalence between $E(\mathscr{A})$ being a Riesz family and (2.21) follows from

$$\langle Gc, c \rangle = \langle Kc, Kc \rangle = \left\| \sum_{i \in I} c_i t_i \right\|^2_{l^2} \qquad for \quad c = (c_i)_{i \in I} \in l^2(I),$$

and Theorem 2.3(ii). Because G is a non-negative definite operator, (2.21) and (2.22) are equivalent. Moreover, $E(\mathscr{A})$ is a Riesz basis if additionally $\ker \tilde{G}(x) = J(x)^\perp = \{0\}$; i.e., $0 \notin \sigma(\tilde{G}(x))$ for a.e. $x \in \mathbb{T}^n$.   ∎

3. THE DECOMPOSITION

The decomposition theorems for finitely generated shift invariant spaces into quasi regular spaces were obtained by de Boor *et al.* [BDR1].

A quasi-regular space, by definition, has a dimension function equal to some finite constant or zero. For more properties see [BDR1].

DEFINITION 3.1. Suppose we have a PSI, $V = S(\varphi)$, where $\varphi \in L^2(\mathbb{R}^n)$. We say that a function $\varphi_0 \in V$ is a *tight frame generator* or *quasi-orthogonal generator* of V if

$$\|f\|^2 = \sum_{k \in \mathbb{Z}^n} |\langle T_k \varphi_0, f \rangle|^2 \qquad \forall f \in S(\varphi). \tag{3.1}$$

We remark that Definition 3.1 establishes the convention that the zero function, $\varphi_0 = 0$, is a quasi-orthogonal generator of the trivial space $V = \{0\}$. This is needed to state correctly Theorem 3.3. Without this convention the orthogonal sum in (3.2) could consist of a finite number of components, e.g., when the dimension function of V is bounded. The following fact is an immediate consequence of Proposition 1.5 and Theorem 2.3 and is due to de Boor *et al.* in [BDR1]. For a more straightforward argument see [BL].

Fact 3.2. Suppose V is a PSI and $\varphi_0 \in V$. Then the following are equivalent:

 (i) φ_0 is a quasi orthogonal generator of V,
 (ii) $\|\mathscr{T}\varphi_0(x)\| = \mathbf{1}_{\sigma(V)}(x)$ for a.e. $x \in \mathbb{T}^n$.

The next theorem provides the decomposition of any shift invariant subspace of $L^2(\mathbb{R}^n)$ into an orthogonal sum of PSI spaces.

THEOREM 3.3. *Suppose V is a shift invariant subspace of $L^2(\mathbb{R}^n)$. Then V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i), \tag{3.2}$$

where φ_i is a quasi orthogonal generator of $S(\varphi_i)$, and $\sigma(S(\varphi_{i+1})) \subset \sigma(S(\varphi_i))$ for all $i \in \mathbb{N}$. Moreover, $\dim_{S(\varphi_i)}(x) = \|\mathscr{T}\varphi_i(x)\|$ for $i \in \mathbb{N}$, and

$$\dim_V(x) = \sum_{i \in \mathbb{N}} \|\mathscr{T}\varphi_i(x)\| \qquad for\ a.e.\quad x \in \mathbb{T}^n. \tag{3.3}$$

Proof. For any shift invariant space W we will construct function $\Phi = \Phi(W) \in \mathscr{T}W$ using the following procedure. Choose a bijection $\pi: \mathbb{N} \to \mathbb{Z}^n$. Let J denotes the range function of W, and P the corresponding projections. If $W = \{0\}$, then $\Phi(W) = 0$. Otherwise, define $A_k = \{x \in \mathbb{T}^n : P(x) e_{\pi(k)} \neq 0\}$ for $k \in \mathbb{N}$. Consider $\eta_k \in L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ defined by

$$\eta_k(x) = \begin{cases} P(x) e_{\pi(k)} / \|P(x) e_{\pi(k)}\| & x \in A_k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Define $\{B_k\}_{k \in \mathbb{N}}$ inductively by $B_1 = A_1$, $B_{k+1} = A_{k+1} \setminus \bigcup_{j=1}^k A_j$. Finally, define $\Phi \in L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ by

$$\Phi = \sum_{k \in \mathbb{N}} \eta_k \mathbf{1}_{B_k}. \tag{3.5}$$

Clearly $\Phi(x) \in J(x)$ for a.e. $x \in \mathbb{T}^n$, and $\|\Phi(x)\| = \mathbf{1}_{\sigma(W)}(x)$ for a.e. $x \in \mathbb{T}^n$, since $\sigma(W) = \bigcup_{k \in \mathbb{N}} A_k$ (modulo sets of measure zero). Hence, by Fact 3.2, $\varphi = \mathcal{T}^{-1}\Phi$ is a quasi orthogonal generator of the subspace $S(\varphi) \subset W$, and $\sigma(S(\varphi)) = \sigma(W)$. Clearly

$$\mathcal{T}(W \ominus S(\varphi)) = \{ \Psi \in L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n)) : \Psi(x) \in J(x), \langle \Phi(x), \Psi(x) \rangle$$
$$= 0 \text{ a.e. } x \in \mathbb{T}^n\}. \tag{3.6}$$

Moreover, if $\Psi \in \mathcal{T}(W \ominus S(\varphi))$ then

$$\langle \Psi(x), e_{\pi(k)} \rangle = \langle \Psi(x), P(x) e_{\pi(k)} \rangle = 0 \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad k = 1, ..., k_0, \tag{3.7}$$

where $k_0 = \min\{k \in \mathbb{N} : |A_k| \neq 0\}$.

Now we are ready to define the sequence of quasi orthogonal generators $(\varphi_k)_{k \in \mathbb{N}}$ by induction. Define $\varphi_1 = \mathcal{T}^{-1}\Phi(V)$. Suppose we constructed functions $\varphi_1, ..., \varphi_k$ having the properties

    (i) $\varphi_i \in V$ is a quasi orthogonal generator of $S(\varphi_i)$ for $i = 1, ..., k$,

    (ii)   the subspaces $S(\varphi_i)$ are mutually orthogonal for $i = 1, ..., k$,

    (iii) $\langle \Psi(x), e_{\pi(i)} \rangle = 0$ for all $i = 1, ..., k$, $\Psi \in \mathcal{T}V_k$ and a.e. $x \in \mathbb{T}^n$, where

$$V_k = V \ominus \left(\bigoplus_{i=1}^k S(\varphi_i) \right). \tag{3.8}$$

In this case set $\varphi_{k+1} = \mathcal{T}^{-1}\Phi(V_k)$. By the construction it is obvious that $\{\varphi_1, ..., \varphi_{k+1}\}$ satisfies (i)–(iii). Indeed, (i) follows from $\|\mathcal{T}\varphi_{k+1}(x)\| = \mathbf{1}_{\sigma(V_k)}(x)$ by Fact 3.2. (ii) is a consequence of $S(\varphi_{k+1}) \subset V_k$ and (3.8), whereas (iii) follows from (3.7).

Take $\Psi \in \mathcal{T}(\bigcap_{i \in \mathbb{N}} V_i)$. By (iii) we have $\langle \Psi(x), e_{\pi(i)} \rangle = 0$ for a.e. $x \in \mathbb{T}^n$, $i \in \mathbb{N}$. Thus $\Psi = 0$ and we have

$$\bigcap_{i \in \mathbb{N}} V_i = \{0\}. \tag{3.9}$$

Therefore (3.2) follows. Moreover, since $V_{i+1} \subset V_i$ we have $\sigma(S(\varphi_{i+1})) = \sigma(V_{i+1}) \subset \sigma(V_i) = \sigma(S(\varphi_i))$. Formula (3.3) is an immediate consequence of (3.2). ∎

Remark (i).   The decomposition in Theorem 3.2 of a shift invariant space V is not unique unless $\dim_V(x) \leqslant 1$ for a.e. $x \in \mathbb{T}^n$. Nevertheless, if the essential supremum of $\dim_V$ is equal to $N \in \mathbb{N}$ then the decomposition in (3.2) has N nontrivial components $S(\varphi_1), ..., S(\varphi_N)$, and $S(\varphi_i) = \{0\}$ for $i \geqslant N+1$. If $\dim_V$ is essentially unbounded then all components $S(\varphi_i)$, $i \in \mathbb{N}$ are nontrivial. Therefore, Theorem 3.3 yields always optimal (minimal) number of nontrivial components in any decomposition of V as in (3.2).

Remark (ii).   Any nontrivial PSI space $V = S(\varphi)$ can be decomposed into any, e.g. infinite, number of orthogonal PSI components. Without loss of generality, assume φ is a quasi-orthogonal generator of V. Let $D = \sigma(V) \subset \mathbb{T}^n$, $|D| > 0$. Partition $D = \bigcup_{i \in \mathbb{N}} D_i$, where $D_i \subset \mathbb{T}^n$ are mutually disjoint and $|D_i| > 0$ for $i \in \mathbb{N}$. Define $\Phi = \mathcal{T}\varphi$ and $\Phi_i = \mathbf{1}_{D_i}\Phi$ for $i \in \mathbb{N}$. Finally, let $\varphi_i = \mathcal{T}^{-1}\Phi_i$ for $i \in \mathbb{N}$. By Proposition 3.2 φ_i is a quasi orthogonal generator of $S(\varphi_i)$. We have $S(\varphi_i) \perp S(\varphi_j)$ for $i \neq j$, since $\sigma(S(\varphi_i)) \cap \sigma(S(\varphi_i)) = D_i \cap D_j = \varnothing$ for $i \neq j$. Therefore

$$V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i).$$

Remark (iii).   Our result extends a theorem in [P], where the decomposition of the form (3.2) was obtained for a shift invariant space associated with a wavelet basis in $L^2(\mathbb{R})$; that is, the space V_0 described in the Example below. As a consequence, one can drop the last condition in the definition of generalized frame multiresolution analysis (GFMRA) introduced in [P].

EXAMPLE.   Assume we have n by n *dilation* matrix A preserving $\mathbb{Z}^n$, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$, and $A\mathbb{Z}^n \subset \mathbb{Z}^n$. A finite set $\Psi = \{\psi^1, ..., \psi^L\} \subset L^2(\mathbb{R}^n)$ is called an *orthonormal multiwavelet* if the system $\{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, ..., L\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Here for $\psi \in L^2(\mathbb{R}^n)$ we use the convention

$$\psi_{j,k} = |\det A|^{j/2} \psi(A^j x - k) \qquad \text{for all} \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n. \qquad (3.10)$$

For each $j \in \mathbb{Z}$ define space

$$W_j := \overline{\text{span}}\{\psi^l_{j,k} : k \in \mathbb{Z}^n, l = 1, ..., L\}. \qquad (3.11)$$

Since spaces W_j are shift invariant for $j \geqslant 0$, so is

$$V_0 := \bigoplus_{j<0} W_j = \left(\bigoplus_{j \geqslant 0} W_j \right)^\perp. \tag{3.12}$$

Therefore

$$V_0 = \overline{\mathrm{span}}\{\psi^l(A^j(\cdot - k)): j < 0, k \in \mathbb{Z}^n, l = 1, ..., L\}. \tag{3.13}$$

By Proposition 1.5

$$V_0 = \{f \in L^2(\mathbb{R}^n) : \mathscr{T}f(x) \in J(x) \text{ for a.e. } x \in \mathbb{T}^n\}, \tag{3.14}$$

for the range function J which is given by

$$J(x) = \overline{\mathrm{span}}\{\Psi_{l,j}(x): j \geqslant 1, l = 1, ..., L\}, \tag{3.15}$$

by (3.13), where $\Psi_{l,j}(x) = |\det A|^{-j/2} \mathscr{T}\psi^l_{-j,0}(x) = (\hat{\psi}^l(B^j(x+k)))_{k \in \mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$ a.e., $B = A^T$, $l = 1, ..., L$, $j \geqslant 1$. The dimension function of V_0 can be computed explicitly, see Theorem 2.9 in [BRS],

$$\dim_{V_0}(x) = \dim J(x) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \|\Psi_{l,j}(x)\|_{l^2}$$

$$= \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} |\hat{\psi}^l(B^j(x+k))|^2 \qquad \text{a.e.} \quad x \in \mathbb{T}^n,$$

where $B = A^T$. By Theorem 3.3 the space V_0 given by (3.12) can be represented as

$$V_0 = \bigoplus_{i \in \mathbb{N}} S(\varphi_i),$$

where φ_i is a quasi orthogonal generator of $S(\varphi_i)$ for each $i \in \mathbb{N}$. Therefore, a multiwavelet $\Psi = \{\psi^1, ..., \psi^L\}$ is associated with a multiresolution analysis if and only if φ_1 is an orthogonal generator, ($\{T_k \varphi_1 : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $S(\varphi_1)$), and $\varphi_i = 0$ for $i \geqslant 2$, i.e. $\dim_{V_0}(x) = 1$ for a.e. $x \in \mathbb{T}^n$. By Proposition 2.7 in [BRS] this may happen only if $L = |\det A| - 1$.

4. SHIFT PRESERVING OPERATORS

DEFINITION 4.1.   A bounded linear operator $L: V \to L^2(\mathbb{R}^n)$ defined on a shift invariant space V is *shift preserving* if $LT_k = T_k L$ for all $k \in \mathbb{Z}^n$.

PROPOSITION 4.2. *Suppose φ is a quasi orthogonal generator of $S(\varphi)$, and a bounded operator $L: S(\varphi) \to L^2(\mathbb{R}^n)$ is shift preserving. Then for every $m \in L^2(\mathbb{T}^n)$*

$$(\mathscr{T} \circ L \circ \mathscr{T}^{-1})(m\Phi)(x) = m(x)(\mathscr{T} \circ L \circ \mathscr{T}^{-1})\,\Phi(x) \qquad \text{for a.e.} \quad x \in \mathbb{T}^n,$$
(4.1)

where $\Phi = \mathscr{T}\varphi$.

Proof. If we denote $F = \mathscr{T}f$ then by (1.7) we have

$$\begin{aligned}
(\mathscr{T} \circ L \circ \mathscr{T}^{-1})(e^{-2\pi i \langle \cdot, k \rangle} F) &= (\mathscr{T} \circ L \circ \mathscr{T}^{-1})(e^{-2\pi i \langle \cdot, k \rangle} \mathscr{T}f) \\
&= (\mathscr{T} \circ L \circ \mathscr{T}^{-1})(\mathscr{T}T_k f) \\
&= (\mathscr{T} \circ L \circ T_k)\,f = (\mathscr{T} \circ T_k \circ L)(\mathscr{T}^{-1}\mathscr{T}f) \\
&= e^{-2\pi i \langle \cdot, k \rangle}(\mathscr{T} \circ L \circ \mathscr{T}^{-1})\,F.
\end{aligned}$$

Therefore, by linearity, (4.1) holds for all polynomials $p(x) = \sum_{k \in \mathbb{Z}^n} a_k e^{-2\pi i \langle x, k \rangle} \in L^2(\mathbb{T}^n)$. Since $\mathscr{T}$ is an isometry we have $\|(\mathscr{T} \circ L \circ \mathscr{T}^{-1})\| = \|L\| = C < \infty$,

$$\begin{aligned}
\int_{\mathbb{T}^n} |p(x)|^2 \|(\mathscr{T} \circ L \circ \mathscr{T}^{-1})\,\Phi(x)\|^2\, dx &= \int_{\mathbb{T}^n} \|(\mathscr{T} \circ L \circ \mathscr{T}^{-1})(p\Phi)(x)\|^2\, dx \\
&\leqslant C^2 \int_{\mathbb{T}^n} |p(x)|^2 \|\Phi(x)\|^2\, dx \\
&= C^2 \int_{\mathbb{T}^n} |p(x)|^2 \,\mathbf{1}_{\sigma(S(\varphi))}(x)\, dx,
\end{aligned}$$
(4.2)

because $\|\Phi(x)\| = \mathbf{1}_{\sigma(S(\varphi))}(x)$ by Fact 3.2. By Luzin's Theorem, for any $r \in L^\infty(\mathbb{T}^n)$ we can find a sequence of polynomials $(p_i)_{i \in \mathbb{N}}$, so that

$$\|p_i\|_\infty \leqslant \|r\|_\infty \qquad \text{for all} \quad i \in \mathbb{N},$$
$$p_i(x) \to r(x) \quad \text{as} \quad i \to \infty \qquad \text{for a.e.} \quad x \in \mathbb{T}^n.$$

By the Lebesgue Dominated Convergence Theorem we can strengthen (4.2) to

$$\int_{\mathbb{T}^n} |r(x)|^2 \|(\mathscr{T} \circ L \circ \mathscr{T}^{-1})\,\Phi(x)\|^2\, dx \leqslant C^2 \int_{\mathbb{T}^n} |r(x)|^2 \,\mathbf{1}_{\sigma(S(\varphi))}(x)\, dx.$$

Since $r \in L^\infty(\mathbb{T}^n)$ is arbitrary we obtain

$$\|(\mathcal{T} \circ L \circ \mathcal{T}^{-1}) \, \Phi(x)\| \leqslant C \, \|\Phi(x)\| \qquad \text{for a.e.} \quad x \in \mathbb{T}^n. \qquad (4.3)$$

Finally, take polynomials $(p_i)_{i \in \mathbb{N}}$ so that $p_i \to m$ in $L^2(\mathbb{T}^n)$. By taking a subsequence we can additionally require that

$$p_i(x) \to m(x), \; (\mathcal{T} \circ L \circ \mathcal{T}^{-1})(p_i \Phi)(x) \to (\mathcal{T} \circ L \circ \mathcal{T}^{-1})(m\Phi)(x) \quad \text{as } i \to \infty, \qquad (4.4)$$

for a.e. $x \in \mathbb{T}^n$. Since (4.1) holds for polynomials then by (4.3) and (4.4) it holds for a general function $m \in L^2(\mathbb{T}^n)$. ∎

As an immediate corollary of Propositions 1.5 and 4.2 we obtain

COROLLARY 4.3. *Suppose $V \subset L^2(\mathbb{R}^n)$ is shift invariant and that a bounded operator $L: V \to L^2(\mathbb{R}^n)$ is shift preserving. For every $\Phi \in \mathcal{T}V$ and measurable function m such that $m\Phi \in L^2(\mathbb{T}^n, l^2)$ (and hence $m\Phi \in \mathcal{T}V$) we have*

$$(\mathcal{T} \circ L \circ \mathcal{T}^{-1})(m\Phi)(x) = m(x)(\mathcal{T} \circ L \circ \mathcal{T}^{-1}) \, \Phi(x) \qquad \text{for a.e.} \quad x \in \mathbb{T}^n.$$

We now introduce the concept of a range operator as a family of operators defined on fibers of the range function and satisfying the natural measurability condition.

DEFINITION 4.4. Suppose V is a shift invariant subspace of $L^2(\mathbb{R}^n)$ with the range function J and associated projection P. A *range operator* on J is a mapping

$R: \mathbb{T}^n \to$ set of bounded operators defined on closed subspaces of $l^2(\mathbb{Z}^n)$,

so that the domain of $R(x)$ equals $J(x)$ for a.e. $x \in \mathbb{T}^n$. R is *measurable* if $x \mapsto R(x) \, P(x)$ is weakly operator measurable, i.e. $x \mapsto \langle R(x) \, P(x) \, a, b \rangle$ is measurable scalar function for each $a, b \in l^2(\mathbb{Z}^n)$.

THEOREM 4.5. *Suppose $V \subset L^2(\mathbb{R}^n)$ is shift invariant and J is its range function. For every shift preserving operator $L: V \to L^2(\mathbb{R}^n)$ there exists a measurable range operator R on J such that*

$$(\mathcal{T} \circ L) \, f(x) = R(x)(\mathcal{T}f(x)) \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad f \in V. \qquad (4.5)$$

Conversely, given a measurable range operator R on J with $\sup \mathrm{ess}_{x \in \mathbb{T}^n} \|R(x)\| < \infty$ there is a bounded shift preserving operator $L: V \to L^2(\mathbb{R}^n)$ such that (4.5)

holds. The correspondence between L and R is one-to-one under the convention that the range operators are identified if they are equal a.e. Moreover, we have $\|L\| = \sup \operatorname{ess}_{x \in \mathbb{T}^n} \|R(x)\|$.

Proof. First decompose V as in Theorem 3.3, and denote $V_k = \bigoplus_{i=1}^k S(\varphi_i)$, $\Phi_i = \mathscr{T}\varphi_i$. Let J_k be the range function of the space V_k. Naturally, the set $\{\Phi_1(x), ..., \Phi_k(x)\} \setminus \{0\}$ forms an orthonormal basis of $J_k(x)$ for a.e. $x \in \mathbb{T}^n$. Note that this set might be empty, if $\Phi_1(x) = 0$, i.e., if $x \notin \sigma(V)$. Define $R_k(x): J_k(x) \to l^2(\mathbb{Z}^n)$ by

$$R_k(x)\left(\sum_{i=1}^k \alpha_i \Phi_i(x)\right) = \sum_{i=1}^k \alpha_i (\mathscr{T} \circ L \circ \mathscr{T}^{-1}) \, \Phi_i(x). \tag{4.6}$$

By virtue of (4.3) this is well defined even in the case $\Phi_i(x) = 0$ for some $i = 1, ..., k$. Take any $f \in V_k$ and write it as $f = f_1 + \cdots + f_k$, where $f_i \in S(\varphi_i)$. Then

$$\mathscr{T}f = \mathscr{T}f_1 + \cdots \mathscr{T}f_k = m_1 \Phi_1 + \cdots m_k \Phi_k,$$

for some $m_i \in L^2(\mathbb{T}^n)$, $i = 1, ..., k$. Proposition 4.2 yields

$$\begin{aligned}
(\mathscr{T} \circ L) \, f(x) &= (\mathscr{T} \circ L \circ \mathscr{T}^{-1}) \left(\sum_{i=1}^k m_i \Phi_i\right)(x) \\
&= \sum_{i=1}^k m_i(x)(\mathscr{T} \circ L \circ \mathscr{T}^{-1}) \, \Phi_i(x) \\
&= \sum_{i=1}^k m_i(x) \, R_k(x)(\Phi_i(x)) = \sum_{i=1}^k R_k(x)(m_i(x) \, \Phi_i(x)) \\
&= \sum_{i=1}^k R_k(x)(\mathscr{T}f(x)). \tag{4.7}
\end{aligned}$$

It is clear that the range operator R_k is measurable.

We claim that $\|R_k(x)\| \leqslant C$ for a.e. $x \in \mathbb{T}^n$, where $C = \|L\| < \infty$. Indeed, for any $s = (s_1, ..., s_k) \in S^{k-1} = \{ y \in \mathbb{C}^k : \|y\| = 1 \}$ define $\Psi_s \in L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ by

$$\Psi_s(x) = \sum_{i=1}^k s_i \Phi_i(x). \tag{4.8}$$

We will show that for any $s \in S^{k-1}$,

$$\sup_{x \in \mathbb{T}^n} \operatorname{ess} \|R_k(x)(\Psi_s(x))\| \leqslant C. \tag{4.9}$$

If not there would exist $\varepsilon > 0$, and a measurable set $D \subset \mathbb{T}^n$ with $|D| > 0$, so that $\|R_k(x)(\Psi_s(x))\| > C + \varepsilon$ for $x \in D$. Consider $\Psi = \Psi_s \mathbf{1}_D$, and $\psi = \mathscr{T}^{-1}\Psi \in V_k$. Since $\mathscr{T}$ is an isometry

$$\|(\mathscr{T} \circ L)\psi\| \leqslant C \|\psi\| = C \|\Psi\|.$$

On the other hand

$$\|(\mathscr{T} \circ L)\psi\|^2 = \int_{\mathbb{T}^n} \|R_k(x)(\Psi(x))\|^2 \, dx = \int_D \|R_k(x)(\Psi_s(x))\|^2 \, dx$$

$$\geqslant \int_D (C + \varepsilon)^2 \|\Psi_s(x)\|^2 \, dx = (C + \varepsilon)^2 \|\Psi\|^2,$$

which is a contradiction. Finally, let $\{s_i\}_{i \in \mathbb{N}} \subset S^k$ be a dense subset of S^k,

$$\sup_{x \in \mathbb{T}^n} \text{ess} \, \|R_k(x)\| = \sup_{x \in \mathbb{T}^n} \text{ess} \sup_{s \in S^k} \|R_k(x)(\Psi_s(x))\|$$

$$= \sup_{x \in \mathbb{T}^n} \text{ess} \sup_{i \in \mathbb{N}} \|R_k(\Psi_{s_i}(x))\| \leqslant C,$$

by (4.9).

The operators $R_k(x)$ are compatible; that is, for any $l \leqslant k \in \mathbb{N}$ we have $R_l(x) = R_k(x)|_{J_l(x)}$. Hence we can define $R(x) \colon \bigcup_{l \in \mathbb{N}} J_l(x) \to l^2(\mathbb{Z}^n)$ by $R(x)(a) = R_l(x)(a)$ if $a \in J_l(x)$ for some $l \in \mathbb{N}$. Since $\|R_l(x)\| \leqslant C$, then

$$\|R(x)(a)\| \leqslant C \|a\| \qquad \text{for} \quad a \in \bigcup_{l \in \mathbb{N}} J_l(x). \tag{4.10}$$

Since

$$\overline{\bigcup_{l \in \mathbb{N}} J_l(x)} = J(x),$$

we can extend $R(x)$ uniquely to $R(x) \colon J(x) \to l^2(\mathbb{Z}^n)$ with $\|R(x)\| \leqslant C$. Furthermore,

$$(\mathscr{T} \circ L)f(x) = R(x)(\mathscr{T}f(x)) \qquad \text{for} \quad f \in V. \tag{4.11}$$

Indeed, take any $f \in V$ and a sequence $f_k \to f$ in $L^2(\mathbb{R}^n)$, $f_k \in V_k$, so that $\mathscr{T}f_k(x) \to \mathscr{T}f(x)$, and $(\mathscr{T} \circ L)f_k(x) \to (\mathscr{T} \circ L)f(x)$ for a.e. $x \in \mathbb{T}^n$ as $k \to \infty$. Then we have

$$(\mathscr{T} \circ L)f(x) \leftarrow (\mathscr{T} \circ L)f_k(x) = R(x)(\mathscr{T}f_k(x))$$

$$\to R(x)(\mathscr{T}f(x)) \qquad \text{as} \quad k \to \infty,$$

pointwise for a.e. $x \in \mathbb{T}^n$.

Conversely, suppose we have a measurable range operator R. Take any $f \in V$. Since $x \mapsto \mathcal{T}f(x)$ is measurable and R is measurable, $\tilde{F}(x) = R(x)(\mathcal{T}f(x))$ is also measurable, and

$$\|\tilde{F}\| = \int_{\mathbb{T}^n} \|\tilde{F}(x)\|^2 \, dx \leqslant \sup_{x \in \mathbb{T}^n} \operatorname{ess} \|R(x)\|^2 \int_{\mathbb{T}^n} \|\mathcal{T}f(x)\|^2 \, dx = C^2 \|f\|^2,$$

where $C = \sup \operatorname{ess}_{x \in \mathbb{T}^n} \|R(x)\|$. Define operator $L: V \to L^2(\mathbb{R}^n)$, by $Lf = \mathcal{T}^{-1}\tilde{F}$. L is linear, bounded $\|Lf\| \leqslant C \|f\|$, and shift preserving, because

$$\begin{aligned}
(\mathcal{T} \circ L) \, T_k f(x) &= R(x)(\mathcal{T} T_k f(x)) = R(x)(e^{-2\pi i \langle x, k \rangle} \mathcal{T}f(x)) \\
&= e^{-2\pi i \langle x, k \rangle} R(x)(\mathcal{T}f(x)) = e^{-2\pi i \langle x, k \rangle} \mathcal{T}(Lf)(x) \\
&= (\mathcal{T} \circ T_k \circ L) \, f(x).
\end{aligned}$$

The uniqueness of the correspondence between L and R is an immediate consequence of (4.5). ∎

THEOREM 4.6. *Suppose L is a shift preserving operator on V and R is its corresponding range operator on J, as in Theorem 4.5. Then L is bounded from below with a constant $c > 0$; i.e.,*

$$\|Lf\| \geqslant c \|f\| \qquad \text{for all} \quad f \in V, \tag{4.12}$$

if and only if for a.e. $x \in \mathbb{T}^n$

$$\|R(x) \, a\| \geqslant c \|a\| \qquad \text{for all} \quad a \in J(x). \tag{4.13}$$

Proof. Note that by (4.5)

$$\|Lf\|^2 = \int_{\mathbb{T}^n} \|R(x)(\mathcal{T}f(x))\|^2 \, dx \qquad \text{for} \quad f \in V. \tag{4.14}$$

If (4.13) holds then by (4.14) for any $f \in V$

$$\|Lf\|^2 \geqslant \int_{\mathbb{T}^n} c^2 \|\mathcal{T}f(x)\|^2 \, dx \geqslant c^2 \|f\|^2.$$

Conversely, assume (4.12). Let $\{d_1, d_2, ...\}$ be a dense subset of $l^2(\mathbb{Z}^n)$. Our aim is to show

$$\|R(x)(P(x) \, d_i)\| \geqslant c \|P(x) \, d_i\| \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad i \in \mathbb{N}, \tag{4.15}$$

where P denotes the projection onto J. If (4.15) fails then there exists a measurable set $D \subset \mathbb{T}^n$ with $|D| > 0$, $i_0 \in \mathbb{N}$ and $\varepsilon > 0$ so that

$$\|R(x)(P(x) \, d_{i_0})\| \leqslant (c - \varepsilon) \|P(x) \, d_{i_0}\| \qquad \text{for} \quad x \in D.$$

Consider $f \in V$ given by $\mathscr{T}f(x) = \mathbf{1}_D(x) P(x) d_{i_0}$. By (4.14)

$$\|Lf\|^2 \leqslant (c - \varepsilon)^2 \int_D \|P(x) d_{i_0}\|^2 \, dx = (c - \varepsilon)^2 \int_{\mathbb{T}^n} \|\mathscr{T}f(x)\|^2 \, dx$$

$$= (c - \varepsilon)^2 \|f\|^2,$$

which contradicts (4.12). ∎

As an immediate consequence of Theorems 4.5 and 4.6 we obtain

COROLLARY 4.7. *A shift preserving operator L is an isometry if and only if its corresponding range operator $R(x)$ is an isometry for a.e. $x \in \mathbb{T}^n$.*

THEOREM 4.8. *Suppose $V \subset L^2(\mathbb{R}^n)$ is a shift invariant space with its associated range function J and that $L: V \to V$ is a shift preserving operator with its corresponding range operator R.*

(o) *The dual operator $L^*: V \to V$ is shift preserving and its corresponding range operator is R^* given by $R^*(x) = (R(x))^*$ for a.e. $x \in \mathbb{T}^n$.*

(i) *Let $A \leqslant B$ be two real numbers. L is self-adjoint and $\sigma(L) \subset [A, B]$ if and only if $R(x)$ is self-adjoint and $\sigma(R(x)) \subset [A, B]$ for a.e. $x \in \mathbb{T}^n$.*

(ii) *L is unitary if and only if $R(x)$ is unitary for a.e. $x \in \mathbb{T}^n$.*

Proof of (o). Clearly R^* is a measurable and uniformly bounded range operator on J. By the virtue of Theorem 4.5 there exists a corresponding shift preserving operator $\tilde{L}$; i.e. $(\mathscr{T} \circ \tilde{L}) f(x) = (R(x))^* (\mathscr{T}f(x))$ for $f \in V$. Take any $f, g \in V$ then

$$\langle Lf, g \rangle = \langle (\mathscr{T} \circ L) f, \mathscr{T}g \rangle = \int_{\mathbb{T}^n} \langle R(x)(\mathscr{T}f(x), \mathscr{T}g(x)) \rangle \, dx$$

$$= \int_{\mathbb{T}^n} \langle \mathscr{T}f(x), R(x)^* (\mathscr{T}g(x)) \rangle \, dx = \langle \mathscr{T}f, (\mathscr{T} \circ \tilde{L}) g \rangle = \langle f, \tilde{L}g \rangle;$$

hence $\tilde{L} = L^*$. ∎

Proof of (i). By (o) L is self-adjoint if and only if $R(x)$ is self-adjoint for a.e. $x \in \mathbb{T}^n$. Assume that $\sigma(L) \subset [A, B]$; that is,

$$A \|f\|^2 \leqslant \langle Lf, f \rangle = \int_{\mathbb{T}^n} \langle R(x)(\mathscr{T}f(x)), \mathscr{T}f(x) \rangle \, dx$$

$$\leqslant B \|f\|^2 \qquad \text{for all} \quad f \in V. \tag{4.16}$$

Let $\{d_1, d_2, ...\}$ be a dense subset of $l^2(\mathbb{Z}^n)$. Our aim is to show

$$A \|P(x) d_i\|^2 \leq \langle R(x)(P(x) d_i), P(x) d_i \rangle$$
$$\leq B \|P(x) d_i\|^2 \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad i \in \mathbb{N}, \qquad (4.17)$$

where P is the projection onto J. If (4.17) fails then, similarly to the proof of (2.3), there exists a measurable set $D \subset \mathbb{T}^n$ with $|D| > 0$, $i_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that at least one of the following two happens

$$\langle R(x)(P(x) d_i), P(x) d_{i_0} \rangle > (B + \varepsilon) \|P(x) d_{i_0}\|^2 \qquad \text{for} \quad x \in D,$$
$$\langle R(x)(P(x) d_i), P(x) d_{i_0} \rangle < (A - \varepsilon) \|P(x) d_{i_0}\|^2 \qquad \text{for} \quad x \in D.$$

By considering $f \in V$ given by $\mathcal{T}f(x) = \mathbf{1}_D(x) P(x) d_{i_0}$ we obtain a contradiction with (4.16).

Conversely, assume that $\sigma(R(x)) \subset [A, B]$ for a.e. $x \in \mathbb{T}^n$. Hence for all $f \in V$

$$A \|\mathcal{T}f(x)\|^2 \leq \langle R(x)(\mathcal{T}f(x)), \mathcal{T}f(x) \rangle \leq B \|\mathcal{T}f(x)\|^2 \qquad \text{for a.e.} \quad x \in \mathbb{T}^n.$$

An integration of the above over $\mathbb{T}^n$ yields (4.16). ∎

Proof of (ii). The operator LL^* (or L^*L) is shift preserving and self-adjoint and its corresponding range operator is $R(\cdot) R(\cdot)^*$ (or $R(\cdot)^* R(\cdot)$). Hence, by (i), L is unitary, i.e. $\sigma(LL^*) = \sigma(L^*L) = \{1\}$ if and only if $\sigma(R(x) R(x)^*) = \sigma(R(x)^* R(x)) = \{1\}$ for a.e. $x \in \mathbb{T}^n$; that is $R(x)$ is unitary for a.e. $x \in \mathbb{T}^n$. ∎

At the end of this section we investigate properties of the dimension function of a shift invariant space under the action of a shift preserving operator.

THEOREM 4.9. *Suppose* $V \subset L^2(\mathbb{R}^n)$ *is shift invariant and* $L: V \to L^2(\mathbb{R}^n)$ *is shift preserving. Let* $V' = \overline{L(V)}$ *then*

$$\dim_{V'}(x) \leq \dim_V(x) \qquad \textit{for a.e.} \quad x \in \mathbb{T}^n. \qquad (4.18)$$

Proof. By Theorem 3.3, $V = S(\mathscr{A})$ for some $\mathscr{A} = \{\varphi_1, \varphi_2, ...\}$ so that the dimension function $\dim_V(x) = \#\{i \in \mathbb{N} : \mathcal{T}\varphi_i(x) \neq 0\}$. Since $V' = S(\mathscr{B})$, where $\mathscr{B} = \{L\varphi_1, L\varphi_2, ...\}$, by Proposition 1.5 and Theorem 4.5 the range function of V' satisfies for a.e. $x \in \mathbb{T}^n$

$$J'(x) = \overline{\operatorname{span}}\{\mathcal{T}\psi(x): \psi \in \mathscr{B}\} = \overline{\operatorname{span}}\{(\mathcal{T} \circ L) \varphi(x) : \varphi \in \mathscr{A}\}$$
$$= \overline{\operatorname{span}}\{R(x)(\mathcal{T}\varphi(x)): \varphi \in \mathscr{A}\} = \overline{R(x)(J(x))}.$$

Therefore $\dim J'(x) \leq \dim J(x)$. ∎

THEOREM 4.10. *Suppose $V, V' \subset L^2(\mathbb{R}^n)$ are shift invariant. Then there is a shift preserving isomorphism (or isometry) $L: V \to V'$ if and only if $\dim_V(x) = \dim_{V'}(x)$ for a.e. $x \in \mathbb{T}^n$.*

Proof. Suppose we have a shift preserving isomorphism $L: V \to V'$. Then by Theorem 4.9 $\dim_{V'} \leqslant \dim_V$. The inverse $L^{-1}: V' \to V$ is also shift preserving hence $\dim_V \leqslant \dim_{V'}$. Therefore the dimension functions must be equal.

Conversely, suppose we have shift invariant spaces V and V' with $\dim_V = \dim_{V'}$. Decompose V, V' as in Theorem 3.3,

$$V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i), \qquad V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i'),$$

where φ_i, φ_i' are quasi-orthogonal, and by Theorem 3.3 $\sigma(S(\varphi_i)) = \sigma(S(\varphi_i'))$ (modulo sets of measure zero) for all $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$ define $L_i: S(\varphi_i) \to S(\varphi_i')$ by $L_i(T_k \varphi_i) = T_k \varphi_i'$. That definition extends to the whole $S(\varphi_i)$ since for any $c \in l^2(\mathbb{Z}^n)$ with finite number of nonzero coordinates

$$\left\| \sum_{k \in \mathbb{Z}^n} c_k T_k \varphi_i \right\|^2 = \left\| \sum_{k \in \mathbb{Z}^n} c_k e^{-2\pi i \langle \cdot, k \rangle} \mathscr{T} \varphi_i \right\|^2$$

$$= \int_{\mathbb{T}^n} \left| \sum_{k \in \mathbb{Z}^n} c_k e^{-2\pi i \langle x, k \rangle} \right|^2 \| \mathscr{T} \varphi_i(x) \|^2 \, dx$$

$$= \int_{\mathbb{T}^n} \left| \sum_{k \in \mathbb{Z}^n} c_k e^{-2\pi i \langle x, k \rangle} \right|^2 \| \mathscr{T} \varphi_i'(x) \|^2 \, dx$$

$$= \left\| \sum_{k \in \mathbb{Z}^n} c_k T_k \varphi_i' \right\|^2$$

$$= \left\| L_i \left(\sum_{k \in \mathbb{Z}^n} c_k T_k \varphi_i \right) \right\|^2,$$

because $\| \mathscr{T} \varphi_i(x) \| = \dim_{S(\varphi_i)}(x) = \dim_{S(\varphi_i')}(x) = \| \mathscr{T} \varphi_i'(x) \|$ for a.e. $x \in \mathbb{T}^n$ by Theorem 3.3. Thus L_i extends to a shift preserving isometry between $S(\varphi_i)$ and $S(\varphi_i')$. Therefore L, given by

$$L = \bigoplus_{i \in \mathbb{N}} L_i: \bigoplus_{i \in \mathbb{N}} S(\varphi_i) \to \bigoplus_{i \in \mathbb{N}} S(\varphi_i'),$$

is a shift preserving isometry of V and V'. ∎

Remark. One can "reverse" Theorem 4.9. Suppose we have two shift invariant spaces V and V' satisfying (4.18). Then there are a shift preserving operator from V onto V' and another one-to-one shift preserving operator from V' into V. To see this decompose V and V' as in the proof of Theorem 4.10 and note that we have $\sigma(S(\varphi'_i)) \subset \sigma(S(\varphi_i))$ (modulo sets of measure zero) for $i \in \mathbb{N}$.

5. DUAL FRAMES

Suppose $\mathscr{A} = \{\varphi_i : i \in I\}$ is a countable family of functions in $L^2(\mathbb{R}^n)$ and $E(\mathscr{A}) = \{T_k\varphi : k \in \mathbb{Z}, i \in I\}$ is a Bessel family. Define operator $F: S(\mathscr{A}) \to l^2(\mathbb{Z}^n \times I)$ by

$$Ff = (\langle f, T_k\varphi_i \rangle)_{(k, i) \in \mathbb{Z}^n \times I} \qquad \text{for} \quad f \in S(\mathscr{A}). \tag{5.1}$$

The dual of F is $F^*: l^2(\mathbb{Z}^n \times I) \to S(\mathscr{A})$ given by

$$F^*c = \sum_{(k, i) \in \mathbb{Z}^n \times I} c_{k, i} T_k\varphi_i \qquad \text{for} \quad c = (c_{k, i}) \in l^2(\mathbb{Z}^n \times I). \tag{5.2}$$

Clearly $L := F^*F$ is a self-adjoint (non-negative definite) operator on $S(\mathscr{A})$ which is sometimes referred as a *frame operator*. Indeed, $E(\mathscr{A})$ is a frame with constants A, B if and only if

$$A \|f\|^2 \leqslant \langle Lf, f \rangle \leqslant B \|f\|^2 \qquad \text{for} \quad f \in S(\mathscr{A}),$$

if and only if $\sigma(L) \subset [A, B]$.

THEOREM 5.1. *Suppose $E(\mathscr{A})$ is a Bessel family and J is the range function of $S(\mathscr{A})$. The operator $L = F^*F$ is self-adjoint and shift preserving with the corresponding range operator $R(x) := \tilde{G}(x)|_{J(x)}$, where $\tilde{G}(x)$ is the dual Gramian of $\{\mathscr{T}\varphi_i(x) : i \in I\}$ for a.e. $x \in \mathbb{T}^n$.*

Proof. Note that

$$Lf = \sum_{(k, i) \in \mathbb{Z}^n \times I} \langle f, T_k\varphi_i \rangle T_k\varphi_i, \tag{5.3}$$

where the convergence is unconditional in $L^2(\mathbb{R}^n)$. By a simple calculation $LT_l = T_lL$ for all $l \in \mathbb{Z}^n$. Let $\tilde{R}$ denotes the range operator of L. Therefore

$$\|Ff\|^2 = \langle Lf, f \rangle = \int_{\mathbb{T}^n} \langle \tilde{R}(x)(\mathscr{T}f(x)), \mathscr{T}f(x) \rangle \, dx \qquad \text{for} \quad f \in S(\mathscr{A}). \tag{5.4}$$

On the other hand (2.1) says that

$$\|Ff\|^2 = \int_{\mathbb{T}^n} \langle \tilde{G}(x)(\mathscr{T}f(x)), \mathscr{T}f(x) \rangle \, dx. \tag{5.5}$$

Consider the (self-adjoint) range operator $R(x) := \tilde{G}(x)|_{J(x)}$. By (5.4) and (5.5) we have

$$\int_{\mathbb{T}^n} \langle (\tilde{R}(x) - R(x))(\mathscr{T}f(x)), \mathscr{T}f(x) \rangle dx = 0 \qquad \text{for all} \quad f \in \mathscr{S}(A).$$

Therefore by Theorem 4.8(i) the self-adjoint shift preserving operator associated with $\tilde{R}(\cdot) - R(\cdot)$ is identically zero and thus $\sigma(\tilde{R}(x) - R(x)) = \{0\}$ for a.e. $x \in \mathbb{T}^n$; i.e., $\tilde{R} = R$. ∎

By virtue of Theorem 5.1 the result about dual Gramian analysis of shift invariant systems due to Ron and Shen; i.e., Theorem 2.5(ii) can be thought as a special case of the general result about range operators, i.e., Theorem 4.8(i).

Finally we present one result about a dual frame to a given shift invariant frame $E(\mathscr{A})$. Some other results about dual frames are contained in [RS1, RS4], where the notion of a *mixed Gramian* is introduced. For generalities about frames and its duals we refer the reader to Section 3.2 in Daubechies' book [D].

THEOREM 5.2. *Suppose $E(\mathscr{A})$ is a frame with constants A, B. Its dual frame with constants B^{-1}, A^{-1} is of the form $E(\tilde{\mathscr{A}})$, where $\tilde{A} = \{\tilde{\varphi}_i : i \in I\}$ and $\tilde{\varphi}_i = (F^*F)^{-1} \varphi_i$, F and F^* are given by (5.1) and (5.2). Moreover,*

$$\mathscr{T}\tilde{\varphi}_i(x) = R(x)^{-1} (\mathscr{T}\varphi_i(x)) \qquad \text{for a.e.} \quad x \in \mathbb{T}^n, \quad i \in I, \tag{5.6}$$

where R is the range operator associated with F^*F, i.e. $R(x) = \tilde{G}(x)|_{J(x)}$ and $\tilde{G}(x)$ is the dual Gramian of $\{\mathscr{T}\varphi_i(x) : i \in I\}$ for a.e. $x \in \mathbb{T}^n$.

Proof. The *dual frame* of $E(\mathscr{A})$ is a system $\{\tilde{\varphi}_{k,i} : k \in \mathbb{Z}^n, i \in I\}$, where $\tilde{\varphi}_{k,i} = L^{-1}T_k\varphi_i$, $L := F^*F$. By Theorem 5.1 L is shift preserving, so is L^{-1} and thus the dual frame of $E(\mathscr{A})$ is also a shift invariant system of the form $E(\tilde{\mathscr{A}})$, where $\tilde{\mathscr{A}} = \{\tilde{\varphi}_i = L^{-1}\varphi_i : i \in I\}$. By Proposition 3.2.3 in [D], $E(\tilde{\mathscr{A}})$ is a frame with constants B^{-1}, A^{-1} and the following reconstruction formula holds

$$\sum_{(k,i) \in \mathbb{Z}^n \times I} \langle f, T_k\tilde{\varphi}_i \rangle T_k\varphi_i$$

$$= f = \sum_{(k,i) \in \mathbb{Z}^n \times I} \langle f, T_k\varphi_i \rangle T_k\tilde{\varphi}_i \qquad \text{for all} \quad f \in S(\mathscr{A}), \tag{5.7}$$

where the convergence is unconditional in $L^2(\mathbb{R}^n)$. Moreover, by Theorem 5.1 the range operator of L^{-1} is $R(\cdot)^{-1}$ and hence we have (5.6). ∎

Remark. In particular, if $E(\mathscr{A})$ is a Riesz family with constants A, B then its dual system $E(\tilde{\mathscr{A}})$ defined in Theorem 5.2 is also a Riesz family with constants B^{-1}, A^{-1}. Furthermore, by (5.7) we have

$$\langle T_k \varphi_i, T_l \tilde{\varphi}_j \rangle = \delta_{k,l} \delta_{i,j} \qquad \text{for} \quad k, l \in \mathbb{Z}^n, \quad i, j \in I.$$

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