TIGHT FRAMES AND GEOMETRIC PROPERTIES OF WAVELET SETS

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1 Introduction

1.1 Background and wavelet sets

$\widehat{\mathbb{R}}^d$ is Euclidean space, $\mathbb{R}^d$, considered as the domain of Fourier transforms F of functions defined on $\mathbb{R}^d$. $\mathbb{1}_K$ denotes the characteristic function of a measurable set $K \subseteq \widehat{\mathbb{R}}^d$ of finite measure. $K \subseteq \widehat{\mathbb{R}}^d$ is a *wavelet set* if the function $\psi \equiv \check{\mathbb{1}}_K$ is a *single dyadic orthonormal wavelet*, i.e., if the family of dilates of translates

$$\left\{\psi_{m,n}(\cdot) \equiv 2^{md/2}\psi(2^m \cdot - n) : m \in \mathbb{Z}, n \in \mathbb{Z}^d\right\}$$

is an orthonormal basis (ONB) for $L^2(\mathbb{R}^d)$. $\check{\mathbb{1}}_K$ denotes the inverse Fourier transform of $\mathbb{1}_K$, and the inverse Fourier transform of $F \in L^2(\widehat{\mathbb{R}}^d)$ is formally defined as

$$\check{F}(x) \equiv \int_{\widehat{\mathbb{R}}^d} F(\xi) e^{2\pi i x \cdot \xi}\, d\xi \quad \text{for } x \in \mathbb{R}^d.$$

We shall prove geometric properties of wavelet sets and construct associated tight frames (tight frames are a generalization of ONBs). Specific results are listed in Subsection 1.3.

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Wavelet ONBs defined in terms of wavelet sets provide a different approach to multidimensional wavelet theory than the *multiresolution analysis* (MRA) of Meyer, Mallat, Daubechies, as well as the engineering community, e.g., [Dau92], [Mal85], [Mal89], [Mey86], [Mey87], [Mey89], [Mey90]. On the other hand, a famous example of a single dyadic orthonormal wavelet in $L^2(\mathbb{R})$, constructed from an MRA, is the *Shannon wavelet* defined by the wavelet set $\left[-1,-\frac{1}{2}\right) \cup \left[\frac{1}{2},1\right)$. It should also be pointed out the recent theory of generalized frame multiresolution analysis (GFMRA) is a means of unifying the MRA and wavelet set theories, but it does not change the fact that single dyadic orthonormal wavelets in $L^2(\mathbb{R}^d)$, $d > 1$, are not an output of any MRA of $L^2(\mathbb{R}^d)$. The fundamental ideas of GFMRA are formed in [PS98], [Pap01], [Pap03], as well as in [BMM99], [BM99], [Cou99] which are also cited below only in the context of wavelet sets.

Remark 1. A set of functions $\psi^1, \psi^2, \ldots, \psi^M \in L^2(\mathbb{R}^d)$ is a *wavelet collection* for $L^2(\mathbb{R}^d)$ if

$$\left\{\psi^i_{m,n}(\cdot) \equiv 2^{md/2}\psi^i(2^m \cdot -n) : m \in \mathbb{Z}, n \in \mathbb{Z}^d, i = 1, \ldots, M\right\}$$

is an ONB for $L^2(\mathbb{R}^d)$. Auscher [Aus95] proved that every wavelet collection $\{\psi^1, \psi^2, \ldots, \psi^M\}$ for $L^2(\mathbb{R}^d)$, whose members satisfy a weak smoothness and decay condition on the Fourier transform side, must come from an MRA. Further, it is known, e.g., see [Aus95], [Gri95], [Mey86], that for a given dyadic MRA there is a wavelet collection consisting of $2^d - 1$ elements. There is an analogous assertion for the expansive matrix case.

Because of this remark, and notwithstanding Journé's celebrated example of a non-MRA wavelet basis for $L^2(\mathbb{R})$, e.g., [Dau92], there was some question during the mid-1990s about the existence of multidimensional single dyadic orthonormal wavelets. Dai, Larson, and Speegle [DLS97] proved the existence of such wavelets in $L^2(\mathbb{R}^d)$, $d > 1$. Their proof depended on wavelet sets and used operator algebra methods. Some of the initial reaction was a combination of disbelief and disinterest, the latter response due to the prevailing intuition that such wavelets would be difficult to implement in an effective way.

Remark 2. a. The wavelets constructed in [DLS97] are a consequence of a major theory developed by Dai and Larson [DL98]. The first draft of [DL98] appeared in 1993, and the authors defined the term *wavelet set* in print in 1992. Dai, Larson, and Speegle [DLS98] as well as Soardi and Weiland [SW98] later constructed several wavelet sets in $\widehat{\mathbb{R}}^2$; there are also examples from the same period due to Zakharov [Zak96].

b. Baggett, Medina, and Merrill [BMM99] introduced a theoretical paradigm to obtain all wavelet sets and subspace wavelet sets. Their analysis begins with a *complementary pair* of maps satisfying some intertwining properties; see [BS02] for a comparison of this method with the neighborhood-mapping construction of Subsection 1.2. There is also subsequent work by Baggett and Merrill [BM99] and Courter [Cou99].

The neighborhood-mapping construction requires the following theorem for its role in constructing wavelet sets.

Theorem 3. *A necessary and sufficient condition for the existence of a wavelet set K in $\widehat{\mathbb{R}}^d$ is given by the tiling properties:*

$$\left\{K+n : n \in \mathbb{Z}^d\right\} \text{ is a partition for } \widehat{\mathbb{R}}^d \text{ a.e.,} \tag{1 a}$$

$$\{2^m K : m \in \mathbb{Z}\} \text{ is a partition for } \widehat{\mathbb{R}}^d \text{ a.e.} \tag{1 b}$$

For an elementary geometric proof see [BL99]. The existence of such sets is not obvious.

1.2 The neighborhood-mapping construction

A general, elementary, and implementable construction of wavelet sets in $\widehat{\mathbb{R}}^d$ was introduced by Benedetto and Leon [BL99], [BL01] in 1997. They begin with a fixed number N and a neighborhood of the origin $K_0 \subseteq [-N, N]^d$, which is τ*-congruent* to the unit cube $\mathrm{Q} \equiv [-\frac{1}{2}, \frac{1}{2}]^d$. This means that there exist countable measurable partitions $\{P_m : m \in \mathbb{Z}\}$ and $\{Q_m : m \in \mathbb{Z}\}$ of K_0 and Q, respectively, and a sequence $\{n_m : m \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that $P_m = Q_m + n_m$ for all $m \in \mathbb{Z}$. The concept of δ-congruence will also be used in this manuscript.

We say that P is *δ-congruent* to Q if there exist countable measurable partitions $\{P_m : m \in \mathbb{Z}\}$ and $\{Q_m : m \in \mathbb{Z}\}$ of P and Q, respectively, and a sequence $\{j_m : m \in \mathbb{Z}\} \subseteq \mathbb{Z}$ such that $P_m = 2^{j_m} Q_m$ for all $m \in \mathbb{Z}$.

K_0 is then successively transformed by a given measurable injective $\mathbb{Z}^d$-translated map,

$$\begin{aligned} T : K_0 &\to \overline{[-2N, 2N]^d \setminus [-N, N]^d} \\ \gamma &\mapsto \gamma + n_\gamma \quad \text{for some } n_\gamma \in \mathbb{Z}^d, \end{aligned}$$

in the following way. We first define

$$A_0 = K_0 \cap \left[\bigcup_{j \geq 1} 2^{-j} K_0 \right] \quad \text{and} \quad K_1 = [K_0 \setminus A_0] \cup T(A_0).$$

For a given integer $n \geq 1$, A_n and K_{n+1} are defined by induction as follows.

$$\begin{aligned} A_n &= K_n \cap \left[\bigcup_{j \geq 1} 2^{-j} K_n \right] \\ K_{n+1} &= (K_n \setminus A_n) \cup T(A_n) \\ &= \left[K_0 \setminus \bigcup_{k=0}^{n} A_k \right] \cup \left[\bigcup_{k=0}^{n} T(A_k) \right]. \end{aligned}$$

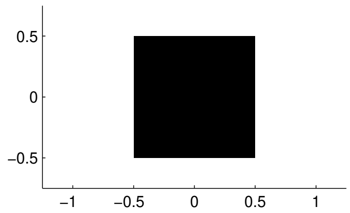
The last equality follows from the assumption that the range of T is contained in $\overline{[-2N, 2N]^d \setminus [-N, N]^d}$. Finally, we define

$$K = \left[K_0 \setminus \bigcup_{k=0}^{\infty} A_k \right] \cup \left[\bigcup_{k=0}^{\infty} T(A_k) \right]. \tag{2}$$

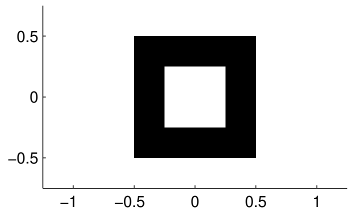
It is shown that K satisfies tiling conditions in (1 a) and (1 b), and hence K is a wavelet set. This is the construction in [BL01], and we shall call it the *neighborhood-mapping construction.*

When $K_0 = \mathrm{Q}$ and $T(\gamma_1, \gamma_2) = (\gamma_1 - \operatorname{sign}(\gamma_1), \gamma_2)$, the sets K_n and $K_n \setminus A_n$ are illustrated in Figure 1, and the sets $\bigcup_{n=0}^{6} A_n$ and $\bigcup_{n=0}^{6} T(A_n)$ are shown in Figures 2 and 3 respectively.

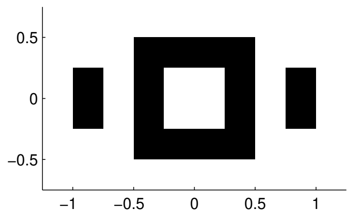
The theme of this paper is the neighborhood-mapping construction; and the tight frames and geometric properties which we construct and prove revolve around this theme.



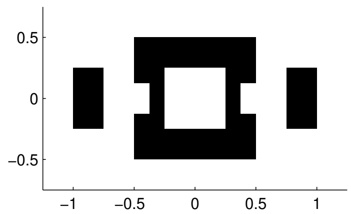
(a) K_0



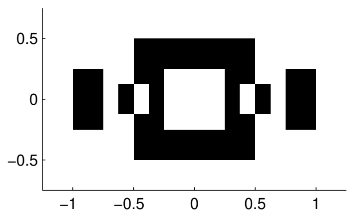
(b) $K_0 \setminus A_0$



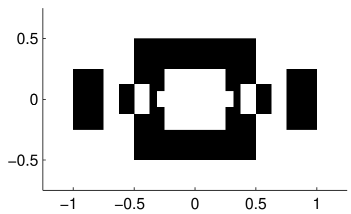
(c) K_1



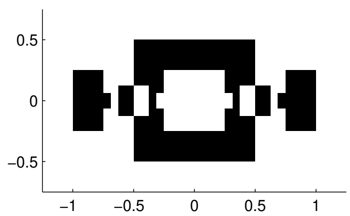
(d) $K_1 \setminus A_1$



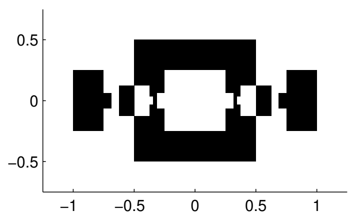
(e) K_2



(f) $K_2 \setminus A_2$



(g) K_3



(h) $K_3 \setminus A_3$

Figure 1: A wavelet set

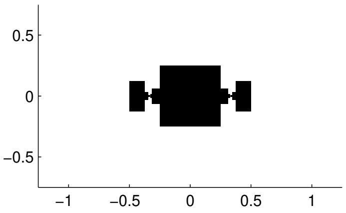


Figure 2: $\bigcup_{n=0}^{6} A_n$

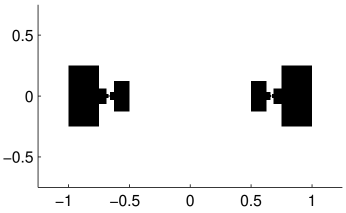


Figure 3: $\bigcup_{n=0}^{6} T(A_n)$

1.3 Results and outline

In Section 2, the neighborhood-mapping construction of wavelet sets is proved to work in a more general setting than previously known in [BL99], [BL01]. In particular, the wedding cake set in [DLS98] is constructible by the neighborhood-mapping construction. Also, Journé's non-MRA wavelet, which is the inverse Fourier transform of the characteristic function of the wavelet set $\left[-2\frac{2}{7}, -2\right) \cup \left[-\frac{1}{2}, -\frac{2}{7}\right) \cup \left[\frac{2}{7}, \frac{1}{2}\right) \cup \left[2, 2\frac{2}{7}\right)$ on the Fourier transform side, is constructible by the neighborhood-mapping construction. We then present a d-dimensional Journé wavelet set and more examples of multidimensional wavelet sets, one of which is a union of two connected sets with connected interiors.

In Section 3, we prove that each finite iteration in the neighborhood-mapping construction of wavelet sets gives rise to a tight frame wavelet set. As such, even though the generally complicated wavelet sets corresponding to the ONBs of Section 2 may be suspect as to their genuine applicability, these tight frames can have simple frequency components at early stages of the recursion.

Section 4 is devoted to the proof that a wavelet set in $\widehat{\mathbb{R}}^d$ can not be a union of d or fewer convex sets. In Section 5 we quantify the generality of the neighborhood-mapping construction for wavelet sets contained in $[-1, 1]^d$; and we consider some special properties of wavelet sets in $[-1, 1]^d$ in Section 6.

It remains to analyze the generality of the neighborhood-mapping construction (beyond what we did in Section 5) because of its simplicity vis a vis the generality and relative abstractness of [BMM99]. It should also be pointed out that this construction can certainly be extended to more general dilations and translations. However, it is not at all clear how general this intuitive construction can go. Recently, Speegle [Spe02] and Wang [Wan02] have studied the existence problem of wavelet sets associated with large classes of dilations and translations in $\widehat{\mathbb{R}}^d$. Another question is motivated by our result in Section 4. We know that a d-dimensional wavelet set can not be a union of d or fewer convex sets. Though this does not prevent a union of more than d convex sets to be a wavelet set, we could not produce an example. In fact, we believe that a wavelet set in $\widehat{\mathbb{R}}^d$ can not be a finite union of convex sets.

Our notation is standard, but we point out that $\dot{\bigcup}$ represents a disjoint union, $|A|$ denotes the Lebesgue measure of $A \subseteq \widehat{\mathbb{R}}^d$, $\operatorname{sign}(\gamma) \equiv (\operatorname{sign}(\gamma_1), \ldots, \operatorname{sign}(\gamma_d))$ for $\gamma = (\gamma_1, \ldots, \gamma_d) \in \widehat{\mathbb{R}}^d$, and $Q \equiv \left[-\frac{1}{2}, \frac{1}{2}\right]^d$.

2 A generalization of the neighborhood-mapping construction

It is assumed in the neighborhood-mapping construction that K_0 has to be in $[-N, N]^d$ and the range of the map T has to be in $\overline{[-2N, 2N]^d \setminus [-N, N]^d}$. In this section we prove that the procedure produces wavelet sets for a more general class of maps T, thereby obtaining wavelet sets that we had not been able to obtain by the original construction.

Let K_0 be a bounded neighborhood of the origin that is τ-congruent to the unit cube $\mathrm{Q} \equiv \left[-\frac{1}{2}, \frac{1}{2}\right]^d$. We shall consider measurable maps $T : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ satisfying the following properties.

I. T is a $\mathbb{Z}^d$-translated map, i.e.,

$$\forall \gamma \in \widehat{\mathbb{R}}^d,\ \exists\, n_\gamma \in \mathbb{Z}^d \text{ such that } T(\gamma) = \gamma + n_\gamma.$$

II. T is injective.

III. The range of $T - \mathrm{I}$ is bounded, where I is the identity map on $\widehat{\mathbb{R}}^d$.

IV. $\left[\bigcup_{i=1}^{\infty} T^i(K_0)\right] \cap \left[\bigcup_{j=0}^{\infty} 2^{-j} K_0\right] = \emptyset$, where $T^0 = \mathrm{I}$ and $T^i \equiv \underbrace{T \circ \cdots \circ T}_{i\text{-fold}}$.

Let T satisfy conditions I-IV. According to [BL01, BS02], we iteratively construct a series of sets K_n, each of which is τ-congruent to Q and hence tiles $\widehat{\mathbb{R}}^d$ by $\mathbb{Z}^d$-translates, as follows. For each $n = 0, 1, \ldots$, we define

$$A_n = K_n \cap \left[\bigcup_{j=1}^{\infty} 2^{-j} K_n \right] \tag{3}$$

and

$$K_{n+1} = (K_n \setminus A_n) \cup T(A_n). \tag{4}$$

The union in (3) need not be disjoint as is easily seen by letting $K_0 = \mathrm{Q}$. By property IV above and Proposition 4b below, $T(A_n)$ is necessarily disjoint from K_0. Since the sets A_n are mutually disjoint (by Proposition 4a), T is one-to-one, and K_{n+1} is contained in $K_0 \cup [\bigcup_{i=0}^{n} T(A_i)]$, we can assert that the union in (4) is a disjoint union. When (4) is applied successively, one has

$$\begin{aligned} K_{n+1} = & \Big(K_0 \setminus A_0 \setminus \ldots A_n \Big) \\ & \cup \Big((T(A_0) \setminus A_1 \setminus \cdots \setminus A_n) \cup (T(A_1) \setminus A_2 \setminus \cdots \setminus A_n) \cup \cdots \cup T(A_n) \Big) \\ = & \left[K_0 \setminus \bigcup_{i=0}^{n} A_i \right] \cup \left[\bigcup_{k=0}^{n} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n} A_i \right) \right]. \end{aligned} \tag{5}$$

Because of (5), we define the set

$$K = \left[K_0 \setminus \bigcup_{i=0}^{\infty} A_i \right] \cup \left[\bigcup_{k=0}^{\infty} \left(T(A_k) \setminus \bigcup_{i>k} A_i \right) \right]. \tag{6}$$

K is a *wavelet set* because of Proposition 7 below.

The definitions of A_n and K_{n+1} in (3) and (4) are exactly the same as those in [BL99, BL01]. However, the calculation (5) gives rise to our generalization of the neighborhood-mapping construction.

Proposition 4. *Let K_0 be a bounded neighborhood of $\mathbf{0} \in \widehat{\mathbb{R}}^d$ that is τ-congruent to* Q. *Let T satisfy conditions I-IV, and define A_n, K_n by (3) and (4), respectively. The auxiliary sets A_n and K_n satisfy the following properties.*

a. *The sets A_n, $n = 0, 1, \ldots$, are mutually disjoint.*

b. For each $n \geq 0$, $A_n \subseteq K_n \subseteq \bigcup_{i=0}^{n} T^i(K_0)$.

c. Each K_n, and hence each A_n, is a bounded set.

d. K is bounded.

e. $\left|\bigcup_{n=0}^{\infty} A_n\right| \leq 1$, and so $\lim_{n\to\infty} |A_n| = 0$.

f. For each $n \geq 1$,

$$A_n = K_n \cap \bigcup_{j=1}^{\infty} 2^{-j} \bigcup_{k=0}^{n-1} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n-1} A_i \right). \tag{7}$$

Proof. a. The disjointness of the A_n follows from (3) and (4).

b. This follows by induction. Clearly, $A_0 \subseteq K_0$. Then $A_{n+1} \subseteq K_{n+1} \subseteq K_n \cup T(A_n)$ which is a subset of $\bigcup_{i=0}^{n+1} T^i(K_0)$ by induction hypothesis.

c. Since K_0 is bounded and T translates a point by an integer in a bounded set (property III above in our definition of the maps T), the inclusion $K_n \subseteq \bigcup_{i=0}^{n} T^i(K_0)$ implies that each K_n is bounded.

To prove the boundedness of K_n and K, we let

$$M = \max\left(\sup_{\gamma \in K_0} \|\gamma\|, \sup_{\gamma \in \widehat{\mathbb{R}}^d} \|T(\gamma) - \gamma\| \right).$$

For $R \geq 0$, let $B_0(R) = \left\{ \gamma \in \widehat{\mathbb{R}}^d : \|\gamma\| \leq R \right\}$ denote the closed ball of radius R centered at the origin. Then we claim that $K_n \subseteq B_0(2M)$ and $A_n \subseteq B_0(M)$. This is clear when $n = 0$. By definition and induction hypothesis,

$$K_{n+1} \subseteq K_n \cup T(A_n) \subseteq B_0(2M) \cup T(B_0(M)) \subseteq B_0(2M)$$

and

$$A_{n+1} \subseteq \bigcup_{j=1}^{\infty} 2^{-j} K_{n+1} \subseteq \bigcup_{j=1}^{\infty} 2^{-j} B_0(2M) \subseteq B_0(M).$$

d. The wavelet set K is also contained in $B_0(2M)$. This follows from the fact that $K \subseteq K_0 \cup \bigcup_{i=0}^{\infty} T(A_i) = \bigcup_{i=0}^{\infty} K_i$.

e. First observe that $\bigcup_{j=1}^{\infty} 2^{-j} A_n \subseteq \bigcup_{j=1}^{\infty} 2^{-j}(K_n \setminus A_n)$ for each $n \geq 0$. In fact, if $\gamma \in 2^{-j} A_n$, then there is $k = k_n \geq 1$ such that $2^{j+k}\gamma \in K_n$. The boundedness of the K_n ensures the existence of the largest such k, and hence $\gamma \in 2^{-(j+k)} \left(K_n \setminus \bigcup_{l=1}^{\infty} 2^{-l} K_n\right) = 2^{-(j+k)}(K_n \setminus A_n)$. Therefore,

$$\bigcup_{j=1}^{\infty} 2^{-j} K_n = \bigcup_{j=1}^{\infty} 2^{-j}(K_n \setminus A_n) \subseteq \bigcup_{j=1}^{\infty} 2^{-j} K_{n+1}.$$

Since $A_i \subseteq \bigcup_{j=1}^{\infty} 2^{-j} K_i$, $\bigcup_{i=0}^{n} A_i$ must be a subset of $\bigcup_{j=1}^{\infty} 2^{-j} K_n$ which has measure no greater than $\sum_{j=1}^{\infty} 2^{-j} |K_n| = 1$. Thus $\left|\bigcup_{i=0}^{\infty} A_i\right| \leq 1$.

f. Finally, let us note that

$$\left(K_0 \setminus \bigcup_{i=0}^{n-1} A_i\right) \cap \bigcup_{j=1}^{\infty} 2^{-j} \left(K_0 \setminus \bigcup_{i=0}^{n-1} A_i\right) = \emptyset$$

and

$$\bigcup_{k=0}^{n-1} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n-1} A_i\right) \cap \bigcup_{j=1}^{\infty} 2^{-j} \left(K_0 \setminus \bigcup_{i=0}^{n-1} A_i\right) = \emptyset.$$

The first intersection is empty since it is a subset of $A_0 = K_0 \cap \bigcup_{j=1}^{\infty} 2^{-j} K_0$ and at the same time does not intersect A_0. The second equality follows from the fact that the range of T and the union $\bigcup_{j=1}^{\infty} 2^{-j} K_0$ are disjoint. Then (7) easily follows. $\square$

Remark 5. It follows immediately from Proposition 4b that

$$T\left(\bigcup_{n=0}^{\infty} A_n\right) = \bigcup_{n=0}^{\infty} T(A_n) \subseteq \bigcup_{i=1}^{\infty} T^i(K_0).$$

Since the map T is applied only to the points in $\bigcup_{n=0}^{\infty} A_n$ in the construction, the "range of T" will for all practical purposes refer to the set $T\left(\bigcup_{n=0}^{\infty} A_n\right)$. Property IV above in the definition of the maps T then implies that the range of T does not intersect $\bigcup_{j=0}^{\infty} 2^{-j} K_0$.

Proposition 6. *For each* $n = 0, 1, \ldots,$

1. K_n *tiles* $\widehat{\mathbb{R}}^d$ *by* $\mathbb{Z}^d$*-translation, and*

2. $K_n \setminus A_n$ *tiles* $\widehat{\mathbb{R}}^d$ *by dyadic dilation.*

Proof. Since T is an injective $\mathbb{Z}^d$-translated map with range in the complement of K_0, it is easily proved by induction that $K_n = (K_{n-1} \setminus A_{n-1}) \cup T(A_{n-1})$ is τ-congruent to K_0, and hence is τ-congruent to Q. Therefore $\{K_n + m : m \in \mathbb{Z}^d\}$ tiles $\widehat{\mathbb{R}}^d$.

The second statement is also proved by induction. Since $K_0 \setminus A_0 = K_0 \setminus \bigcup_{j=1}^{\infty} 2^{-j} K_0$, it is immediate that $2^k (K_0 \setminus A_0) \cap 2^{k'} (K_0 \setminus A_0) = \emptyset$ for all integers $k \neq k'$. Let $\gamma \in \widehat{\mathbb{R}}^d \setminus \{\mathbf{0}\}$ be given. There exists an integer k for which $2^{-k}\gamma$ is in the neighborhood K_0 of the origin. The smallest such integer, denoted by k_0, exists because of the boundedness of the set K_0. Thus

$$\gamma \in 2^{k_0} K_0 \setminus \bigcup_{j=1}^{\infty} 2^{k_0 - j} K_0 \subseteq \bigcup_{k \in \mathbb{Z}} 2^k \left(K_0 \setminus A_0\right).$$

Let $n \geq 0$ and assume that $\left\{2^k \left(K_n \setminus A_n\right)\right\}_{k \in \mathbb{Z}}$ tiles $\widehat{\mathbb{R}}^d$. We can then rewrite A_{n+1} in (3) as

$$\begin{aligned} A_{n+1} &= \left[(K_n \setminus A_n) \cap \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right] \dot\cup \left[T(A_n) \cap \bigcup_{j=1}^{\infty} 2^{-j} K_{n+1} \right] \\ &= \left[(K_n \setminus A_n) \cap \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right] \\ &\quad \dot\cup \left[T(A_n) \cap \bigcup_{j=1}^{\infty} 2^{-j} \bigcup_{k=0}^{n} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n} A_i \right) \right]. \end{aligned} \tag{8}$$

The first equation follows from the induction hypothesis while the second is from (5) and the fact that the range of T is disjoint from $\bigcup_{j=1}^{\infty} 2^{-j} K_0$. Thus,

$$\begin{aligned} K_{n+1} \setminus A_{n+1} = &\left[(K_n \setminus A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right] \\ &\dot\cup \left[T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} \bigcup_{k=0}^{n} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n} A_i \right) \right]. \end{aligned}$$

We then observe that $(K_n \setminus A_n) \cap \bigcup_{k=1}^{\infty} 2^{-k} T(A_n)$ can be rewritten as the disjoint union

$$\bigcup_{k=1}^{\infty} (K_n \setminus A_n) \cap \left(2^{-k} T(A_n) \setminus \bigcup_{j>k} 2^{-j} T(A_n) \right), \tag{9}$$

which is δ-congruent to

$$\begin{aligned}
&\bigcup_{k=1}^{\infty} 2^k \left[(K_n \setminus A_n) \cap \left(2^{-k} T(A_n) \setminus \bigcup_{j>k} 2^{-j} T(A_n) \right) \right] \\
&= \bigcup_{k=1}^{\infty} 2^k (K_n \setminus A_n) \cap \left(T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right) \\
&= \left(T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right) \setminus \left(\bigcup_{k=0}^{\infty} 2^{-k} (K_n \setminus A_n) \cap \left(T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right) \right) \\
&= \left(T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right) \setminus \left(\bigcup_{j=1}^{\infty} 2^{-j} \bigcup_{k=0}^{n-1} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n} A_i \right) \cap \left(T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right) \right) \\
&= \left(T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} T(A_n) \right) \setminus \bigcup_{j=1}^{\infty} 2^{-j} \bigcup_{k=0}^{n-1} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n} A_i \right) \\
&= T(A_n) \setminus \bigcup_{j=1}^{\infty} 2^{-j} \bigcup_{k=0}^{n} \left(T(A_k) \setminus \bigcup_{i=k+1}^{n} A_i \right).
\end{aligned}$$

Here we use the induction hypothesis in the second equation. The third equation is justified by the assumptions that the range of T is outside $\bigcup_{j=1}^{\infty} 2^{-j} K_0$ and that T is one-to-one. Thus $K_{n+1} \setminus A_{n+1}$ is δ-congruent to $K_n \setminus A_n$, and hence $\{K_n \setminus A_n\}$ tiles $\widehat{\mathbb{R}}^d$ by dyadic dilations. ☐

Proposition 7. *The set K tiles $\widehat{\mathbb{R}}^d$ both by $\mathbb{Z}^d$-translation and by dyadic dilation.*

Proof. The difference sets between K and K_n and between K and $K_n \setminus A_n$ are given by

$$\begin{aligned}
K \triangle K_n &= [K \setminus K_n] \cup [K_n \setminus K] \\
&= \left[\bigcup_{k=n}^{\infty} T(A_k) \setminus \bigcup_{i>k} A_i \right] \cup \left[\left(K_0 \cap \bigcup_{i=n}^{\infty} A_i \right) \cup \left(\bigcup_{k=0}^{n-1} T(A_k) \cap \bigcup_{i=n}^{\infty} A_i \right) \right] \\
&\subseteq T \left(\bigcup_{i=n}^{\infty} A_i \right) \cup \bigcup_{i=n}^{\infty} A_i
\end{aligned}$$

and

$$
\begin{aligned}
K \triangle (K_n \setminus A_n) &= [K \setminus (K_n \setminus A_n)] \cup [(K_n \setminus A_n) \setminus K] \\
&= \left[\bigcup_{k=n}^{\infty} T(A_k) \setminus \bigcup_{i>k} A_i\right] \cup \left[\left(K_0 \cup \bigcup_{k=0}^{n-1} T(A_k)\right) \cap \bigcup_{i=n+1}^{\infty} A_i\right] \\
&\subseteq T\left(\bigcup_{i=n}^{\infty} A_i\right) \cup \bigcup_{i=n+1}^{\infty} A_i .
\end{aligned}
$$

Since the map T is measure preserving, Proposition 4e implies that both K_n and $K_n \setminus A_n$ converge to K, almost everywhere and in $\mathrm{L}^p(\widehat{\mathbb{R}}^d)$, $1 \le p < \infty$ (the latter assertion in terms of the characteristic functions of the appropriate sets).

Now that K_n and $K_n \setminus A_n$ each converges to K, we have that $\sum_{k\in\mathbb{Z}^d} \mathbb{1}_{K_n}(\cdot + k)$ converges to $\sum_{k\in\mathbb{Z}^d} \mathbb{1}_K(\cdot+k)$ and $\sum_{j\in\mathbb{Z}} \mathbb{1}_{K_n\setminus A_n}(2^j\cdot)$ converges to $\sum_{j\in\mathbb{Z}} \mathbb{1}_K(2^j\cdot)$ almost everywhere. Therefore, by Proposition 6, K tiles $\widehat{\mathbb{R}}^d$ by $\mathbb{Z}^d$-translation and dyadic dilation. □

Example 8. A generalization of the Journé wavelet set.
Since the Journé wavelet set can be constructed by the neighborhood-mapping construction with $K_0 = \left[-\frac{1}{2}, \frac{1}{2}\right)$ and $T(\gamma) = \gamma + 2\,\mathrm{sign}(\gamma)$, one of its d-dimensional versions can be produced by setting

$$K_0 = \left[-\frac{1}{2}, \frac{1}{2}\right)^d \text{ and } T(\gamma_1, \ldots, \gamma_d) = (\gamma_1 + 2\,\mathrm{sign}(\gamma_1), \ldots, \gamma_d + 2\,\mathrm{sign}(\gamma_d)),$$

see Figure 4 for the corresponding wavelet set in $\widehat{\mathbb{R}}^2$.

Example 9. The wedding cake set.
This wavelet set was defined in [DL98, DLS98], and it can be constructed by our method with $K_0 = \mathrm{Q}$ and $T(\gamma_1, \gamma_2) = (\gamma_1 + \mathrm{sign}\,\gamma_1, \gamma_2)$ for $(\gamma_1, \gamma_2) \in K_0$. It was introduced as a simple wavelet set consisting of 3 connected sets, see Figure 5. The interior of each component is connected.

Example 10. The wedding night set.
Let $K_0 = \left\{\left[-\frac{1}{2}, 0\right) \times \left[-\frac{1}{3}, \frac{2}{3}\right)\right\} \cup \left\{\left[0, \frac{1}{2}\right) \times \left[-\frac{2}{3}, \frac{1}{3}\right)\right\} \subset \widehat{\mathbb{R}}^2$ and let $T : K_0 \to \widehat{\mathbb{R}}^2$ be defined by $T(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2 - 1)$. The wavelet set that this pair of

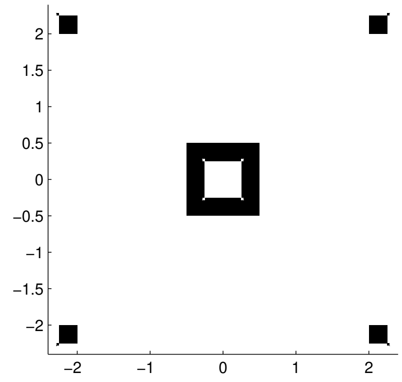


Figure 4: A generalization of Journé wavelet set

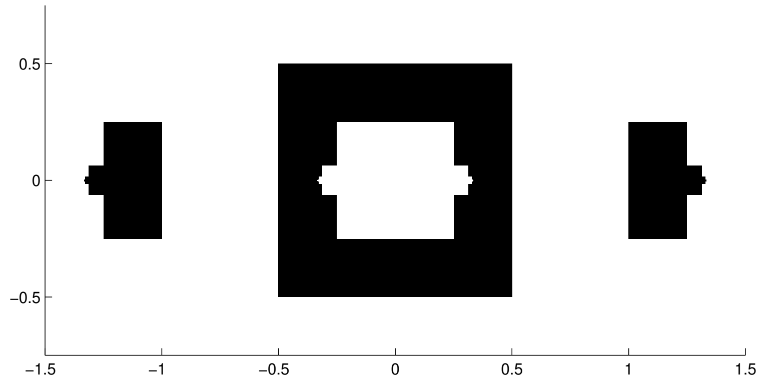


Figure 5: Wedding cake set

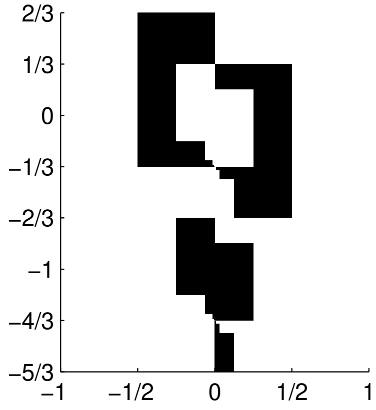


Figure 6: Wedding night set

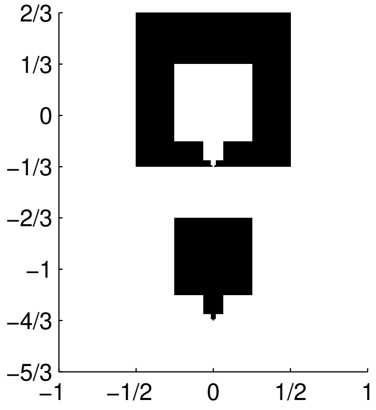


Figure 7: Another wedding cake set

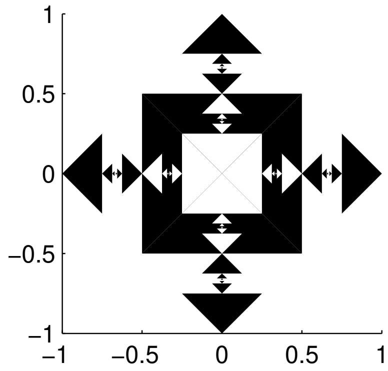


Figure 8: Connected wavelet set from [BL01]

neighborhood and mapping produces consists of 2 connected sets, see Figure 6. In this example the interior of each component is not connected.

Example 11. Another wedding cake set.
This wavelet set consists of 2 connected sets; and it is constructed by letting $K_0 = \left[-\frac{1}{2}, \frac{1}{2}\right) \times \left[-\frac{1}{3}, \frac{2}{3}\right)$ and defining $T : K_0 \to \widehat{\mathbb{R}}^2$ to be the map $(\gamma_1, \gamma_2) \mapsto (\gamma_1, \gamma_2 - 1)$. The associated wavelet set is the union of only 2 connected sets, see Figure 7. The interior of each component is connected.

An example of a connected wavelet set, in which the interior consists of infinitely many components, was given in [BL01], see Figure 8. It is believed that there is no connected wavelet set with connected interior.

It should be noted that our construction produces only the wavelet sets that are bounded away from the origin and infinity, i.e., they have holes at the origin and are bounded sets. Related work can be found in [Li02].

3 Tight frames

The concept of frame introduced by Duffin and Schaeffer [DS52] is a natural generalization of an orthonormal basis. A countable family $\{\phi_i\}_{i\in\mathbb{Z}}$ of functions in $\mathrm{L}^2(\mathbb{R}^d)$ is a *frame* if there exist constants $0 < A \leq B < \infty$ called *frame* *bounds* for which

$$\forall f \in \mathrm{L}^2(\mathbb{R}^d), \quad A\|f\|^2 \leq \sum_{i\in\mathbb{Z}} |\langle f, \phi_i\rangle|^2 \leq B\|f\|^2 .$$

When the frame bounds A and B coincide, $\{\phi_i\}$ is called a *tight frame.* If $A = B = 1$, the inequality becomes the Parseval identity and $\{\phi_i\}$ is aptly called a *Parseval frame*, i.e.,

$$\forall f \in \mathrm{L}^2(\mathbb{R}^d), \quad \sum_{i\in\mathbb{Z}} |\langle f, \phi_i\rangle|^2 = \|f\|^2 .$$

An $\mathrm{L}^2(\mathbb{R}^d)$ function ψ is a *frame wavelet*, respectively, *tight frame wavelet* and *Parseval frame wavelet*, if the generated family $\{\psi_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is a frame, respectively, tight frame and Parseval frame.

In recent years, frame wavelets, in particular tight and Parseval frame wavelets, have been studied extensively, e.g., see [Han97], [DDG01], [DDG02], [DDGH02], and [DDG03].

It is a natural question to ask whether the sets K_n constructed from finite iterations in the neighborhood-mapping construction give rise to frame wavelets, respectively, tight frame wavelets and Parseval frame wavelets, i.e., whether the functions $\check{\mathbb{1}}_{K_n}$ are frame wavelets, respectively, tight frame wavelets and Parseval frame wavelets. It turns out that $\check{\mathbb{1}}_{K_n}$ is a frame wavelet with frame bounds 1 and 2, while we obtain Parseval frame wavelets from the auxiliary sets $K_n \setminus A_n$ (Theorem 15 below). Let us first recall a characterization of tight frame wavelets due to [Bow00, CS00, CCMW02](Theorem 12 below) and a sufficient condition for a function to be a frame wavelet for $\mathrm{L}^2(\mathbb{R}^d)$ [Chr01, Dau92, HW96, KZ95](Theorem 13 below). The proof for $\mathbb{R}$ of Theorem 5.1 in [Chr01] can be easily generalized to $\mathbb{R}^d$, and this generalization is Theorem 13.

Theorem 12. *Let $\psi \in \mathrm{L}^2(\mathbb{R}^d)$. The family*

$$\left\{\psi_{m,n} = D_m T_n \psi = 2^{md/2}\psi(2^m \cdot -n) : m \in \mathbb{Z}, n \in \mathbb{Z}^d\right\}$$

is a Parseval frame if and only if

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j\xi)\right|^2 = 1 \quad \textit{and} \quad t_q(\xi) = \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q))} = 0$$

for almost every $\xi \in \widehat{\mathbb{R}}^d$ *and for all* $q \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d$.

Theorem 13. *Let* $a > 1$, $b > 0$, *and* $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ *be given. Suppose that*

$$A \equiv \inf_{\|\xi\| \in [1,a]} \left[\sum_{n\in\mathbb{Z}} \left|\widehat{\psi}(a^n\xi)\right|^2 - \sum_{k\neq\mathbf{0}} \sum_{n\in\mathbb{Z}} \left|\widehat{\psi}(a^n\xi)\widehat{\psi}(a^n\xi + k/b)\right| \right] > 0,$$

$$B \equiv \sup_{\|\xi\| \in [1,a]} \left[\sum_{n\in\mathbb{Z}} \left|\widehat{\psi}(a^n\xi)\right|^2 + \sum_{k\neq\mathbf{0}} \sum_{n\in\mathbb{Z}} \left|\widehat{\psi}(a^n\xi)\widehat{\psi}(a^n\xi + k/b)\right| \right] < \infty.$$

Then $\left\{a^{jd/2}\psi(a^j \cdot -kb)\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a frame for* $\mathrm{L}^2(\mathbb{R}^d)$ *with frame bounds* A/b^d, B/b^d.

We begin the proof of Theorem 15 with the following lemma which uses Theorem 12.

Lemma 14. *If a set* L *tiles* $\widehat{\mathbb{R}}^d$ *by dyadic dilation and* $L \subseteq K$ *for some set* K *that tiles* $\widehat{\mathbb{R}}^d$ *by* $\mathbb{Z}^d$*-translation, then the function* $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ *defined by* $\widehat{\psi} = \mathbb{1}_L$ *is a Parseval frame wavelet.*

Proof. Since L tiles $\widehat{\mathbb{R}}^d$ by dyadic dilation a.e., we have

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}\left(2^j\xi\right)\right|^2 = \sum_{j\in\mathbb{Z}} \mathbb{1}_L\left(2^j\xi\right) = \mathbb{1}_{\bigcup_{j\in\mathbb{Z}} 2^{-j}L}(\xi) = 1 \quad \text{a.e.}$$

We then compute

$$t_q(\xi) = \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q))} = \sum_{j=0}^{\infty} \mathbb{1}_{2^{-j}L}(\xi)\mathbb{1}_{2^{-j}L-q}(\xi) = \sum_{j=0}^{\infty} \mathbb{1}_{2^{-j}[L\cap(L-2^jq)]}(\xi).$$

From the second assumption, $L \cap (L - 2^jq) \subseteq K \cap (K - 2^jq) = \emptyset$. Therefore $t_q = 0$. Hence, by Theorem 12, ψ is a Parseval frame wavelet. $\square$

Theorem 15. *For each* $n \geq 0$, $K_n \setminus A_n$ *is a Parseval frame wavelet set, and* K_n *is a frame wavelet set with frame bounds* 1 *and* 2.

Proof. By Proposition 6, Lemma 14, and the inclusion $K_n \setminus A_n \subseteq K_n$, it is clear that $K_n \setminus A_n$ is a Parseval frame wavelet set.

Let $\widehat{\psi} = \mathbb{1}_{K_n}$. Then

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j\xi)\right|^2 = \sum_{j\in\mathbb{Z}} \mathbb{1}_{K_n\setminus A_n}(2^j\xi) + \sum_{j\in\mathbb{Z}} \mathbb{1}_{A_n}(2^j\xi)$$
$$= 1 + \mathbb{1}_{\bigcup_{j\in\mathbb{Z}} 2^{-j}A_n}(\xi).$$

It is straightforward from the definition that the sets $2^{-j}A_n$, $j \in \mathbb{Z}$, are mutually disjoint. This justifies the second equation. Therefore $\sup_{\xi\in\widehat{\mathbb{R}}^d} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j\xi)\right|^2 = 2$ and $\inf_{\xi\in\widehat{\mathbb{R}}^d} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j\xi)\right|^2 = 1$. Since K_n tiles $\widehat{\mathbb{R}}^d$ by $\mathbb{Z}^d$-translation, we have $\widehat{\psi}(2^j\xi)\widehat{\psi}(2^j\xi + k) = \mathbb{1}_{K_n}(2^j\xi)\mathbb{1}_{K_n-k}(2^j\xi) = 0$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Hence, we can invoke Theorem 13 to assert that K_n is a frame wavelet set with frame bounds 1 and 2. □

Remark 16. 1. In the proof of Theorem 15, $\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j\xi)\right|^2 = 2$ only when $\xi \in \bigcup_{j\in\mathbb{Z}} 2^{-j}A_n$.

2. $\left\{\psi_{m,n} = \left(\mathbb{1}_{K_i\setminus A_i}\right)\check{}_{m,n}\right\}$ is not a Riesz basis whenever $|A_i| > 0$. Indeed, for $(m', n') \in \mathbb{Z} \times \mathbb{Z}^d$ and $f \in \mathrm{L}^2(\mathbb{R}^d)$,

$$\sum_{m\neq m', n\neq n'} |\langle f, \psi_{m,n}\rangle|^2 = \|f\|^2 - |\langle f, \psi_{m',n'}\rangle|^2 \leq \|f\|^2 .$$

By the Cauchy-Schwarz inequality, $|\langle f, \psi_{m',n'}\rangle| \leq \|f\|^2 \|\psi_{m',n'}\|^2$. Thus

$$\left(1 - \|\psi\|^2\right) \|f\|^2 \leq \sum_{(m,n)\neq(m',n')} |\langle f, \psi_{m,n}\rangle|^2 \leq \|f\|^2 .$$

Since $\|\psi\|^2 = |K_i \setminus A_i| = 1 - |A_i| < 1$, $\{\psi_{m,n} : (m, n) \neq (m', n')\}$ is still a frame with frame bounds $1 - |A_i|$ and 1. A natural question is to determine whether or not the tight frame $\{\psi_{m,n}\}$ contains a Riesz basis.

4 Wavelet sets and convex sets

From the "self-similar" properties of wavelet sets and the iterative nature of the wavelet set constructions, both introduced by Benedetto and Leon [BL99, BL01] and Baggett, Medina, and Merrill [BMM99], it seems apparent that wavelet sets in $\widehat{\mathbb{R}}^d$ with $d > 1$ are *not* "simple" objects. All the d-dimensional wavelet sets explicitly constructed in the literature consist of infinitely many convex sets (whose union may be 2 or 3 simple-looking connected non-convex sets such as illustrated in Figures 5-7). We shall now prove that a convex set is never a dyadic wavelet set. We shall also prove the more general result that a wavelet set in $\widehat{\mathbb{R}}^d$ can not be a union of d or less convex sets.

Recall that a set C is a *convex set* if for every point γ and ζ in C, the straight line joining γ and ζ lies completely in C.

Lemma 17. *A convex set C can not partition $\widehat{\mathbb{R}}^d$ by dilations, that is, the set $\left\{2^j C : j \in \mathbb{Z}\right\}$ can not be a partition of $\widehat{\mathbb{R}}^d$.*

Proof. i. The statement is obvious in $\widehat{\mathbb{R}}$, and, for simplicity, we shall only prove the result in the case $d = 2$. The proof can be easily generalized to $\widehat{\mathbb{R}}^d$ by replacing lines in $\widehat{\mathbb{R}}^2$ with hyperplanes in $\widehat{\mathbb{R}}^d$.

ii. Suppose that C is a convex set and that $\left\{2^j C : j \in \mathbb{Z}\right\}$ is a partition of $\widehat{\mathbb{R}}^2$. Define $S = \left\{\gamma \in \widehat{\mathbb{R}}^2 : \gamma \in C \text{ and } -\gamma \in C\right\} = C \cap (-C) \subseteq C$. It is clear from the definition that the set S is symmetric, i.e., $\gamma \in S$ if and only if $-\gamma \in S$. We claim that S is also convex. In fact, if $\gamma, \zeta \in S$, then, for each $t \in [0,1]$, $t\gamma + (1-t)\zeta \in C$ and $-(t\gamma + (1-t)\zeta) = t(-\gamma) + (1-t)(-\zeta) \in C$, and so $t\gamma + (1-t)\zeta \in S$.

We shall prove that S is a null set. We do this by assuming $|S| > 0$. With this assumption we shall show that S, and hence C, must contain a neighborhood of the origin. This contradicts the disjointness of $2^j C$, $j \in \mathbb{Z}$, and thus $|S| = 0$.

To show that S contains a neighborhood of 0 if $|S| > 0$, let us first observe that there exist positive numbers α_1 and α_2 such that both $(\alpha_1, 0)$ and $(0, \alpha_2)$ are in S. To this end we begin by noting that if $(\alpha_1, 0) \notin S$ for all $\alpha_1 > 0$, then $(\alpha_1, 0) \notin S$ for all $\alpha_1 \neq 0$ by the symmetry of S. This makes it impossible for two non-parallel symmetric lines passing through the origin to lie in S. Indeed, if there are two non-parallel symmetric lines $\overline{\mu(-\mu)}$ and $\overline{\nu(-\nu)}$ lying in S, for which the first coordinates of μ and ν have opposite signs, then the line joining μ and ν, which is in S by its convexity, must cross the first coordinate axis $\widehat{\mathbb{R}} \times \{0\}$. This contradicts the above statement that $\left(\widehat{\mathbb{R}} \setminus \{0\}\right) \times \{0\} \cap S = \emptyset$.

Therefore, there is at most one line in S, which contradicts the assumption that $|S| > 0$. Hence, we have proved that there exist positive numbers α_1 and α_2 such that both $(\alpha_1, 0)$ and $(0, \alpha_2)$ are in S. By the symmetry of S, $(\pm\alpha_1, 0)$, $(0, \pm\alpha_2) \in S$. Therefore, the convex hull of S, and hence S itself, contains a neighborhood of the origin.

iii. Next, we claim that

$$\left(2^j C\right) \cap \left(-2^i C\right) \subseteq \begin{cases} 2^j S & \text{if } j \geq i \\ 2^i S & \text{if } j \leq i. \end{cases} \tag{10}$$

Let γ belong to $\left(2^j C\right) \cap \left(-2^i C\right)$. Equivalently, both $2^{-j}\gamma$ and $-2^{-i}\gamma$ are in C. If $j \leq i$, we have $2^{-i}\gamma = t_0(-2^{-i}\gamma) + (1 - t_0)(2^{-j}\gamma)$ where $t_0 = \left(2^{-j} - 2^{-i}\right) / \left(2^{-j} + 2^{-i}\right)$ is in $[0, 1)$. Therefore, $\pm 2^{-i}\gamma$ are in C, and hence $\gamma \in 2^i S$. Similarly, if $j \geq i$, then $-2^{-j}\gamma = t_0(2^{-j}\gamma) + (1 - t_0)(-2^{-i}\gamma)$, where $t_0 = \left(2^{-i} - 2^{-j}\right) / \left(2^{-i} + 2^{-j}\right) \in [0, 1)$, and thus $\gamma \in 2^j S$.

Finally, we consider the union $\bigcup_{j \in \mathbb{Z}} 2^j C = \widehat{\mathbb{R}}^2$.

$$\begin{aligned} \left|\widehat{\mathbb{R}}^2\right| &= \left| \left(\bigcup_{j \in \mathbb{Z}} 2^j C \right) \cap \left(\bigcup_{i \in \mathbb{Z}} -2^i C \right) \right| \\ &= \left| \bigcup_{i,j \in \mathbb{Z}} 2^j C \cap \left(-2^i C\right) \right| \\ &\leq \sum_{i,j \in \mathbb{Z}} \left| 2^{\max(i,j)} S \right| = 0, \end{aligned} \tag{11}$$

by (10) and since $|S| = 0$. (11) is obviously false, and so the proof is complete.

□

Theorem 18. *A convex set is not a wavelet set.*

Proof. This is immediate from Lemma 17 and the fact that a wavelet set has to tile $\widehat{\mathbb{R}}^d$ by dilations, that is, if K is a wavelet set, it is necessary that $\left\{2^j K : j \in \mathbb{Z}\right\}$ be a tiling of $\widehat{\mathbb{R}}^d$. □

A more general result than Theorem 18 is Theorem 21, which in turn requires the following lemmas.

Lemma 19 (Separation Theorem). *If K_1 and K_2 are disjoint closed convex subsets of $\widehat{\mathbb{R}}^d$, then there exists a hyperplane H separating K_1 and K_2.*

Proof. See, for example, [BP63], cf., Theorem 3.4 in [Rud73] for the standard separation between compact and closed convex sets in locally convex spaces. $\square$

Lemma 20. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be halfspaces in $\widehat{\mathbb{R}}^d$ with non-parallel boundaries $H_1, \ldots, H_n$. If $\bigcap_{i=1}^n \mathcal{H}_i \neq \emptyset$, by which we mean that the Lebesgue measure of $\bigcap_{i=1}^n \mathcal{H}_i$ is non-zero, hence positive, and $\bigcup_{i=1}^n \mathcal{H}_i = \widehat{\mathbb{R}}^d$, then $n \geq d+1$.*

Proof. Suppose $n \leq d$ and $|\bigcap_{i=1}^n \mathcal{H}_i| > 0$. Let $A \subseteq \bigcap_{i=1}^n \mathcal{H}_i$ be such that $|A| > 0$. Since $n \leq d$ and $H_1, \ldots, H_n$ are non-parallel hyperplanes in $\widehat{\mathbb{R}}^d$, we have that $\bigcap_{i=1}^n H_i \neq \emptyset$ is a translate of $(d-n)$-dimensional vector subspace. Let us pick a point $\zeta \in \bigcap_{i=1}^n H_i$ and define an isometry $R_\zeta : \bigcap_{i=1}^n \mathcal{H} \longrightarrow \bigcap_{i=1}^n \mathcal{H}^c$ mapping $\gamma \mapsto 2\zeta - \gamma$.

For $\gamma \in \mathcal{H}_i$, we have the line joining γ and $2\zeta - \gamma$ passing through H_i at ζ. Therefore $2\zeta - \gamma \notin \mathcal{H}_i$ for all i and hence $R_\zeta(A) \subseteq \bigcap_{i=1}^n \mathcal{H}^c$ and $|R_\zeta(A)| = |A| > 0$. Thus $\bigcup_{i=1}^n \mathcal{H}_i \neq \widehat{\mathbb{R}}^d$. $\square$

Theorem 21. *Let $C_1, C_2, \ldots,$ and C_n be convex sets in $\widehat{\mathbb{R}}^d$. If the disjoint union*

$$\dot{\bigcup}_{i=1}^{n} \dot{\bigcup}_{j\in\mathbb{Z}} 2^j C_i = \widehat{\mathbb{R}}^d,$$

then $n \geq d+1$. In particular, if $C_1, C_2, \ldots,$ and C_n are convex sets in $\widehat{\mathbb{R}}^d$ such that $\bigcup_{i=1}^n C_i$ is a wavelet set, then $n \geq d+1$.

Proof. Assume $\bigcup_{j\in\mathbb{Z}} 2^j C_i = \widehat{\mathbb{R}}^d$ and $\left\{2^j C_i : j \in \mathbb{Z}, i = 1, \ldots, n\right\}$ is a disjoint collection of dilates of C_i. For each $1 \leq i \leq n$, since $2^j C_i$, $j \in \mathbb{Z}$, are disjoint, there must exist $\varepsilon > 0$ such that $B(0,\varepsilon) \cap C_i$ is a null set. In fact, if $C_i \cap B(0,\varepsilon)$ has positive measure for all $\varepsilon > 0$, then we have sequences $\left\{\gamma_k^{(1)}\right\}_{k=1}^\infty, \ldots, \left\{\gamma_k^{(d)}\right\}_{k=1}^\infty$ for which $\gamma_k^{(1)}, \ldots, \gamma_k^{(d)}$ lie in $C_i \cap B\left(0, \frac{1}{2^k}\right)$ and are not simultaneously on the same hyperplane containing 0. Consequently the convex hull $\operatorname{conv}\left(\left\{\gamma_k^{(1)}\right\} \cup \cdots \cup \left\{\gamma_k^{(d)}\right\}\right)$ is contained in $\overline{C_i}$. Moreover, $\operatorname{conv}\left(\bigcup_i \left\{\gamma_k^{(i)}\right\}\right)$, and hence $\overline{C_i}$, contains the polygon-cone with vertices $\gamma_1^{(1)}, \ldots, \gamma_1^{(d)}$, and 0. This contradicts the assumption that the sets $2^j\overline{C_i}, j \in \mathbb{Z}$, are disjoint. (By a *polygon-cone* with vertices $\alpha_1, \ldots, \alpha_d$, and α_0, we mean the cone with vertex α_0 and base defined by the polygon circumscribed by $\alpha_1, \ldots, \alpha_d$.)

We have shown that there exists a positive number ε such that $B(0,\varepsilon) \cap C_i$ is a null set for all $i = 1, \ldots, n$. By Lemma 19 there exist hyperplanes $H_1, \ldots, H_n$ separating $\overline{B(0,\varepsilon/2)}$ and $\overline{C_1}, \ldots, \overline{C_n}$, respectively. Since the distance between $\overline{B(0,\varepsilon/2)}$ and each $\overline{C_i}$ is positive and since each $\overline{C_i}$ is bounded set, we can choose H_is so that they are pairwise non-parallel. Let $\mathcal{H}_i$ be the halfspace on the side of H_i that contains 0. Clearly $\bigcap_{i=1}^n \mathcal{H}_i \supseteq \overline{B(0,\varepsilon/2)}$ is not empty. To show that $\bigcup_{i=1}^n \mathcal{H}_i = \widehat{\mathbb{R}}^d$, let $\gamma \in \widehat{\mathbb{R}}^d$. Since $\dot{\bigcup}_{i,j} 2^j C_i = \widehat{\mathbb{R}}^d$, γ belongs to $2^j C_i$ for some unique j and i.

If $\gamma \in \mathcal{H}_i$, we are done. Otherwise, we consider $-\gamma \in 2^{j'} C_{i'}$ for some j', i'. In the first case when $-\gamma \in \mathcal{H}_{i'}$, since $-2^{j'}\gamma$ is in $C_{i'}$ but not in $\mathcal{H}_{i'}$, the line joining $-2^{j'}\gamma$ and γ crosses the hyperplane $H_{i'}$. Therefore $\gamma \in \mathcal{H}_{i'}$. In the other case when $-\gamma \in \mathcal{H}_{i'}$, γ must be in $\mathcal{H}_{i'}$ because the origin is in $\mathcal{H}_{i'}$ by definition.

Therefore, we have obtained n halfspaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ with pairwise non-parallel boundaries for which the intersection $\bigcap_{i=1}^n \mathcal{H}_i$ has measure zero and the union $\bigcup_{i=1}^n \mathcal{H}_i$ covers all of $\widehat{\mathbb{R}}^d$. By Lemma 20, $n \geq d+1$. $\square$

So far, we have only used only one tiling property of wavelet sets, viz., tiling by dilations. It remains to see how tiling by translations would help us obtain more results concerning "irregularity" of wavelet sets. Instead of working with two tiling conditions, we anticipate that it would be easier to work with one condition. One candidate is the consistency equation introduced by Baggett et al. [BMM99] in their characterization of subspace wavelet sets. By definition, $W \subset \widehat{\mathbb{R}}^d$ is a *subspace wavelet set* if the family $\left\{ (\check{\mathbb{1}}_W)_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}^d \right\}$ is an orthonormal basis for a subspace of $\mathrm{L}^2(\mathbb{R}^d)$.

Theorem 22. *The set $W \subset \widehat{\mathbb{R}}^d$ is a subspace wavelet set if and only if the characteristic function $\mathbb{1}_E$ of the set $E = \bigcup_{j<0} 2^j W$ satisfies the following con-*

sistency equation:

$$1 + \sum_{k \in \mathbb{Z}^d} \mathbb{1}_E(\gamma + k) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_E\left(\frac{1}{2}(\gamma + k)\right) \quad a.e. \tag{12}$$

In particular, W is a wavelet set for all of $\mathbb{R}^d$ if and only if, in addition, $\bigcup_{j\in\mathbb{Z}} 2^j E$ contains, up to a set of measure zero, a neighborhood of the origin.

5 Generality of the neighborhood-mapping construction

It is natural to investigate how general the neighborhood-mapping construction of wavelet sets is. In fact, the algorithm can construct every wavelet set in the annulus $2\,\mathrm{Q} \setminus \left(\frac{1}{2}\,\mathrm{Q}\right) = [-1,1]^d \setminus [-\frac{1}{4}, \frac{1}{4}]^d$. We prove the following theorem.

Theorem 23. *Every wavelet set in $[-1,1]^d \setminus [-\frac{1}{4}, \frac{1}{4}]^d$ can be constructed by means of the neighborhood-mapping construction.*

Proof. Let K be a wavelet set contained in $[-1,1]^d \setminus [-\frac{1}{4}, \frac{1}{4}]^d$. Define the $\mathbb{Z}^d$-translated map $T : [-\frac{1}{2}, \frac{1}{2}]^d \to [-1,1]^d \setminus [-\frac{1}{2}, \frac{1}{2}]^d$ by

$$T(\gamma) = \begin{cases} \gamma - n & \text{if } \gamma \in K + n, \\ \gamma - \operatorname{sign}(\gamma) & \text{if } \gamma \in K. \end{cases}$$

T is well-defined almost everywhere because $\left\{K + n : n \in \mathbb{Z}^d\right\}$ is a partition of $\mathbb{R}^d$ a.e. Since the domain of T is the unit cube Q, it is clearly injective. Observe that for almost every $\gamma \in \mathrm{Q} = [-\frac{1}{2}, \frac{1}{2}]^d$,

$$\gamma \in K \text{ if and only if } T(\gamma) \notin K,$$

and for almost every $\gamma \in 2\,\mathrm{Q} \setminus \mathrm{Q}$,

$$\gamma \in K \text{ if and only if } \frac{1}{2}\gamma \notin K.$$

Let $S = T\left(\frac{1}{2}\cdot\right)$. From the neighborhood-mapping construction,

$$\tilde{K} \equiv \left[\mathrm{Q} \setminus \bigcup_{k=0}^{\infty} \frac{1}{2} S^k(\mathrm{Q})\right] \dot{\cup} \left[\bigcup_{k=0}^{\infty} T\left(\frac{1}{2} S^k(\mathrm{Q})\right)\right]$$

is a wavelet set. In order to prove that $K = \tilde{K}$ a.e., it suffices to show that

$$K \cap \mathrm{Q} \subseteq \mathrm{Q} \setminus \bigcup_{k=0}^{\infty} \frac{1}{2} S^k(\mathrm{Q}) \quad \text{a.e.} \tag{13}$$

and

$$K \setminus \mathrm{Q} \subseteq \bigcup_{k=1}^{\infty} S^k(\mathrm{Q}) \quad \text{a.e.} \tag{14}$$

In fact, if $K \subseteq \tilde{K}$ and $|K| = |\tilde{K}| = 1$ then $K = \tilde{K}$ a.e.

Since $\frac{1}{2}S^k(\mathrm{Q}) \subseteq \mathrm{Q}$, it is easy to check that (13) is equivalent to saying that $\bigcup_{k=0}^{\infty} \left(K \cap \frac{1}{2}S^k(\mathrm{Q})\right) = \emptyset$ a.e. We show that $K \cap \frac{1}{2}S^k(\mathrm{Q}) = \emptyset$ a.e. by induction on k. The case $k = 0$ is clear by the assumption. If $\frac{1}{2}S^k(\mathrm{Q}) \cap K = \emptyset$ a.e., then $S^{k+1}(\mathrm{Q}) = T\left(\frac{1}{2}S^k(\mathrm{Q})\right) \subseteq K$ a.e. and so $\frac{1}{2}S^{k+1}(\mathrm{Q}) \cap K = \emptyset$ a.e.

In order to prove (14), we first define the mappings $\mathcal{T}$, $\mathcal{S}$, and the set $\mathcal{M}$. For a given measurable set $E \subseteq \mathrm{Q}$, we set

$$\mathcal{T}(E) \equiv \bigcup \left\{T(E) | T : \mathrm{Q} \to 2\,\mathrm{Q} \setminus \mathrm{Q} \text{ is a measurable 1-1 } \mathbb{Z}^d\text{-translated map}\right\},$$

$$\mathcal{S}(E) \equiv \mathcal{T}\left(\frac{1}{2}E\right),$$

and

$$\mathcal{M} \equiv \bigcup_{k=1}^{\infty} \mathcal{S}^k(\mathrm{Q}).$$

It is shown in [Sum03] that the complement $(2\,\mathrm{Q} \setminus \mathrm{Q}) \setminus \mathcal{M}$ of $\mathcal{M}$ in $2\,\mathrm{Q} \setminus \mathrm{Q}$ is a fractal set, and hence a null set. The result in $\mathbb{R}^2$ was stated and proved in [BS02].

Returning to the proof of (14), first note that $(K \setminus \mathrm{Q}) \setminus \left(\bigcup_{k=1}^{\infty} \mathcal{S}^k(\mathrm{Q})\right) \subseteq (2\,\mathrm{Q} \setminus \mathrm{Q}) \setminus \mathcal{M} = \emptyset$. Thus, it is sufficient to show, almost everywhere, that $\bigcup_{k=1}^{\infty} (K \setminus \mathrm{Q}) \cap \mathcal{S}^k(\mathrm{Q}) = (K \setminus \mathrm{Q}) \cap \left(\bigcup_{k=1}^{\infty} \mathcal{S}^k(\mathrm{Q})\right)$ is a subset of $\bigcup_{k=1}^{\infty} S^k(\mathrm{Q})$. To this end, we begin by making $\bigcup_{k=1}^{\infty} \mathcal{S}^k(\mathrm{Q})$ a disjoint union as follows:

$$\begin{aligned}
Q_1 &= \mathcal{S}(\mathrm{Q}), \\
Q_2 &= \mathcal{S}^2(\mathrm{Q}) \setminus \mathcal{S}(\mathrm{Q}), \\
&\vdots \\
Q_k &= \mathcal{S}^k(\mathrm{Q}) \setminus \bigcup_{l<k} \mathcal{S}^l(\mathrm{Q}),
\end{aligned}$$

and then show by induction on k that $(K \setminus \mathrm{Q}) \cap Q_k \subseteq S^k(\mathrm{Q})$ a.e.

Let $(\gamma_1, \gamma_2) \in (K \setminus \mathrm{Q}) \cap \mathcal{S}(\mathrm{Q})$. Then $\gamma_1^{(1)} = \gamma_1^{(2)}$ and $\gamma_2^{(1)} = \gamma_2^{(2)}$. From the surjectivity of T onto $(2\,\mathrm{Q} \setminus \mathrm{Q}) \cap K = K \setminus \mathrm{Q}$, $(\gamma_1, \gamma_2) = T\left(\frac{1}{2}(\zeta_1, \zeta_2)\right)$ for some $(\zeta_1, \zeta_2) \in 2\,\mathrm{Q}$. Since the first two binary digits of $\zeta_1/2$ and $\zeta_2/2$ are equal, i.e., $(\zeta_1/2)^{(1)} = (\zeta_1/2)^{(2)}$ and $(\zeta_2/2)^{(1)} = (\zeta_2/2)^{(2)}$, and $-\frac{1}{2} \leq \frac{\zeta_1}{2}, \frac{\zeta_2}{2} \leq \frac{1}{2}$, the first two digits of both $\zeta_1/2$ and $\zeta_2/2$ must be zero. Therefore $-\frac{1}{4} \leq \frac{\zeta_1}{2}, \frac{\zeta_2}{2} \leq \frac{1}{4}$ and $(\gamma_1, \gamma_2) \in T\left(\frac{1}{2}\,\mathrm{Q}\right) = S(\mathrm{Q})$.

Assume that $(K \setminus \mathrm{Q}) \cap Q_k \subseteq S^k(\mathrm{Q})$ a.e. and let $(\gamma_1, \gamma_2) \in (K \setminus \mathrm{Q}) \cap Q_{k+1}$ be given. By the definition of Q_{k+1}, $k+1$ is the first (smallest) i such that $\gamma_1^{(i)} = \gamma_1^{(i+1)}$, $\gamma_2^{(i)} = \gamma_2^{(i+1)}$, and there exists $(\zeta_1, \zeta_2) \in 2\,\mathrm{Q}$ for which $(\gamma_1, \gamma_2) = T(\frac{1}{2}(\zeta_1, \zeta_2))$. In fact $(\zeta_1, \zeta_2) \in K$ since K tiles $\mathbb{R}^d$ by both integer translation and dyadic dilation and $K \subseteq [-1,1]^d \setminus [-\frac{1}{4}, \frac{1}{4}]^d$. Further, $(\zeta_1, \zeta_2) \in Q_k$. To verify that $(\zeta_1, \zeta_2) \notin \mathrm{Q} = [-\frac{1}{2}, \frac{1}{2}]^d$, we observe that (γ_1, γ_2) not being in $Q_1 = \mathcal{S}(\mathrm{Q})$ implies that $(\zeta_1, \zeta_2) = 2T^{-1}(\gamma_1, \gamma_2) \notin \mathrm{Q}$. Thus $(\zeta_1, \zeta_2) \in (K \setminus \mathrm{Q}) \cap Q_k$ which is contained in $S^k(\mathrm{Q})$ by the assumption. Consequently $(\gamma_1, \gamma_2) = T\left(\frac{1}{2}(\zeta_1, \zeta_2)\right) \in S^{k+1}(\mathrm{Q})$ as desired. $\square$

Corollary 24. *For every wavelet set K in $[-1,1]^d$,*

$$\left|K \cap [-c, c]^d\right| > 0 \quad \text{for all } c > \frac{1}{4}.$$

Proof. If $\left|K \cap \left[-\frac{1}{4}, \frac{1}{4}\right]^d\right| > 0$, then we are done. Otherwise, by Theorem 23, the wavelet set K can be constructed by the neighborhood-mapping construction. By definition of the map T, it can be shown that A_2 is the only A_n that intersects the "square annulus" $\overline{\left[-\frac{5}{16}, \frac{5}{16}\right]^d \setminus \left[-\frac{1}{4}, \frac{1}{4}\right]^d}$. Again, since T is a $\mathbb{Z}^d$-translated map, it is not possible for A_2 to cover $[-c, c]^d \setminus \left[-\frac{1}{4}, \frac{1}{4}\right]^d$ for any $c > \frac{1}{4}$. This completes the proof. $\square$

6 Wavelet sets in $[-1,1]^d$

It is a consequence of the geometrical characterization of wavelet sets in (1) that a wavelet set can not be contained in a closed proper sub-cube of $2\,\mathrm{Q} = [-1,1]^d$. In fact, we have the following result.

Proposition 25. *For any $\alpha < 1$, a wavelet set K can not be contained in $[-\alpha, \alpha]^d$.*

Proof. Suppose that $K \subseteq [-\alpha, \alpha]^d$ is a wavelet set. By (1), the $\mathbb{Z}^d$-translates of K will cover $\widehat{\mathbb{R}}^d$, i.e.,

$$\dot{\bigcup}_{k \in \mathbb{Z}^d} (K + k) = \widehat{\mathbb{R}}^d.$$

Observe that, for each $i = 1, \ldots, d$, the union of all translates $K + (k_1, \ldots, k_d)$ with $k_i \neq 0$ leaves out the band $B_i \equiv \{(\gamma_1, \gamma_2, \ldots, \gamma_d) \in \mathbb{R}^d : \alpha - 1 < x_i < 1 - \alpha\}$, i.e., for each $i = 1, 2, \ldots, d$,

$$\dot{\bigcup}_{k_i \neq 0} (K + k) \cap B_i = \emptyset.$$

Therefore,

$$\dot{\bigcup}_{k \neq (0, \ldots, 0)} (K + k) \cap B = \emptyset,$$

where $B \equiv \bigcap_{i=1}^d B_i = (\alpha - 1, 1 - \alpha)^d$. This clearly implies that $B \subseteq K$ and that $\emptyset \neq B \subseteq K \cap 2K$, a contradiction to the second condition in (1). $\square$

To construct wavelet sets K by the neighborhood-mapping construction in Section 1.2, we note that N must always be at least $\frac{1}{2}$ since K_0 has to be τ-congruent to $[-\frac{1}{2}, \frac{1}{2}]^d$ and contained in $[-N, N]^d$. Further, if $K \subseteq [-1, 1]^d$, then it is also necessary that $N = \frac{1}{2}$. In fact, it is impossible to map a point in $[-N, N]^d \setminus [-\frac{1}{2}, \frac{1}{2}]^d$ into $\overline{[-1, 1]^d \setminus [-N, N]^d}$ by $\mathbb{Z}^d$-translation; and this is a requirement for the map T. Hence, the only possible K_0 in this case is $\mathrm{Q} \equiv [-\frac{1}{2}, \frac{1}{2}]^d$, and the function T is a mapping from Q to $\overline{2\,\mathrm{Q} \setminus \mathrm{Q}}$ defined by $T(\gamma) = \gamma + n_\gamma$ for some $n_\gamma \in \Lambda \equiv \{-1, 0, 1\}^d \setminus \{(0, \ldots, 0)\}$.

Proposition 26. *Let $T : \mathrm{Q} \to 2\,\mathrm{Q} \setminus \mathrm{Q}$ be a measurable injective $\mathbb{Z}^d$-translated map. Let A_n and K_n be the auxiliary sets in the neighborhood-mapping construction and let K be the wavelet set that is constructed. Then, for each $n = 0, 1, \ldots,$*

$$A_n = \frac{1}{2} S^n(\mathrm{Q}) = R^n\left(\frac{1}{2}\mathrm{Q}\right), \tag{15}$$

where $S(\gamma) \equiv T\left(\frac{1}{2}\gamma\right)$ and $R(\gamma) \equiv \frac{1}{2} T(\gamma)$.

Proof. First, it is clear by definition that $A_0 = \frac{1}{2}\,\mathrm{Q} = \left[-\frac{1}{4}, \frac{1}{4}\right]^d$. Since $2^{-j}T(E) \subseteq \left[-\frac{1}{4}, \frac{1}{4}\right]^d$ for all $j \geq 2$, $E \subseteq \mathrm{Q}$, we also have

$$A_n = \left[K_0 \setminus \bigcup_{k=0}^{n-1} A_k\right] \cap \left[\frac{1}{2}T(A_{n-1})\right].$$

It is then left to show that $\frac{1}{2}T(A_{n-1})$ does not intersect $\bigcup_{k=0}^{n-1} A_k$, from which we can conclude that $A_n = \frac{1}{2}T(A_{n-1})$. This in turn follows from the limitation of the map T, that it can only translate by at most 1 both "horizontally" and "vertically", and an induction argument showing that each A_n is contained in a "square annulus" $B_n = \overline{[-b_n, b_n]^d \setminus [-a_n, a_n]^d}$, where $b_n - a_n = 2^{-n-2}$ and the B_ns are mutually disjoint. ☐

Therefore, combining (2) and (15), we obtain

$$K = \left[\mathrm{Q} \setminus \bigcup_{k=0}^{\infty} \frac{1}{2}S^k(\mathrm{Q})\right] \cup \left[\bigcup_{k=0}^{\infty} T\left(\frac{1}{2}S^k(\mathrm{Q})\right)\right]. \qquad (16)$$

When T is regular enough, e.g., $|\cup_{k\in\Lambda}\partial\Omega_k| = 0$ where $\Omega_k \equiv \{\gamma \in \mathrm{Q} : T(\gamma) = \gamma + n_\gamma\}$, we also have

$$K = \left[\mathrm{Q} \setminus \bigcup_{k=0}^{\infty} \frac{1}{2}\overline{S^k(\mathrm{Q})}\right] \cup \left[\bigcup_{k=1}^{\infty} \overline{S^k(\mathrm{Q})}\right] \quad \text{a.e.} \qquad (17)$$

In light of this reformulation of the wavelet set K in the case $K_0 = \mathrm{Q}$, we set

$$M \equiv \bigcup_{k=1}^{\infty} \overline{S^k(\mathrm{Q})} \subseteq \overline{2\,\mathrm{Q} \setminus \mathrm{Q}}, \qquad (18)$$

where, of course, S and M both depend on the map T.

Example 27. Consider the map $T_1 : \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \to [-1,1]^2 \setminus \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ defined by

$$T_1(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2) - (\operatorname{sign}\gamma_1, \operatorname{sign}\gamma_2);$$

and write $S_1 \equiv S_{T_1}$ and $M_1 \equiv M_{T_1}$.

The wavelet set associated with T_1 and constructed by (17) is illustrated in Figure 9. It is a generalization to $\mathbb{R}^2$ of the wavelet set $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$, whose corresponding wavelet is usually designated as the Shannon or Littlewood-Paley wavelet. Also observe that the map $S_1 \equiv T_1\left(\frac{1}{2}\cdot\right)$ is a similarity map

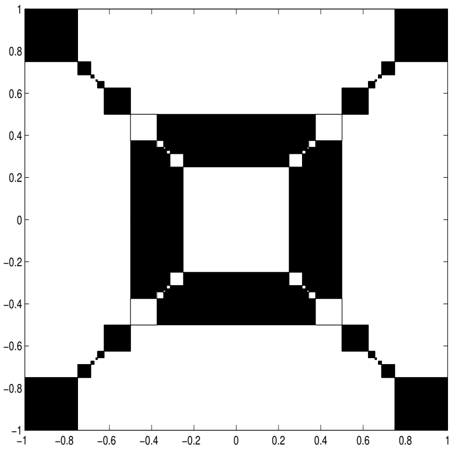


Figure 9: The set K_7

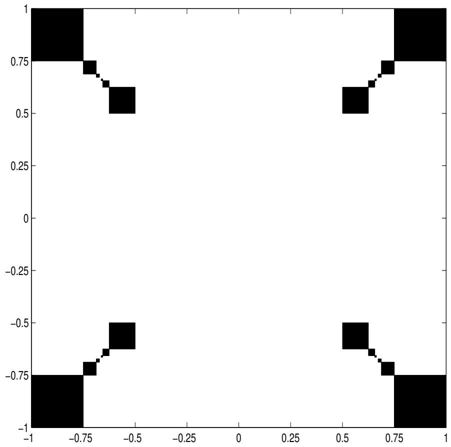


Figure 10: The set $\bigcup_{k=1}^{6} \overline{S_1^k(\mathrm{Q})}$

from one quadrant to the opposite with ratio $\frac{1}{2}$ when restricted to each quadrant. The resulting set M_1 in Figure 10 is self-similar at the four limit points $F_1 \equiv \left\{\left(\pm\frac{2}{3}, \pm\frac{2}{3}\right)\right\}$ in the sense that there is a compact subset P_1, viz., $P_1 = \left(\left[-1, -\frac{3}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right)^2$ consisting of the four largest squares in the outer corners of M_1, such that

$$M_1 = S_1(M_1) \cup P_1 \quad \text{and} \quad F_1 = S_1(F_1).$$

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References

[Aus95] Pascal Auscher. Solution of two problems on wavelets. *J. Geom. Analysis*, 5(2):181–236, 1995.

[BL99] John J. Benedetto and Manuel T. Leon. The construction of multiple dyadic minimally supported frequency wavelets on $\mathbb{R}^d$. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, volume 247 of *Contemp. Math.*, pages 43–74. Amer. Math. Soc., Providence, RI, 1999.

[BL01] John J. Benedetto and Manuel T. Leon. The construction of single wavelets in D-dimensions. *J. Geom. Analysis*, 11(1):1–15, 2001.

[BM99] Lawrence W. Baggett and Kathy D. Merrill. Abstract harmonic analysis and wavelets in $\mathbb{R}^n$. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, pages 17–27. Amer. Math. Soc., Providence, RI, 1999.

[BMM99] Lawrence W. Baggett, Herbert A. Medina, and Kathy D. Merrill. Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$. *J. Fourier Analysis Appl.*, 5(6):563–573, 1999.

[Bow00] Marcin Bownik. A characterization of affine dual frames in $\mathrm{L}^2(\mathbb{R}^n)$. *Appl. Comput. Harmon. Analysis*, 8(2):203–221, 2000.

[BP63] Errett Bishop and Robert R. Phelps. The support functionals of a convex set. In *Proc. Sympos. Pure Math., Vol. VII*, pages 27–35. Amer. Math. Soc., Providence, R.I., 1963.

[BS02] John J. Benedetto and Songkiat Sumetkijakan. A fractal set constructed from a class of wavelet sets. In *Inverse Problems, Image Anlysis, and Medical Imaging (New Orleans, LA, 2001)*, pages 19–35. Amer. Math. Soc., Providence, RI, 2002.

[CCMW02] Charles K. Chui, Wojciech Czaja, Mauro Maggioni, and Guido Weiss. Characterization of general tight wavelet frames with matrix dilations and tightness preserving oversampling. *J. Fourier Analysis Appl.*, 8(2):173–200, 2002.

[Chr01] Ole Christensen. Frames, Riesz bases, and discrete Gabor/wavelet expansions. *Bull. Amer. Math. Soc. (N.S.)*, 38(3):273–291 (electronic), 2001.

[Cou99] Jennifer Courter. Construction of dilation-d wavelets. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, volume 247 of *Contemp. Math.*, pages 183–205. Amer. Math. Soc., Providence, RI, 1999.

[CS00] Charles K. Chui and Xianliang Shi. Orthonormal wavelets and tight frames with arbitrary real dilations. *Appl. Comput. Harmon. Analysis*, 9(3):243–264, 2000.

[Dau92] Ingrid Daubechies. *Ten Lectures on Wavelets.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[DDG01] Xingde Dai, Yuanan Diao, and Qing Gu. Frame wavelet sets in $\mathbb{R}$. *Proc. Amer. Math. Soc.*, 129(7):2045–2055 (electronic), 2001.

[DDG02] Xingde Dai, Yuanan Diao, and Qing Gu. Subspaces with normalized tight frame wavelets in $\mathbb{R}$. *Proc. Amer. Math. Soc.*, 130(6):1661–1667 (electronic), 2002.

[DDG03] Xingde Dai, Yuanan Diao, and Qing Gu. Frame wavelets with frame set support in the frequency domain. Preprint, 2003.

[DDGH02] Xingde Dai, Yuanan Diao, Qing Gu, and Deguang Han. Frame wavelets in subspaces of $\mathrm{L}^2(\mathbb{R}^d)$. *Proc. Amer. Math. Soc.*, 130(11):3259–3267 (electronic), 2002.

[DL98] Xingde Dai and David R. Larson. Wandering vectors for unitary systems and orthogonal wavelets. *Memoirs Amer. Math. Soc.*, 134(640):viii+68, 1998.

[DLS97] Xingde Dai, David R. Larson, and Darrin M. Speegle. Wavelet sets in $\mathbb{R}^n$. *J. Fourier Analysis Appl.*, 3(4):451–456, 1997.

[DLS98] Xingde Dai, David R. Larson, and Darrin M. Speegle. Wavelet sets in $\mathbb{R}^n$. II. In *Wavelets, Multiwavelets, and their Applications (San Diego, CA, 1997)*, pages 15–40. Amer. Math. Soc., Providence, RI, 1998.

[DS52] Richard J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[Gri95] Gustaf Gripenberg. A necessary and sufficient condition for the existence of a father wavelet. *Studia Math.*, 114(3):207–226, 1995.

[Han97] Bin Han. On dual wavelet tight frames. *Appl. Comput. Harmon. Analysis*, 4(4):380–413, 1997.

[HW96] Eugenio Hernández and Guido Weiss. *A First Course on Wavelets.* CRC Press, Boca Raton, FL, 1996.

[KZ95] Tharmarajah Kugarajah and Qinghua Zhang. Multidimensional wavelet frames. *IEEE Transactions on Neural Networks*, 6(6):1552–1556, November 1995.

[Li02] Yunzhang Li. On the holes of a class of bidimensional nonseparable wavelets. 2002.

[Mal85] Stéphane G. Mallat. An efficient image representation for multiscale analysis. Technical report, GRASP Laboratory, Dept. of Computer and Information Science, U. of Pennsylvania, Philadelphia, (about 1985).

[Mal89] Stéphane G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. *Trans. Amer. Math. Soc.*, 315(1):69–87, September 1989.

[Mey86] Yves Meyer. Ondelettes, fonctions splines et analyses graduées. Lectures given at the University of Torino, Italy, 1986.

[Mey87] Yves Meyer. Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs. *Astérisque*, (145-146):4, 209–223, 1987. Séminaire Bourbaki, Vol. 1985/86.

[Mey89] Yves Meyer. Ondelettes, filtres miroirs en quadrature et traitement numérique de l'image. *Gaz. Math.*, (40):31–42, 1989.

[Mey90] Yves Meyer. *Ondelettes et Opérateurs*. Hermann, Paris, 1990.

[Pap01] Manos Papadakis. Frames of translates in abstract Hilbert spaces and the generalized frame multiresolution analysis. In *Trends in approximation theory (Nashville, TN, 2000)*, Innov. Appl. Math., pages 353–362. Vanderbilt Univ. Press, Nashville, TN, 2001.

[Pap03] Manos Papadakis. Generalized frame multiresolution analysis of abstract Hilbert spaces. In *Sampling, Wavelets, and Tomography*. Birkhäuser, Boston, MA, 2003.

[PS98] Manos Papadakis and Theodoros Stavropoulos. On the multiresolution analyses of abstract Hilbert spaces. *Bull. Greek Math. Soc.*, 40:79–92, 1998.

[Rud73] Walter Rudin. *Functional Analysis*. McGraw-Hill Book Co., New York, 1973. McGraw-Hill Series in Higher Mathematics.

[Spe02] Darrin M. Speegle. On the existence of wavelets for non-expansive dilation matrices. Preprint, 2002.

[Sum03] Songkiat Sumetkijakan. Fractal sets in the neighborhood-mapping construction of wavelet sets. Preprint, 2003.

[SW98] Paolo M. Soardi and David Weiland. Single wavelets in n-dimensions. *J. Fourier Analysis Appl.*, 4(3):299–315, 1998.

[Wan02] Yang Wang. Wavelets, tiling, and spectral sets. *Duke Math. J.*, 114(1):43–57, 2002.

[Zak96] V. Zakharov. Nonseparable multidimensional Littlewood-Paley like wavelet bases. Preprint, 1996.