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**TITLE:** IMPORTANCE OF ZAK TRANSFORMS FOR HARMONIC ANALYSIS

**ABSTRACT:** In engineering and applied mathematics, Zak transforms have been effectively used for over 50 years in various applied settings. As Andre Weil observed in the 1940s and as I. Gelfand noted in a 1950 paper, an exceedingly elementary proof of the Plancherel Theorem for LCA groups uses only the Fourier series ideas later incorporated in Zak transforms; in brief, Zak transforms are Fourier series expressions and the

Fourier transform on any non-compact LCA group is an average of Zak transforms. It is remarkable that only a small handful of mathematicians know this proof and that all textbooks continue to give much harder and less transparent proofs for even the case of the group $\mathbb{R}$. Generalized Zak transforms arise naturally as intertwining operators for various representations of Abelian groups and allow formulation of many appealing theorems.

*Remark : The results discussed below represent joint work by the speaker with E. Hernandez, H. Sikic, and G. Weiss.*

## 1. The Abelian Group Plancherel Theorem

**1.1 Overview.** In textbooks on real analysis, one can find a variety of proofs of the Plancherel Theorem for $\mathbb{R}^n, n \in \mathbb{N}$. All are lengthy, non-elementary, and technical, *e.g.* :

$*$ use of complex analysis to compute Fourier transforms of Gaussian functions followed by use of approximate identities defined by Gaussians to extend the Fourier transform from $\mathrm{L}^1(\mathbb{R}^n) \cap \mathrm{L}^2(\mathbb{R}^n)$ to a unitary operator on $\mathrm{L}^2(\mathbb{R}^n)$;

$*$ use of the Hermite function orthonormal basis for $L^2(\mathbb{R}^n)$ and the computation that theHermite functions are eigenfunctions of the Fourier transform having eigenvalues which are fourth roots of 1;

$*$ reversion to the 19th century interpretation of Fourier transforms as Riemann sum limits of rescaled Fourier series expressions and justification of this approach by somewhat delicate dominated convergence arguments.

In fact, as we'll show, none of this is necessary. The proof of the Plancherel Theorem for $\mathbb{R}^n$ via Zak transforms uses only basic Fourier series ideas and applies with only superficial changes in notation to every locally compact, Abelian group (LCA group).

1.2. **Notations and Definitions.** Let G=(G,+) be an additive LCA group.

($i$) The dual group $\widehat{\mathrm{G}} = (\widehat{\mathrm{G}}, +)$ of G is the essentially unique LCA group for which there is a continuous, bi-additive homomorphism $(\xi, x) \mapsto \xi \cdot x$ from $\widehat{\mathrm{G}} \times \mathrm{G}$ into $\mathbb{R}/\mathbb{Z}$ such that, with $e_\xi(x) = e_x(\xi) \equiv e^{2\pi i \xi \cdot x}$, every continuous homomorphism from G (*respectively*, $\widehat{\mathrm{G}}$) into the multiplicative group $\{z \in \mathbb{C} : |z| = 1\}$ is of the form

$e_\xi$ for some $\xi \in \widehat{\mathrm{G}}$ ($respectively,$ of the form $e_x$for some $x \in$ G). Existence of $\widehat{\mathrm{G}}$ is shown in many texts. [For G=$\mathbb{R}^n$ = additive group of $n \times 1$ real column matrices, it's convenient to take $\widehat{\mathrm{G}}$ to be the group of $1 \times n$ real row matrices with $\xi \cdot x$ a matrix product and similarly with all other Abelian Lie groups]

$(ii)$ A lattice in G is a topologically discrete subgroup $\mathcal{L} \subset$ G for which $\mathrm{T}_{\mathcal{L}} =$ G/$\mathcal{L}$ is compact in the quotient topology ($e.g.$, the integer lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$).

Existence of lattices follows from Weil's structural theorem: The connected component $\mathrm{G}_0$of 0 in G is the direct sum of the unique maximal, connected, compact, subgroup K of G and a non-unique subgroup isomorphic and homeomorphic to $\mathbb{R}^n$, $n \geq 0$, with G/$\mathrm{G}_0$ discrete.

$(iii)$ Given a lattice $\mathcal{L}$, there is a unique Haar measure $\mu = \mu_{\mathcal{L}}$ on G assigning mass 1 to every $\mathcal{L}$-tiling domain C $\subset$ G (thus C is Borel measurable and G is the disjoint union of the translates of C by members of $\mathcal{L}$--if we wish we can take C to have compact closure). Then $\mathcal{L}^\perp = \{j \in \widehat{\mathrm{G}} : \forall k \in \mathcal{L},\ j \cdot k$ is the zero element in $\mathbb{R}/\mathbb{Z}\}$ is a lattice in $\widehat{\mathrm{G}}$ called the lattice dual of $\mathcal{L}$ and there is a unique Haar measure $\widehat{\mu} = (\widehat{\mu})_{\mathcal{L}^\perp}$ on $\widehat{\mathrm{G}}$ assigning mass 1 to every $\mathcal{L}^\perp$ – tiling domain. Also, $\mu$ (respectively, $\widehat{\mu}$ )induces normalized Haar measure on the compact group $\mathrm{T}_{\mathcal{L}} = \mathrm{G}/\mathcal{L}$ (respectively, on $\mathrm{T}_{\mathcal{L}^\perp} = \widehat{\mathrm{G}}/\mathcal{L}^\perp$ ) and $\{e_j : j \in \mathcal{L}^\perp\}$ is an orthonormal basis for $\mathrm{L}^2(\mathrm{T}_{\mathcal{L}})$ while $\{e_k : k \in \mathcal{L}\}$ is an orthonormal basis for $\mathrm{L}^2(\mathrm{T}_{\mathcal{L}^\perp})$.

[In the $\mathbb{R}^n$ case with $\mathcal{L}$=$\mathbb{Z}^n$, $[0,1)^n$ is a $\mathcal{L}$-tiling domain and the matrix transpose map takes $\mathbb{Z}^n$ to $(\mathbb{Z}^n)^\perp$, so $\mu$ is Lebesgue measure on $\mathbb{R}^n$ and $\widehat{\mu}$ is Lebesgue measure on $(\mathbb{R}^n)^\wedge$. When G is an Abelian Lie group with finitely many connected components, each of the compact groups $\mathrm{T}_{\mathcal{L}} = \mathrm{G}/\mathcal{L}$ is isomorphic and homeomorphic to the product of the standard $n$ – torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 0$, and a finite group.]

$(iv)$ Using the notations in $(iii)$, let $\mathcal{M}_{\mathrm{T}_{\mathcal{L}} \times \mathrm{T}_{\mathcal{L}^\perp}}$ and $(\mathcal{M}_{\mathrm{T}_{\mathcal{L}} \times \mathrm{T}_{\mathcal{L}^\perp}})^\sim$ be the spaces of $\mathbb{C}$-valued measurable functions $\Phi$ and $\Phi^\sim$ on $\mathrm{G} \times \widehat{\mathrm{G}}$ such that, for all $(k,j) \in \mathcal{L} \times \mathcal{L}^\perp$ and all $(x,\xi) \in \mathrm{G} \times \widehat{\mathrm{G}}$,

$$\Phi(x+k, \xi+j) = e_k(\xi)\Phi(x,\xi) \ a.e., \qquad (1)$$

$$\Phi^\sim(x+k, \xi+j) = e_{-j}(x)\Phi(x,\xi) \ a.e., \quad (2)$$

and the $\mathcal{L} \times \mathcal{L}^{\perp}$ periodic functions $|\Phi|$, $|\Phi^{\sim}|$ are in $\mathrm{L}^2(\mathrm{T}_{\mathcal{L}} \times \mathrm{T}_{\mathcal{L}^{\perp}})$.

Note that the magnitudes of $\Phi$ and $\Phi^{\sim}$ are periodic in both variables by (1) and (2). The norms of these functions are understood to be the $\mathrm{L}^2$ norms of their magnitudes as functions on the compact group $\mathrm{T}_{\mathcal{L}} \times \mathrm{T}_{\mathcal{L}^{\perp}}$ relative to the normalized Haar measure induced by $\mu \times \widehat{\mu}$.

($v$) Using the notations in ($iii$), for $f \in \mathrm{L}^2(\mathrm{G}, \mu)$ and $g \in \mathrm{L}^2(\widehat{\mathrm{G}}, \widehat{\mu})$, the Zak transforms $\mathrm{Z}_{\mathcal{L}} f$ and $\mathrm{Z}^{\sim}_{\mathcal{L}^{\perp}} g$ of $f$ and $g$ are the $a.e.$ well defined Fourier series expressions

$$(\mathrm{Z}_{\mathcal{L}} f)(x, \,\cdot\,) = \sum_{k \in \mathcal{L}} f(x+k) e_{-k}(\,\cdot\,) \tag{3}$$

$$(\mathrm{Z}^{\sim}_{\mathcal{L}^{\perp}} g)(\,\cdot\,, \xi) = \sum_{j \in \mathcal{L}^{\perp}} g(\xi + j)\, e_j(\,\cdot\,) \tag{4}$$

Note that not only are the roles of $x$ and $\xi$ reversed in (3) and (4) but, as in (1) and (2), we also have a change of sign in the exponents.

**1.3. Theorem.** Using the above notations, for each choice of $\mathcal{L}$,

($i$) $f \mapsto \mathrm{Z}_{\mathcal{L}} f$ is a unitary map from $\mathrm{L}^2(\mathrm{G},\mu)$ onto $\mathcal{M}_{\mathrm{T}_{\mathcal{L}} \times \mathrm{T}_{\mathcal{L}^{\perp}}}$ whose inverse is $a.e.$ well defined by

$f(x) = ((Z_{\mathcal{L}})^{-1}\Phi)(x) = \int_{T_{\mathcal{L}^\perp}} \Phi(x,\xi)d\widehat{\mu}(\xi);$ (5)

$(ii) g \mapsto Z^{\sim}_{\mathcal{L}^\perp} g$ is a unitary map from $L^2(\widehat{G},\widehat{\mu})$ onto $(\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^\perp}})^{\sim}$ whose inverse is $a.e.$ well defined by

$g(\xi) = ((Z^{\sim}_{\mathcal{L}^\perp})^{-1}\Phi^{\sim})(x) = \int_{T_{\mathcal{L}}} \Phi(x,\xi)d\mu(x);$ (6)

$(iii)$ $\Phi^{\sim}(x,\xi) = (\mathcal{U}\Phi)(x,\xi) = e^{-2\pi i\xi x}\Phi(x,\xi)$ (7) defines a unitary map $\mathcal{U}$ from $\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^\perp}}$ onto $(\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^\perp}})^{\sim}$.

**Proof :** $(i)$ For each $\mathcal{L}$-tiling domain $C \subset G$, translation invariance of $\mu$ gives

$$||f||^2_{L^2(G,\mu)} = \int_C \sum_{k\in\mathcal{L}} |f(x+k)|^2 \qquad (8)$$

so $(f(x+k))_{k\in\mathcal{L}} \in l^2(\mathcal{L})$ for $a.e.$ $x \in G$. Since $\{e_{-k} : k \in \mathcal{L}\}$ is an orthonormal basis for $L^2(T_{\mathcal{L}^\perp})$, $(Z_{\mathcal{L}}f)(x,\cdot) \in L^2(T_{\mathcal{L}^\perp})$ for $a.e.$ $x$ and a simple change of summation index argument shows that $Z_{\mathcal{L}} f$ satisfies the transformation condition (1) with (8) then yielding $Z_{\mathcal{L}} f \in \mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^\perp}}$ and $||Z_{\mathcal{L}} f||^2 = ||f||^2_{L^2(G,\mu)}$. Finally, given $\Phi \in \mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^\perp}}$ with $f$ defined by (5), the transformation law (1) implies that, for $a.e.\ x$, $f(x+k)$ is the $(-k)^{th}$ Fourier coefficient of the the $L^2(T_{\mathcal{L}^\perp})$ function $\Phi(x,\ \cdot\ )$ from which it follows that $f \in L^2(G,\mu)$ with $\Phi = Z_{\mathcal{L}}\, f$.

$(ii)$ We merely repeat the arguments in $(ii)$ with the roles of $x$ and $\xi$ reversed and use the transformation law (2) in place of (1).

$(iii)$ is merely an elementary computation showing that $\mathcal{U}$ converts the transformation law (1) for $\Phi$ to the transformation law (2) for $\Phi^{\sim}$ along with the trivial observation that $\mathcal{U}$ doesn't effect magnitudes.

1.4 **Corollary 1 (The Plancherel Theorem for LCA Groups).** Using the above notations, for each choice of a lattice $\mathcal{L} \subset G$ and corresponding dual lattice $\mathcal{L}^\perp \subset \widehat{G}$,

$(i)$ the unitary map $\mathcal{F}_G = (Z^{\sim}_{\mathcal{L}^\perp})^{-1} \circ \mathcal{U} \circ Z_{\mathcal{L}}$ from $L^2(G,\ \mu)$ onto $L^2(\widehat{G},\ \widehat{\mu})$ is described on the dense subspace $L^1(G,\ \mu) \cap L^2(G,\ \mu) \subset L^2(G,\mu)$ by

$$(\mathcal{F}_G f)(\xi) = \widehat{f}(\xi) = \textstyle\int_G f(x) e_{-\xi}(x) d\mu(x); \quad (9)$$

$(ii)$ when $g \in L^1(\widehat{G},\ \widehat{\mu}) \cap L^2(\widehat{G},\ \widehat{\mu})$,

$$((\mathcal{F}_{\mathrm{G}})^{-1}g)(x) = ((\mathrm{Z}_{\mathcal{L}})^{-1}\circ\mathcal{U}^{-1}\circ\mathrm{Z}_{\mathcal{L}^{\perp}}g)(x)$$
$$= (\mathcal{F}_{\widehat{\mathrm{G}}}g)(-x) = \int_{\widehat{\mathrm{G}}}g(\xi)e_x(\xi)d\widehat{\mu}(\xi) \quad (10)$$

[In particular, of course, $(i)$ proves the existence of a unique unitary extension to $\mathrm{L}^2(\mathrm{G})$ of the Fourier transform $f \mapsto \widehat{f}$ on $(\mathrm{L}^1 \cap \mathrm{L}^2)$ $(\mathrm{G}, \mu)$ and gives an explicit expression for this extension, $(ii)$ gives the standard formula relating $\mathcal{F}_{\widehat{\mathrm{G}}}$ to the inverse of $\mathcal{F}_{\mathrm{G}}$, and $(i)$ and $(ii)$ show that the only pairs of Haar measures $\mu$, $\widehat{\mu}$ on G, $\widehat{\mathrm{G}}$ for which the Plancherel Theorem holds are $\mu = \mu_{\mathcal{L}}$, $\widehat{\mu} = (\widehat{\mu}\,)_{\mathcal{L}^{\perp}}$ for some dual lattice pair $\mathcal{L}$, $\mathcal{L}^{\perp}$. ]

**Proof:** $(i)$ For $f \in \mathrm{L}^1(\mathrm{G}, \mu) \cap \mathrm{L}^2(\mathrm{G}, \mu)$, we use the definitions of $\mathrm{Z}_{\mathcal{L}}f$ and $\mathcal{U}$ in (3) and (7) along with the inversion formula (6) for $\mathrm{Z}_{\mathcal{L}^{\perp}}$ to obtain, for each choice of a $\mathcal{L}$-tiling domain $\mathrm{C} \subset \mathrm{G}$,

$$((\mathrm{Z}_{\mathcal{L}^{\perp}}^{\sim})^{-1}\circ\mathcal{U}\circ\mathrm{Z}_{\mathcal{L}}\, f)(\xi)$$
$$= \int_{\mathrm{C}}e^{-2\pi i\xi\cdot x}(\mathrm{Z}_{\mathcal{L}}\, f)(x,\xi)d\mu(x)$$

$$= \int_{\mathrm{C}}\sum_{k\in\mathcal{L}} f(x+k)e^{-2\pi i\xi\cdot(x+k)}$$
$$= \text{(by translation invariance of } \mu)$$
$$\int_{\mathrm{G}}f(y)e_{-\xi}(y)d\mu(y)$$

$= \widehat{f}(\xi).$

$(ii)$ follows from a similar computation using (4) and (5) in place of (3) and (6), the only changes being reversal of the roles of G and $\widehat{\mathrm{G}}$ and the sign changes in the exponents for $\mathrm{Z}^{\sim}_{\mathcal{L}^{\perp}}$ and $\mathcal{U}^{-1}$.

**1.5 Corollary 2** (Poisson Summation Formula) When $f$ satisfies the smoothness and decay properties needed to have both $\mathrm{Z}_{\mathcal{L}}f$ and $\mathrm{Z}^{\sim}_{\mathcal{L}^{\perp}}\widehat{f}$ pointwise well – defined and jointly continuous in an open neighborhood of (0,0),

$$\sum_{k\in\mathcal{L}} f(k) = \sum_{j\in\mathcal{L}^{\perp}} \widehat{f}(j). \tag{11}$$

**Proof.** From Theorem 1.3 and Corollary 1, $\mathrm{Z}^{\sim}_{\mathcal{L}^{\perp}}\widehat{f} = \mathcal{U}\mathrm{Z}_{\mathcal{L}}f$. Since $(\mathcal{U}\mathrm{Z}_{\mathcal{L}}f)(0,0)$ $=(\mathrm{Z}_{\mathcal{L}}f)(0,0)=\sum_{k\in\mathcal{L}} f(k)$ while $(\mathrm{Z}^{\sim}_{\mathcal{L}^{\perp}}\widehat{f})(0,0) = \sum_{j\in\mathcal{L}^{\perp}} \widehat{f}(j)$, we obtain (11).

1.6 **Remarks.** Corollary 2 is not surprising since all standard proofs of the Poisson Summation Formula rest on lattice periodization of Fourier integrals and that is precisely what is going on with Zak transforms. Zak transforms can be viewed as discretizations of Fourier integrals and, for the case G=$\mathbb{R}$, can compared with other discretizations such as the short-time Fourier transform and the Discrete Cosine transform. However, Corollary 1 yields the intriguing converse statement that Fourier integrals are just averages of Zak transforms for any choice of a lattice. What is surprising is that, since periodization techniques have been used for over 100 years in harmonic analysis, and since A. Weil's 1940 book on integration on locally compact Hausdorff spaces alludes to a proof of the Plancherel Theorem for Abelian groups via Fourier series ideas with I. Gelfand being sufficiently impressed by this approach to sketch Weil's argument for $\mathbb{R}$ in a 1950 paper on eigenfunction expansions, only a very small handful of mathematicians have paid any attention. Perhaps part of the reason is that the Zak transform for $\mathbb{R}$ is often presented as a somewhat arcane way to turn $L^2(\mathbb{R})$ into $L^2(\mathbb{T}^2)$ and its applications are customarily described as part of the discretization machinery germane to certain problems in mathematical physics (Zak's original motivation in his 1967 paper for introducing a transform motivated by his reading of Weil's famous 1964 Acta paper but which others later labeled the Zak transform) and applied harmonic analysis. The above discussion is intended to suggest that the Zak transform ought to be seen as a fundamental tool for every aspect of Abelian harmonic analysis with the Fourier transform being just a by-product of Zak transforms and with passage to Zak transform image spaces for calculations equivalent to but often considerably less technical than passage to Fourier domains. To say the least, this substantially changes the perspective on Fourier transforms and suggests that introductory courses in real analysis should follow-up standard coverage of elementary measure theory and Fourier series for $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ with definition of the Zak transform $Z_{\mathbb{Z}}$ and at least a sketch of the above argument showing how $Z_{\mathbb{Z}}$ leads quickly and "painlessly" to the Plancherel Theorem for $\mathbb{R}$.

## 2. Generalized Zak Transforms for Abelian Group Representations.

**2.1 Overview.** The isometry and transformation condition properties discussed above for $Z_{\mathcal{L}}$ are succinctly expressed in the language of group representations by saying that the unitary map $Z_{\mathcal{L}}$ intertwines the restriction to $\mathcal{L}$ of the regular representation $f(\,\cdot\,) \mapsto f(x+\,\cdot\,)$ of G on $L^2(G,\mu)$ with the modulation representation $\Phi(x,\,\cdot\,) \mapsto e_l(\,\cdot\,)\Phi(x,\,\cdot\,)$of $\mathcal{L}$

on the Zak space $\mathcal{M}_{\mathrm{T}_{\mathcal{L}}\times\mathrm{T}_{\mathcal{L}^\perp}}$. This suggests going on to define and apply generalized Zak transforms intertwining certain unitary representations of discrete LCA groups with modulation representations. One can also look at operator-valued analogs for non Abelian discrete groups, the limitation being that a non-Abelian discrete group $\mathcal{L}$ has a Plancherel Formula if and only if $\mathcal{L}$ is a finite extension of an Abelian group. We won't take time below to discuss non-Abelian generalizations.

## 2.2 **General Setting for (Abelian) Zak transforms:**

$(i)$ $(l, x) \mapsto l \cdot x$ is a free action of a countable additive group $\mathcal{L}$ on a set X. Thus, $k \cdot (l \cdot x)$ $= (k + l) \cdot x$ for all $x \in$ X and all $k, l \in \mathcal{L}$ with $l \cdot x = x \Leftrightarrow l = 0$.

$(ii)$ There is a $\sigma$-finite measure $\nu$ on X for which $\mathrm{L}^2$(X,$\nu$) is a separable Hilbert space and for which $\nu$ is quasi $\mathcal{L}$-invariant in the sense that, for each $l \in \mathcal{L}$, $x \mapsto l \cdot x$ is measurable and we have a Radon-Nikodym derivative $\mathrm{J}_l(x) = \frac{d\nu(l\cdot x)}{d\nu(x)}$ defined and >0 for $a.e.\, x$. Then, by the chain rule for Radon-Nikodym derivatives,

$$J_{l+k}(x) = J_l(k \cdot x) J_k(x) \; a.e. \text{ and}$$
$$(D_l f)(x) = J_l(x)^{\frac{1}{2}} f(l \cdot x)$$

defines a unitary representation D of $\mathcal{L}$ on $L^2(X,\nu)$.

$(iii)$ The action is regular in the sense that there exists a measurable set C such that X is the disjoint union of the sets $l \cdot C$, $l \in \mathcal{L}$. Hence, X is also the disjoint union of the orbits $\mathcal{L} \cdot x$ as $x$ ranges over C. (Obviously, C plays the role of a $\mathcal{L}$-tiling domain for the special case when $\mathcal{L}$ is a lattice in X=G and $l \cdot x = l + x$).

*Remarks:* $(i)$ *In practice, we start with a continuous action of a non-discrete LCA group G on a locally compact, Hausdorff space* $X'$ *(perhaps a topological manifold), take* $\mathcal{L}$ *to be a lattice in G, take* $C$ *to be a Borel subset of* $X'$ *with* $l \cdot C \cap C = \emptyset$ *for each* $l \in \mathcal{L} \backslash \{0\}$*, then take X=*$\mathcal{L} \cdot C$*. But C could then be replaced by any measurable subset of X containing exactly one point from each* $\mathcal{L}$*-orbit.*

$(ii)$ *In general, when a quasi* $\mathcal{L}$*-invariant measure* $\nu$ *exists, one can construct a finite* $\mathcal{L}$*-invariant measure* $\mu$ *which is equivalent to* $\nu$ *in the usual measure sense. But, for examples of actions of discrete Abelian matrix groups on* $\mathbb{R}^n$ *and, more generally, actions by commuting manifold* *diffeomorphisms along the integral curves of commuting vector fields, there will be a natural choice for* $\nu$, *e.g. Lebesgue measure on* $\mathbb{R}^n$ *in the first case and the measure defined by a Riemannian volume form in the manifold case. In such cases, replacement of* $\nu$ *by* $\mu$ *is artificial and doesn't add anything new.*

2.3 **Notations and Definitions.** In the above general setting:

$(i)$ $(\widehat{\mathcal{L}},\ +)$ is the compact, additive LCA group dual to $\mathcal{L}$ and $\widehat{\mu}$ is normalized Haar measure on $\widehat{\mathcal{L}}$ with $\{e_l : l \in \mathcal{L}\}$ the orthonormal basis of $L^2(\widehat{\mathcal{L}}, \widehat{\mu})$ defined as in §1 by $e_l(\xi) = e^{2\pi i l \cdot x}$.

$(ii)$ For $\psi \in L^2(X, \nu)$, the generalized Zak transform $Z\psi$ of $\psi$ relative to the action of $\mathcal{L}$ and the measure $\nu$ is the $L^2(\widehat{\mathcal{L}}, \widehat{\mu})$ – valued function on X well defined $\nu$-*a.e.* by

$$Z\psi(x,\ \cdot\ ) = \sum_{l \in \mathcal{L}} (D_l\psi)(x)\, e_{-l}(\ \cdot\ ) \qquad (12)$$

2.4 **Remarks:** The computations we made earlier in the case of the translation action of $\mathcal{L} \subset G$ on G generalize easily to yield the following:

$(i)$ $(ZD_l\psi)(x, \xi) = e_l(\xi)\, (Z\psi)(x, \xi)\ a.e.$ (13)

so $\psi \mapsto Z\psi$ intertwines the unitary D of $\mathcal{L}$ on $L^2(X, \mu)$ and the modulation representation of $\mathcal{L}$ on the image under Z of $L^2(X,\mu)$;

(*ii*) For any choice of an orbit-cross section set $C \subset X$ as above,

$$\int_C \int_{\widehat{\mathcal{L}}} |Z\psi(x,\xi)|^2 \, d\widehat{\mu}(\xi) d\nu(x) =$$

$$\int_C \sum_{l \in \mathcal{L}} J_l(x) |\psi(l \cdot x)|^2 d\nu(x) \quad =$$

$$\sum_{l \in \mathcal{L}} \int_{l \cdot C} |\psi(y)|^2 d\nu(y) \quad = ||\psi||^2_{L^2(X,\nu)} \qquad (13)$$

Defining the initial expression in (13) to be $||Z\psi||^2_{\mathcal{M}}$, it follows that Z is an isometry from $L^2(X,\nu)$ onto the Hilbert space $\mathcal{M}$ of measurable functions $\Phi$ from $X \times \widehat{\mathcal{L}}$ into $\mathbb{C}$ satisfying the transformation condition $D_l(\Phi(\,\cdot\,,\xi)) = e_l(\xi)\Phi(\,\cdot\,,\xi)$ *a.e.* and $||\Phi||^2_{\mathcal{M}} = \int_C \int_{\widehat{\mathcal{L}}} |\Phi(x,\xi)|^2 \, d\widehat{\mu}(\xi) d\nu(x) < \infty$. Indeed, for $\Phi \in \mathcal{M}$, $f = Z^{-1}\Phi$ is *a.e.* well defined by

$$f(x) = \int_{\widehat{\mathcal{L}}} \Phi(x,\xi) d\widehat{\mu}(\xi).$$

2.5 **More Notations and Definitions.** In the context of 2.2-2.4, for $\phi, \psi \in L^2(X, \nu)$,

$(i)$ $[\phi,\psi] = [\phi,\psi]_D$ is the member of $L^1(\widehat{\mathcal{L}}, \widehat{\mu})$ well defined $a.e.$ by

$$[\phi,\psi](\xi) = \int_C Z\phi(x,\xi)\overline{Z\psi(x,\xi)}\, d\nu(x); \qquad (14)$$

[Note that by the computations in 2.4 and use of the Cauchy-Schwartz inequality, $(\phi,\psi) \mapsto [\phi,\psi]$ is a bounded, sesquilinear, Hermitian symmetric map from $L^2(X, \nu) \times L^2(X, \nu)$ into $L^1(\widehat{\mathcal{L}},\widehat{\mu})$ and has the positive semi-definite property $[\psi,\psi] \geq 0$.]

$(ii)$ $p_\psi$ is the $L^1(\widehat{\mathcal{L}}, \widehat{\mu})$ weight function $[\psi,\psi]$;

$(iii)$ $supp\, p_\psi = \{\xi : p_\psi(\xi) \neq 0\}$ (well defined modulo a $\widehat{\mu}$ – null set);

$(iv)$ when $\psi \in L^2(X, \nu)\backslash\{0\}$, the D-cyclic subspace $<\psi>_D$ is the closure in $L^2(X, \nu)$ of the span of $\mathcal{B}_\psi = \{D_l\psi : l \in \mathcal{L}\}$.

**Theorem 2.6.** Using the above notations, for $\phi,\psi$ non-zero members of $L^2(X,\nu)$ and $l \in \mathcal{L}$

$(i)$ $[D_l\phi, \psi] = e_l[\phi, \psi] = [\phi, D_{-l}\psi]$ ;

$(ii)$ $< \mathrm{D}_l\phi, \psi >_{L^2(X,\nu)} = \int_{\widehat{\mathcal{L}}} e_l[\phi, \psi] d\widehat{\mu}$
$= < \phi, \mathrm{D}_l\psi >_{L^2(X,\nu)}$;

$(iii)$ $< \phi >_D \perp < \psi >_D \Leftrightarrow [\phi, \psi] = 0 \; a.e.$;

$(iv)$ $\phi \mapsto \mathcal{J}_\psi(\phi) = \frac{[\phi,\psi]}{p_\psi^{1/2}} \chi_{supp\, p_\psi}$ is a unitary map from $<\psi>_{\mathrm{D}}$ onto the closed subspace $\mathcal{H}_\psi$ of $\mathrm{L}^2(\widehat{\mathcal{L}}, \widehat{\mu})$ consisting of members of $\mathrm{L}^2(\widehat{\mathcal{L}}, \widehat{\mu})$ which vanish $a.e.$ off $supp\, p_\psi$. In particular, using $(i)$, $\mathcal{J}_\psi$ intertwines D on $<\psi>_{\mathrm{D}}$ with the modulation representation of $\mathcal{L}$ on $\mathcal{H}_\psi$.

**Sketch of the Proof.** $(i)$ and $(ii)$ are easy calculations using the transformation and isometry properties of Z, $(iii)$ follows easily from $(ii)$, and $(iv)$ is another routine calculation using $(i)$, $(iii)$, and the "inner product" properties of $[\,\cdot\,,\cdot\,]$.

**Corollary 2.7.** For each non-zero $\psi \in \mathrm{L}^2(\mathrm{X},\mu)$,

the spanning set $\mathcal{B}_\psi$ for $<\psi>_D$
$(i)$ is an orthonormal basis $\Leftrightarrow p_\psi = 1 \; a.e.$;
$(ii)$ is a Riesz basis $\Leftrightarrow$ both $||p_\psi||_\infty$ and $||\frac{1}{p_\psi}||_\infty$ are finite;

$(iii)$ is a frame $\Leftrightarrow$ there are positive constants A,B with

$$\mathrm{A}\chi_{supp\, p_\psi} \le p_\psi \le \mathrm{B}\chi_{supp\, p_\psi}\ a.e.$$

$(when\, we\, can\, take\ A = B = 1,\ \mathcal{B}_\psi$ is said to be a Parseval frame)

$Proof$ : Immediate from the properties of modulation representations and the fact that $\mathcal{J}_\psi(\mathrm{D}_l\psi) = e_l p_\psi^{1/2}$.

*Remark. The above list of connections between properties of the generating set* $\mathcal{B}_\psi$ *for* $< \psi >_D$ *and properties of the weight function* $p_\psi$ *can be expanded considerably to discuss other properties. Also, as discussed in the paper* [1] by *Heil and Powell, the non-averaged weight function* $|Z\psi|$ *controls the properties of the Gabor system generated by* $\psi$ *in the* $\mathbb{R}^n$ *case.*

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