Multifractal analysis of Cantor-like measures

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ABSTRACT. In this course we will study generalized Cantor sets and measures. We will see that they share many properties in common with self-similar sets and measures, although new geometric ideas are often needed in the proofs to replace the combinatorial structure of self-similar sets/measures. In particular, under a suitable separation condition the multifractal spectrum of generalized Cantor measures (the set of local dimensions) can be shown to be a closed interval, with one specific local dimension being attained at almost every point of the Cantor set.
Surprisingly, the property that the multifractal spectrum is a closed interval need not be true for convolutions of (even self-similar) Cantor measures. This seems to be a consequence of 'overlap' in their construction and was established first for certain examples of self-similar Cantor measures and subsequently for generalized Cantor measures. We will see that it is typically the case that the multifractal spectrum of a sufficiently large number of convolutions of fairly arbitrary, continuous measures admits an isolated point. This argument was motivated by the geometric ideas used in proving a special case of this property for generalized Cantor measures.

1. Introduction

Often in analysis one is interested in subsets of $\mathbb{R}$ of Lebesgue measure zero and the singular measures[1] concentrated on these sets. Many of the problems that arise have to do with quantifying the size of the set or the singularity of the measure; for such problems fractal dimensions can be very helpful.

The classical middle-third Cantor set and its associated uniform measure is an important example of such a set and measure. The Cantor set and measure are often introduced in real analysis courses to illustrate unusual ideas or pathological behaviour. In this course, we will discuss generalizations of the classical Cantor set and measure, and investigate fractal concepts that help to quantify their singularity, such as local dimension and multifractal spectrum. These generalizations have interesting and unusual properties.

Generalized Cantor sets and measures are typically not self-similar and thus need not have the same symmetry or uniformity as the classical Cantor set/measure. Consequently, the concentration of the measure can vary at different points in its support, meaning general Cantor measures typically take on a range of different

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[1]By a measure, we mean a finite, positive, regular, compactly supported, Borel measure on $\mathbb{R}$.

local dimensions. These different values are known as the multifractal spectrum. The study of the multifractal spectrum and the 'size' of the sets on which a given local dimension is attained is known as multifractal analysis.

For self-similar measures arising from an IFS which satisfies the open set condition, it is well known that the multifractal spectrum is a closed interval and formulas have been established for the Hausdorff dimension of the sets on which a given local dimension occurs. We will modify this argument to show that a similar result can be obtained for generalized Cantor measures, under reasonably weak assumptions. Another interesting fact we will establish is that the 'average' value of the local dimensions is attained at almost every point. These results can be found in Section 3.

Convolutions of the classical Cantor measure are again self-similar measures. However, they are not necessarily generated by an IFS that satisfies the open set condition so the general multifractal theory does not apply. In fact, the theory can fail in a striking way: the multifractal spectrum of the 3-fold convolution of the classical Cantor measure contains an isolated point. Here we will see that convolutions of quite general, continuous, probability measures typically admit isolated points in their multifractal spectrum, provided the number of convolutions is sufficiently large. In particular, this is the case for many generalized Cantor measures. These ideas are the content of Section 4.

Most of the proofs given in this note can be found in the literature, as detailed in the final section. There are many other important research papers on related topics; we have only mentioned those most relevant for the material discussed in the course.

2. Notation and Basic Facts

2.1. The classical Cantor set and measure. The *classical middle-third Cantor set* C is a fascinating set which is often used in analysis to construct interesting examples. It is compact, totally disconnected, perfect (meaning, every point is an accumulation point), uncountable and of Lebesgue measure zero. By the *classical Cantor measure* we mean the singular, probability measure on $\mathbb{R}$ that is uniformly distributed on C. This measure, μ, can be defined in several equivalent ways:

(1) As the self-similar measure that arises from the iterated function system (IFS) with contractions $F_i(x) = x/3 + 2i/3$, $i = 0, 1$ and probabilities $1/2, 1/2$. This means the measure is invariant in the sense that

$$\mu(E) = \frac{1}{2}\left(\mu \circ F_0^{-1}(E) + \mu \circ F_1^{-1}(E)\right) \text{ for all Borel sets } E.$$

The classical Cantor set C is the self-similar set associated with this IFS.

(2) As the Borel measure supported on C that assigns mass 2^{-k} to the Cantor intervals that arise at step k in the construction of the Cantor set.

(3) As the weak limit of the discrete probability measures $\mu_K = 2^{-K} \sum_{j=1}^{2^K} \delta_{x_j}$, where $x_1, ..., x_{2^K}$ are the left end points of the 2^K Cantor intervals that are constructed at step K in the standard Cantor set construction. By a weak limit, we mean that for all continuous functions f on $[0, 1]$ it is the case that $\int_0^1 f d\mu = \lim_K \int_0^1 f d\mu_K$.

(4) As the probability measure whose cumulative distribution function is the Cantor ternary function.

From these different (but equivalent) descriptions of the Cantor measure one can easily establish many properties of the Cantor set/measure. Definition (2), for example, is useful in calculating the Hausdorff dimension of the set. From definition (3) it can be seen that the Fourier transform of μ is given by $\widehat{\mu}(y) = \prod_{k=1}^{\infty}(1+e^{-4\pi i 3^{-k}y})/2$ for all y. Since the Cantor ternary function is a continuous function, it follows immediately from definition (4) that the Cantor measure is a continuous (or non-atomic) measure, meaning the measure of any singleton is 0.

The classical Cantor set and measure has been generalized in many ways. One obvious generalization is to consider the self-similar set arising from the IFS with contractions $F_i(x) = rx + i(1-r)$, $i = 0, 1$ where $0 < r < 1/2$. This is the Cantor set with ratio of dissection r, (rather than 1/3rd, as in the classical case), meaning that at each step in the standard Cantor set construction one keeps the two outer closed intervals whose length is r times that of the parent interval. We will denote this Cantor set as $C(r)$, so that with this notation the classical Cantor set is $C(1/3)$. We can again define the associated *uniform Cantor measure* that assigns mass 2^{-k} to the Cantor intervals at step k, which in this case are of length r^k. This is the self-similar measure generated by the IFS given above, with probabilities $1/2, 1/2$.

Alternatively, rather than the uniform Cantor measure, we could consider the self-similar measure generated by the same iterated function systems again, but with probabilities p and $1-p$, where $0 \leq p \leq 1$. We call this the *p-Cantor measure* on $C(r)$. If $p = 0$ or 1, the p-Cantor measure is the point mass measure at 0 or 1, respectively. In all other cases, it is a continuous, singular, probability measure.

2.2. Cantor sets and measures with varying ratios of dissection. In fractal geometry one is often interested in studying self-similar sets and measures arising from quite general iterated function systems. The IFS structure makes it possible to compute many important quantities and deduce various properties of the sets and measures. At the same time, the structure limits the kinds of examples that arise. If we relax this structure, we can create many other intriguing examples. One such variation is to allow the ratios of dissection in the construction of the Cantor set to vary at each step. We could also allow the probabilities to vary at different steps.

2.2.1. *Cantor sets with varying ratios of dissection.* Let $0 < r_j < 1/2$. We denote by $C(r_j)$[2] the Cantor set with varying ratios of dissection, r_j at step j, given by the following iterative Cantor-like construction: Let $C_0 = [0, 1]$. Remove from C_0 the open middle interval of length $1 - 2r_1$, leaving two closed intervals of lengths r_1. Call these intervals the *Cantor intervals* of step one and their union C_1. At step j in the construction assume we have inductively constructed C_j as a union of 2^j closed intervals of length $r_1 \cdots r_j$, the Cantor intervals of step j. Remove the open middle interval of length $(1 - 2r_{j+1})r_1 \cdots r_j$ from each of the step j intervals and let C_{j+1} be the union of the remaining 2^{j+1} closed intervals of length $r_1 \cdots r_{j+1}$.

[2]More properly, we should write $C(\{r_j\})$, but we prefer $C(r_j)$ for simplicity. This should not cause any confusion with the notation $C(r)$ for the Cantor set with fixed ratio of dissection r.

Finally, define the Cantor set $C(r_j)$ by

$$C(r_j) = \bigcap_{j=1}^{\infty} C_j.$$

As with the classical Cantor set, $C(r_j)$ is compact, perfect, totally disconnected and uncountable. Its Lebesgue measure is $\liminf_{n\to\infty} 2^{-n} r_1 \cdots r_n$ and hence is zero if, for instance, the r_j are bounded away from $1/2$.

2.2.2. *Labelling Cantor intervals and the elements of the Cantor set.* The Cantor intervals from this construction can be labelled by finite words with letters from $\{0,1\}$. The Cantor intervals of step one will be denoted I_0 (left interval) and I_1 (right interval). In general, if the Cantor interval of step n is labelled by the word w of length n, then its two descendents are I_{w0} and I_{w1}. Each $x \in C(r_j)$ belongs to a unique Cantor interval of step n for each n and these intervals are descendents of one another. Thus x corresponds to an infinite word w with the property that if $w|n$ denotes the truncation of w to length n, then $I_{w|n}$ is the step n Cantor interval to which x belongs. When we write $x = (w_j)$ we mean this correspondence.

2.2.3. *Uniform and p-Cantor measures.* Given $0 \le p \le 1$, by the *p-Cantor measure* associated with $C(r_j)$, we mean the probability measure μ with the property that

$$\mu(I_{w0}) = \mu(I_w)p \text{ and } \mu(I_{w1}) = \mu(I_w)(1-p).$$

Thus if $w = (w_1, ..., w_n)$ with $w_i \in \{0,1\}$, then $\mu(I_{w_1 \cdots w_n}) = p^{n_0}(1-p)^{n-n_0}$ where $n_0 = card\{i : w_i = 0\}$. As in the case for Cantor sets with fixed ratio of dissection, the p-Cantor measure μ is a singular measure whose support is the Cantor set $C(r_j)$. It is continuous provided $p \neq 0, 1$. If $p = 1/2$ we call μ the *uniform Cantor measure* on $C(r_j)$.

More generally, given a sequence of weights $\{p_j\}$, $0 \le p_j \le 1$, we could define a Cantor measure by the rule $\mu(I_{w_1 \dots w_n}) = p_{w_1 1} p_{w_2 2} \cdots p_{w_n n}$ where $p_{0j} = p_j$ and $p_{1j} = 1 - p_j$.

One could consider still more general Cantor sets and measures by removing from $[0,1]$, k_1 equally spaced, open intervals of length g_1 at step one, so that C_1 is the union of $k_1 + 1$ closed interals of length r_1 where $(k_1 + 1)r_1 + k_1 g_1 = 1$. Then inductively remove from each Cantor interval of step j, k_j equally spaced open intervals of length g_j so that C_j is the union of $\prod_{i=1}^{j}(k_j + 1)$ closed intervals of length $r_1 \cdots r_j$ where $(k_j + 1)r_j + k_j g_j = 1$. We can also define a general Cantor measure by putting weights p_{ij} on the $i = 1, ..., k_j + 1$ descendents at step j. In this note, we will focus on p-Cantor measures on $C(r_j)$, but much of what is said here is true for these very general Cantor sets and measures, at least under suitable assumptions. The technical details will be left for the reader.

2.3. Hausdorff dimension. Let $\delta > 0$. By a δ-cover of a non-empty Borel subset $E \subseteq \mathbb{R}$ we mean a countable collection of sets $\{U_i\}$ of diameter at most δ, whose union contains E. We write $|U_i|$ to denote the diameter of the set U_i. Given $s \ge 0$, we define

$$H^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}$$

and put

$$H^s(E) = \sup_{\delta > 0} H^s_\delta(E) = \lim_{\delta \to 0^+} H^s_\delta(E).$$

$H^s(\cdot)$ is a measure known as the *s-dimensional Hausdorff measure.* $H^s(E)$ is a decreasing function of s and can be positive and finite for at most one choice of s. The *Hausdorff dimension* of E, denoted $\dim_H E$, is defined to be the unique index s such that $H^t(E) = 0$ if $t > s$ and $H^t(E) = \infty$ for $t < s$. Thus

$$\begin{aligned}\dim_H F &= \inf\{s : H^s(F) = 0\} \\ &= \sup\{s : H^s(F) = \infty\}.\end{aligned}$$

A useful fact is the Mass distribution principle: If there are a measure μ on E and real numbers $c, \delta > 0$ such that $\mu(U) \leq c|U|^s$ for all Borel sets U with diameter at most δ, then $H^s(E) \geq \mu(E)/c$ and $\dim_H E \geq s$.

We leave it as an exercise to verify that the Hausdorff dimension of $C = C(r_j)$ is given by the formula

$$\dim_H C = \liminf_{n\to\infty} \frac{\log 2}{\frac{1}{n}|\log r_1 \cdots r_n|}.$$

Exercise 1. *Establish the formula given for the Hausdorff dimension of $C(r_j)$.*

Exercise 2. *Show that for every $s \leq 1$ there is a Cantor set with Hausdorff dimension s.*

Exercise 3. *Construct a Cantor-like set, $C(r_j)$, with Hausdorff dimension one and Lebesgue measure zero.*

3. Multifractal analysis of p-Cantor measures

3.1. Local Dimension. In many problems one is interested in quantifying the singularity of a measure, i.e., to specify, in some sense, how concentrated the measure is. One way to quantify this is through the *Hausdorff dimension of the measure μ.* This is defined as

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}.$$

This quantity provides global information on the singularity of the measure μ. For measures that are not uniformly distributed it is also of interest to quantify their local singularity. The local dimension is useful for this.

Definition 1. *By the local dimension at x of a probability measure μ on $\mathbb{R}$ we mean the quantity*

$$dim_{loc}\mu(x) = \lim_{r\to 0^+} \frac{\log(\mu(B(x,r)))}{\log r}$$

where $B(x,r)$ is the ball centred at x with radius r, provided this limit exists.

The upper and lower dimensions, denoted $\overline{dim}_{loc}\mu(x)$ and $\underline{dim}_{loc}\mu(x)$, are obtained by replacing the limit in the definition above with $\limsup$ *and* $\liminf$ *respectively.*

The local dimension at x describes the power law behaviour of $\mu(B(x,r))$ for small r. Notice that if $x \notin \text{supp}\mu$, then $dim_{loc}\mu(x) = \infty$, while if μ is Lebesgue measure on $[0,1]$, $dim_{loc}\mu(x) = 1$ at all $x \in [0,1]$.

One can prove that

$$\dim_H \mu = \sup\{s : \underline{dim}_{loc}\mu(x) \geq s \text{ for } \mu \text{ a.e. } x\}.$$

Moreover, the following is true.

Proposition 1. *Suppose μ is a probability measure, $F \subseteq \mathbb{R}$ is a Borel set and $0 < c < \infty$.*

(a) $H^s(F) \geq \mu(F)/c$ if

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r))}{r^s} \leq c \text{ for all } x \in F.$$

(b) $H^s(F) \leq 10^s \mu(\mathbb{R})/c$ if

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r))}{r^s} \geq c \text{ for all } x \in F.$$

Proof. (a) Fix $\varepsilon > 0$ and for each n let

$$F_n = \{x \in F : \mu(B(x,r)) \leq (c+\varepsilon) r^s \text{ for all } r \leq 1/n\}.$$

The sets F_n are increasing and the assumption of (a) guarantees that their union is all of F.

Temporarily fix n and let $\{U_i\}$ be a $1/2n$-cover of F and hence also of F_n. Each set U_i has diameter less than $1/n$ and thus $\mu(B(x,|U_i|)) \leq (c+\varepsilon)|U_i|^s$ for all $x \in F_n$. Notice that if $x \in U_i \cap F_n$, then $B(x,|U_i|) \supseteq U_i$ and $\mu(U_i) \leq (c+\varepsilon)|U_i|^s$. Thus

$$\mu(F_n) \leq \sum_{i : U_i \cap F_n \neq \text{empty}} \mu(U_i) \leq (c+\varepsilon) \sum |U_i|^s .$$

This is true for all $1/2n$-covers of F and consequently $\mu(F_n) \leq (c+\varepsilon) H^s_{1/2n}(F)$. But as $n \to \infty$, $\mu(F_n) \to \mu(F)$ and $H^s_{1/2n}(F) \to H^s(F)$. Since $\varepsilon > 0$ was arbitrary, $\mu(F) \leq c H^s(F)$.

(b) Fix $\varepsilon, \delta > 0$ and consider the collection of all balls, $B(x,r)$, with $x \in F$, $0 < r < \delta$ and $\mu(B(x,r)) \geq (c-\varepsilon) r^s$. By assumption, every $x \in F$ belongs to such a ball for arbitrarily small r. By the Vitali covering lemma there are countably many disjoint balls from the collection, $\{B_i\}$, such that $\mu(F \backslash \bigcup_i B_i) = 0$ and every ball in the collection is contained in the union of the sets $\widetilde{B_i}$, where $\widetilde{B_i}$ is a ball concentric with B_i and having five times the radius. Thus $F \subseteq \bigcup_i \widetilde{B_i}$ and $\left|\widetilde{B_i}\right|^s \leq 10^s \mu(B_i)/(c-\varepsilon)$. As $\left|\widetilde{B_i}\right| \leq 10\delta$ and the sets B_i are disjoint,

$$\begin{aligned} H^s_{10\delta}(F) &\leq \sum_i \left|\widetilde{B_i}\right|^s \leq \frac{10^s}{c-\varepsilon} \sum_i \mu(B_i) \\ &= \frac{10^s}{c-\varepsilon} \mu(\bigcup_i B_i) = \frac{10^s}{c-\varepsilon} \mu(F). \end{aligned}$$

□

Corollary 1. *If there is a probability measure μ, concentrated on E, such that $\dim_{loc} \mu(x) = s$ for all $x \in E$, then $\dim_H E = s$.*

Proof. One can deduce from the previous proposition that for any $\varepsilon > 0$, $H^{s-\varepsilon}(E) > 0$ and $H^{s+\varepsilon}(E) < \infty$, from whence the result follows. □

We remark that there is a partial converse to this.

PROPOSITION 2. *If* $\dim_H E > s$*, then there exists a probability meausre* μ*, concentrated on* E*, such that* $\underline{\dim}_{loc}\mu(x) \geq s$ *for all* $x \in E$*. Similarly, if* $\dim_H E < s$*, then there exists a probability measure* μ*, concentrated on* $\overline{E}$*, such that* $\underline{\dim}_{loc}\mu(x) \leq s$ *for all* $x \in E$*.*

The proof of this is more sophisticated and can be found in the literature; see section 5.

It is an easy calculation to check that if μ is the uniform Cantor measure on the Cantor set $C(r)$, then

$$dim_{loc}\mu(x) = \frac{\log 2}{|\log r|} = \dim_H C(r) \text{ at all } x \in C(r).$$

In contrast, for measures that are not uniform the local dimension can vary at different points in the support of the measure. This is the case with the p-Cantor measures, for example, when $p \neq 1/2$. Indeed, suppose $C = C(r_j)$ and μ is the p-Cantor measure on C. To avoid technicalities we will also assume $\lim_n \frac{1}{n} \log (r_1 \cdots r_n) = \log r_0$. If $r = r_1 \cdots r_n$, then

$$\frac{\log (\mu(B(0,r)))}{\log r} = \frac{\log (\mu(I_{0 \cdots 0}))}{\log r} = \frac{n \log p}{\log r_1 \cdots r_n} \to \frac{\log p}{\log r_0},$$

while

$$\frac{\log (\mu(B(1,r)))}{\log r} = \frac{\log (\mu(I_{1 \cdots 1}))}{\log r} = \frac{n \log (1-p)}{\log r_1 \cdots r_n} \to \frac{\log (1-p)}{\log r_0}.$$

Thus

$$\begin{aligned} dim_{loc}\mu(0) &= \frac{\log p}{\log r_0} \text{ and} \\ dim_{loc}\mu(1) &= \frac{\log (1-p)}{\log r_0}. \end{aligned}$$

In the next subsection, we will see that under a suitable separation assumption, these are the extreme values of the set of local dimensions and all numbers in between arise as local dimensions.

3.2. Multifractal spectrum. Given $\alpha \geq 0$, we will denote

$$E_\alpha(\mu) = E_\alpha = \{x : dim_{loc}\mu(x) = \alpha\}.$$

The set of all α such that E_α is non-empty is known as the *multifractal spectrum* of μ. For measures that are not uniform it is of interest to determine the multifractal spectrum and the 'size' of the sets E_α, the so-called, *multifractal analysis.*

The multifractal analysis is well understood for self-similar measures generated by an IFS which satisfies the open set condition. In this section, we will establish similar results for p-Cantor measures supported on Cantor sets $C(r_j)$, under a suitable separation condition that plays the role of the open set condition, namely,

$$\text{Assumption: } \sup r_j < 1/2.$$

3.2.1. *Local dimensions are constant almost everywhere.* First, we will show that the local dimension is constant at almost all points of the support of μ.

THEOREM 1. *Suppose μ is a p-Cantor measure on the Cantor set $C = C(r_j)$ that satisfies $\sup r_j < 1/2$, and assume $\lim \frac{1}{n} \log(r_1 \cdots r_n) = \log r_0$. Then for μ a.e. $x \in C$,*

$$\dim_{loc} \mu(x) = \frac{p \log p + (1-p) \log(1-p)}{\log r_0}.$$

REMARK 1. *Assuming $\lim \frac{1}{n} \log(r_1 \cdots r_n)$ exists is a convenience. Similar results can be proved with the local dimension of μ at x replaced by the upper or lower local dimensions and with $\lim \frac{1}{n} \log(r_1 \cdots r_n)$ replaced by $\limsup \frac{1}{n} \log(r_1 \cdots r_n)$ (or $\liminf$).*

The proof has two parts, a geometric and a probabilistic part. We begin with a geometric lemma which will have other applications. Its significance is to show that under the assumption $\sup r_j < 1/2$ we may replace balls by Cantor intervals in the definition of local dimension.

NOTATION 1. *If $x \in C$, by $I^{(k)}(x)$ we mean the unique Cantor interval of step k that contains x.*

Of course, $I^{(k)}(x) = I_{w_1 \ldots w_k}$ where x is associated with the infinite word whose first k letters are $w_1, ..., w_k$.

LEMMA 1. *Assume $\sup r_j < 1/2$, μ is a p-Cantor measure on $C(r_j)$ and $x \in C(r_j)$. Then*

$$\dim_{loc} \mu(x) = \lim_{k \to \infty} \frac{\log\left(\mu(I^{(k)}(x))\right)}{\log\left|I^{(k)}(x)\right|}.$$

PROOF. Fix $x \in C$. Given $r > 0$, choose the minimum integer k so that $B(x,r)$ contains the Cantor interval of step k that contains x. As $I^{(k)}(x) \subseteq B(x,r)$, we must have

$$r_1 \cdots r_k = \left|I^{(k)}(x)\right| \leq 2r.$$

On the other hand, as $I^{(k-1)}(x) \subsetneq B(x,r)$,

$$r_1 \cdots r_{k-1} = \left|I^{(k-1)}(x)\right| \geq r.$$

Assume $x = (w_j)$. Then $I^{(k)}(x) = I_{w_1 \ldots w_k}$ and if t_k is the number of indices i such that $w_i = 0$ for $i = 1, ..., k$, then putting $p_j = p$ if $j = 0$ and $p_j = 1 - p$ if $j = 1$ we have

$$\mu(I^{(k)}(x)) = p_{w_1} \cdots p_{w_k} = p^{t_k}(1-p)^{k-t_k}.$$

Since $B(x,r)$ does not contain $I^{(k-1)}(x)$, it must be the case that $B(x,r) \cap C$ is contained in the union of at most two Cantor intervals of step $k-1$. If it is actually the case that $B(x,r) \cap C \subseteq I^{(k-1)}(x)$, then $\mu(B(x,r)) \leq \mu(I^{(k-1)}(x))$ and similar arguments to those used below, but easier, will complete the proof.

So assume $B(x,r) \cap C \subseteq I^{(k-1)}(x) \bigcup I^*$ and that the gap between these two step $k-1$ intervals was removed at step L in the construction, where $L \leq k-1$. This means both $I^{(k-1)}(x)$ and I^* are descendents of a (common) step $L-1$ interval I. Furthermore, the step L gap is contained in $B(x,r)$ and thus

$$r_1 \cdots r_{L-1}(1 - 2r_L) \leq r \leq r_1 \cdots r_{k-1}.$$

By assumption there exists $\varepsilon > 0$ such that $r_j \leq 1/2 - \varepsilon$ for all j. Consequently,

$$r_1 \cdots r_{L-1} 2\varepsilon \leq r \leq r_1 \cdots r_{L-1}(1/2)^{k-L}.$$

Hence there must be some integer m (depending only on ε) such that $k - L \leq m$, in other words, $I^{(k-1)}(x)$ and I^* are both descendents of the Cantor interval $I = I_{w_1 \cdots w_{k-m}}$, of step $k - m$ and $B(x,r) \bigcap C \subseteq I$. Thus

$$p_{w_1} \cdots p_{w_k} \leq \mu(B(x,r)) \leq p_{w_1} \cdots p_{w_{k-m}}$$

and

$$\frac{\log p_{w_1} \cdots p_{w_{k-m}}}{\log r_1 \cdots r_k/2} \leq \frac{\log\left(\mu(B(x,r))\right)}{\log r} \leq \frac{\log p_{w_1} \cdots p_{w_k}}{\log r_1 \cdots r_{k-1}}.$$

Since m is bounded, we obtain the same limiting behaviour on both the left and right hand side as $r \to 0$, (or $k \to \infty$) and therefore

$$\dim_{loc} \mu(x) = \lim_{k\to\infty} \frac{\log p_{w_1} \cdots p_{w_k}}{\log r_1 \cdots r_k} = \lim_{k\to\infty} \frac{\log\left(\mu(I^{(k)}(x))\right)}{\log\left|I^{(k)}(x)\right|}. \tag{3.1}$$

$\square$

Remark 2. *It follows easily from (3.1) that the local dimensions at* 0 *and* 1 *are the extreme values.*

Proof. (of Theorem) Define independent and identically distributed random variables on C by

$$X_k(x) = \begin{cases} 1 & \text{if } w_k = 0 \\ 0 & \text{if } w_k = 1 \end{cases} \quad \text{where } x = (w_k).$$

As the expected value of X_k is p, the Strong law of large numbers states that if $t_k(x)$ is the number of $0's$ occuring in the first k digits of x, then

$$\frac{t_k(x)}{k} = \frac{1}{k}\sum_{j=1}^{k} X_j(x) \to p \ \mu \text{ a.s.}$$

Thus, for μ almost all x,

$$\begin{aligned} \frac{\log\left(\mu(I^{(k)}(x))\right)}{\log\left|I^{(k)}(x)\right|} &= \frac{\log p^{t_k}(1-p)^{k-t_k}}{\log r_1 \cdots r_k} \\ &= \frac{t_k \log p + (k-t_k)\log(1-p)}{\log r_1 \cdots r_k} \\ &\to \frac{p\log p + (1-p)\log(1-p)}{\log r_0}. \end{aligned}$$

$\square$

3.2.2. *Multifractal formalism for p-Cantor measures.* An important feature of self-similar measures arising from an IFS that satisfies the open set condition is that the multifractal spectrum is a closed interval and the Hausdorff dimension of the sets E_α can be computed. Here we will see that the same property holds for many p-Cantor measures supported on Cantor sets with varying ratios of dissection. In place of the open set condition, we will assume that $\sup r_n < 1/2$. We will also continue to assume that $\frac{1}{n}\log(r_1 \cdots r_n) \to \log r_0$ so that we can work with limits, but related results can again be obtained using $\limsup$ or $\liminf$.

THEOREM 2. *Suppose μ is the p-Cantor measure supported on the Cantor set $C = C(r_j)$ which satisfies $\sup r_n < 1/2$ and $\lim_n \frac{1}{n} \log (r_1 \cdots r_n) = \log r_0$. Without loss of generality, assume $p \geq 1-p$. Then the set $E_\alpha = \{x \in C : \dim_{loc} \mu(x) = \alpha\}$ is non-empty if and only if*

$$\alpha \in \left[\frac{\log p}{\log r_0}, \frac{\log(1-p)}{\log r_0}\right]$$

and $\dim_H E_\alpha = f(\alpha)$ where

$$f(\alpha) = \inf_{q \in \mathbb{R}} \left(q\alpha - \frac{\log(p^q + (1-p)^q)}{\log r_0} \right).$$

The proof we sketch below is similar to that known for self-similar sets satisfying the strong separation condition. Indeed, p-Cantor measures on Cantor sets with fixed ratio of dissection are examples of self-similar measures satisfying this separation property.

PROOF. (Sketch) The fact that E_α is non-empty only for the specified α is clear from (3.1).

For each $q \in \mathbb{R}$ we define the set function, ν_q, on C by

$$\nu_q(I_w) = (p_{w_1} \cdots p_{w_k})^q \ \ (p^q + (1-p)^q)^{-k} \text{ if } w = w_1 ..., w_k.$$

One can check that ν_q is a probability measure concentrated on C and

$$\log \nu_q(I_w) = q \log \mu(I_w) - k \log(p^q + (1-p)^q)$$

Applying (a variant of) Lemma 1 to both μ and ν_q shows that

$$\begin{aligned} \dim_{loc} \nu_q(x) &= \lim_{k \to \infty} \frac{\log \nu_q(I^{(k)}(x))}{\log \left|I^{(k)}(x)\right|} \\ &= q \dim_{loc} \mu(x) - \frac{\log(p^q + (1-p)^q)}{\log r_0}. \end{aligned}$$

Thus, if $x \in E_\alpha$,

$$\dim_{loc} \nu_q(x) = q\alpha - \frac{\log(p^q + (1-p)^q)}{\log r_0}.$$

It is a routine calculus exercise to check that $f(\alpha)$ is achieved with the choice of $q = q(\alpha)$ satisfying

$$\alpha = \frac{p^q \log p + (1-p)^q \log(1-p)}{(p^q + (1-p)^q) \log r_0},$$

Thus $\dim_{loc} \nu_{q(\alpha)}(x) = f(\alpha)$ for all $x \in E_\alpha$.

If we can establish that $\nu_{q(\alpha)}$ is actually concentrated on E_α, then it will follow from Cor. 1 that $\dim_H E_\alpha = f(\alpha)$. To see this, fix $\varepsilon > 0$ and let $\delta > 0$ be small. Note that

$$\nu_{q(\alpha)} \left\{ x : \mu(I^{(k)}(x)) \geq \left| I^{(k)}(x) \right|^{\alpha - \varepsilon} \right\}$$

$$\begin{aligned}
&\le \int \left(\mu(I^{(k)}(x))\right)^{\delta} \left|I^{(k)}(x)\right|^{-(\alpha-\varepsilon)\delta} d\nu(x) \\
&= \sum_{|w|=k} \mu(I_w)^{\delta}(r_1\cdots r_k)^{-(\alpha-\varepsilon)\delta}\nu(I_w) \\
&= \sum_{|w|=k} (p_{w_1}\cdots p_{w_k})^{\delta+q}\left(\prod_{j=1}^{k} r_j^{-(\alpha-\varepsilon)\delta}\right)(p^q+(1-p)^q)^{-k} \\
&= \prod_{j=1}^{k}\left(\left(p^{q+\delta}+(1-p)^{q+\delta}\right) r_j^{-(\alpha-\varepsilon)\delta}(p^q+(1-p)^q)^{-1}\right) \equiv \Phi_1^{(k)}(\alpha).
\end{aligned}$$

Similarly,

$$\nu_{q(\alpha)}\left\{x : \mu(I^{(k)}(x)) \le \left|I^{(k)}(x)\right|^{\alpha+\varepsilon}\right\}$$

$$\le \prod_{j=1}^{k}\left(\left(p^{q+\delta}+(1-p)^{q+\delta}\right) r_j^{(\alpha+\varepsilon)\delta}(p^q+(1-p)^q)^{-1}\right) \equiv \Phi_2^{(k)}(\alpha).$$

Using Taylor series (in the variable δ), one can verify that for sufficiently large k (say, $k \ge k_1$) and suitable positive constants C_1, C_2,

$$\Phi_j^{(k)}(\alpha) \le \exp(-k\delta C_j\varepsilon) \text{ for } j = 1, 2.$$

Thus

$$\sum_{k\ge k_1} \nu_{q(\alpha)}\{x : \mu(I^{(k)}(x)) \ge \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}\} \le \sum_{k\ge k_1} \exp(-k\delta C_1\varepsilon) < \infty$$

and similarly for $\sum \nu_{q(\alpha)}\{x : \mu(I^{(k)}(x)) \le \left|I^{(k)}(x)\right|^{\alpha+\varepsilon}\}$. By the Borel Cantelli lemma, the probability that $\mu(I^{(k)}(x)) \ge \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}$ occurs infinitely often is zero and similarly for $\mu(I^{(k)}(x)) \le \left|I^{(k)}(x)\right|^{\alpha+\varepsilon}$. Thus for $\nu_{q(\alpha)}$ a.e. x and large enough k,

$$\left|I^{(k)}(x)\right|^{\alpha+\varepsilon} \le \mu(I^{(k)}(x)) \le \left|I^{(k)}(x)\right|^{\alpha-\varepsilon}.$$

Hence, for large enough k,

$$\alpha+\varepsilon \ge \frac{\log(\mu(I^{(k)}(x)))}{\log\left|I^{(k)}(x)\right|} \ge \alpha-\varepsilon$$

for $\nu_{q(\alpha)}$ a.e. x. As $\varepsilon > 0$ was arbitrary, it follows that $\dim_{loc}\mu(x) = \alpha$ for $\nu_{q(\alpha)}$ a.e. x, in other words, $\nu_{q(\alpha)}$ is concentrated on E_α as we desired to show.

We will leave it to the reader to check that $f(\alpha) \ne 0$ for $\alpha \in (\log p/\log r_0, \log(1-p)/\log r_0)$. As the endpoints of this interval are the local dimensions at 0 and 1 respectively, it follows that E_α is non-empty if and only if α belongs to the closure of the interval above. □

EXERCISE 4. *Determine for which α the set E_α has maximal Hausdorff dimension and find that dimension.*

4. Isolated points in the multifractal spectrum

4.1. Isolated points in the spectrum of convolutions of Cantor measures. An important operation in many branches of analysis is convolution. Convolution is a binary operation on the space of measures on $\mathbb{R}$ defined in the following way.

DEFINITION 2. *If μ, ν are measures on $\mathbb{R}$, then their convolution, $\mu * \nu$, as defined as the measure with the property that for any Borel set $E \subseteq \mathbb{R}$,*

$$\mu * \nu(E) = \int \mu(E - x) d\nu(x).$$

One can verify that the support of $\mu * \nu$ is contained in the sum of the supports of μ and ν.

Given a measure μ, we will write μ^m for the m-fold convolution of μ. When μ is the uniform Cantor measure on $C(r)$, then μ^m is a self-similar measure generated by the IFS with contractions $F_i(x) = rx + (1 - r)i$ for $i = 0, 1, ..., m$ and weights $p_i = 2^{-m}\binom{m}{i}$. The support of the invariant measure is the m-fold sum of $C(r)$. For example, if μ is the classical Cantor measure and $m \geq 2$, then μ^m is supported on $[0, m]$. In this case, the IFS satisfies the open set condition if and only if $m \leq 2$.

EXERCISE 5. *Determine the multifractal spectrum of $\mu * \mu$ for the classical Cantor measure μ.*

In striking contrast to the case of self-similar measures associated with IFS satisfying the open set condition, the multifractal spectrum of μ^3 is known to consist of the union of a closed interval and an isolated point:

$$\{\alpha : E_\alpha(\mu^3) \neq \phi\} = \left[\frac{\log 8/3}{\log 3}, \frac{\log 8/\sqrt{b}}{\log 3}\right] \bigcup \left\{\frac{\log 8}{\log 3}\right\},$$

where $b = \left(7 + \sqrt{13}\right)/2$. It can be checked that $\log 8/\sqrt{b}/\log 3 \sim 1.1335$ and $\log 8/\log 3 \sim 1.89278$. It is also known that

$$\begin{aligned}
\frac{\log 8/3}{\log 3} &= \dim_{loc} \mu^3(x) \text{ for } x = (w_i) \text{ where } w_i \in \{1, 2\} \\
\frac{\log 8/\sqrt{b}}{\log 3} &= \dim_{loc} \mu^3(x) \text{ for } x = (w_i) \text{ where } w_{2i} = 0, w_{2i+1} = 1 \\
\frac{\log 8}{\log 3} &= \dim_{loc} \mu^3(0) = \dim_{loc} \mu^3(3) \text{ (and at no other } x).
\end{aligned}$$

The proof of these facts make strong use of the elegant combinatorial structure of the the Cantor set and its 3-fold sum.

Similar results have been obtained for m-fold convolutions of the uniform Cantor measures on the Cantor sets $C(1/d)$ when $d \in \mathbb{N}$ and, more generally, for self-similar measures generated by an IFS consisting of contractions $F_i(x) = x/d + (d - 1)i/d$ for $i = 0, 1, ..., m$ and probabilities $p_i > 0$, where $p_0, p_m \leq p_i$ for all $i \neq 0, m$ and $d \geq 3$ is an integer. The algebraic and combinatorial structure of these self-similar measures can again be used to show that if $m \geq d$, then the multifractal spectrum is the union of a closed interval and one (or two) isolated points, the local dimensions at $0, m$. The significance of $m \geq d$ is that the support of μ^m is $[0, m]$.

A similar result holds, as well, for convolutions of p-Cantor measures μ supported on the Cantor sets $C(r_j)$. Provided $\inf r_j > 0$, the spectrum of μ^m has also been shown to have an isolated point for sufficiently large m, either $\dim_{loc} \mu^m(0)$ or $\dim_{loc} \mu^m(m)$, depending on whether p or $1-p$ is larger. Again, a key idea in the proof of this result is that the Cantor sets, $C(r_j)$, have the property that the M-fold sum of $C(r_j)$ (the support of μ^M) is the interval $[0,M]$ if $M+1 \geq \sup 1/r_j$.

4.2. Isolated points in the spectrum of convolutions of general measures. It turns out a much more general result is true for convolutions of probability measures: If μ is any continuous, probability measure supported on $[0,1]$ and there is some integer M with the M-fold sum of the support of μ equal to $[0,M]$, then under rather mild assumptions, it is guaranteed that there will be an isolated point in the spectrum of μ^m for sufficiently large m.

THEOREM 3. *Suppose μ is a continuous, probability measure supported on $[0,1]$ with $0,1 \in supp\mu$ and assume $(M)supp\mu = [0,M]$ for some integer M. Assume, also, that*

(1) $\overline{\dim}_{loc}\mu(0) > 0$ *and*
(2) $\sup\{\overline{\dim}_{loc}\mu(x) : x \in supp\mu\} < \infty$.

Then there is an integer n_0 such that for all $n \geq n_0$, $\overline{\dim}_{loc}\mu^n(0)$ is isolated in the set of local dimensions of μ^n.

A similar statement holds with upper local dimensions replaced by lower local dimensions. We begin with two preliminary lemmas.

LEMMA 2. *Suppose μ, ν are measures with $supp\nu = [0,n]$ and $0,1 \in supp\mu \subseteq [0,1]$.*

(i) If $\overline{\dim}_{loc}\nu(x) \leq \lambda < \infty$ for all $x \in [0,n]$, then $\overline{\dim}_{loc}\nu * \mu(z) \leq \lambda$ for all $z \in (0,n+1)$.

(ii) If, in addition, μ is a continuous measure, then the same conclusion holds under the weaker assumption that $\overline{\dim}_{loc}\nu(x) \leq \lambda$ for all $x \in (0,n)$.

PROOF. (i) Fix $z \in (0,n+1)$ and let $I = [0,1] \cap [z-n,z]$. Notice that at least one of 0 or 1 belongs to I and that I has non-empty interior. As $0,1$ both belong to suppμ it follows that $\mu(I) = \eta > 0$.

Fix $\delta > 0$. If $x \in I$, then $z - x \in [0,n]$ =suppν, hence $\overline{\dim}_{loc}\nu(x) \leq \lambda$. This means that for every $x \in I$, there exists $r_x > 0$ such that if $r < r_x$ then

$$\frac{\log(\nu(B(z-x,r)))}{\log r} \leq \lambda + \varepsilon.$$

Let $A_n = \{x \in I : r_x \geq 1/n\}$. As $\bigcup A_n = I$, by continuity of measure there is some n such that $\mu(A_n) > \eta/2$. For all $x \in A_n$,

$$\nu(B(z-x,r)) \geq r^{\lambda+\varepsilon} \text{ for all } r \leq 1/n,$$

thus

$$\begin{aligned} \nu * \mu(B(z,r)) &= \int \nu(B(z-x,r))d\mu(x) \geq \int_{A_n} r^{\lambda+\varepsilon} d\mu(x) \\ &\geq r^{\lambda+\varepsilon}\mu(A_n) \geq r^{\lambda+\varepsilon}\eta/2. \end{aligned}$$

Hence

$$\frac{\log(\nu * \mu(B(z,r)))}{\log r} \leq \frac{\log \eta/2}{\log r} + \lambda + \varepsilon \text{ for all } r \leq 1/n.$$

Letting $r \to 0$, it follows that $\overline{\dim}_{loc}\nu * \mu(z) \leq \lambda + \varepsilon$ and since that holds for all $\varepsilon > 0$, we conclude that $\overline{\dim}_{loc}\nu * \mu(z) \leq \lambda$.

(ii) Under the weaker assumption of (ii), it is still true that $\overline{\dim}_{loc}\nu(x) \leq \lambda$ for all $x \neq z, z-n$. But $\mu\{z\} = \mu\{z-n\} = 0$, hence the sets $I \backslash \{z, z-n\}$ and I have the same positive μ-measure. Let $A_n = \{x \in I\backslash\{z, z-n\} : r_x \geq 1/n\}$ and choose n such that $\mu(A_n) > \mu(I)/2$. We conclude the proof as in the first part. $\square$

LEMMA 3. *Assume μ is supported on $[0,1]$.*
(i) Then $\overline{\dim}\mu^n(0) = n\overline{\dim}\mu(0)$.
(ii) If $x_j \in$ suppμ and $x = \sum_{j=1}^n x_j$, then $\overline{\dim}\mu^n(x) \leq \sum_{j=1}^n \overline{\dim}\mu^n(x_j)$.

PROOF. (i) This follows easily from the fact that

$$(\mu(B(0,r/n)))^n \leq \mu^n(B(0,r)) \leq (\mu(B(0,r)))^n .$$

(ii) is similar. $\square$

PROOF. (of Theorem) By assumption, if $x \in [0, M]$, then there are real numbers $x_j \in$suppμ such that $\sum_{j=1}^M x_j = x$. As $\overline{\dim}\mu(z) \leq \lambda$ for all $z \in$suppμ, the previous lemma (ii) implies $\overline{\dim}\mu^M(x) \leq M\lambda$ for all $x \in [0, M] =$suppμ^M.

Now apply Lemma 2 (either (i) or (ii)) with $\nu = \mu^M$ and $M = n$ to deduce that $\overline{\dim}\mu^{M+1}(x) \leq M\lambda$ for all $x \in (0, M+1)$.

Since supp$\mu^{M+T} = [0, M+T]$, we can repeatedly apply this argument (but with part (ii) of the lemma as we have only the weaker hypothesis satisfied) to deduce that $\overline{\dim}\mu^n(x) \leq M\lambda$ for all $x \in (0, n)$ and any $n \geq M$.

As Lemma 3 (i) implies $\overline{\dim}\mu^n(0) \to \infty$ as $n \to \infty$, $\overline{\dim}\mu^n(0)$ will be isolated in the spectrum for large enough n. $\square$

EXERCISE 6. *For what n can you be sure μ^n has an isolated point in its spectrum when μ is the uniform Cantor measure on the Cantor set $C(r)$?*

5. Credits

The size of Cantor sets and their sums was explored in [**3**]. There the formula is given for the Hausdorff dimension of $C(r_k)$ and it is proven that if $\inf r_k > 0$, then some n-fold sum of $C(r_k)$ is the interval $[0, n]$.

An excellent exposition on local dimensions, including the proofs of Prop. 1 and 2, the probabilistic ideas in the proof of Theorem 1, and the multifractal analysis for self-similar measures arising from IFS satisfying the strong separation condition, can be in Falconer's books, [**5**] and [**6**], (particularly, chapters 17 and 10, 11 respectively). This is based in part upon the earlier work of Cawley and Mauldin [**4**], Mandelbrot [**14**], Riedi [**17**] and others. We refer the reader to the bibliographies given in [**5**] and [**6**] for further papers. In particular, Olsen in [**15**] developed a strong mathematical foundation for multifractal analysis.

Motivated in part by [**16**], the multifractal analysis of p-Cantor measures on $C(r_k)$ is investigated in [**11**]. There one can find the proofs of Theorem 2 and the geometric result, Lemma 1.

Hu and Lau in [**12**] established the existence of an isolated point in the spectrum of the 3-fold convolution of the uniform Cantor measure on $C(1/3)$. This fact was extended to various self-similar measures with overlap, generated by IFS with contraction factors $1/d$, $d \in \mathbb{N}$, in [**18**], [**8**] and [**2**]. In the latter paper, formulas are given for the spectrum in the case of convolutions of uniform Cantor measures on

Cantor sets $C(1/d)$. A proof of the existence of an isolated point in the spectrum of convolutions of very general Cantor measures is given in [**10**]. Theorem 3 is proven in [**1**]. Pathological examples are constructed in [**3**] and [**19**].

Hu and Lau have extensively investigated the multifractal analysis of self-similar measures with overlap in a series of papers, including [**7**], [**8**], [**9**] and [**13**].

References

[1] C. Bruggeman and K. E. Hare, *Multi-fractal analysis of convolution powers of measures*, Real Anal. Exch., to appear.

[2] C. Bruggeman, K. E. Hare and C. Mak, *Multifractal spectrum of self-similar measures with overlap*, 2013.

[3] C. Cabrelli, K.E. Hare and U. Molter, *Sums of Cantor sets*, Ergodic theory and dynamic systems **17(**1997), 1299-1313.

[4] R. Cawley and R.D. Mauldin, *Multi-fractal decompositions of Moran fractals*, Adv. Math. **92**(1992), 196-236.

[5] K. Falconer, *Mathematical foundations and applications*, John Wiley and Sons, New York, 1990.

[6] K. Falconer, *Techniques in fractal geometry*, John Wiley and Sons, New York, 1997.

[7] A-H. Fan, K-S. Lau and S-M. Ngai, *Iterated function systems with overlaps*, Asian J. Math. **4**(2000), 243-250.

[8] D-J. Feng and K-S. Lau, *Multi-fractal formalism for self-similar measures with weak separation condition*, J. Math. Pures Appl. **92**(2009), 407-428.

[9] D-J. Feng, K-S. Lau and X-Y. Wang, *Some exceptional phenomena in multi-fractal formalism: Part II*, Asian J. Math. **9**(2005), 473-488.

[10] V. P-W. Fong, K. E. Hare and D. Johnstone, *Multi-fractal analysis for convolutions of overlapping Cantor measures*, Asian J. Math. **15**(2011), 53-70.

[11] K. E. Hare and S. Yazdani, *Quasi self-similarity and multifractal analysis of Cantor measures*, Real Anal. Exch.**27**(2001/2), 287-307.

[12] T-Y. Hu and K-S. Lau, *Multi-fractal structure of convolution of the Cantor measure*, Adv. App. Math. **27**(2001), 1-16.

[13] K-S. Lau and X-Y. Wang, *Some exceptional phenomena in multi-fractal formalism: Part I*, Asian J. Math **9**(2005), 275-294.

[14] B. B. Mandelbrot, *An introduction to multifractal distribution functions*, in Fluctuations and pattern formation, Ed. H. Stanley and N. Ostrowsky, Kluwer Academic, Dordrecht, 1988.

[15] L. Olsen, *Multi-fractal geometry*, Progress in probability **46**(2000), 3-37.

[16] O'Neil, T., *The multifractal spectrum of quasi self-similar measures*, J. Math. Anal. and Appl. **211**(1997), 233-257.

[17] R. Riedi, *An improved multi-fractal formalism and self-similar measures,* J. Math. Anal. Appl. **189**(1995), 462-490.

[18] P. Shmerkin, *A modified multi-fractal formalism for a class of self-similar measures with overlap*, Asian J. Math **9**(2005), 323-348.

[19] B. Testud, *Phase transitions for the multi-fractal analysis of self-similar measures*, Nonlinearity **19**(2006), 1201-1217.

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