A Tutorial on Compressive Sensing

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New Trends in Applied Harmonic Analysis

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Part 3: Uniform and Nonuniform Recovery, Optimality

This lecture draws attention to the difference between uniform and nonuniform guarantees in sparse recovery. We give a number of nonuniform results concerning ℓ_1-minimization and we introduce partial Fourier matrices in passing. We also show that the uniform results from Compressive Sensing are essentially optimal. This relies on a close connection with the geometry of Banach spaces, in particular with properties of the unit balls of ℓ_1-spaces.

Uniform vs. Nonuniform Guarantees

Empirical Performances: HTP and ℓ_1-Minimization

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Gaussian matrices with Rademacher then with Gaussian vectors

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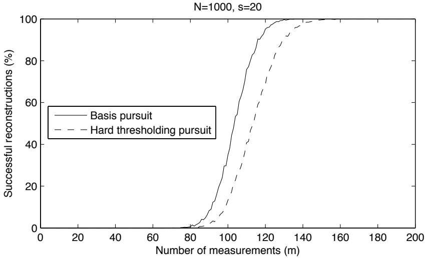

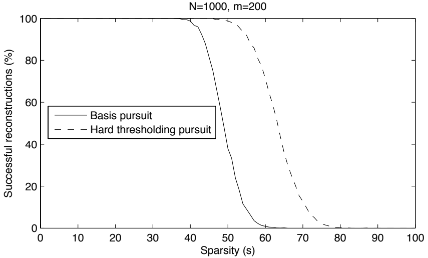

Empirical Performances: Phase Transitions

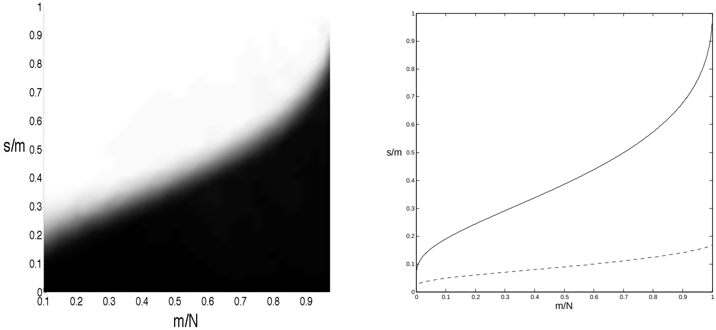

Figure: L: empirically observed weak threshold
R: strong (dashed) and weak (solid) thresholds (courtesy of J. Tanner)

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Using Gaussian matrices, nonuniform s-sparse recovery results hold for exactly s iterations of OMP in the regime $m \geq cs \ln(N/s)$, but uniform results do not hold in this regime.

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$$\left| \sum_{j \in S} \overline{\mathrm{sgn}(x_j)} v_j \right| < \|\mathbf{v}_{\overline{S}}\|_1 \qquad \text{for all } \mathbf{v}_{\neq \mathbf{0}} \in \ker A.$$

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$$(A^*\mathbf{h})_j = \mathrm{sgn}(x_j),\ j \in S, \qquad \left|(A^*\mathbf{h})_\ell\right| < 1,\ \ell \in \overline{S}.$$

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(which is replaced by $m \gtrsim 2s\big(\sqrt{\ln(eN/s)} + \sqrt{\ln(1/\varepsilon)/s}\big)^2$ for Gaussian matrices using Gordon's *escape through the mesh*).

Bounded Orthonormal Systems

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Let $\mathcal{D} \in \mathbb{R}^d$ be endowed with a probability measure ν. A bounded orthonormal system (BOS) with constant $K \geq 1$ is a system $(\phi_1, \ldots, \phi_N)$ of function of $\mathcal{D}$ satisfying

$$\int_{\mathcal{D}} \phi_j(t)\overline{\phi_k(t)} d\nu(t) = \delta_{j,k},$$

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2. discrete orthonormal systems — let $U \in \mathbb{C}^{N \times N}$ be a unitary matrix with $\sqrt{N}|U_{k,j}| \leq K$ for all $k, j \in \{1, \dots, N\}$ (e.g. Fourier or Hadamard matrix):
   take $\phi_k(t) = \sqrt{N}U_{k,t}$ for $t \in \{1, \dots, N\}$, $\nu(B) = \dfrac{\text{card}(B)}{N}$.

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$$m \geq C\, K^2 s \ln(N) \ln(\varepsilon^{-1}),$$

then $\mathbf{x}$ is the unique minimizer of $\|\mathbf{z}\|_1$ subject to $A\mathbf{z} = A\mathbf{x}$ with probability at least $1 - \varepsilon$.

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(Proof based on the *golfing scheme*).

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(Proof uses *Dudley's inequality*, *empirical method of Maurey*, etc.)

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- Given $q \geq p \geq 1$, a pair (A, Δ) is mixed (ℓ_q, ℓ_p)-instance optimal of order s with constant $C > 0$ if

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- If there is an ℓ_2-instance optimal pair of order $s \geq 1$ with constant C, then

$$m \geq c\, N$$

  for some constant c depending only on C.

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$$\|\mathbf{x} - \Delta_1(A\mathbf{x} + \mathbf{e})\|_2 \leq C\, \sigma_s(\mathbf{x})_2 + D\, \|\mathbf{e}\|_2$$

holds for all $\mathbf{e} \in \mathbb{R}^m$ with probability at least $1 - 5\exp(-c_1 m)$.

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- $\|\mathbf{e}\|_* \leq d\,\sqrt{s_*}\|A^*\mathbf{e}\|_\infty$ for all $\mathbf{e} \in \mathbb{R}^m$.

Optimality of Uniform Guarantees

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For a subset K of a normed space X, define

$$E^m(K,X) := \inf \left\{ \sup_{\mathbf{x} \in K} \|\mathbf{x} - \Delta(A\mathbf{x})\|,\ A : X \stackrel{\text{linear}}{\rightarrow} \mathbb{R}^m, \Delta : \mathbb{R}^m \rightarrow X \right\}$$

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If $-K = K$, then

$$d^m(K,X) \leq E^m(K,X),$$

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$$d^m(K,X) := \inf \left\{ \sup_{\mathbf{x}\in K\cap L^m} \|\mathbf{x}\|,\ \ L^m \text{ subspace of } X,\ \mathrm{codim}(L^m) \le m \right\}$$

If $-K = K$, then

$$d^m(K,X) \le E^m(K,X),$$

and if in addition $K + K \subseteq a\,K$, then

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provided c is chosen small enough.

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$$\geq \frac{c_1'}{c} \frac{\ln(eN/m)}{\min\left\{1, \frac{\ln(eN/m)}{m}\right\}} \geq \frac{c_1'}{c} m > m,$$

provided c is chosen small enough. This is a contradiction.

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A combinatorial lemma

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There exists $n \geq \left(\frac{N}{4s}\right)^{\frac{s}{2}}$ *subsets* $S^1, \ldots, S^n$ *of size* s *such that*

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Insight 2: ℓ_1-recovery and number of measurements

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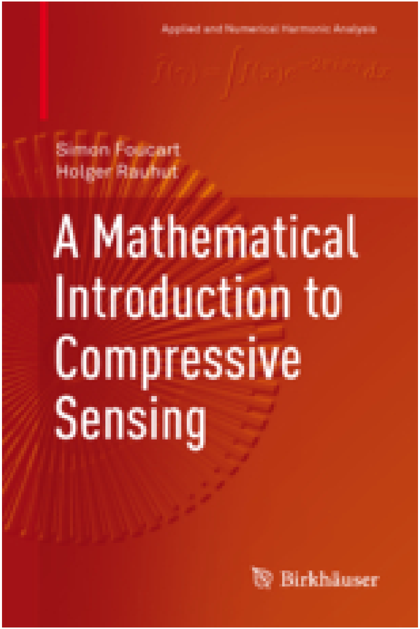
Taking the logarithm yields

$$m \ge \frac{s}{\ln 9} \ln\left(\frac{N}{4s}\right).$$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