A Tutorial on Compressive Sensing

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Part 2: The Restricted Isometry Property

In this lecture, the coherence is replaced by the concept of restricted isometry constant. This allows one to prove the robust null space property, which is equivalent to the robustness of ℓ_1-minimization for sparse reconstruction. It is also shown that the restricted isometry property guarantees the success of other algorithms such as Iterative Hard Thresholding and Orthogonal Matching Pursuit. Finally, the existence of matrices satisfying the restricted isometry property is established.

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In fact, $\delta_s \leq \delta_*$ imposes $m \geq \dfrac{c}{\delta_*^2} s$.

RIP-based Recovery Guarantees

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Note that $\rho := \delta_{2s}/(1-\delta_{2s}) < 1/2$ whenever $\delta_{2s} < 1/3$.

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Objective: for $p \in [1,2]$, for all $\mathbf{x} \in \mathbb{C}^N$ and $\mathbf{e} \in \mathbb{C}^m$ with $\|\mathbf{e}\|_2 \le \eta$:

$$\|\mathbf{x} - \Delta(A\mathbf{x} + \mathbf{e})\|_p \le \overbrace{\frac{C}{s^{1-1/p}} \min_{\mathbf{x}_s\ s-\text{sparse}} \|\mathbf{x} - \mathbf{x}_s\|_1}^{\text{stability}} + \overbrace{D\ s^{1/p-1/2}\ \eta}^{\text{robustness}},$$

where $\Delta(\mathbf{y}) = \Delta_{1,\eta}(\mathbf{y}) := \text{argmin}\, \|\mathbf{z}\|_1 \quad$ subject to $\|A\mathbf{z} - \mathbf{y}\|_2 \le \eta$.

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Taking $\mathbf{x} = \mathbf{v} \in \mathbb{C}^N$, $\mathbf{e} = -A\mathbf{v} \in \mathbb{C}^m$, and $\eta = \|A\mathbf{v}\|_2$ gives

$$\|\mathbf{v}\|_p \leq \frac{C}{s^{1-1/p}} \|\mathbf{v}_{\overline{S}}\|_1 + D\, s^{1/p-1/2} \|A\mathbf{v}\|_2$$

for all $S \subseteq [N]$ with $\mathrm{card}(S) = s$.

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For $q \in [1,2]$, $A \in \mathbb{C}^{m \times N}$ has the ℓ_q-robust null space property of order s (wrto $\|\cdot\|$) with constants $0 < \rho < 1$ and $\tau > 0$ if, for any set $S \subset [N]$ with $\mathrm{card}(S) \leq s$,

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- ℓ_2-RNSP (wrto $\|\cdot\|_2$) holds when $\delta_{2s} < 0.62$.

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- $\|\mathbf{x}_S - \mathbf{x}^{n+1}\|_2 \leq \rho \|\mathbf{x}_S - \mathbf{x}^n\|_2 + (1-\rho)\tau \|A\mathbf{x}_{\overline{S}} + \mathbf{e}\|_2$,

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- For HTP, (pseudo)robustness is achieved in $\leq c\, s$ iterations.

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$$S^n = S^{n-1} \cup \left\{ j^n := \mathrm{argmax}_j \left| \left(A^*(\mathbf{y} - A\mathbf{x}^{n-1}) \right)_j \right| \right\},$$

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- Challenge: natural proof of OMP success in $c\, s$ iterations?

RIP for Random Matrices

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- For a fixed $\mathbf{x} \in \mathbb{R}^N$, note that $(A\mathbf{x})_i = \sum_{j=1}^N a_{i,j} x_j$, hence

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- In fact, $\|A\mathbf{x}\|_2^2$ concentrates around its mean: for $t \in (0,1)$,

(CI) $$\mathbb{P}\big(\big|\|A\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2\big| > t\|\mathbf{x}\|_2^2\big) \le 2\exp(-ct^2 m).$$

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The argument relies on the following fact:

A subset U of the unit ball of $\mathbb{R}^k$ relative to a norm $\|\cdot\|$ has covering and packing numbers satisfying

$$\mathcal{N}(U, \|\cdot\|, t) \leq \mathcal{P}(U, \|\cdot\|, t) \leq \left(1 + \frac{2}{t}\right)^k.$$

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- For Gaussian matrices, more powerful techniques can provide an explicit value for c'.

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Next, we will see that this number of measurement is optimal, in the sense that estimates of type (1) require (2) to hold. We will also examine the gain in replacing *for all* $\mathbf{x}$ in (1) by *for a fixed* $\mathbf{x}$.