A Tutorial on Compressive Sensing

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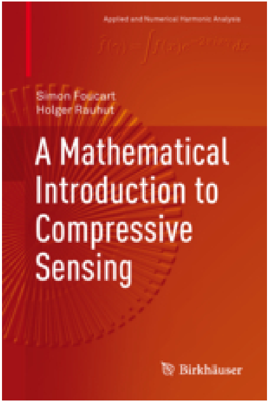
This minicourse acts as an invitation to the elegant theory of Compressive Sensing. It aims at giving a solid overview of the fundamental mathematical aspects. Its content is partly based on a recent textbook coauthored with Holger Rauhut:



Part 1: The Standard Problem and First Algorithms

The lecture introduces the question of sparse recovery, establishes its theoretical limits, and presents an algorithm achieving these limits in an idealized situation. In order to treat more realistic situations, other algorithms are necessary. Basis Pursuit (that is, ℓ_1-minimization) and Orthogonal Matching Pursuit make their first appearance. Their success is proved using the concept of coherence of a matrix.

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  Preferred, but competitive alternatives are available
  (reconstruction process)

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Stability: $\mathbf{x}$ not sparse but compressible,

Robustness: measurement error in $\mathbf{y} = A\mathbf{x} + \mathbf{e}$.

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- Magnetic resonance imaging

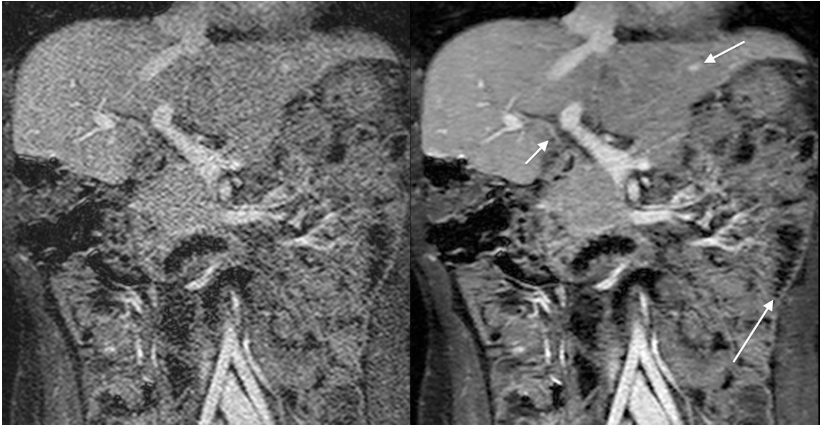

Figure: Left: traditional MRI reconstruction; Right: compressive sensing reconstruction (courtesy of M. Lustig and S. Vasanawala)

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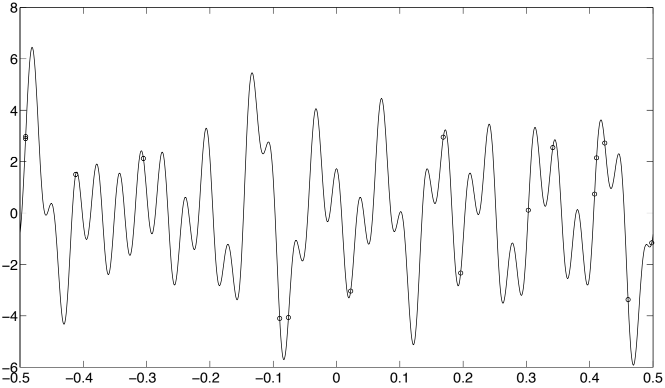

Figure: Time-domain signal with 16 samples.

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- and many more...

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This is a combinatorial problem, NP-hard in general.

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Optimization and Greedy Strategies

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- In the complex setting, recast as a second order cone program

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Real and complex NSPs are in fact equivalent.

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Starting with $S^0 = \emptyset$ and $\mathbf{x}^0 = \mathbf{0}$, iterate

$$(\mathrm{OMP}_1) \qquad S^{n+1} = S^n \cup \{j^{n+1} := \underset{j \in [N]}{\mathrm{argmax}} \{|(A^*(\mathbf{y} - A\mathbf{x}^n))_j|\}\},$$

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- Every vector $\mathbf{x}_{\neq \mathbf{0}}$ supported on S, $\operatorname{card}(S) = s$, is recovered from $\mathbf{y} = A\mathbf{x}$ after at most s iterations of OMP if and only if A_S is injective and

$$(\mathrm{ERC}) \qquad \max_{j \in S} |(A^*\mathbf{r})_j| > \max_{\ell \in \overline{S}} |(A^*\mathbf{r})_\ell|$$

for all $\mathbf{r}_{\neq \mathbf{0}} \in \big\{ A\mathbf{z}, \operatorname{supp}(\mathbf{z}) \subseteq S \big\}$.

First Recovery Guarantees

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- Deterministic matrices with coherence $\mu \leq c/\sqrt{m}$ exist

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- In fact, the Exact Recovery Condition can be rephrased as

$$\|A_S^{\dagger} A_{\overline{S}}\|_{1\to 1} < 1,$$

and this implies the Null Space Property.

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Next, we will introduce new tools to break this *quadratic barrier* (with random matrices).