Scaling and Multifractal: From Theory to Applications.

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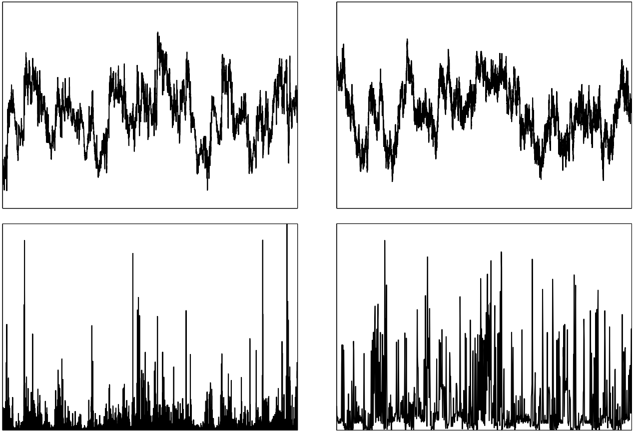
ENS DE LYON

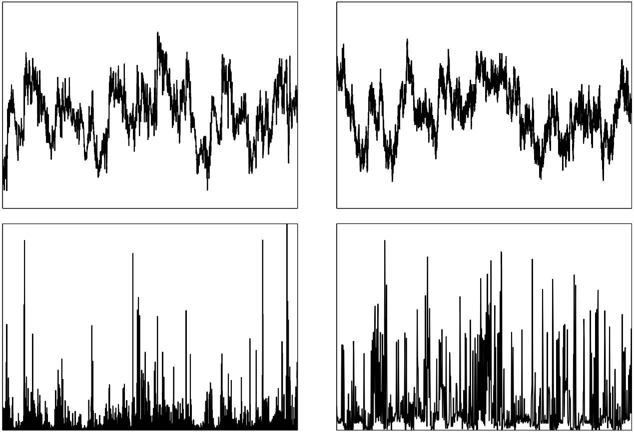
Outline

Empirical data : Images

Empirical data : Signals



Empirical data : Signals



$\Longrightarrow$ describing/analyzing/modeling ?

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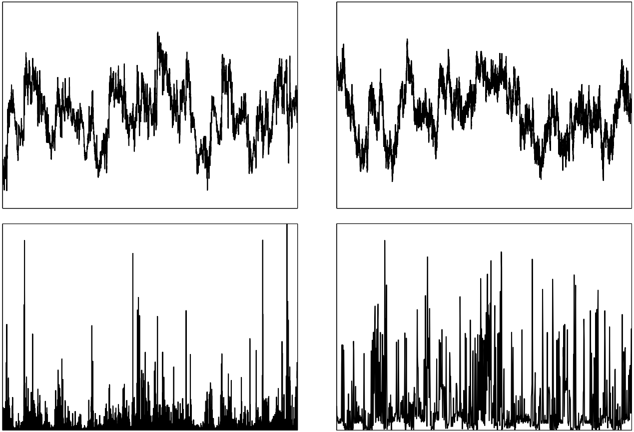
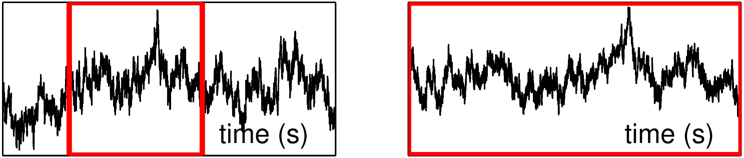


$\Longrightarrow$ describing/analyzing/modeling ?

$\Longrightarrow$ irregularity ? variability ?

Scaling ? Covariance under dilatation



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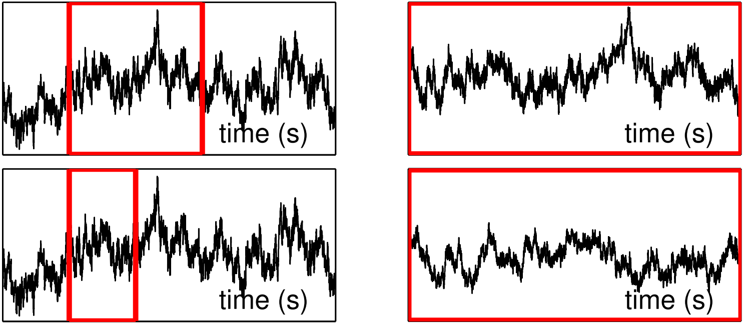

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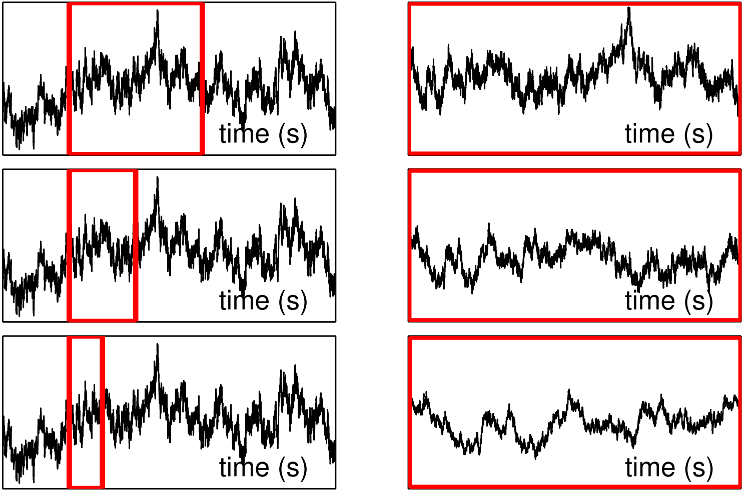

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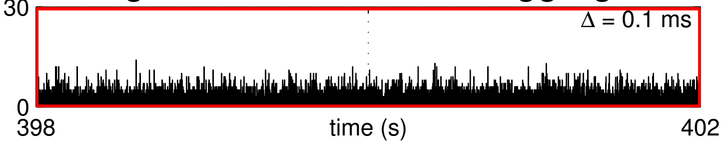

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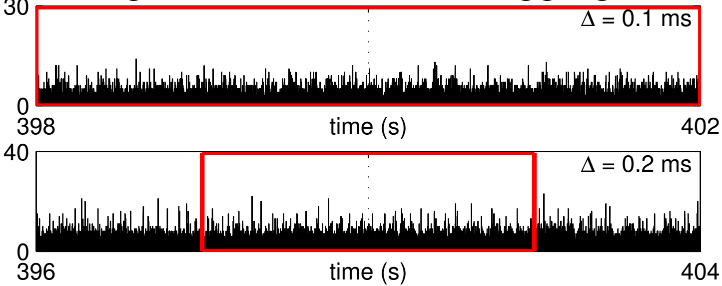

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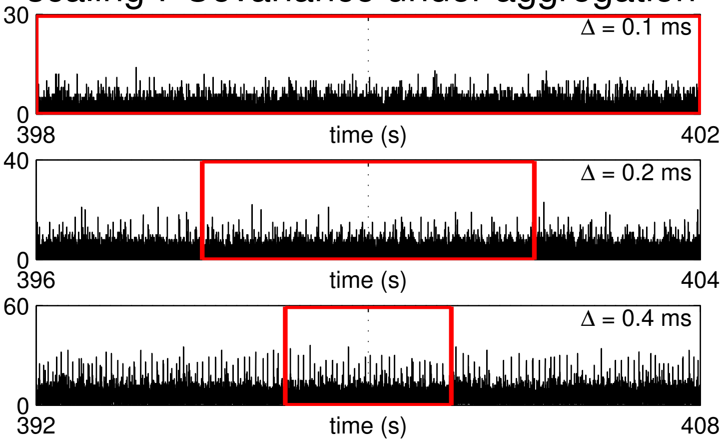

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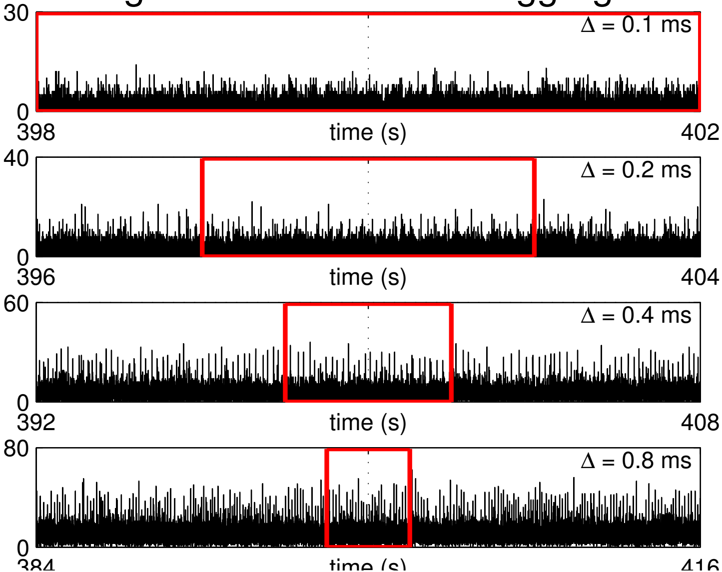
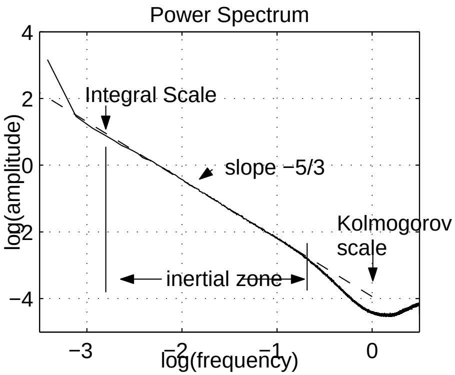
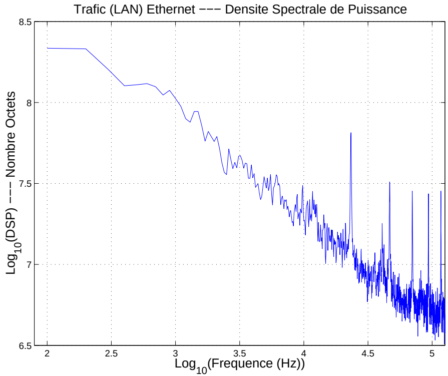


Scaling ? Power-law spectrum.





Scaling : Intuitive definition

- Definition : no characteristic scale !
- Equivalently : all scales are equally characteristic !
- Manifestation : cannot distinguish the whole from the part.

- Modeling :

rather than characteristic scale indentification,
seek for mechanisms relating scales.

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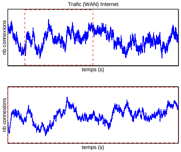

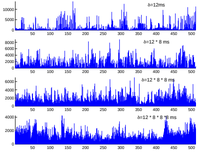

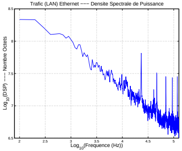

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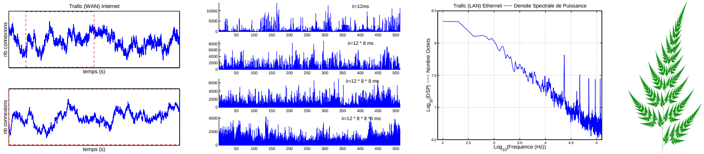

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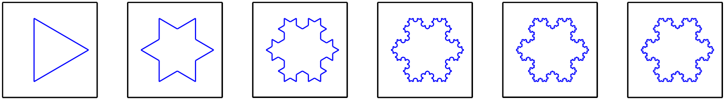

Scaling and power laws

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1D \Rightarrow |t_1 \cdot t_2|^{-\gamma} &= |t_1|^{-\gamma} \cdot |t_2|^{-\gamma}
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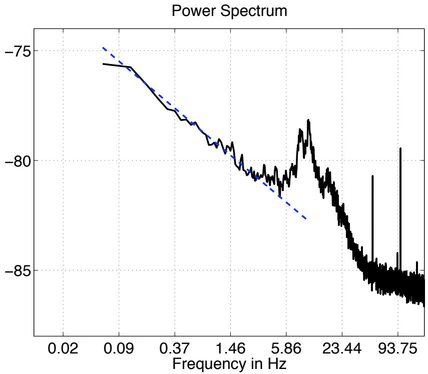
Scaling ⇒ Power Law

Outline

$1/f$-process (Model 1)

- 2nd order stationary $1/f$-process

$$\Gamma_Y(\nu) = C|\nu|^{-\gamma}, \ \gamma > 0, \ \nu_m \leq |\nu| \leq \nu_M, \ \frac{\nu_M}{\nu_m} \gg 1$$



Data MEG, Courtesy, Ph. Ciuciu, Neurospin, France

Scale Invariance Analysis Tool 1 : Spectrum Analysis

- Spectrum Estimation

  (time windowed) Periodogram or Welch estimator
  $|\tilde{Y}_{k,T}(\nu)|^2$ spectral estimate in $t \in [t_k - T/2, t_k + T/2]$
  $\hat{\Gamma}_Y(\nu) = \sum_k |\tilde{Y}_{k,T}(\nu)|^2$

- $1/f$-spectrum :

  $\frac{1}{K}\sum_k |\tilde{Y}_{k,T}(\nu)|^2 = C|\nu|^{-\gamma}$

- Estimation :

  $\hat{\gamma} \rightarrow \log \frac{1}{K}\sum_k |\tilde{Y}_{k,T}(\nu)|^2$ versus $\log|\nu|$
  involve $\hat{\gamma}$ in analysis, detection, classification

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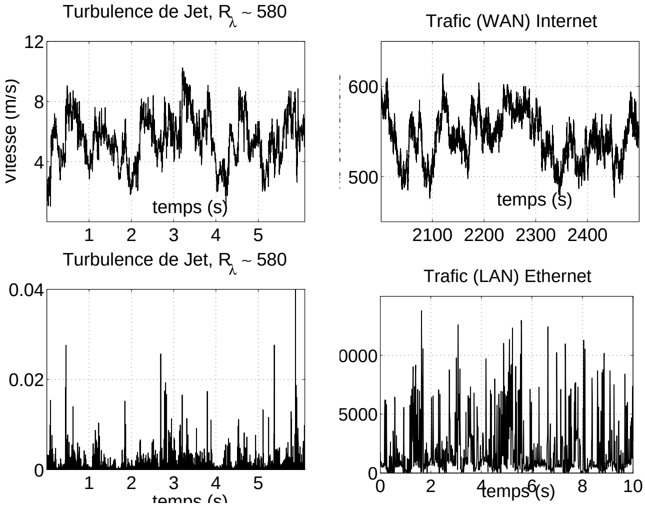
Scaling in applications ?



Scaling in applications ?

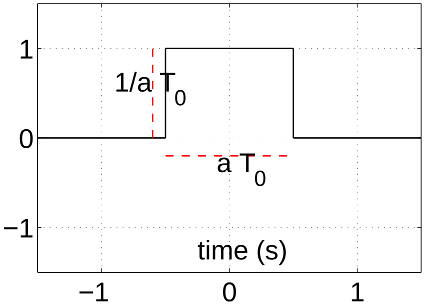
- hydrodynamic turbulence,
- statistical physics (long range interactions),
- astrophysics and cosmology (universe structure),
- geophysics (failures, earthquake),
- hydrology (nivology),
- physiology, biology, body rhythms (heart, gait),
- brain activity (fMRI),
- genomic,
- internet,
- geography (population repartition),
- artificial vision (intelligence),
- financial market,
- . . .

Outline

Scaling analysis : Aggregation

Average within box of size a

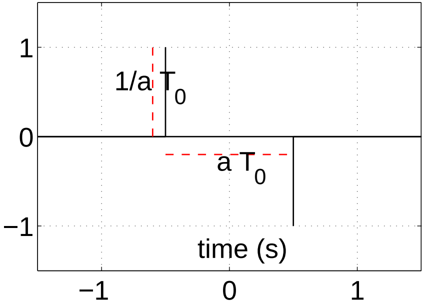
$$T_X(a,t) = \frac{1}{aT_0} \int_t^{t+aT_0} X(u)du$$



Scaling analysis : Increments

Difference over step lag of size a

$$T_X(a,t) = X(t + aT_0) - X(t)$$



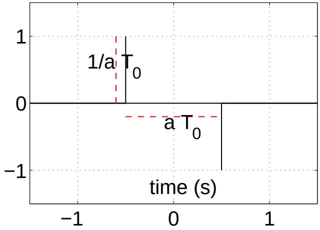
Scaling analysis : multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$

increment

DIFFERENCE

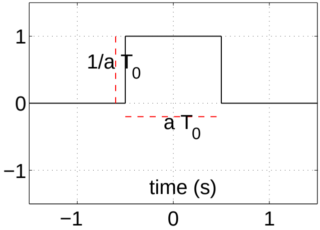
$X(t + aT_0) - X(t)$



Aggregation

AVERAGE

$\frac{1}{aT_0} \int_t^{t+aT_0} X(u)du$



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increment	Aggregation	?
DIFFERENCE	AVERAGE	?
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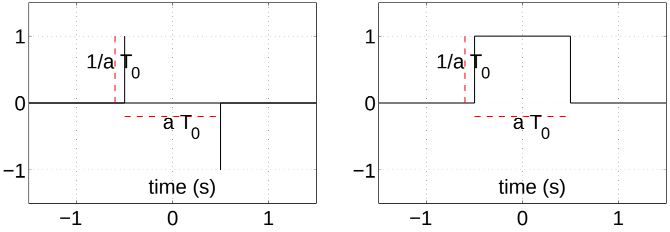
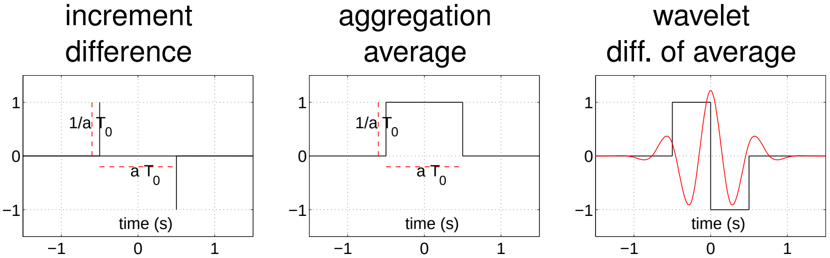


?

Multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$



Continuous Wavelet Transform

- Fourier Transform : $X(t) \Longrightarrow \tilde{X}(\nu) = \langle X, e_\nu \rangle$.
  Fourier Basis : $e_\nu(t) = \exp(\imath 2\pi\nu t)$
  Interpretation : ever lasting pure tone

- Continuous Wavelet Transform : $T_X(a, t) = \langle X, \psi_{a,t} \rangle$
  Mother-wavelet : $\int \psi_0(u) du = 0$,

  Wavelet-basis : $\psi_{a,t}(u) = \frac{1}{|a|}\psi_0(\frac{u-t}{a})$
  Interpretation : Joint time and frequency energy content

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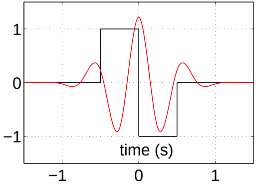

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$$X(t) \rightarrow T_X(a, t) = \langle \frac{1}{a} \psi \left(\frac{u-t}{a} \right) | X \rangle$$

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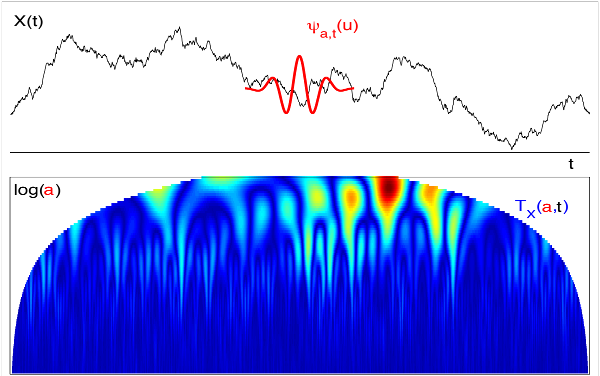
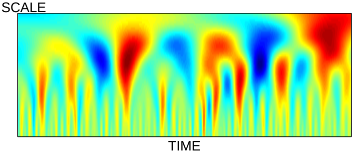


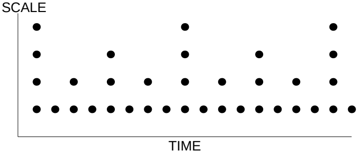
Interpretation : Joint time and frequency energy content

Outline

Discrete Wavelet Transform

- Redundancy (reproducing kernel transform)





- Critical sampling : $(j, k) \Leftarrow (a = 2^j, t = k2^j)$

  $d_X(j, k) = T_X(a = 2^j, t = k2^j)$; $\psi_{j,k}(u) = 2^{-j}\psi_0(2^{-j}u - k)$

- Multiresolution analysis
  - Shift invariant nested spaces (Meyer, Mallat, Daubechies)
  - Critical sampling and frames
  - Orthonormal basis
  - Fast pyramidal recursive algorithm

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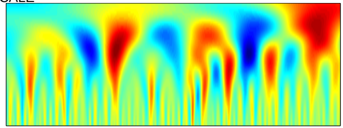

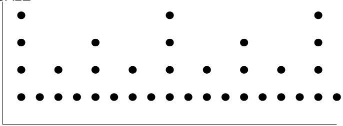

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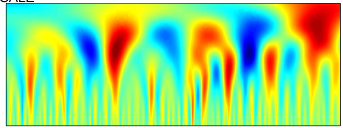

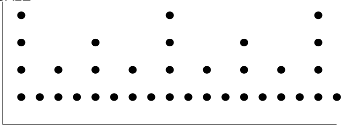

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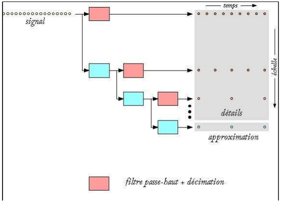
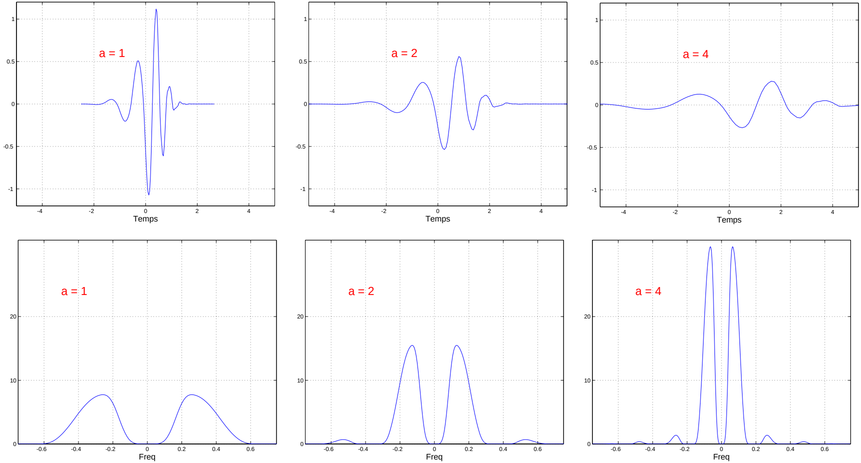


Wavelet and scaling : Dilation operator

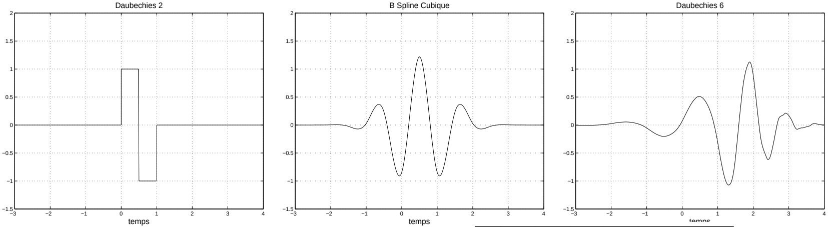
- Dilation (change of scale) operator : $\frac{1}{|a|}\psi_0(\frac{t}{|a|})$



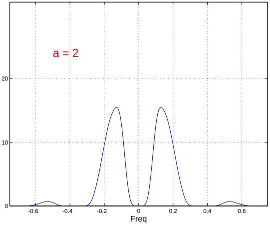
Wavelet and scaling : Vanishing moments

- Number of vanishing moments :

$$N_\psi \geq 1, \int t^k \psi_0(t)dt \equiv 0, \quad k = 0, 1, \ldots, N_\psi - 1.$$



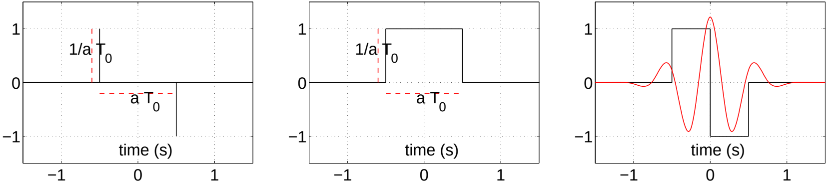
- Fourier transform :
  $|\Psi(\nu)| \sim_{|\nu| \to 0} C|\nu|^{N_\psi}$



Multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$

increment difference	aggregation average	wavelet diff. of average

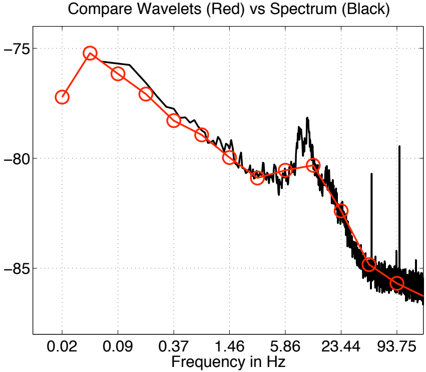


N_ψ

Number of vanishing moments

Spectrum Analysis versus Wavelet Analysis

- Wavelet Analysis : $\mathbf{E}|T_X(a,k)|^2 = \int \Gamma_X(\nu)|\tilde{\Psi}(a\nu)|^2 d\nu$
- $1/f-$process : $\frac{1}{n_a}\sum_{k=1}^{n_a}|T_X(a,k)|^2 \simeq C_q a^{\gamma-1}$
- $1/f-$process : $\hat{\Gamma}_Y(\nu) = \sum_k |\tilde{Y}_{k,T}(\nu|^2 \simeq C|\nu|^{-\gamma}$
- $\nu \simeq \nu_0/a$ and $q=2$



Data FMRI, Courtesy, Ph. Ciuciu, Neurospin, France

Scale invariance, wavelet and multiresolution

- Signal, Image : $X(t) \to$ wavelet coefficients : $T_X(a, t)$,

- Scaling :

  $|T_X(a, k)|$ covariant w.r.t. a change of the analysis scale a

  $\Rightarrow$Power Laws : $\boxed{\frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, k)|^q \simeq C_q a^{\zeta(q)}}$

- Range of scales : $a \in [a_m, a_M], \ a_M / a_m \gg 1$,
- Statistical orders : $q \in [q_m, q_M], \ q_m < 0 < q_M$,

- Scaling exponents : $\zeta(q)$: $\Rightarrow$ estimation,

$\Rightarrow$ detection, identification, classification.

Outline

From $1/f$ processes to ...

- Definition : 2nd order stationary $1/f$-process
$$\Gamma_Y(\nu) = C|\nu|^{-\gamma},\ \gamma > 0,\ \ \nu_m \le |\nu| \le \nu_M,\ \frac{\nu_M}{\nu_m} \gg 1$$

- $\nu_M \to +\infty$:$\Rightarrow$ fractal sample path

- $\nu_m \to 0$:$\Rightarrow$ Long Range dependence, self-similarity

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Long Range Dependence

- Definitions : Y 2nd stationary process with

  Spectrum : $\Gamma_Y(\nu) = c_f|\nu|^{-\gamma}, 0 < \gamma < 1, \nu \to 0.$

  Covariance :

  $c_X(\tau) = \mathbb{E}Y(t)Y(t+\tau) = c_\tau|\tau|^{-\beta}, 0 < \beta < 1, \tau \to +\infty$

  with $\gamma = 1 - \beta$ and $c_f = 2(2\pi)\sin((1-\gamma)\pi/2)c_\tau$.
- Scaling : Power-law $\Rightarrow$ No Characteristic Scale,
- $\int_0^{+\infty} c_X(\tau)d\tau = +\infty$ Sum of Covariance Infinite,

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  Aggregation : $T_X(a,t) = \frac{1}{aT_0}\int_t^{t+aT_0} X(u)du,$

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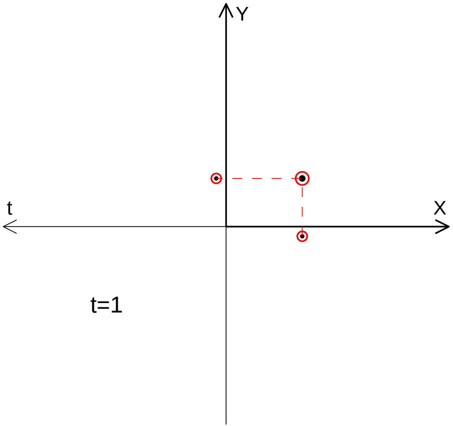

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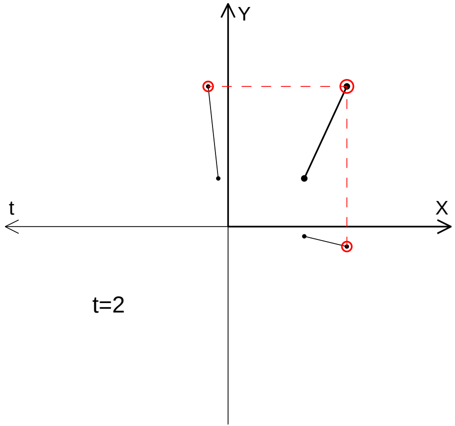

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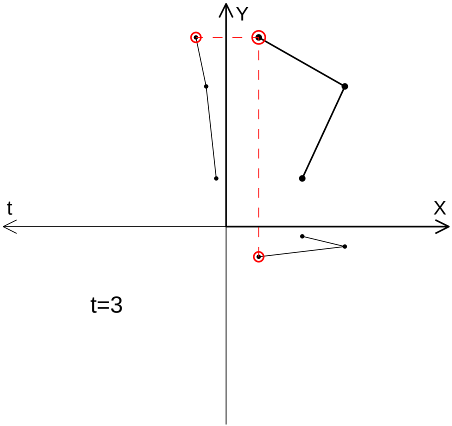

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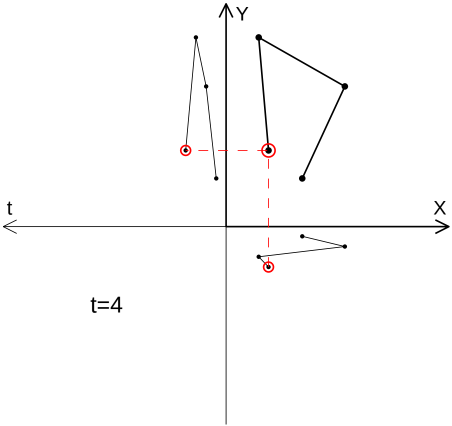

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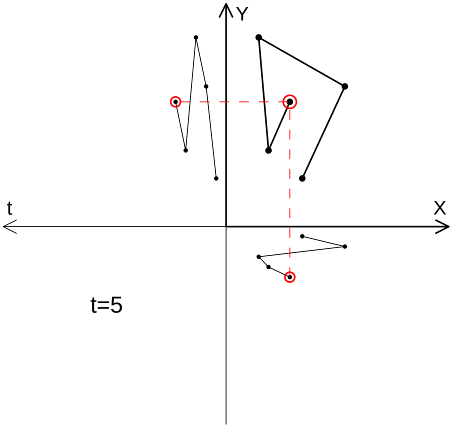

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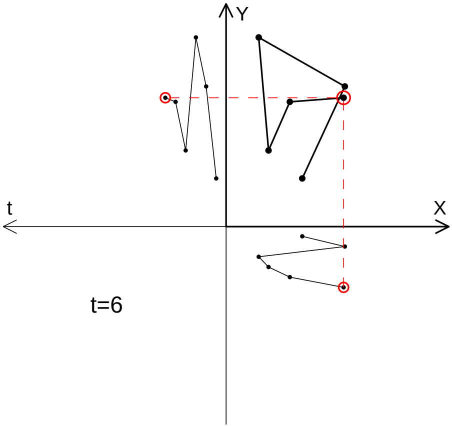

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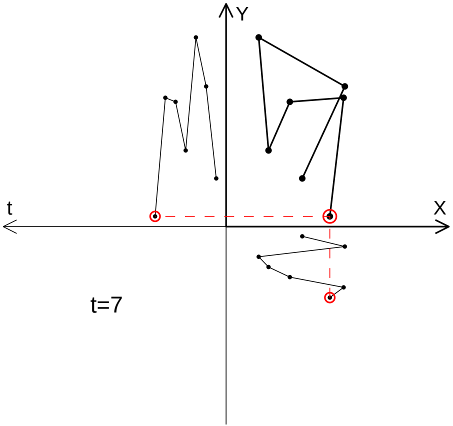

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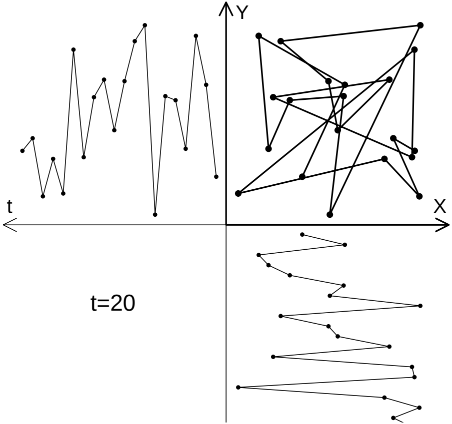

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  $\Rightarrow \mathbf{E}X(t)^2 = 2D|t|$, Einstein relation,
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Self-Similar process

- Definition : $\{X(t)\}_{t\in\mathcal{R}} \overset{fdd}{=} \{a^H X(t/a)\}_{t\in\mathcal{R}}$
  Dilation Factor : $\forall a > 0$,
  Self-Similarity Exponent : $H > 0$.
- Scaling :

Self-Similar process and scale invariance

- power laws :

$\Rightarrow$ $\mathbf{E}|X(t)|^q = \mathbf{E}|X(1)|^q \, |t|^{qH}$

$\Rightarrow$ non stationary processes

Self-Similar process and stationary increments

- Stationary increments :
  $\{X(t+\tau) - X(t)\}_{t \in \mathcal{R}} \overset{fdd}{=} \{X(0+\tau) - X(0)\}_{t \in \mathcal{R}}$

- Finite variance : $\mathbf{E}X(t)^2 < +\infty$

$\Rightarrow \quad 0 < H < 1,$
$\Rightarrow \quad \mathbf{E}X(t)X(s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$
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Self-Similar process and correlation

- Stationary increments and finite variance :

$\Rightarrow$ $\mathbf{E}(X(t+\tau)-X(t))(X(s+\tau)-X(s)) \sim_{|t-s|\gg\tau} \frac{\sigma^2 H(2H-1)}{\tau^{2H}} |t-s|^{2H-2}$,

$\Rightarrow$ $H = 1/2$ no correlation,

$\Rightarrow$ $1 > H > 1/2$ positive correlation,

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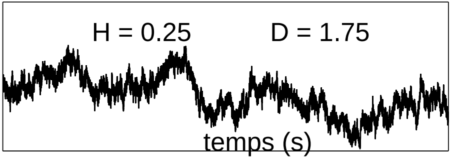

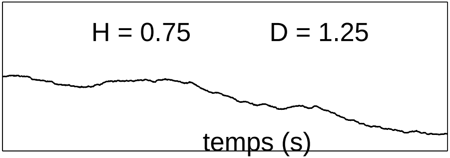

Self-Similar, long range dependence and $1/f$

- Increment process :

$$Y_\tau(t) = X(t+\tau) - X(\tau),$$
$$\mathbf{E} Y_\tau(t) Y_\tau(s) \sim C_\tau |t-s|^{2H-2},$$
$$\Gamma_Y(\nu) = C_f |\nu|^{-(2H-1)}$$
$$\Rightarrow \quad 1 > H > 1/2 \Rightarrow 1 > 2-2H > 0$$

$\Rightarrow$ Long range dependence

$\Rightarrow$ Time averages poorly estimates ensemble averages

- Linear Filter :

$$Y_\tau(t) = X(t+\tau) - X(\tau) = (X * \psi)(t),$$
$$\psi(t) = \delta(t+\tau) - \delta(t),$$
$$\Gamma_Y(\nu) = |1 - \exp(\imath 2\pi\nu\tau)|^2 \Gamma_X(\nu),$$
$$\Gamma_Y(\nu) \sim |\nu|^2 \Gamma_X(\nu) \quad |\nu| \to 0,$$
$$\Rightarrow \quad "\Gamma_X(\nu) \sim C_f |\nu|^{-(2H+1)}", \quad |\nu| \to 0$$

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- Stationary increments and finite variance :

$\Rightarrow$ $0 < H < 1$,

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$\mathbf{E}|X(t + a\tau_0) - X(t)|^q = C_q |a|^{qH}$,
for all scales : $\forall a > 0$.

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- Random Walk : $X(t+\tau) = X(t) + \underbrace{\delta_\tau X(t)}_{\text{Steps or Increments}}$

- Ordinary :

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  Gaussian $\Rightarrow$ Fractional Brownian Motion $0 < H < 1$: $B_H(t)$
  $H = 1/2 \Rightarrow$ (Ordinary) Brownian Motion $B(t)$,

  Increments $\Rightarrow G_H(t) = B_H(t+1) - B_H(t)$ Frac. Gauss. Noise,
  $H = 1/2 \Rightarrow$ White Gaussian Noise.

Outline

Wavelets and self-similar processes

(Flandrin, 89, 92)

- 1. Wavelet and Self Similarity.
- 2. Wavelet and Non Stationarity.
- 3. Wavelet and Long Range Dependence.

Wavelets and Self-Similarity

- Self-Similarity :

$$\{X(t)\} \stackrel{d}{=} \{a^H X(t/a)\} \Rightarrow \{d_X(0,k)\} \stackrel{d}{=} \{2^{-jH} d_X(j,k)\}$$

- Marginal Distributions :

$$P_j(d) = \frac{1}{a} P_{j'}\left(\frac{d}{a}\right), \quad a = \left(\frac{2^{j'}}{2^j}\right)^H.$$

- Sketch of Proof :

$$\begin{aligned} d_X(j,k) &= \int X(u)\psi(2^{-j}u - k)2^{-j}du \\ &= \int X(2^j u)\psi(u-k)du \\ &\stackrel{d}{=} 2^{jH} \int X(u)\psi(u-k)du \\ &= 2^{jH} d_X(0,k). \end{aligned}$$

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Wavelet and non-Stationarity

(with stationary increments)

- $\{d_X(j,k), k \in \mathcal{Z}\}$ stationary sequences for each scale $a = 2^j$.

- Sketch of Proof :

$$\begin{aligned} d_X(0, k+k_0) &= \int X(u)\psi(u-k-k_0)du \\ &= \int X(u+k_0)\psi(u-k)du \\ &= \int [X(u+k_0) - X(k_0)]\psi(u-k)du \\ &\stackrel{d}{=} \int [X(u) - X(0)]\psi(u-k)du \\ &= \int X(u)\psi(u-k)du \\ &= d_X(0,k). \end{aligned}$$

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- Sketch of Proof :
$\mathbf{E}X(t)X(s) = \sigma^2/2 \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$,

$$\begin{aligned}
\mathbf{E} d_X(j,k) d_X(j',k') &= \int\int dt ds \mathbf{E} X(t)X(s) \psi_{j,k}(t)\psi_{j',k'}(t), \\
&= \underbrace{\int ds \psi_{j',k'}(s)}_{0} \int dt |t|^{2H} \psi_{j,k}(t) + \\
&\quad \underbrace{\int dt \psi_{j,k}(t)}_{0} \int ds |s|^{2H} \psi_{j',k'}(s) + \\
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- Sketch of Proof :

$$\mathbf{E}X(t)X(s) = \sigma^2/2\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

$$\mathbf{E}d_X(j,k)d_X(j',k') = \int\int dtds\mathbf{E}X(t)X(s)\psi_{j,k}(t)\psi_{j',k'}(t),$$

$$= \underbrace{\int ds\psi_{j',k'}(s)}_{0}\int dt|t|^{2H}\psi_{j,k}(t) +$$

$$\underbrace{\int dt\psi_{j,k}(t)}_{0}\int ds|s|^{2H}\psi_{j',k'}(s) +$$

$$\int\int dtds|t-s|^{2H}\psi_{j,k}(t)\psi_{j',k'}(s).$$

Wavelets and Long Range Dependence - Con't

$$\mathbf{E}d_X(j,k)d_X(j',k') = \int du|u|^{2H} \underbrace{\int dv\psi_{j,k}(v+u/2)\psi_{j',k'}(v-u/2)}_{\Psi_{j,j',k,k'}(u)}.$$

$$\mathbf{E}d_X(j,k)d_X(j',k') = \int d\nu|\nu|^{-(2H+1)}2^{(j+j')/2}|\tilde{\psi}(2^j\nu)\tilde{\psi}(2^{j'}\nu)|\exp(-\imath 2\pi(2^jk-2^{j'}k')).$$

- But
$$|\nu|^{-(2H+1)}|\tilde{\psi}(2^j\nu)\tilde{\psi}(2^{j'}\nu)| \simeq_{|\nu|\to 0} |\nu|^{-(2H+1)}|(2^j\nu)|^N|(2^{j'}\nu)|^N \simeq_{|\nu|\to 0} 2^{(j+j')N}|\nu|^{2N_\psi-2H-1}.$$

- Hence, when $|2^jk-2^{j'}k'| \to +\infty$,
$$\mathbf{E}d_X(j,k)d_X(j',k') \sim K|2^jk-2^{j'}k'|^{2(H-N_\psi)}.$$

- Key-Points :
  Dilation Operator and Number of vanishing Moments.

Wavelets and Long Range Dependence - Con't

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$\Rightarrow$ $\{d_X(j,k)\}$ are short range dependent as soon as $N_\psi > H + 1/2$.

- Key-Points :
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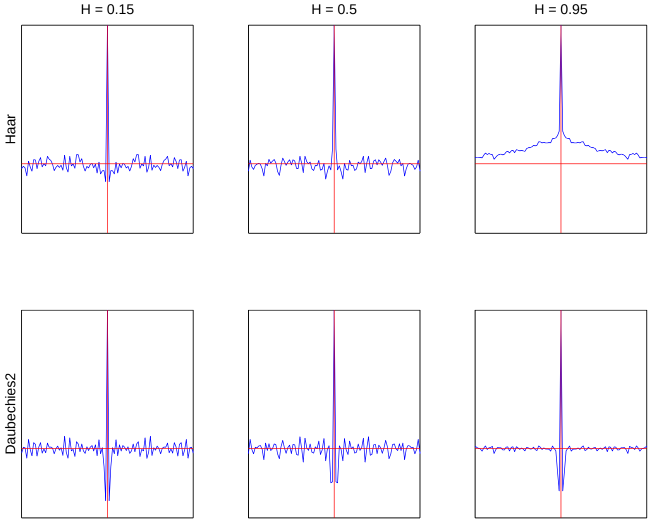
Wavelets and Long Range Dependence - Con't



Wavelets and Self-Similar Processes - Summary

- Self-Similarity :

$$\{X(t)\} \stackrel{d}{=} \{a^H X(t/a)\} \Rightarrow \{d_X(0,k)\} \stackrel{d}{=} \{2^{-jH} d_X(j,k)\}$$

- Non-stationarity (with stationary increments) :
  $\{d_X(j,k), k \in \mathcal{Z}\}$ stationary sequences for each scale 2^j

$$\Rightarrow \mathbf{E}|d_X(j,k)|^q = |d_X(0,0)|^q 2^{jqH}, \ \forall a = 2^j, \ q > -1.$$

- Long range dependence :
  $\{d_X(j,k)\}$ Short Range Dependent if $N > H + 1/2$.
  $|2^j k - 2^{j'} k'| \to +\infty, \ |\mathrm{Cov}\, d_X(j,k) d_X(j',k')| \le D|2^j k - 2^{j'} k'|^{2(H-N_\psi)},$
  Weak correlation amongst wavelet coefficients.

- Interpretations :

$$X(t) = \sum_k a_X(J,k)\varphi_{J,k}(t) + \sum_{j,k} d_X(j,k)\psi_{j,k}(t).$$

Scaling analysis : Logscale Diagrams

- Principle :

$$\mathbf{E}|d_X(j,k)|^q = |d_X(0,0)|^q 2^{jqH} \Rightarrow \text{log-log plots}$$

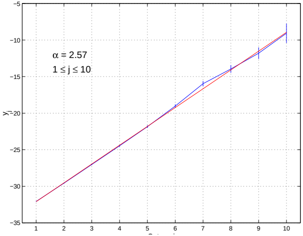
- Estimation : short-range dependence $\Rightarrow$

Ensemble averages $\rightarrow$ Time Averages

$$\mathbf{E}|d_X(j,k)|^q \Rightarrow 1/n_j \sum_k |d_X(j,k)|^q = S(2^j, q)$$

- Logscale Diagrams :

$$\log_2 S(2^j, q) \text{ versus } \log_2 2^j = j \Rightarrow qH$$



Wavelets and $1/f$-processes

- Spectral Analysis :
  Let Y be a 2nd Order stationary process,
  Let ψ have central frequency ν_0 and bandwith $\Delta\nu_0$.

$$\begin{aligned} \mathbf{E}|d_Y(j,k)|^2 &= \int \Gamma_Y(\nu)|\Psi(2^j\nu)|d\nu \\ &\simeq 2^{-j}\Gamma_Y(2^{-j}\nu_0) \text{ within bandwith } 2^{-j}\Delta\nu_0. \end{aligned}$$

- Let Y be Long Range Dependent :
  $\Gamma_Y(\nu) = C|\nu|^{-\gamma},\ 0 < \gamma < 1,\ |\nu| \to 0,$

$\Rightarrow$ Power Law : $\mathbf{E}|d_X(j,k)|^2 \sim C2^{j(\gamma-1)},\ j \to +\infty,$

- Decorrelation :
  $\mathbf{E}d_X(j,k)d_X(j,k')^2 \sim C|k-k'|^{\gamma-1-2N_\psi},\ |k-k'| \to +\infty,$

$\Rightarrow$ Short Range Dependence as soon as $N > \gamma/2$.

$\Rightarrow$ Logscale Diagram :

$$\log_2 S(2^j, 2) \text{ versus } \log_2 2^j = j \Rightarrow \gamma$$

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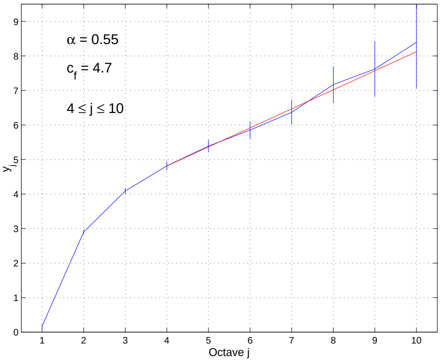


Outline

Linear regression

- Weighted Linear Fits : $\hat{\gamma} = \sum_j w_j \log_2 S_2(2^j, 2)$

$$w_j = \frac{B_0 j - B_1}{a_j (B_0 B_2 - B_1^2)}, \quad B_k = \sum_j j^k / a_j \ a_j > 0 \text{ arbitrary numbers}$$

- Analytical Performances : $q = 2$
  - *i*) Assume : process is Gaussian,
  - *ii*) Idealisation : wavelet coefficients are exactly independent
- Bias :

$$\mathbf{E} \log_2 S(2^j, 2) = \log_2 \mathbf{E} S(2^j, 2) + \underbrace{\Gamma'(n_j/2) - \log_2(n_j/2)}_{g_j}.$$

$$\Rightarrow \mathbf{E}\hat{\gamma} = \gamma + \sum_j w_j g_j.$$

- Variance :

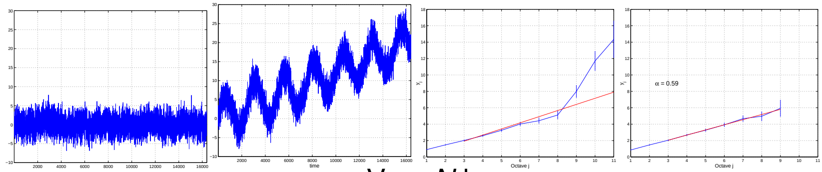
$$\mathrm{Var}\,\hat{\gamma} \simeq \left((2\log_2(e))^2 (\textstyle\sum_j w_j^2 \sigma_j^2)\right) / n,$$

$$\text{min. if } a_j = \sigma_j^2 = \mathrm{Var}\, \log_2 S(2^j, 2) \simeq 2^j.$$

- Actual Performances : close to MLE.
- Conceptual and Practical Simplicity : DWT + Linear Fit

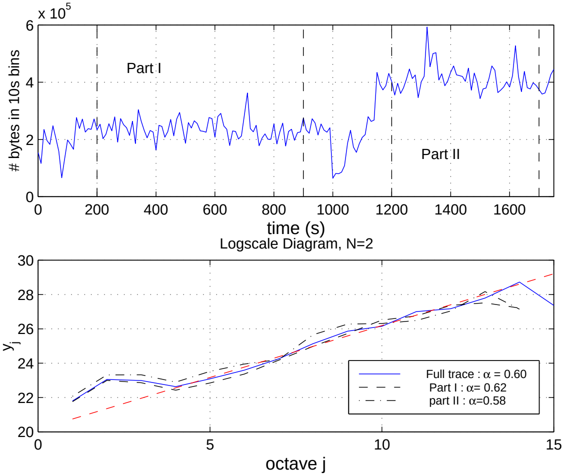
Scaling versus non stationarity : superimposed trends

- Linear transform :
  $Y(t) = X(t) + T(t) \Rightarrow d_Y(j,k) = d_X(j,k) + d_T(j,k)$
- If $T(t)$ Polynomial of degree P, then $d_T \equiv 0$ when $N_\psi > P$,
- If $T(t)$ smooth trend, then the d_T decrease as N increases.

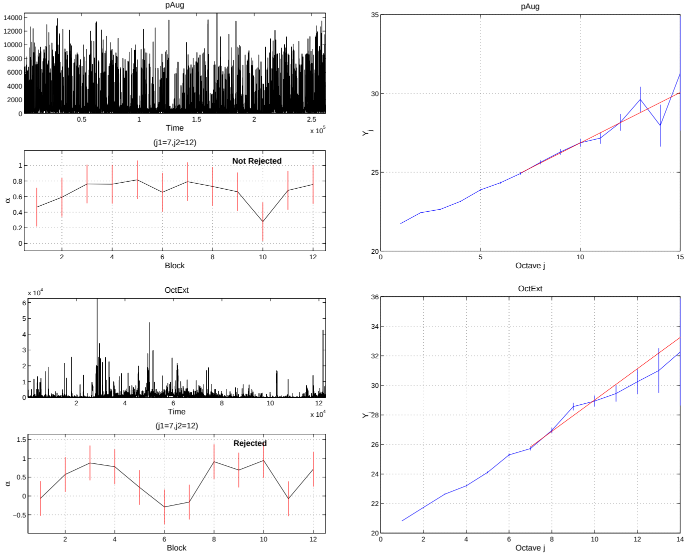


Vary N !

Superimposed Trends - Ethernet Data



Constancy of Scaling along time



Outline

Beyond self-similarity. . . ?

- Self-Similarity :

  Power Laws : $\mathbf{E}|d_X(j,k)|^q = C_q(2)^{jqH}$
  For all scales : $\forall a = 2^j$,
  For all orders : $q > -1$,
  A single parameter qH.

- Beyond :

  Power Laws : $\mathbf{E}|d_X(j,k)|^q = C_q(2)^{j\zeta(q)}$
  $\zeta(q)$ non linear concave function of q,
  For a limited range of scales : $a_m \leq a \leq a_M$,
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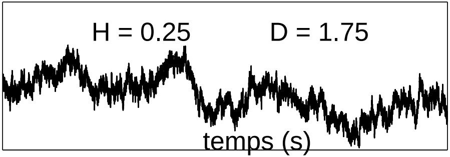
The other end of the spectrum - sample path regularity

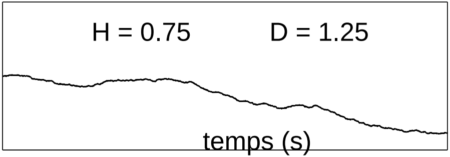
- $1/f$-processes : $\Gamma_Y(\nu) \simeq C|\nu|^{-\gamma}, \ |\nu| \to +\infty$

$\Rightarrow$ $\mathbf{E}Y(t+\tau)Y(t) \sim \sigma^2(1 - C|\tau|^{\gamma})\,\tau \to 0,$

- Self-similar process :
  $\mathbf{E}|X(t + a\tau_0) - X(t)|^q = C_q|a|^{qH}, \ a \to 0.$

$\Rightarrow$ Fractal Sample Path - Hölder Regularity





$\Rightarrow$ Multifractal analysis

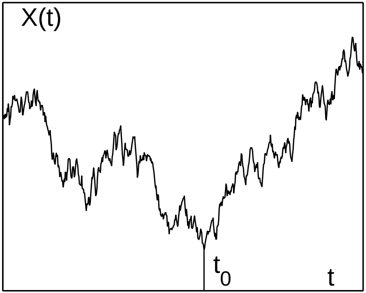
$\Rightarrow$ Multifractal processes

$\Rightarrow$ Multifractal formalism

Multifractal Analysis

- **Local regularity** of $X(t)$ at $t_0 : 0 < \alpha < 1$

  Compare : $|X(t) - X(t_0)| < C|t - t_0|^{\alpha}$



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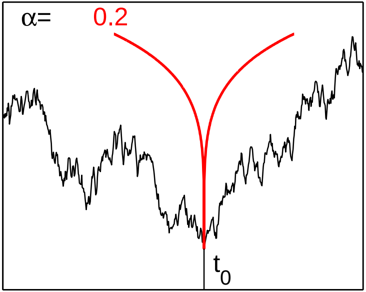

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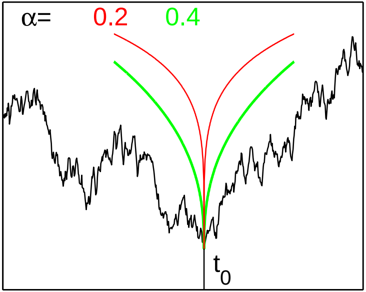

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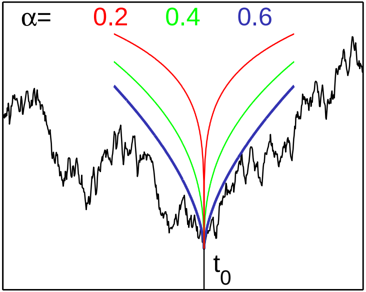

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  Extend differentiability to non integer : $0 < h(t_0) < 1$

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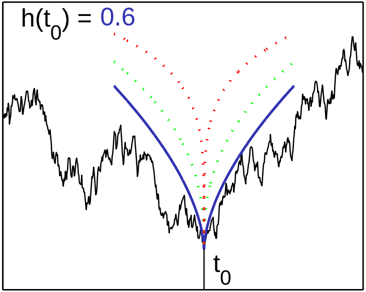

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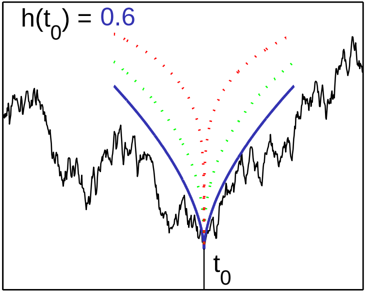
  $h(t_0) \to 1 \Rightarrow$, smooth, very regular,

  $h(t_0) \to 0 \Rightarrow$, rough, very irregular



Hölder exponent and Wavelets (Intuition)

- Local singularity :

$$X(t_0) \in C^{h(t_0)}, h(t_0) \text{ non (even) Integer }, P \leq h(t_0) < P+1,$$
$$X(t) \simeq_{t\to t_0} X(t_0) + \sum_{k=1}^{P} X^{(k)}(t_0)\frac{(t-t_0)^k}{k!} + C|t-t_0|^{h(t_0)}.$$

- Mother Wavelet with $N_\psi \leq P$:

$$T_X(a,t_0) \simeq_{a\to 0} a^{N_\psi} \int_{\mathcal{R}} u^{N_\psi}\psi_0(u)du.$$

- Mother Wavelet with $N_\psi \geq P+1$:

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▸ Hölder and Wavelets (theory)

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Multifractal (or singularity) spectrum

- Data : a collection of singularities
  $|X(\mathbf{t}) - X(\mathbf{t_0})| \leq C|\mathbf{t} - \mathbf{t_0}|^{h(\mathbf{t_0})}$
- Fluctuations of local regularity : $h(\mathbf{t})$?
- not interested in h for each $(\mathbf{t})$!
- Instead, set $E(h)$ of points $\mathbf{t}$ with same h : $h(\mathbf{t}) = h$,
- Fractal dimension of $E(h)$,
- Actually Hausdorff dimension of $E(h)$, ▸ Hausdorff
- Multifractal spectrum : ▸ $D(h)$

$$D(h) = \dim_{\text{Haussdorf}}(E(h)).$$

$$0 \leq D(h) \leq d,$$
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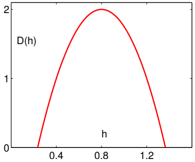

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- How to measure $D(h)$ from a single finite length observation ?

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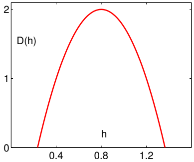

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Outline

Multifractal processes

- Mono-Fractal :

  A single h : $\forall t$, $h(t) = h$, $D(h) = d\delta(h - H)$,
  Scaling exponents are linear in q : $\zeta(q) = qH$,
  Finite variance Self-similar processes,
  Fractional Brownian motion.

- Multi-Fractal :

  Many different h,
  leaving on fractal subset of $\mathcal{R}^d$,
  Bell-shaped $D(h)$,
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  Multiplicative cascades (Mandelbrot)

- Multi-Fractional : is something else !

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1D Examples

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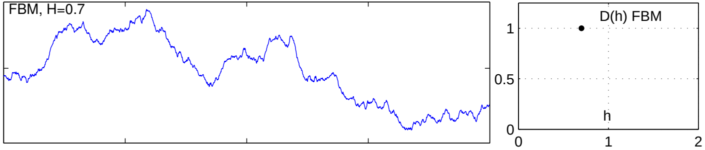
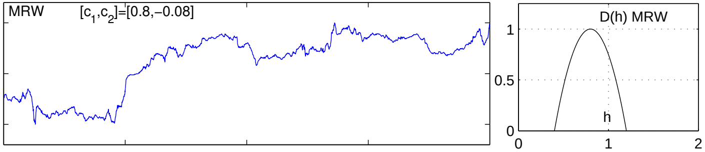
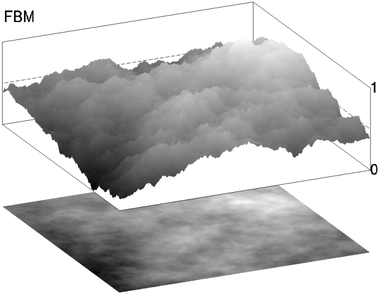


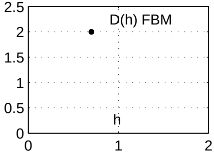
2. Multifractal Random Walk (MF)



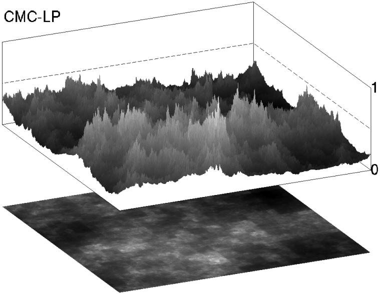
2D Examples

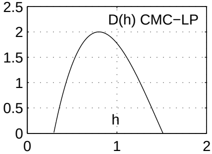
FBM (H-sssi)



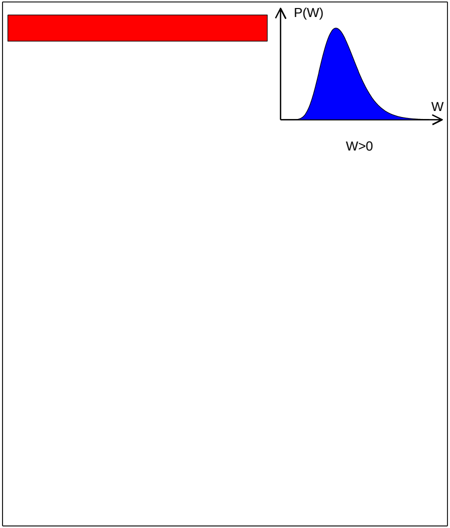


Multiplicative casc. (MF)





Mandelbrot Multiplicative Cascades (Model 3)



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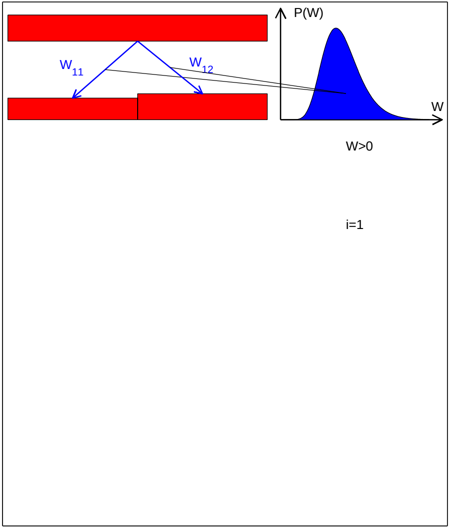

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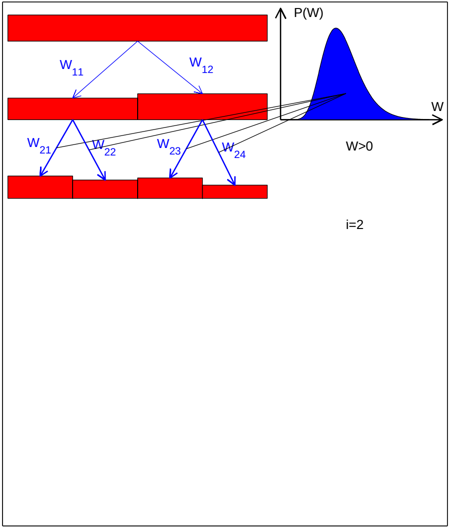

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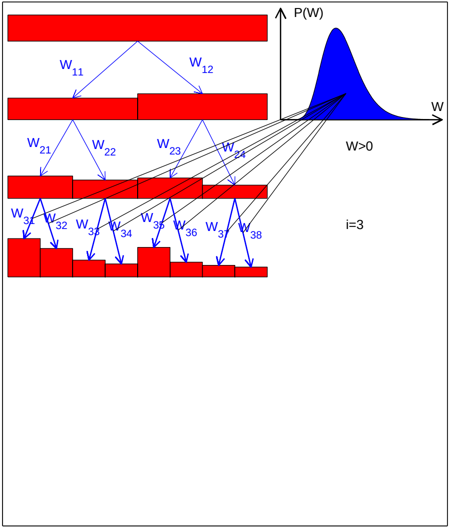

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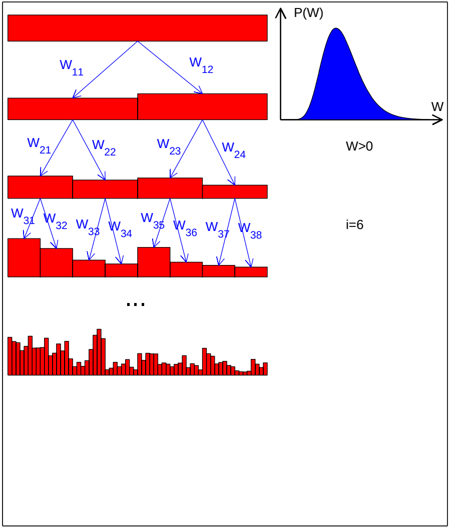

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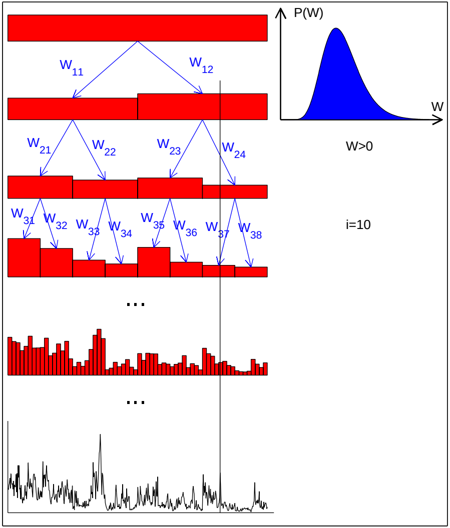
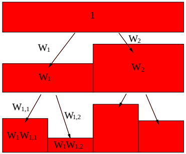
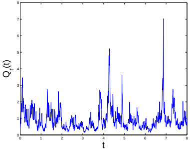


Mandelbrot Multiplicative Cascades (Model 3)

- Definition :

  Split Dyadic Intervals $I_{j,k}$ into two,
  I.I.D. positive (mean one) Multipliers $W_{j,k}$
  $Q_J(t) = \Pi_{\{(j,k):1 \leq j \leq J, t \in I_{j,k}\}} W_{j,k}$,



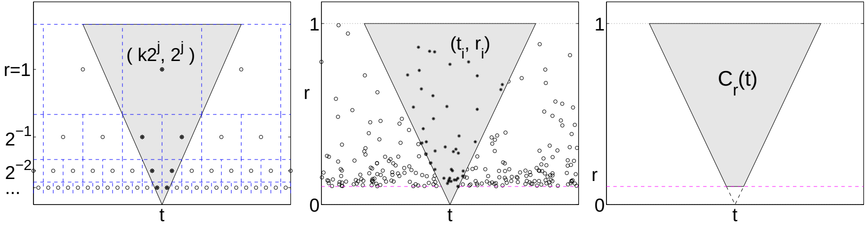


- Implications :

  Cascades, Multiplicative Structure,
  Power Laws,
  $\mathbf{E}\left(1/2^j \int_{k2^j}^{(k+1)2^j\tau_0} X(u)du\right)^q = C_q|2^j|^{\zeta_q}$,
  Multiple Exponents $\zeta_q = q - \log_2 \mathbf{E} W^q$, Non Linear in q,
  Fine Scales $a = 2^j \to 0$, $a \ll L$ Integral Scale,
  No Characteristic Scale of Time beyond an Integral Scale.
  Non Stationarity,
  Local Holder Exponent,
  MultiFractal Sample Paths, MultiFractal Spectrum $D(h)$.

Multiplicative Cascades (Model 3)

MANDELBROT'S CASCADE (CMC) — **COMPOUND POISSON CASCADE (CPC)** — **INF. DIV. CASCADE (IDC)**



$$Q_r(t) = \Pi\ W_{j,k}, \qquad = \Pi\ W_{j,k}, \qquad = \exp \int dM(t', r'),$$

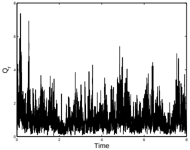
$$\varphi(q) = -\log_2 \mathbf{E} W^q, \quad = -q(1-\mathbf{E}W)+1-\mathbf{E}W^q, \quad = \rho(q)-q\rho(1),$$

$$A(t) = \lim_{r\to 0} \int_0^t Q_r(u)du,$$

For a Range of qs, $\mathbf{E}|A(t+a\tau_0)-A(t)|^q = c_q|a|^{q+\varphi(q)}$,

MultiFractal Processes (Model 3)

- Density $Q_r(t) = \Pi \; W_{j,k}$ $\mathbf{E}\left(\frac{1}{a}\int_t^{t+a\tau_0} Q_r(u)du\right)^q = c_q a^{\varphi(q)}$,



- Measure : $A(t) = \lim_{r\to 0} \int_0^t Q_r(u)du$,

$$\mathbf{E}|A(t+a\tau_0) - A(t)|^q = c_q|a|^{q+\varphi(q)},$$

- FBM in Multifractal Time : $V_H(t) = B_H(A(t))$,

$$\mathbf{E}|V_H(t+a\tau_0) - V_H(t)|^q = c_q|a|^{qH+\varphi(qH)},$$

- MultiFractal Random Walk : $Y_H(t) = \int^t Q_r(s)dB_H(s)$

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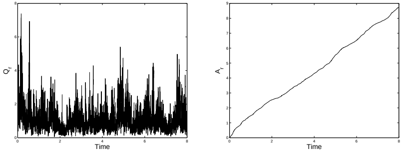

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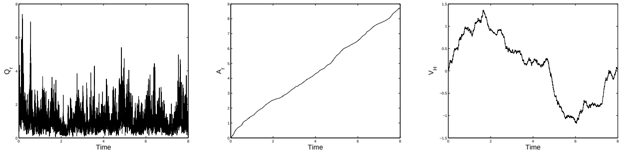

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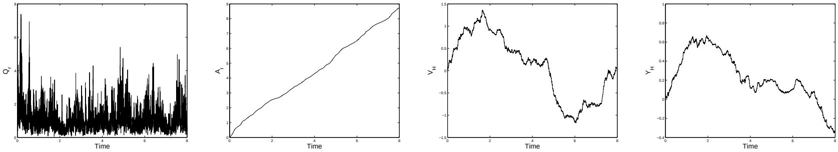

Outline

How to measure the multifractal spectrum ?

- How to measure $D(h)$?
- from a single finite length observation ?

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$\Rightarrow$ Intuition : estimate h for each t !

- Not interested in measuring $h(t)$ for each t
- may vary from one sample path to another for stochastic processes,
- may vary widely between two very close t and t',
- Measuring $h(t)$ for each t makes no sense ! intrinsical obstruction !
- Actual measure are done on real data with finite resolution :
  - $\Rightarrow$ cannot do $|t - t_0| \to 0$,
  - $\Rightarrow$ on any subset, all h are present.
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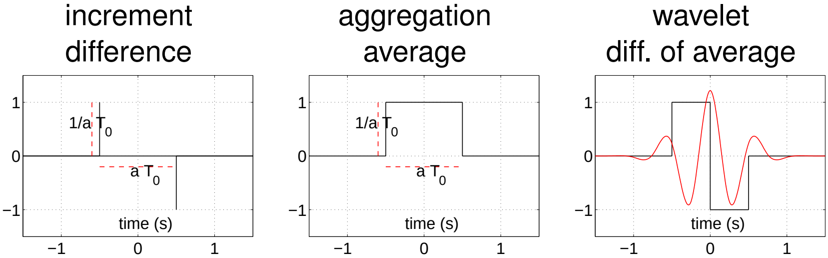
Multiresolution analysis

- $X(t) \rightarrow T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$



Multifractal formalism

- Multiresolution Quantities : $T_X(a, t)$,
- Structure functions : $S(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, t)|^q$,
- Power laws : $S(a, q) \simeq c_q |a|^{\zeta(q)}, \quad a \to 0$,
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$$D(h) = \min_{q \neq 0} (d + qh - \zeta(q))$$

- Multifractal formalism → Scaling analysis :
- Analogies :

Thermodynamic formalism

Rényi entropies

$q \geq 0$ AND $q \leq 0$.

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$$\begin{aligned} S_n(a,q) &\simeq a^d \sum_h a^{-D(h)} a^{hq}, \\ &\simeq \sum_h a^{d-D(h)+hq)}, \\ &\sim_{a\to 0} c_q a^{\zeta(q)} \end{aligned}$$

Saddle-point argument : $\Rightarrow$ Legendre transform

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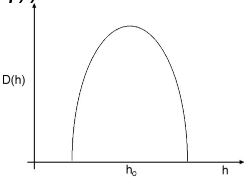

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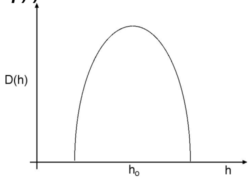

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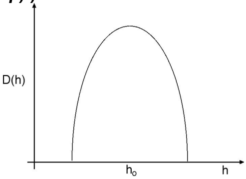

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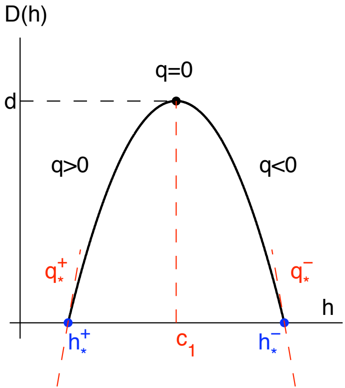
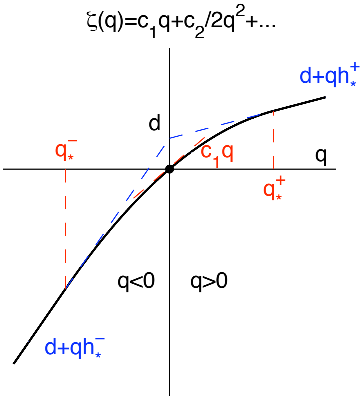
Legendre transform

Linearization effect





Linearization effect - Multiplicative cascades

- Multifractal spectrum for multiplicative cascades

  Moments : $\mathbf{E}|d_X(j,k)|^q = c_q 2^{j\lambda(q)}, q \in [q_c^-, q_c^+]$.
  Multipliers : $\lambda(q) =$ functional $(q, \mathbf{E}W^q)$),
  Legendre : $D_\lambda(h) = min_q(d + qh - \lambda(q))$,
  Multifractal Spectrum

  $D(h) = D_\lambda(h)$, if $D_\lambda(h) \geq 0$,,
  $D(h) = -\infty$, otherwise ;

- Scaling exponents (Structure functions) :

  $S(2^j, q) = (1/n_j)\sum_k |d_X(j,k)|^q = c_q 2^{j\zeta(q)}, q \in [q_*^-, q_*^+]$
  $\zeta(q) = min_h(d + qh - D(h))$,
  $\zeta(q) = \lambda(q), q_*^- \leq q \leq q_*^+$,
  $\zeta(q) = d + qh_M, q_*^+ \leq q$,
  $\zeta(q) = d + qh_m, q \leq q_*^-$.

- Disentangling a confusion : $\zeta(q) \neq \lambda(q)$.

$\Rightarrow$ Multifractal $\neq$ Multiplicative !

Integral scale

- Power Law : $S(a,q) = C_q a^{\zeta(q)}$,
- Hölder inequality $\Rightarrow$ $S(a,q)$ convex in q,
- $\ln S(a,q)$ convex in q,
- $\zeta(q)$ concave in q,

$\Rightarrow$ $\ln a \leq \frac{(\ln C_q)''}{(\zeta(q))''}$,

- Scaling with concave $\zeta(q)$, only at fine scales,
- Multifractal theory : $a \to 0$.

Log-cumulants

- Polynomial expansion :

$$\zeta(q) = \sum_{p\geq 1} c_p \frac{q^p}{p!} = c_1 q + \frac{c_2}{2!} q^2 + \frac{c_3}{3!} q^3 + \frac{c_4}{4!} q^4 + \cdots$$

- $C(j,p)$: **cumulants** of $\ln |d_X((j,\cdot)|$

$$C(j,p) = c_{0,p} + c_p \ln 2^j$$

- $D(h) = d + \frac{c_2}{2!}\left(\frac{h-c_1}{c_2}\right)^2 + \frac{-c_3}{3!}\left(\frac{h-c_1}{c_2}\right)^3 + \frac{-c_4+3c_3{}^2/c_2}{4!}\left(\frac{h-c_1}{c_2}\right)^4 + \ldots$
- $\zeta(q), D(h) \rightarrow (c_1, c_2, c_3, c_4)$

- Discrimination :

  self-similar : $\zeta(q)$ linear , $\Rightarrow \forall p \geq 2 : \; c_p \equiv 0$
  multiplicative cascade : $\zeta(q)$ non linear, $\Rightarrow \exists p \geq 2 : c_p \neq 0$

Log-cumulants

- Polynomial expansion :

$$\zeta(q) = \sum_{p\geq 1} c_p \frac{q^p}{p!} = c_1 q + \frac{c_2}{2!} q^2 + \frac{c_3}{3!} q^3 + \frac{c_4}{4!} q^4 + \cdots$$

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$$C(j,p) = c_{0,p} + c_p \ln 2^j$$

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- $\zeta(q), D(h) \to (c_1, c_2, c_3, c_4)$

- Discrimination :

  self-similar : $\zeta(q)$ linear , $\Rightarrow \forall p \geq 2 : \; c_p \equiv 0$
  multiplicative cascade : $\zeta(q)$ non linear, $\Rightarrow \exists p \geq 2 : c_p \neq 0$

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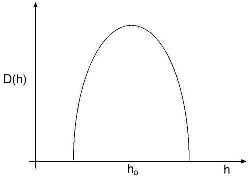
Multifractal formalism

- What Multiresolution Quantities : $T_X(a, t)$?
- Structure functions : $S(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, t)|^q$,
- Power laws : $S(a, q) \simeq c_q |a|^{\zeta(q)}, \quad a \to 0$,
- Scaling function : $\zeta(q) = \liminf_{a \to 0} \frac{\ln S(a,q)}{\ln a}$,
- Legendre transform : $\zeta(q) \to D(h)$.

$$D(h) = \min_{q \neq 0} (d + qh - \zeta(q))$$

- Multifractal formalism → Scaling analysis :

$S(a, q) \simeq C_q a^{\zeta(q)}$,
$a \to 0$,
$q \geq 0$ AND $q \leq 0$.



How well does it work ?

- Usual choices :
  increment, aggregation, wavelet
- Unsatisfactory :
  $q \leq 0 \Rightarrow S(a, q)$ instable
  $\zeta(q) \to D(h)$: not valid in general
- $T_X(a, t)$ need to be hierarchical (Jaffard, 2004).
  oscillations : $K_a = [t - a, t + a]$
  $$T_X(a, t) = \sup_{u \in K_a} X(u) - \inf_{u \in K_a} X(u),$$
  hierarchical : if $a_1 \leq a_2$, then $T_X(a_1, t) \leq T_X(a_2, t)$
- Solution (Jaffard et al., 2006) :
  increments $\to$ oscillations,
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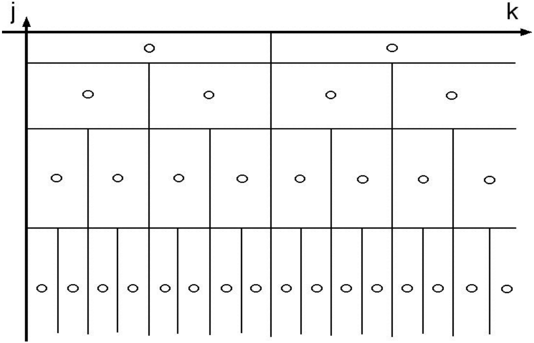
Outline

Wavelet Leaders

- Discrete Wavelet Transform : $\lambda_{j,k} = [k2^j, (k+1)2^j)$

$$d_X(j,k) = \langle \frac{1}{2^j}\psi\left(\frac{t-2^jk}{2^j}\right) | X(t) \rangle ,$$



- Wavelet Leaders : $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$

$$L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{X,\lambda'}|$$

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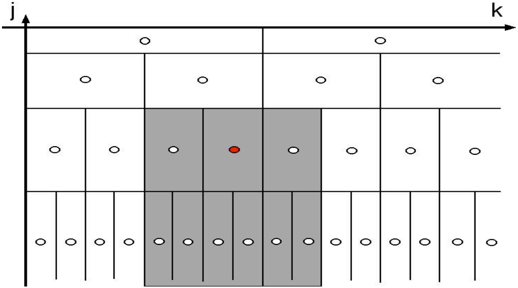

2D Wavelet leaders

- 2D Discrete Wavelet Transform :

$$\lambda_{j,k_1,k_2} = \{[k_1 2^j, (k_1+1)2^j), [k_2 2^j, (k_2+1)2^j)\}$$
$$d_X^{(m)}(j, k_1, k_2),\ m = 1, 2, 3\ (L_1 \text{ norm}).$$

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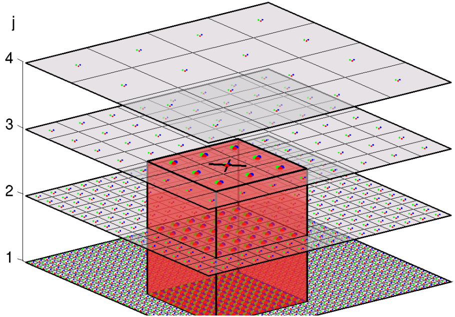

Wavelet Leaders and Limitations

- $q < 0$: Obviously solved : Leaders are large
  $\Rightarrow$ MultiFractal Spectrum over its Entire Range,
- Oscillating (Chirp-type) Singularities :
  Cusp :
  $|X(t) - X(t_0)| \sim_{|t-t_0|\to 0} |t - t_0|^h \Rightarrow |L_X(j,k)| \sim_{|2^j|\to 0} 2^{jh}$
  Chirp : $|X(t) - X(t_0)| \sim_{|t-t_0|\to 0} |t - t_0|^h \sin(\frac{1}{|t-t_0|^\beta}) \Rightarrow$
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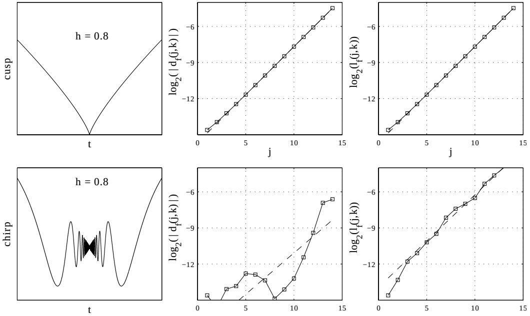

Multifractal formalism

- Multiresolution Quantities : $L_X(a, t)$,
- Structure functions :$S(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |L_X(a, k)|^q$,
- Scaling function : $\zeta(q) = \liminf_{a \to 0} \frac{\ln S(a,q)}{\ln a}$,
- Legendre transform : $\zeta(q) \to D(h) = min_{q \neq 0}(d + qh - \zeta(q)$.

- Valid (Jaffard et al., 2006) :
  for $q \geq 0$ AND $q \leq 0$,
  for all classes of processes,
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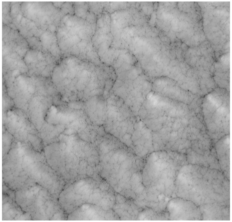
Multifractal Formalism

$X(t) \rightarrow d_X(a,t) \rightarrow L_X(a,t)$



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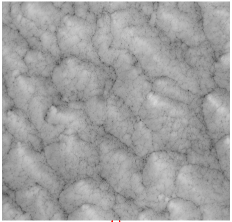

$\Downarrow$

$S(a,q) = \frac{1}{n_a} \sum_{k=1}^{n_a} L_X(a,k)^q$

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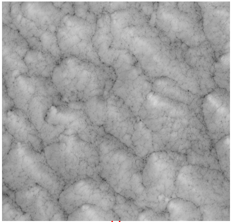

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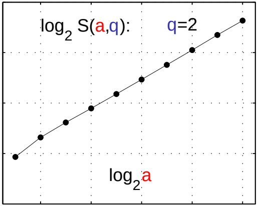

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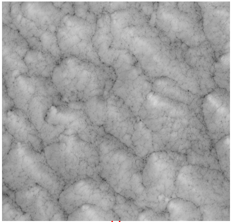

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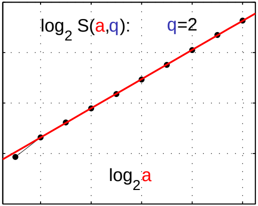

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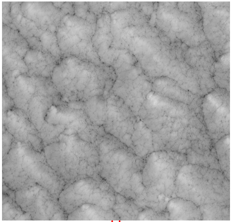

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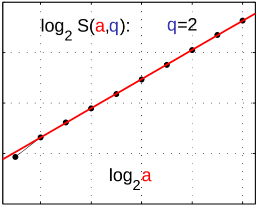
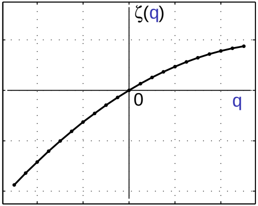


$\Longrightarrow$



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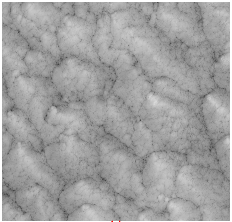

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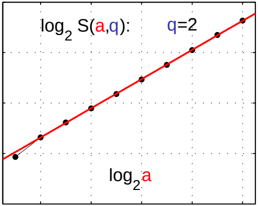

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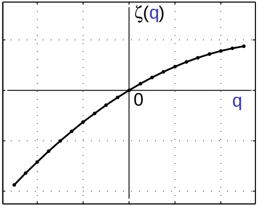

$\Uparrow$

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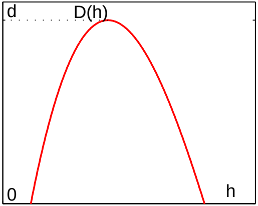

Estimation for multifractal attributes

1. $X(t) \to d_X(j,k) \to L_X(j,k)$

2. Structure functions and log-cumulants :

   $\log_2 S(j,q) \simeq G_q + \zeta(q) \log_2 2^j$
   $C(j,p) = c_p^0 + c_p \ln 2^j$

3. Weighted linear fits :

   $\hat{\zeta}(q) = \sum_{j=j_1}^{j_2} w_j \log_2 S(j,q)$
   $\hat{c}_p = (\log_2 e) \sum_{j=j_1}^{j_2} w_j \hat{C}(j,p)$

4. Multifractal Spectrum :

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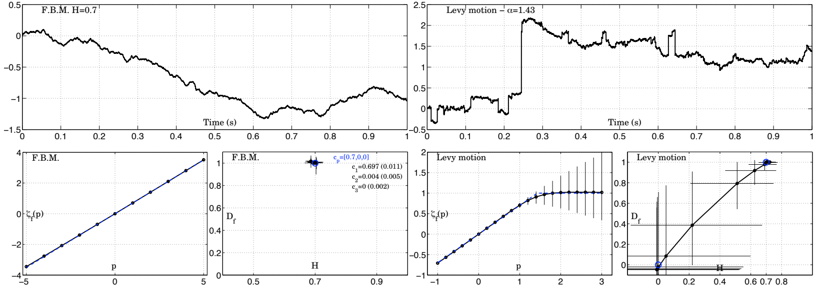
Multifractal formalism at work : 1D



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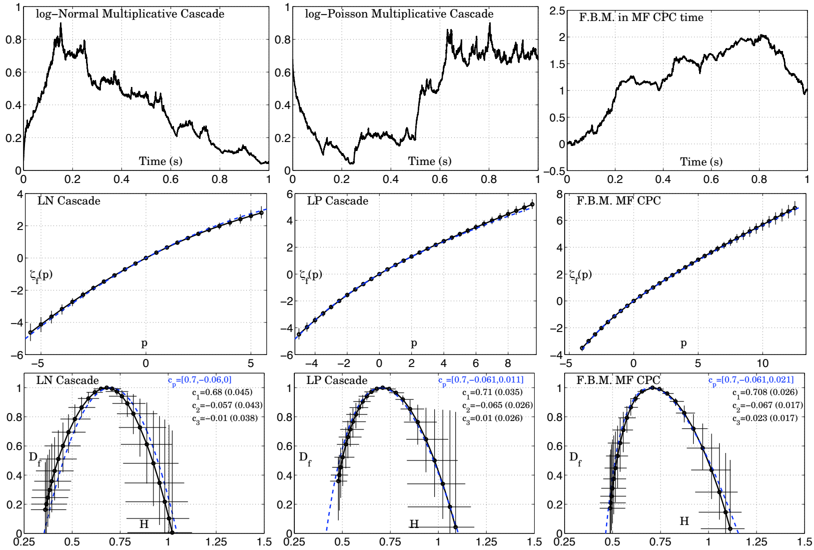
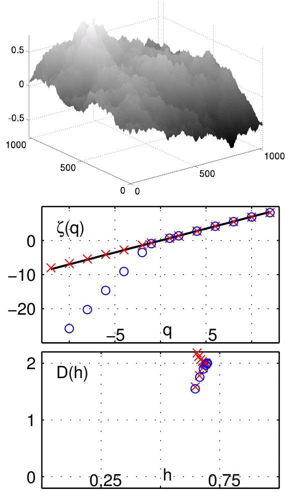
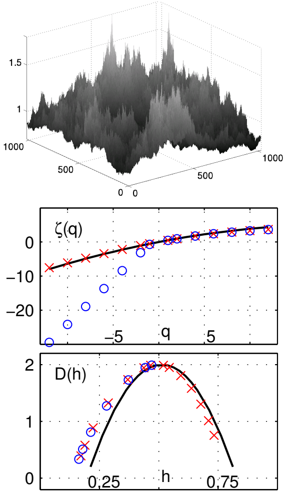


Multifractal formalism at work : 2D

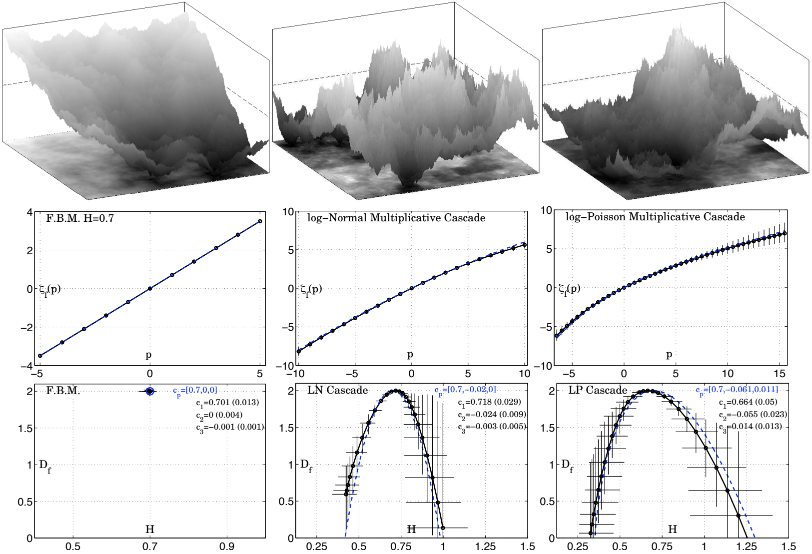
Fractional Brownian motion



Multiplicative cascade



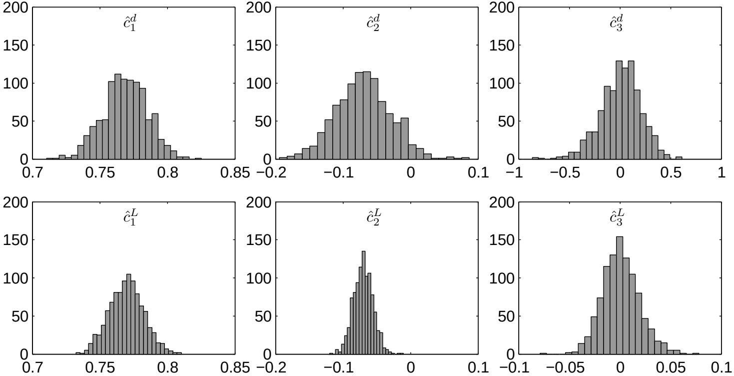
Multifractal formalism at work : 2D



Estimation performance

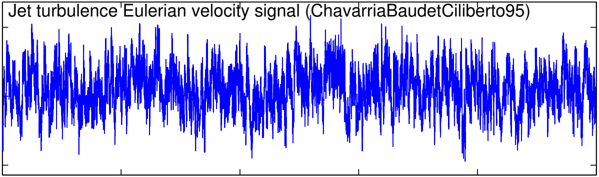
- Leader based estimates outperform wavelet based ones :
- mostly for $c_p, p \geq 2$ (multifractal properties),

$$\zeta(q) = c_1 q + c_2 q^2/2 + c_3 q^3/6 + \dots$$



Multifractal formalism at work : Hydrodynamic Turbulence

- Multifractal attributes estimation,
- from a single finite length observation



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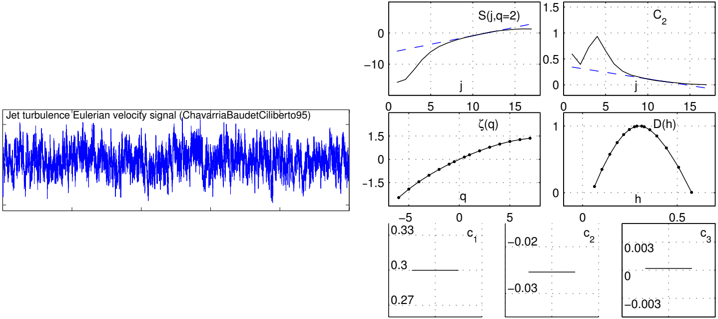

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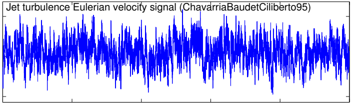
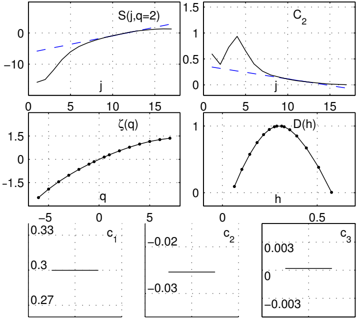
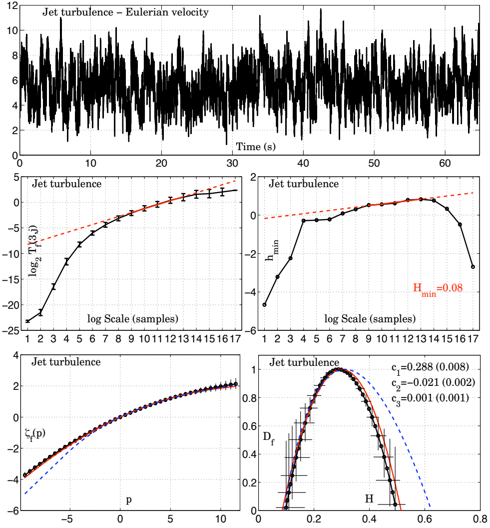




Hydrodynamic Turbulence



Multifractal analysis ?

- Distributions :
  - $h(a,t) = \frac{\ln L(j,k)}{\ln a}$, $a = 2^j$,
  - p_a distribution of $h(a,t)$,
  - Theory : What does p_a look like when $a \to 0$?
  - Practice : How does p_a changes from scale a to scale a' ?
- Moments :
  - $\mathbf{E} L(j,k)^q ? \equiv \mathbf{E} \exp(q \ln L(j,k)) = c_q a^{\zeta(q)}$,
- Cumulants or propagator :
  - $a' < a, p_{a'}(h) = p_a(h) * G'_{a,a'} h)$,
  - $a' < a, \tilde{p}_{a'}(q) = \tilde{p}_a(q) * \tilde{G}_{a,a'}(q)$,
  - $\tilde{G}_{a,a'}(q) = \exp(\tilde{F}(q)(\ln a - \ln a')$, $\tilde{F}(q) = \sum_{p \geq 1} c_p \frac{q^p}{p\,!}$
- Large deviations :
  - $D(h) = -\lim_{a \to 0} \frac{\ln p_a(h)}{\ln a}$,
  - $p_a(h) \sim_{a \to 0} \exp -D(h) \ln a$

Scaling Range Selection

- Multifractal analysis : $a \to 0$
- Application knowledge driven selection :
  - Hydrodynamic turbulence (Integral/injection scale : Geometry - Kolmogorov/Dissipation scale : fluid property)
  - Heart rate variability (LF/HF bands)
- Data driven selection :
  - (sampled) discrete time data
  - noise
  - scaling only exists in a given range

  $\Rightarrow$ Bootstrap-assisted data-driven automated scaling range selection

  $\Rightarrow$ R. Leonarduzzi's poster

Outline

Leaders and Positive Hölder

- Leader MF formalism applies if *X* has positive uniform regularity
  - ⇒ from finite data, Leaders can always be practically computed even if undefined
  - ⇒ uniform regularity needs to be tested a priori
- *X* has uniform Hölder regularity ▸ Unif. Holder
  - ⇒ *X* has positive Hölder exponent only !
- Minimal regularity computed a priori from wav. coefficients

$$h_{min} = \liminf_{2^j \to 0} \frac{\ln \sup_{m,\mathbf{k}} |d_X^{(m)}(j,\mathbf{k})|}{\ln 2^j}$$

$h_{min} > 0 \Rightarrow$ uniform Hölder regularity.

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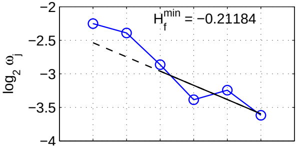

Positive Hölder and (fractional) Integration

- What if $h_{min} < 0$?
  No Multifractal formalism ?
  Issue both for images and signals !
- Solution $\Rightarrow$ (fractional) Integration :
  $(\widehat{I^{\gamma}X})(\xi) = (1 + |\xi|^2)^{\gamma/2}\hat{X}(\xi)$.
  if $\gamma > h_{min}$, $I^{\gamma}X$ is uniformly Hölder function
- Pseudo (fractional) Integration :
  $L_X^{\gamma}(j, k_1, k_2) = \sup_{m,\, \lambda' \subset 3\lambda_{j,k_1,k_2}} |d_X^{(m),\gamma}(\lambda')|$.
  MF spectrum from $L_X^{\gamma}(j, k_1, k_2)$ is identical to that of $(\widehat{I^{\gamma}X})(\xi)$
  vary γ with $\gamma > h_{min}$, Deduce $D(h) = D^{\gamma}(h - \gamma)$

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  $(\widehat{I^\gamma X})(\xi) = (1 + |\xi|^2)^{\gamma/2} \hat{X}(\xi)$.
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  integrate enough then apply wavelet Leader formalism.
  avoid to actually compute fractional integration.
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  compute $d_X^{(m)}(j, k_1, k_2)$ directly from X,
  $d_X^{(m),\gamma}(j, k_1, k_2) = 2^{\gamma j} d_X^{(m)}(j, k_1, k_2)$.
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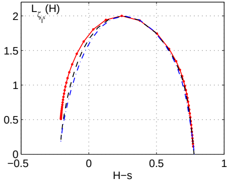

Open issue

- Pseudo (fractional) Integration...

  $d_X^{(m),\gamma}(j, k_1, k_2) = 2^{\gamma j} d_X^{(m)}(j, k_1, k_2).$

  $L_X^{\gamma}(j, k_1, k_2) = \sup_{m,\, \lambda' \subset 3\lambda_{j,k_1,k_2}} |d_X^{(m),\gamma}(\lambda')|.$

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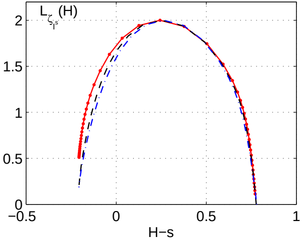

- ... is not always valid : theoretical issue

  chirp (or oscillating) singularities

  MF spectrum not well defined if both negative h and chirp !

Wavelet coefficients or Leaders ?

- Wavelet coefficients :

  Estimate global attributes : self-similarity, LRD, global regularity

  Compute h_{min}, Estimate c_1 and $\zeta(2)$,
  but can hardly see deviation from $\zeta(q) = qH$.

- Wavelet Leaders :

  Estimate multifractal attributes

  Estimate c_p, $p \geq 2$
  $\zeta(q)$ (for both positive and negative qs),
  can perfectly estimate concave $\zeta(q)$,

$\Rightarrow$ Coefficients and Leaders are to be used in a complementary way !

First, wavelet coef. ($h_{min} > 0$, c_1)
Second, Wavelet Leaders.

$\Rightarrow$ even if not truly interested in multifractal properties, local regularity and Hölder exponent, Leaders are needed to measure concave $\zeta(q)$

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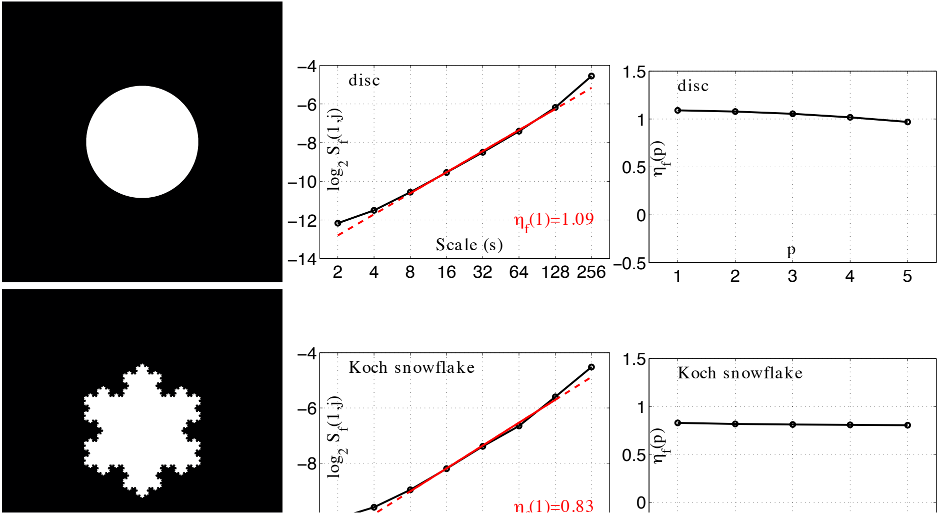
Wavelet scaling function

$$\eta(q) = \frac{1}{n_j} \sum_k |d_X(j,k)|^q \, ; \quad \dim_B(Set) = 2 - \zeta_d(1)$$



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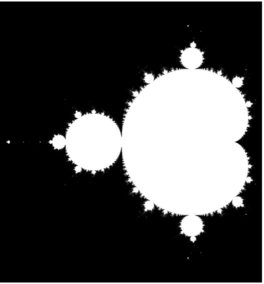

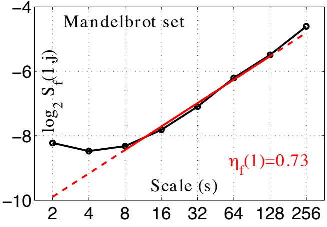

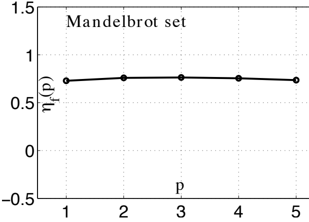

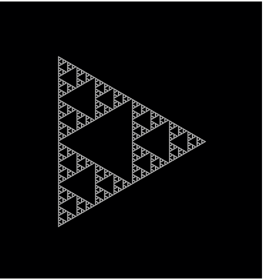

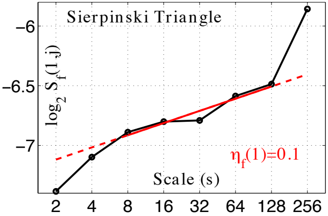

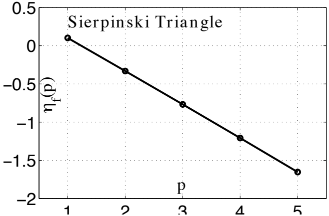

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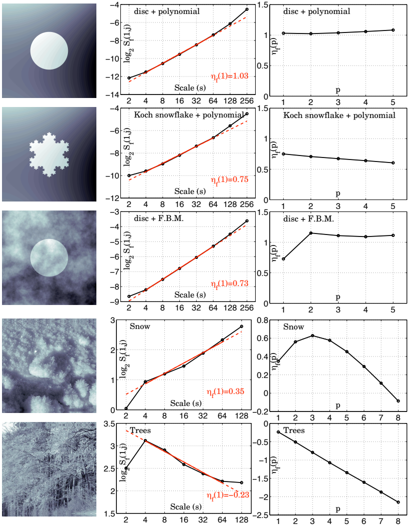
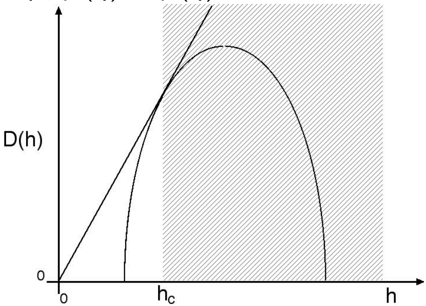


Leader scaling function

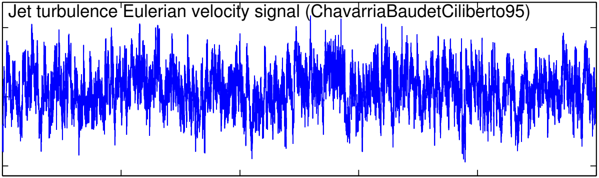
- Leader scaling function :
  - $\zeta(q) = \frac{1}{n_j} \sum_k L_X(j,k)^q$
  - $\dim_B(\text{Graph}X) = max(d, d+1-\zeta_L(1))$
  - if $\zeta_L(2) > d$, then X has bounded quadratic variation
- Leader versus wavelet scaling function :
  - if $q > 0$, $\zeta_d(q) \geq \zeta_L(q)$,
  - $\exists q_c$ such that $q \geq q_c, \zeta_d(q) = \zeta_L(q)$



Outline

Estimation from a single observation

- Multifractal attributes estimation :



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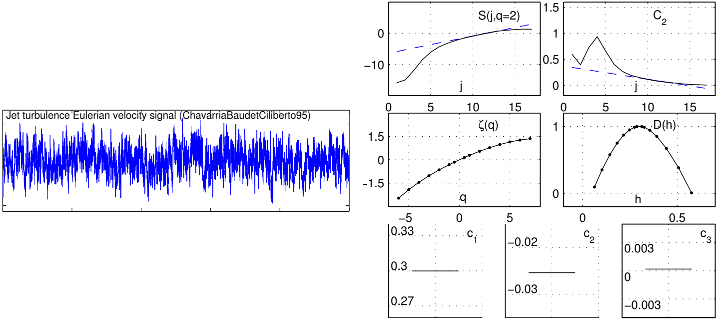

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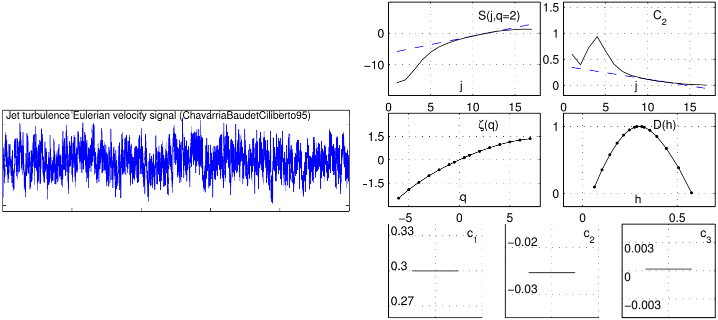

- Confidence intervals, hypothesis tests, . . . ?

Confidence intervals for multifractal estimates ?

- Issues :
  confidence intervals for multifractal attribute estimates ?
  from a single observation,
  from a finite length observation.
- State of the art ?
  no theoretical results so far,
  Gaussian asymptotics perform poorly,
  almost totally overlooked issue.
- Why ?
  multifractal processes : a large class
  amongst which, multiplicative cascades,
  often with heavy tail marginals,
  and intricate (power law type) dependence structures,
- Our proposition : Non parametric bootstrap
  Drawing with replacement procedure

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Bootstrap for Multifractal analysis

Bootstrap Principle

Bootstrap for Multifractal analysis

▸ Bootstrap Principle

▸

Bootstrap for multifractal analysis

- Multiplicative multifractal Processes :

  intricate, power law, dependencies
  non stationary,
  heavy tails.

  $\Rightarrow$ no bootstrap in the time domain

- Wavelet domain :

  decorrelation properties ? as for fBm ?
  time-scale residual dependency

$\Rightarrow$ **Time-scale block bootstrap in the wavelet domain !**

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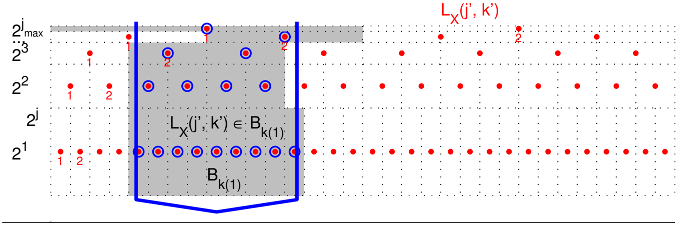
$\Rightarrow$ **Time-scale block bootstrap in the wavelet domain !**

Time-scale block bootstrap

- Time-scale block (1D) :

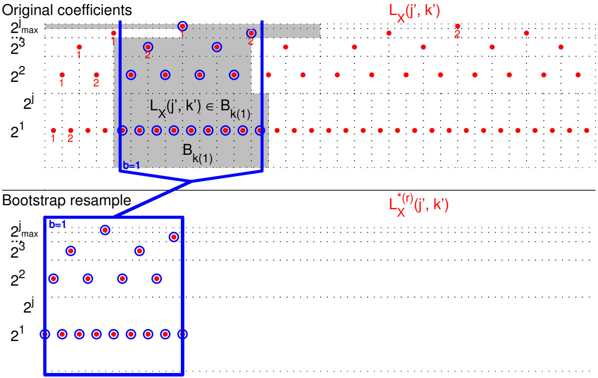
  timescale strip
  of time length $2l$
  through all scales

$$\mathcal{B}_k = \{L_X(j', k') : |k - k'2^{j'}| \leq l, 1 \leq j' \leq j_{max}\}, 1 \leq k \leq n$$



Time-scale block bootstrap

- From the $\{L(j,k)\}$:
  draw with replacement, $B = \lceil \frac{N}{2l} \rceil$ blocks $\mathcal{B}_k$:
  chain them, $\rightarrow \{L^{*(r)}(j,k)\}$



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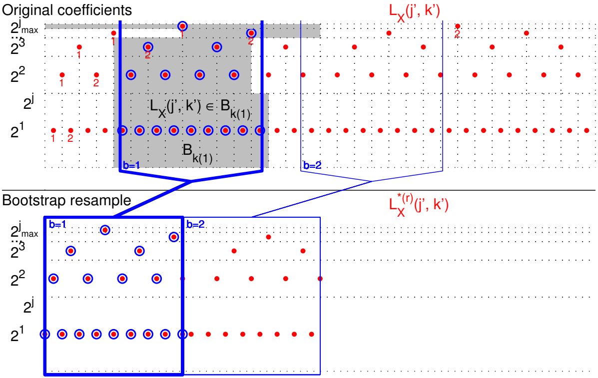

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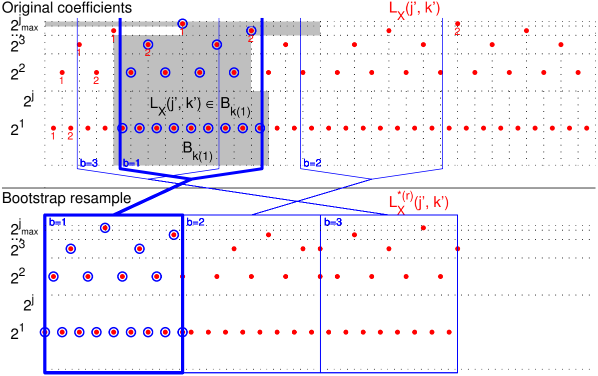

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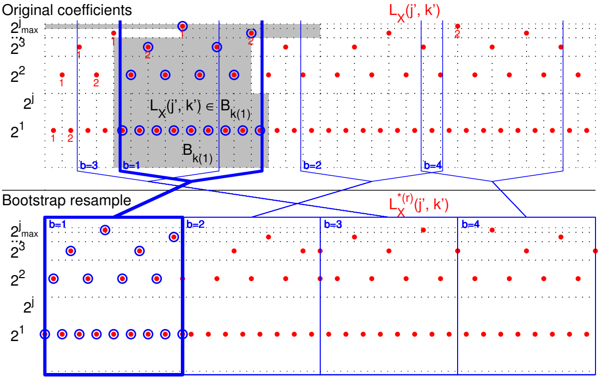


Wavelet leaders time-scale blok bootstrap

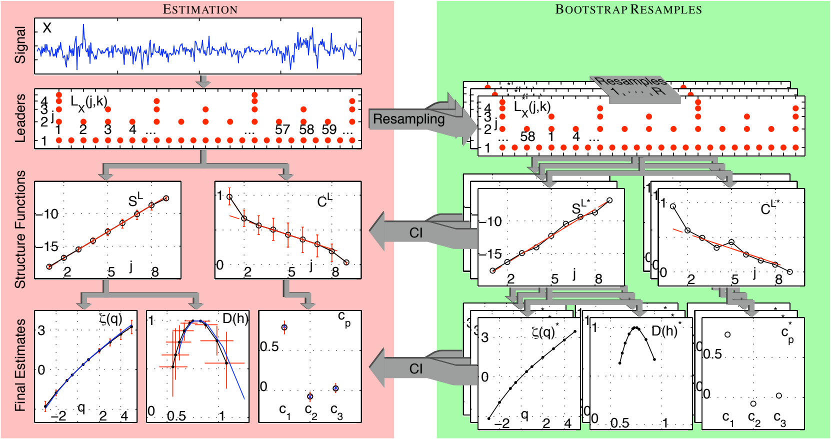
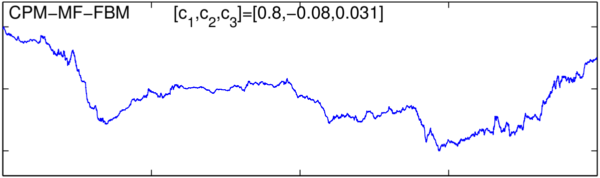


Illustration : 1D example

- fBm in multifractal time :



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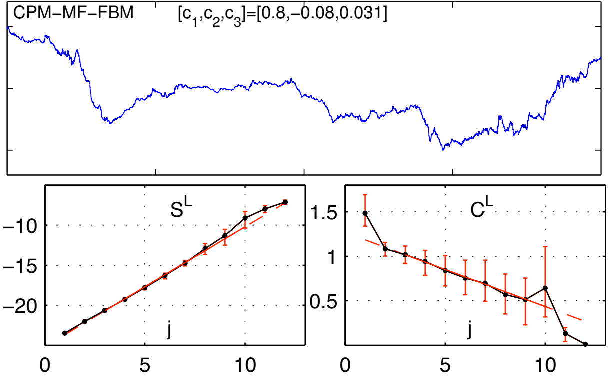

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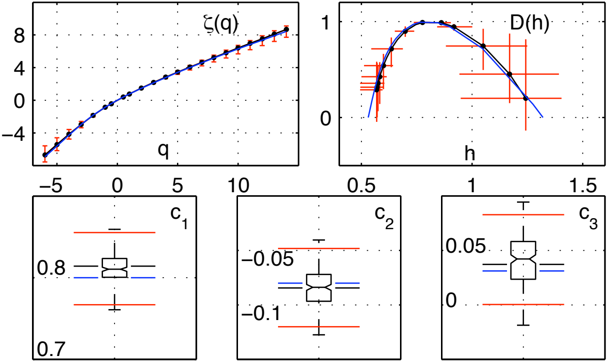
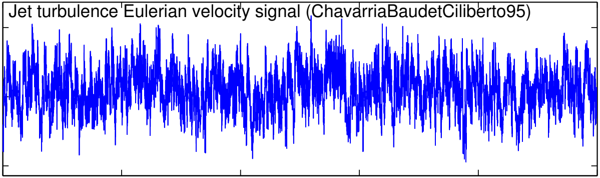


Turbulence

- Estimation and confidence intervals :



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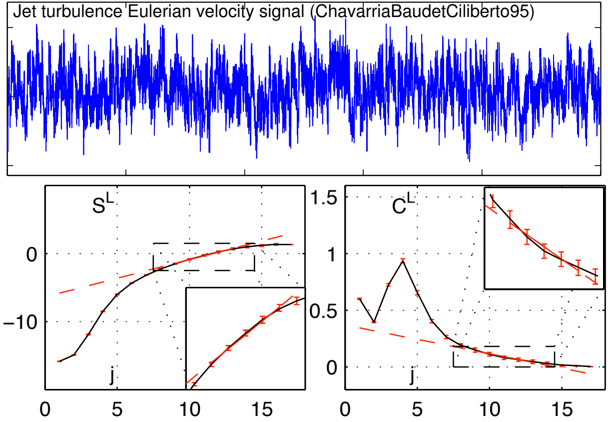

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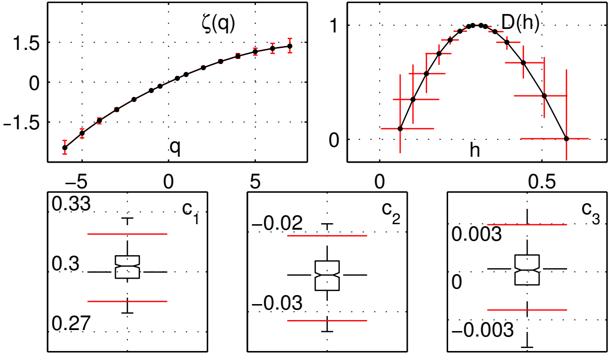

Illustration : 2D Example

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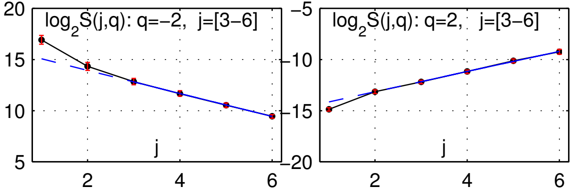

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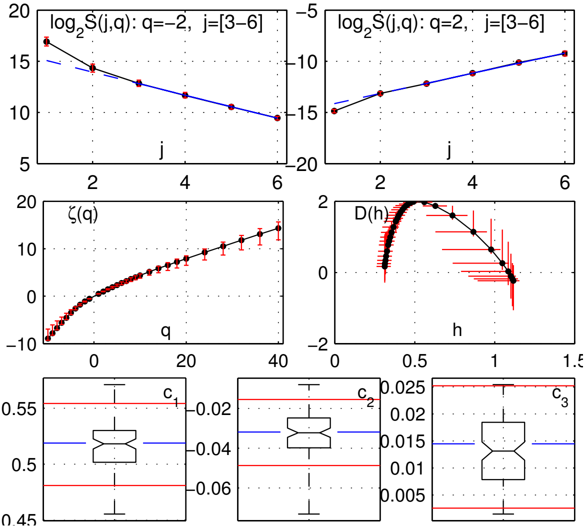

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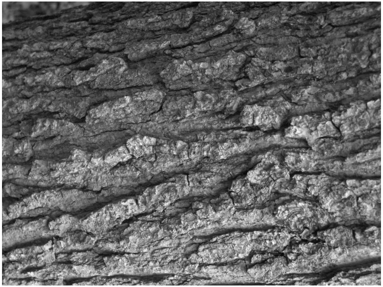

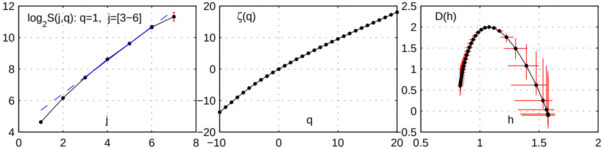

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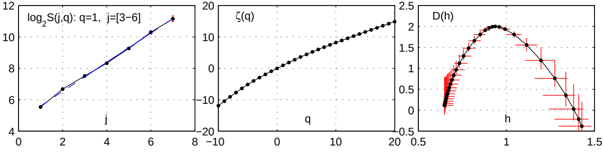

Hypothesis tests

- Multifractality tests :

  test for a prescribed value of a given multifractal attribute
  example 1 : $c_1 = H > 0.5$? Long range dependence ?
  example 2 : $c_2 = 0$ or $c_2 \neq 0$? test for multifractal ?

  $$\zeta(q) = c_1 q + c_2 q^2/2 + \ldots$$

  $\zeta(q)$ linear or not ? $\Rightarrow$ practical test for MF

- Stationarity-type test :

  are the estimated MF attributes constant along time ?

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Open Issues

- Bootstrap procedures validated by numerical simulations.
- Why does Bootstrap theoretically work ?
- What are the actual dependence structure of multiresolution quantities for multifractal processes ?
- How does the number of vanishing moments help ?

⇒ Under current investigations !

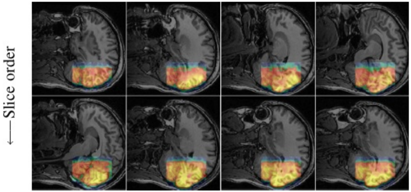
⇒ Preliminary results :

Dependence structure is long range (power-law) slowly decreasing type !
Vanishing moments help much less : decorrelation but no independence !

Outline

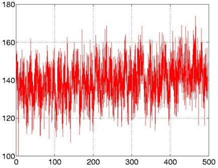
fMRI : Active versus Resting States

- fMRI data - task-related activation of brain regions

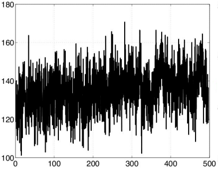


collaboration P. Ciuciu, CEA/NeuroSpin

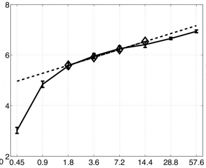
time series [active]



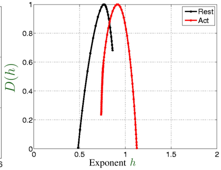
[resting]



scale invariance



multifractal spectra



Texture Classification

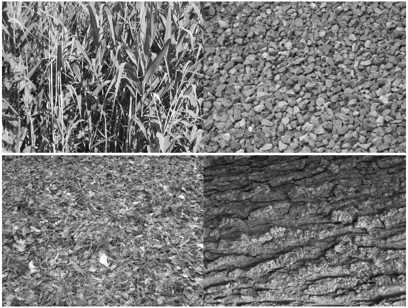
Collaboration J. Hui and S. Zuowei, NUS, Singapore

- texture image database :

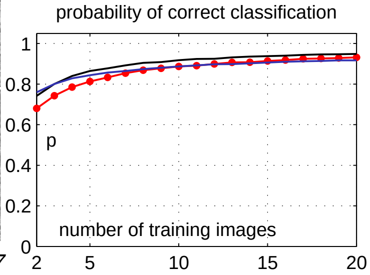
  40 classes × 50 images – size 1280 × 960

  rotation, illumination, angle, viewpoint invariance

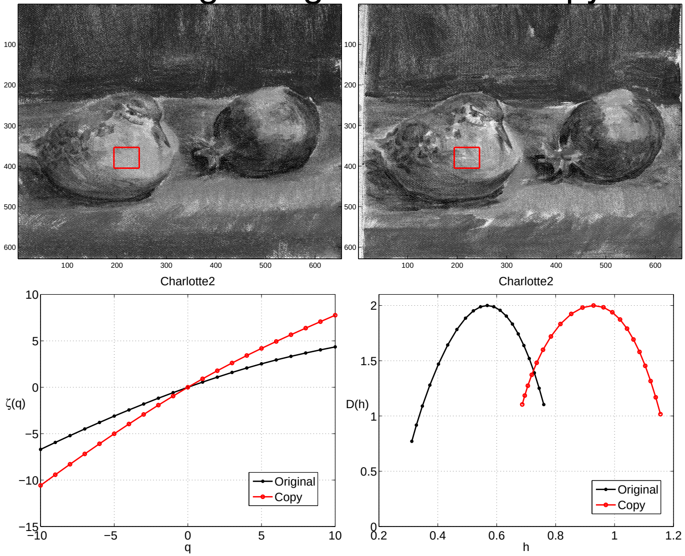
- accurate, robust, meaningful, efficient classification



U Maryland high-resolutions texture database, 2007



Painting : Original versus Copy



Outline

Conclusions

- Scaling :
  no characteristic scales
  all scales are characteristic
  inter-scale relations
  power laws
- Modeling :
  2nd order stationary processes,
  Long range dependence,
  self-similarity, random walks, addition
  multifractality, cascades, multiplication
- Analysis :
  aggregation, increments ⇒ Multiresolution
  wavelet coefficients ($N_\psi > h$, $N_\psi > H$)
  wavelet Leaders
- Misc. :
  Bootstrap,
  1D (signal), 2(images), nD

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More information

- WEB Site :
  patrice.abry@ens-lyon.fr
  ens-lyon.fr/PHYSIQUE/Signal
  perso.ens-lyon.fr/patrice.abry
- References (articles, talks)
- MATLAB Toolboxes (Wavelets, scaling and fractal)

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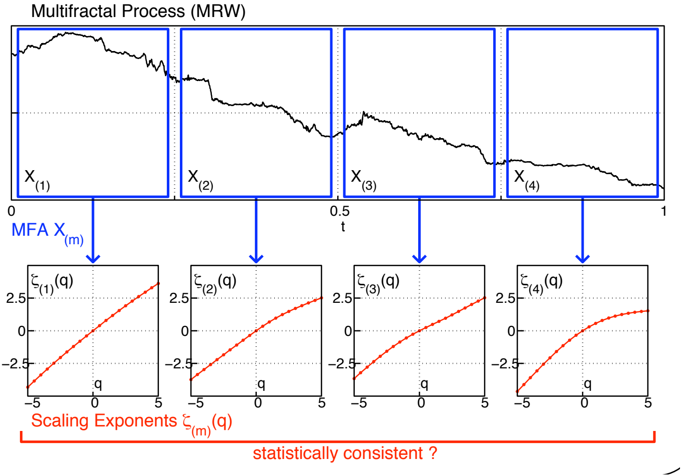
Thanks to (co-authors : 1992-2010)

- P. Flandrin, P. Gonçalvès, SISYPHE, ENS Lyon, France,
- D. Veitch, N. Hohn, Univ. of Melbourne, Australia,
- L. Delbeke, Leuwen Univ., Belgium,
- M. Taqqu, V. Pipiras, Boston Univ., USA,
- P. Borgnat, S. Roux, B. Vedel, ENS Lyon,
- P. Chainais, B. Lashermes, H. Wendt, ENS Lyon,
- S. Jaffard, LAMA, Paris, France,
- R. Riedi, Rice University, USA,
- M.-E. Torres, R. Leonarduzzi, Univ. de Entre-Rios, Argentina,
- Ph. Ciuciu (NeuroSpin, France), M. Doret, E. Pereira (Univ. Hospital, Lyon, France)

Multifractal attribute time constancy test

- Issue ?



Multifractal attribute time constancy test

- Step1 : Split then estimate/bootstrap

  Split data into M adjacent non overlapping blocks

  M estimations $\hat{\theta}_{(m)} \to H_{\text{null}}: \quad \theta_{(1)} = \theta_{(2)} = \cdots = \theta_{(M)}$

$$T_\theta = \sum_{m=1}^{M} \frac{1}{\hat{\sigma}^{2*}_{(m)}} \left(\hat{\theta}_{(m)} - \frac{\sum_{n=1}^{M} \frac{\hat{\theta}_{(m)}}{\hat{\sigma}^{2*}_{(m)}}}{\sum_{n=1}^{M} \frac{1}{\hat{\sigma}^{2*}_{(m)}}} \right)^2$$

- Step2 : Bootstrap then split/estimate

$$T^*_\theta = \sum_{m=1}^{M} \frac{1}{\hat{\sigma}^{2**}_{(m)}} \left(\hat{\theta}^*_{(m)} - \frac{\sum_{n=1}^{M} \frac{\hat{\theta}^*_{(m)}}{\hat{\sigma}^{2**}_{(m)}}}{\sum_{n=1}^{M} \frac{1}{\hat{\sigma}^{2**}_{(m)}}} \right)^2$$

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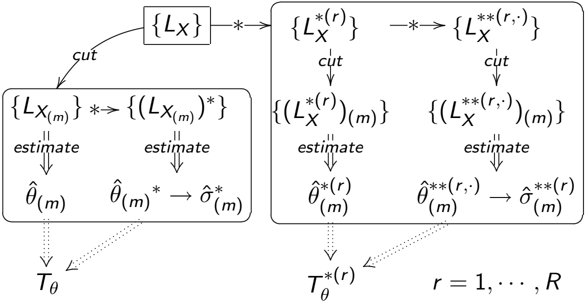

- Test :

  $T^*_{\theta,C}$ - quantile $(1-\alpha)$ of empirical distribution $\{T^*_\theta\}$
  $d_\theta = 1$ if $T_\theta > T^*_{\theta,C}$ and 0 otherwise

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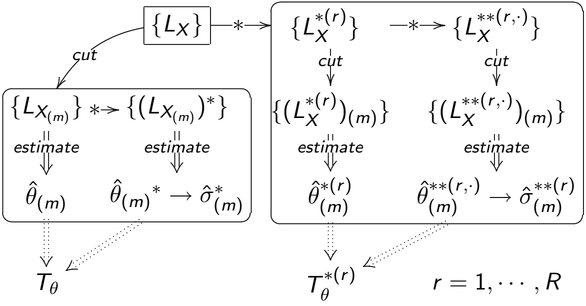

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- Numerical simulation : concatenate two MF processes

- Under H_{null} (MRW) : **Significance level**, $\alpha = 0.1$

$\{c_1, c_2\}$	$\{0.75, -0.01\}$		$\{0.8, -0.02\}$	
θ	c_1	c_2	c_1	c_2
$\bar{d}_\theta^{H_0}$	0.113	0.143	0.075	0.139
$\bar{p}_\theta^{H_0}$	0.478	0.469	0.530	0.485

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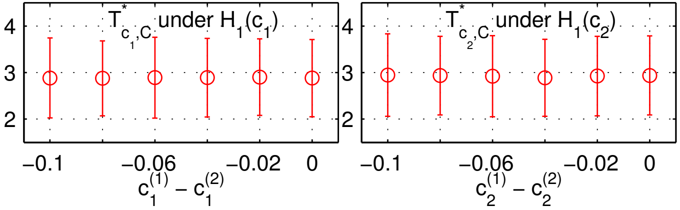
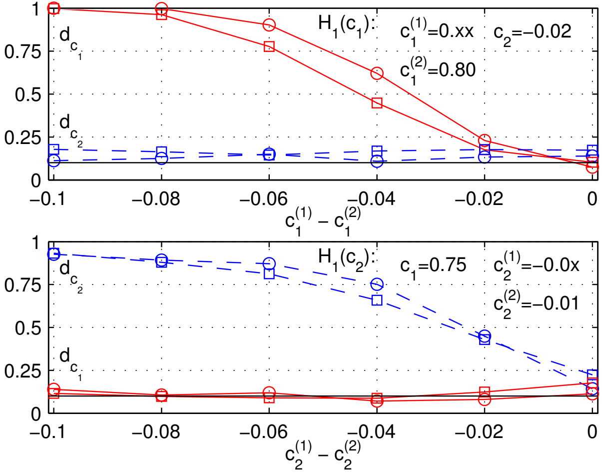


Multifractal attribute time constancy test

- Under H_A (MRW) : **Power** c_2 c_1



2D Discrete Wavelet Transform

- 2D Orthonormal basis

$$
\begin{array}{rrcl}
 & \phi_0(t) & & \text{scaling function (father wavelet)} \\
 & \psi_0(t) & & \text{(mother) wavelet} \\
\text{Dyadic sampling} & \phi_{j,k}(t) & = & 2^{-j}\phi_0(2^{-j}t-k) \\
 & \psi_{j,k}(t) & = & 2^{-j}\psi_0(2^{-j}t-k) \\
 & & & \\
\text{base} & \tilde{\psi}^{(1)}_{j,k_1,k_2}(x,y) & = & \phi_{j,k_1}(x)\psi_{j,k_2}(y) \\
 & \tilde{\psi}^{(2)}_{j,k_1,k_2}(x,y) & = & \psi_{j,k_1}(x)\phi_{j,k_2}(y) \\
 & \tilde{\psi}^{(3)}_{j,k_1,k_2}(x,y) & = & \psi_{j,k_1}(x)\psi_{j,k_2}(y)
\end{array}
$$

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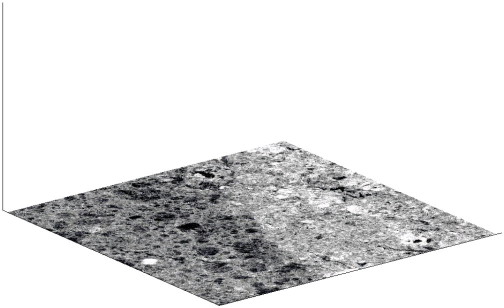
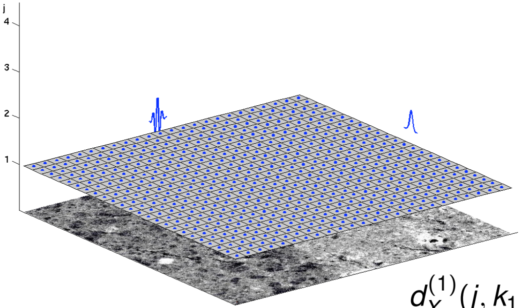


Image : $X(\cdot,\cdot)$

2D Discrete Wavelet Transform



wavelet coefficients :

$$m = 1,\ j = 1$$

$$\begin{aligned} d_X^{(1)}(j,k_1,k_2) &= \langle\, X\,,\ \tilde{\psi}^{(1)}_{j,k_1,k_2}(x,y)\,\rangle \\ \tilde{\psi}^{(1)}_{j,k_1,k_2}(x,y) &= \phi_{j,k_1}(x)\psi_{j,k_2}(y) \end{aligned}$$

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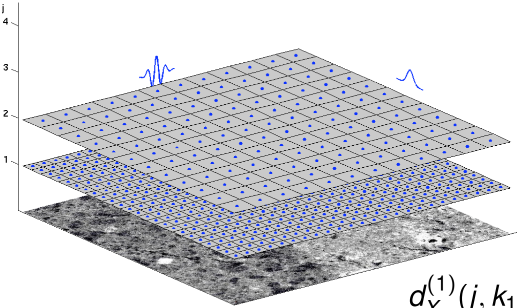
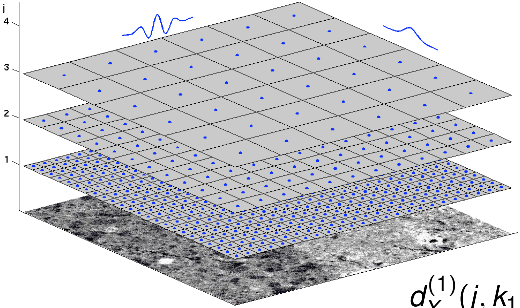


wavelet coefficients :

$m = 1, j = 2$

$$d_X^{(1)}(j, k_1, k_2) = \langle X , \tilde{\psi}^{(1)}_{j,k_1,k_2}(x, y) \rangle$$
$$\tilde{\psi}^{(1)}_{j,k_1,k_2}(x, y) = \phi_{j,k_1}(x)\psi_{j,k_2}(y)$$

2D Discrete Wavelet Transform

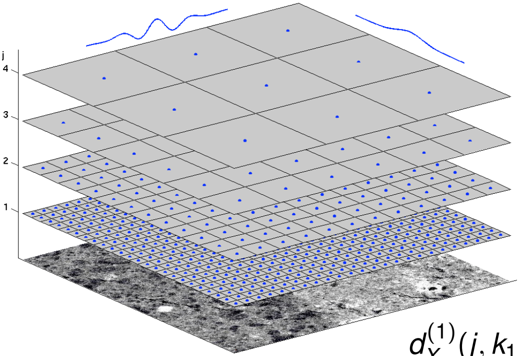


wavelet coefficients :

$$m = 1,\ j = 3$$

$$d_X^{(1)}(j, k_1, k_2) = \langle X , \tilde{\psi}^{(1)}_{j,k_1,k_2}(x, y) \rangle$$
$$\tilde{\psi}^{(1)}_{j,k_1,k_2}(x, y) = \phi_{j,k_1}(x)\psi_{j,k_2}(y)$$

2D Discrete Wavelet Transform

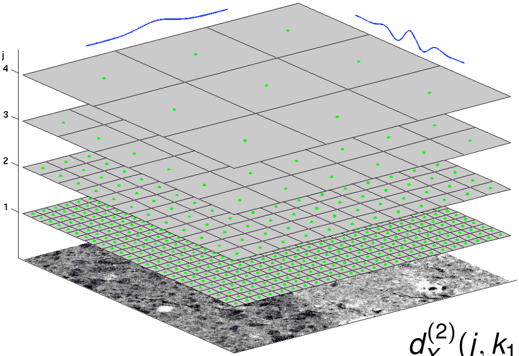


wavelet coefficients :

$$m = 1,\ j = 4$$

$$d_X^{(1)}(j, k_1, k_2) = \langle X, \tilde{\psi}^{(1)}_{j,k_1,k_2}(x, y) \rangle$$
$$\tilde{\psi}^{(1)}_{j,k_1,k_2}(x, y) = \phi_{j,k_1}(x)\psi_{j,k_2}(y)$$

2D Discrete Wavelet Transform

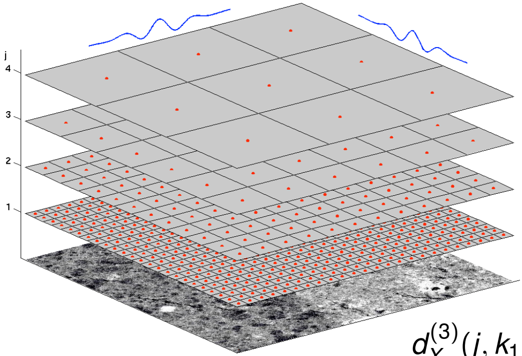


wavelet coefficients :

$m = 2$

$$d_X^{(2)}(j,k_1,k_2) = \langle X \,,\, \tilde{\psi}^{(2)}_{j,k_1,k_2}(x,y) \rangle$$
$$\tilde{\psi}^{(2)}_{j,k_1,k_2}(x,y) = \psi_{j,k_1}(x)\phi_{j,k_2}(y)$$

2D Discrete Wavelet Transform



wavelet coefficients :

$m = 3$

$$d_X^{(3)}(j, k_1, k_2) = \langle X, \tilde{\psi}^{(3)}_{j,k_1,k_2}(x,y) \rangle$$
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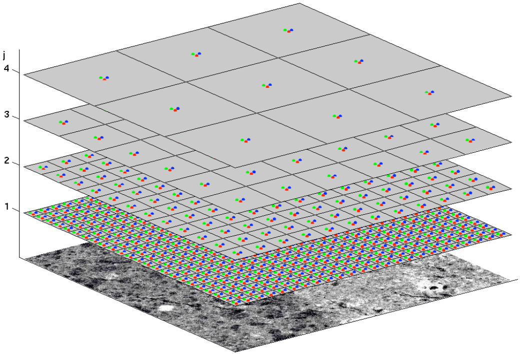
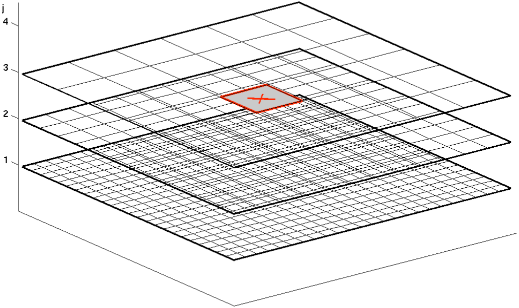


wavelet coefficients :

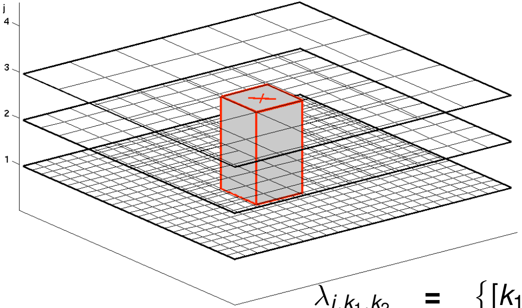
$$\{d_X^{(m)}(j,\cdot,\cdot)\},\ m = 1,2,3$$

2D wavelet Leaders



$j = 3, \quad k_1 = k_2 = 3$

2D wavelet Leaders



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Dyadic cubes

$$\lambda_{j,k_1,k_2} = \{[k_1 2^j, (k_1+1)2^j), [k_2 2^j, (k_2+1)2^j)\}$$

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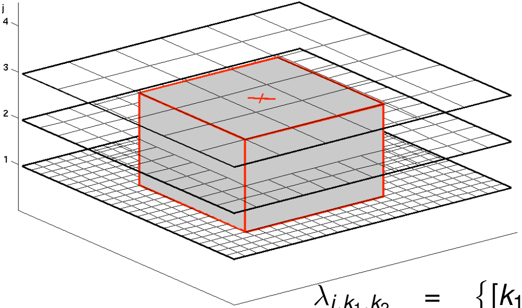

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union of 9 *such cubes*

$$\lambda_{j,k_1,k_2} = \{[k_1 2^j, (k_1+1)2^j), [k_2 2^j, (k_2+1)2^j)\}$$
$$3\lambda_{j,k_1,k_2} = \bigcup_{m,n=\{-1,0,1\}} \lambda_{j,k_1+m,k_2+n}$$

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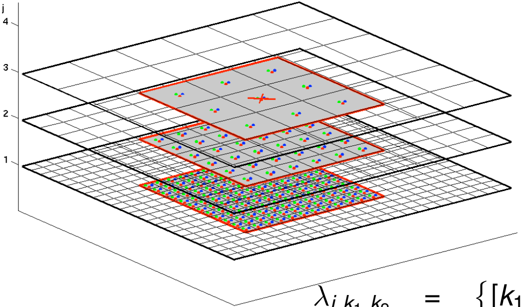

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supremum over coefficients

$$
\begin{aligned}
\lambda_{j,k_1,k_2} &= \{[k_1 2^j, (k_1+1)2^j), [k_2 2^j, (k_2+1)2^j)\} \\
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\end{aligned}
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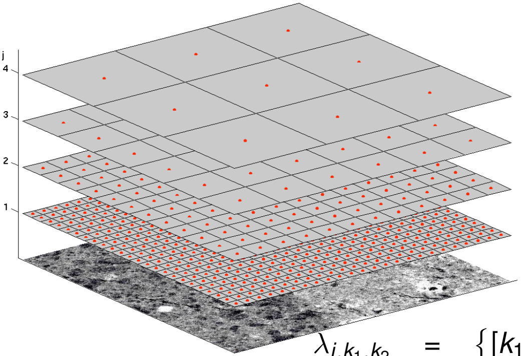
2D wavelet Leaders



$$\{d_X^{(m)}(\cdot,\cdot,\cdot)\} \rightarrow \{L_X(\cdot,\cdot,\cdot)\}$$

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Log-Cumulants

- For certain classes of processes :
  - $\mathbf{E}L_X(j,\cdot)^q = F_q|2^j|^{\zeta(q)}$
- 2^{nd} characteristic function $\ln L_X(j,\cdot)$:
  - $\ln \mathbf{E}e^{q \ln L_X(j,\cdot)} = \sum_p C_p^j \frac{q^p}{p!} = \ln F_q + \zeta(q) \ln 2^j$

    C_p^j : cumulant of order $p \geq 1$ de $\ln L_X(j,\cdot)$
- $\Rightarrow \forall p \geq 1 : \quad C_p^j = c_p^0 + c_p \ln 2^j$
  - $$\ln \mathbf{E}e^{q \ln L_X(j,\cdot)} = \underbrace{\sum_{p=1}^{\infty} c_p^0 \frac{q^p}{p!}}_{\ln F_q} + \underbrace{\sum_{p=1}^{\infty} c_p \frac{q^p}{p!}}_{\zeta(q)} \ln 2^j,$$
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Uniform Hölder regularity

- Uniform Hölder : X is a uniform Hölder function iff

  $\exists \epsilon > 0$ such that $X \in C^{\epsilon}(\mathcal{R}^d)$,

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- Wavelet Coefficients and uniform Hölder function :

  $h > 0$, if X is $C^{h(t_0)}$, then $\exists C > 0$ such that :
  $\forall j \geq 0, \quad |d_X(j,k)| \leq C 2^{jh(t_0)}(1 + |2^{-j}t_0 - k|)^{h(t_0)}$.

- Conversely, if relation above holds and if X is uniform Hölder,

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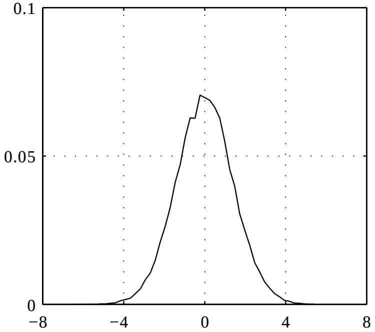
Limitation 1 : Negative *q*s

- Wavelet Coefficients $\Rightarrow d_X(j,k) \simeq 0$,



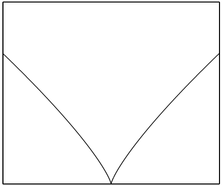
- Structure Functions are numerically instable,

$\Rightarrow$ Decreasing part of $D(h)$ cannot be measured !

Limitation 2 : Oscillating Singularities

- Cusp Singularity : $|X(t) - X(t_0)| \sim_{|t-t_0| \to 0} |t - t_0|^h$
- Chirp (Oscillating) Singularity :

  $|X(t) - X(t_0)| \sim_{|t-t_0| \to 0} |t - t_0|^h \sin(\frac{1}{|t-t_0|^\beta})$



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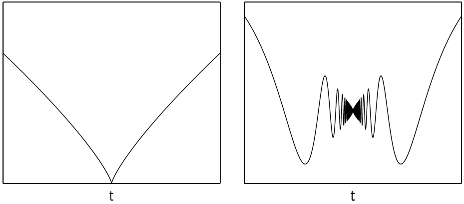

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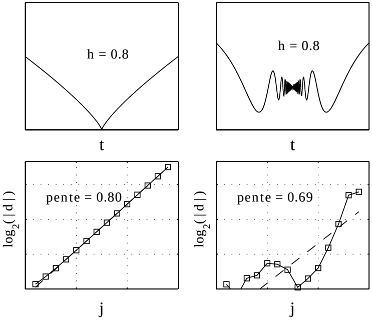

Bootstrap : Principles

- Issues :

  Data $X = \{x_1, \cdots, x_N\}$,
  $x_i \overset{i.i.d.}{\sim} P_X(x)$, unknown !
  $\hat{\theta} = \theta(X)$

  Statistical performance of $\hat{\theta}$? pdf of $\hat{\theta}$?

- Non parametric Bootstrap :

  Drawing with replacement procedure
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  $\rightarrow R$ copies $X^{*(r)}$,
  $\rightarrow R$ estimates $\hat{\theta}^{*(r)} = \theta(X^{*(r)})$,
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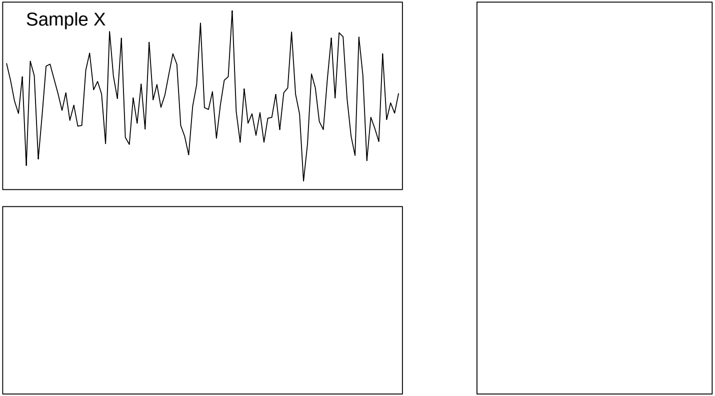
Illustration

Example : $x_i \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, $\theta(X) = \mathrm{Var}\,(X)$



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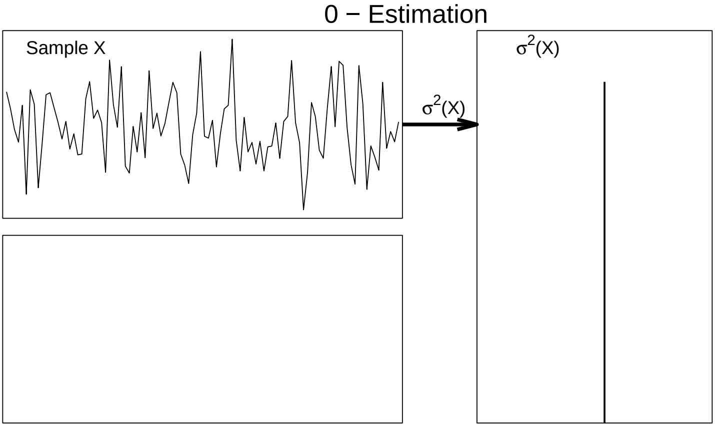

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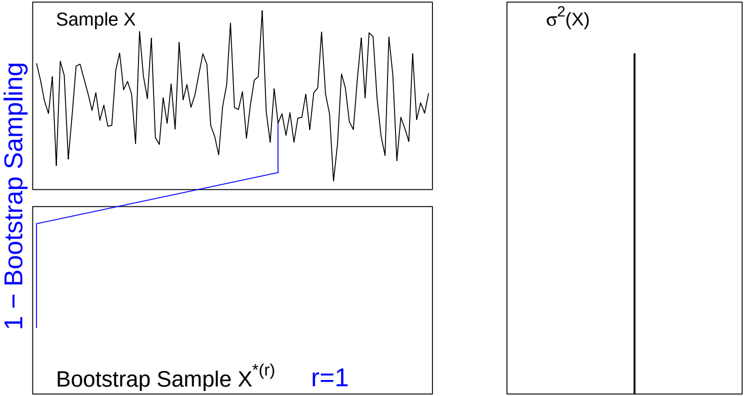

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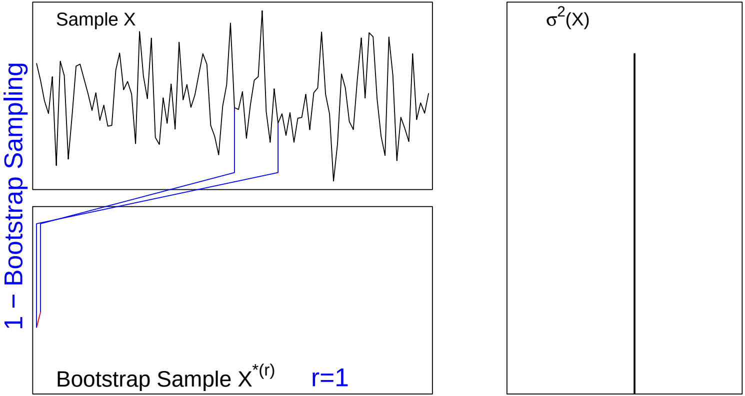

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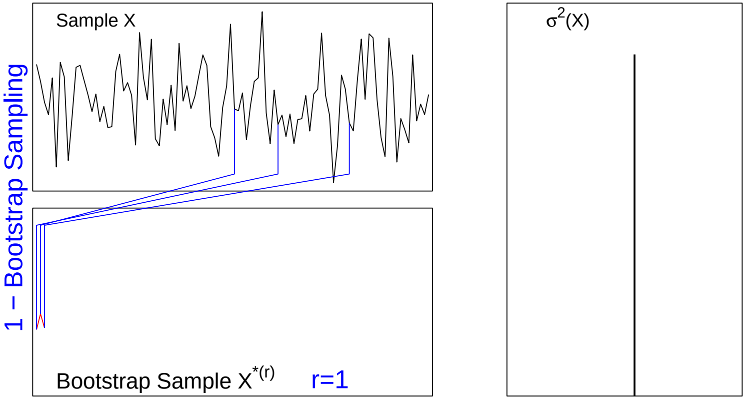

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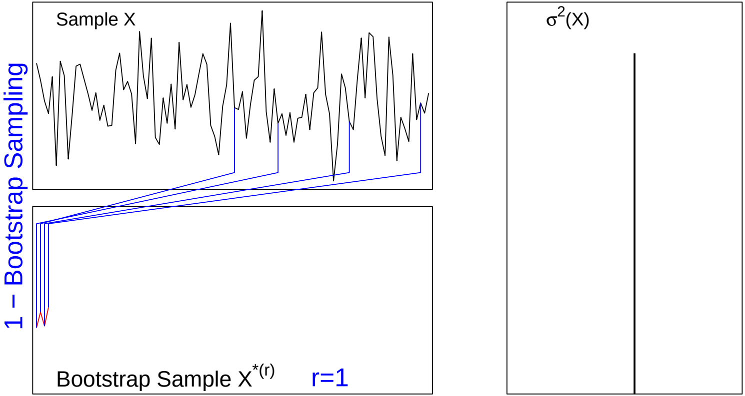

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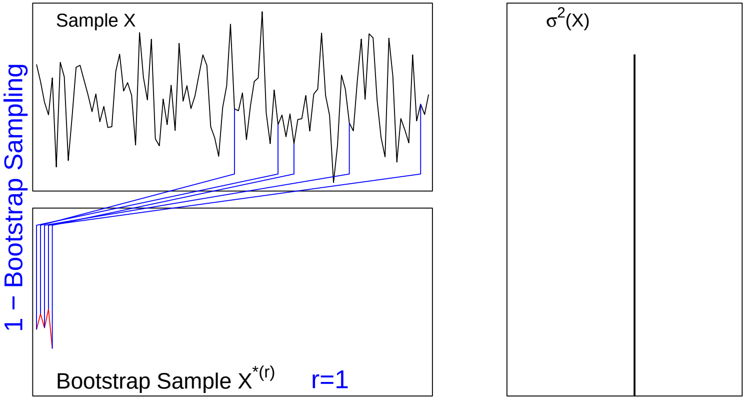

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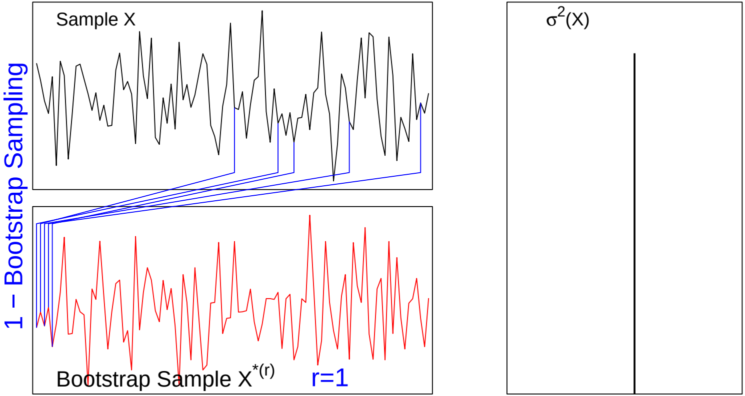

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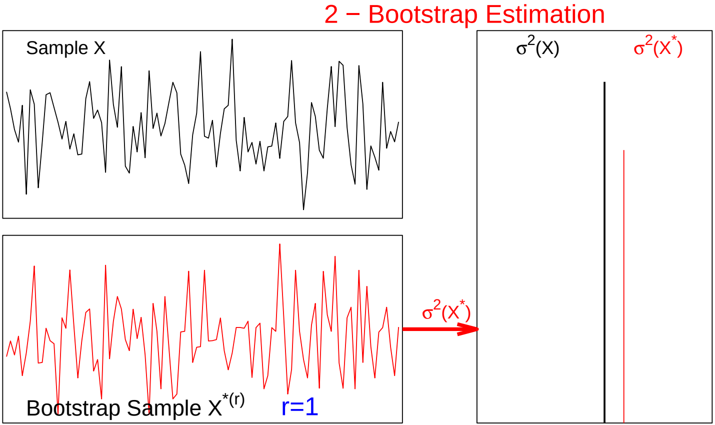

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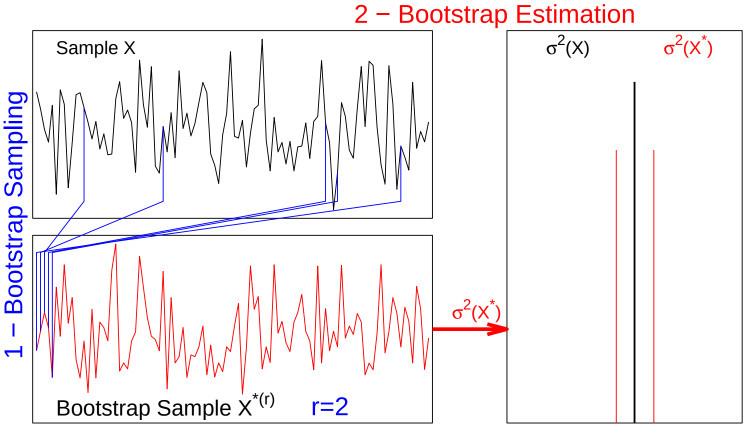

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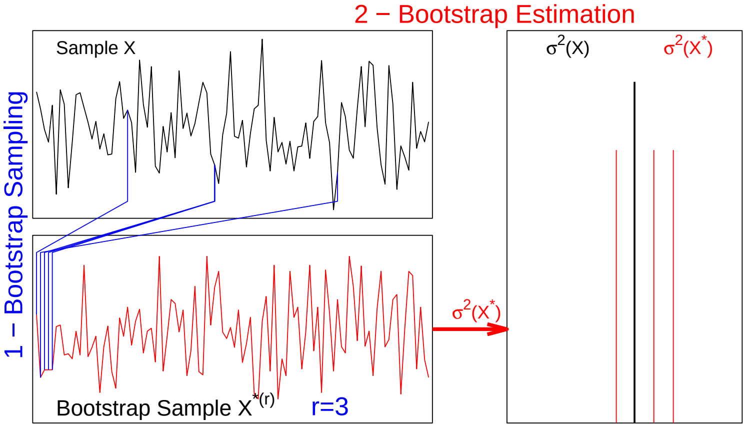

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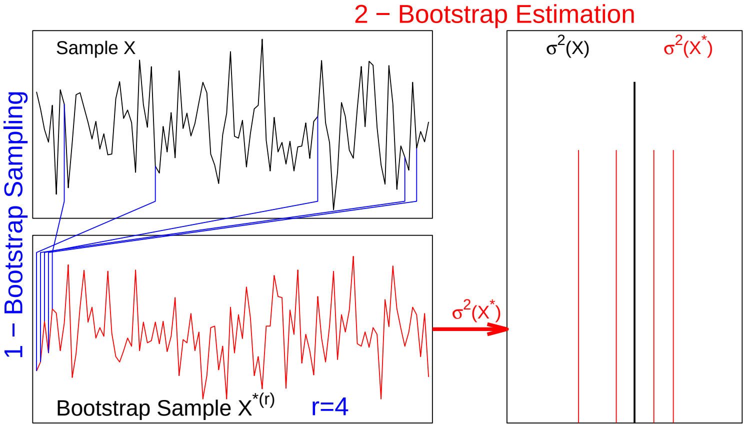

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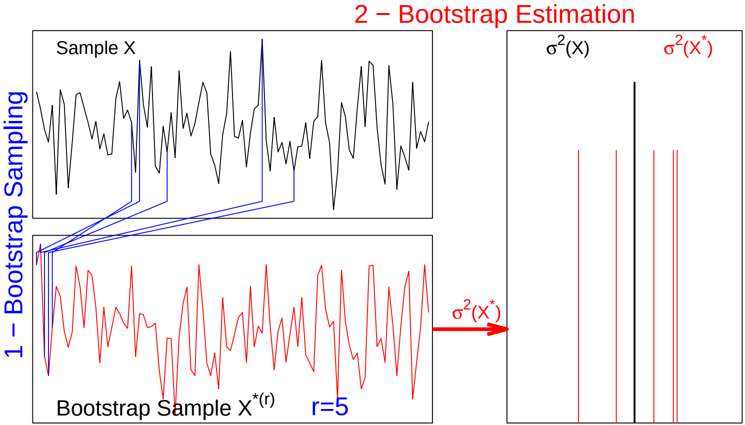

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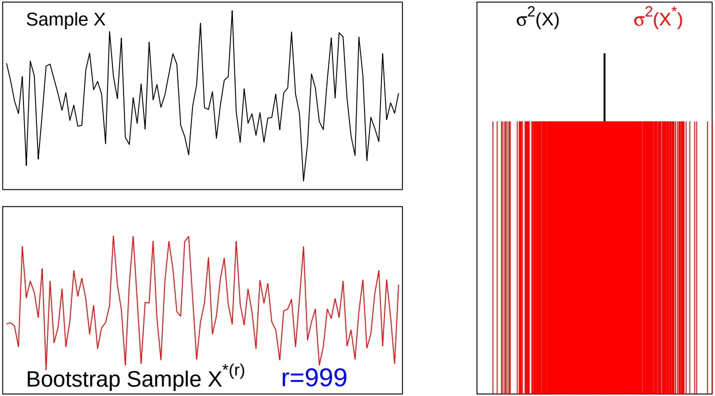

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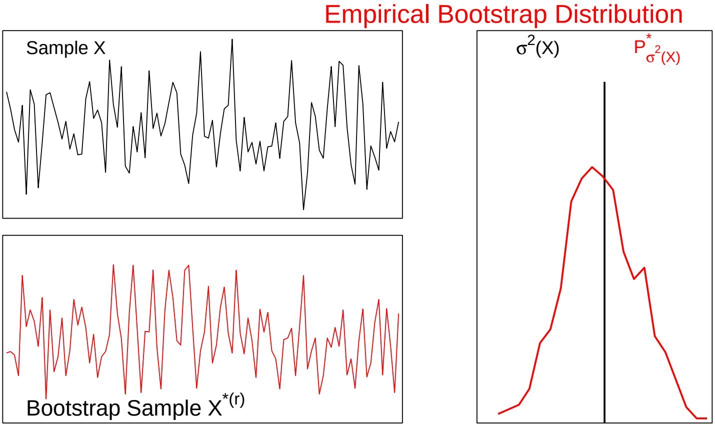

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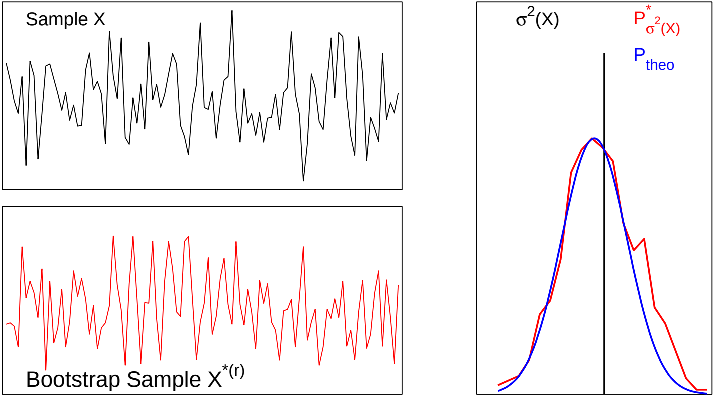
Empirical Bootstrap Distribution



Block Bootstrap

- Dependent data :

  $X = \{x_1, \cdots, x_N\}$, with $P_X(x)$, unknown !
  $\hat{\theta} = \theta(X)$, Statistical performance of $\hat{\theta}$?

- Block Bootstrap : Drawing with replacement procedure,

  Drawing with replacement procedure,

  over blocks of data $\{x_i, \cdots, x_{i+l-1}\}$

- Intuition : preserve dependence structure

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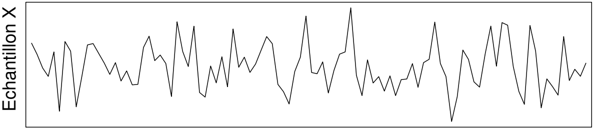

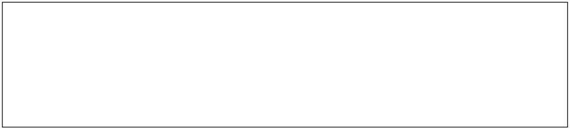

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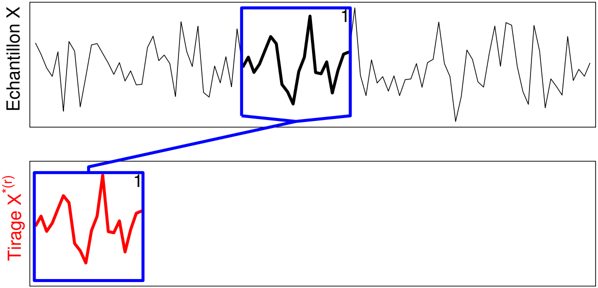

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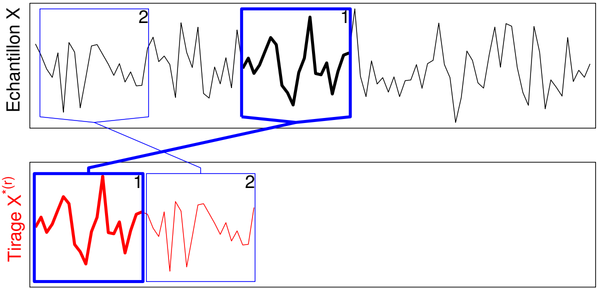

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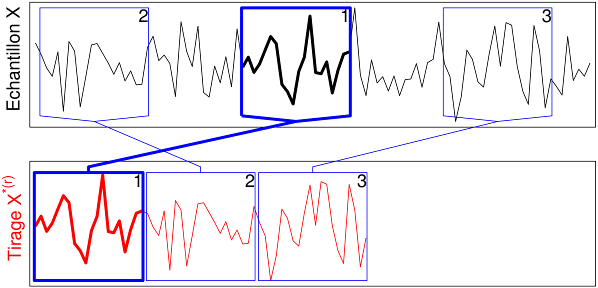

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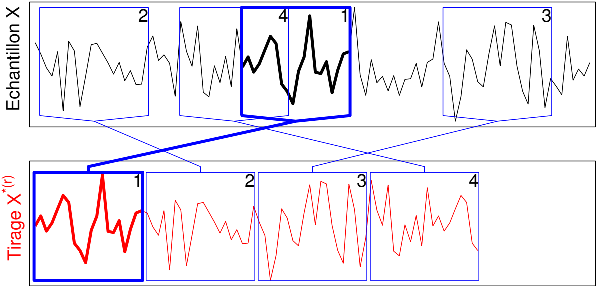

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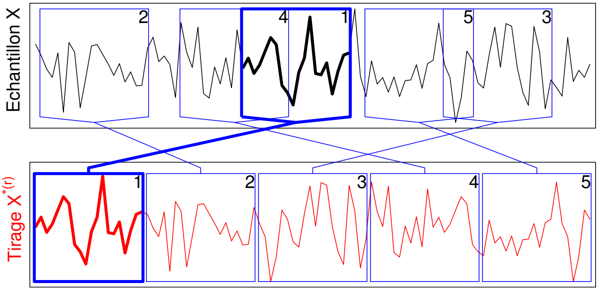

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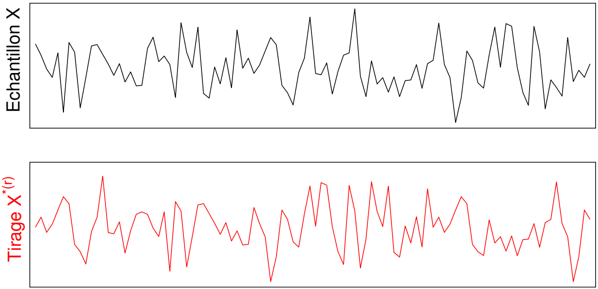
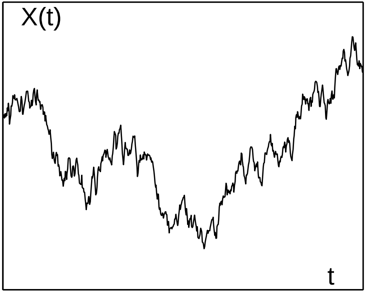
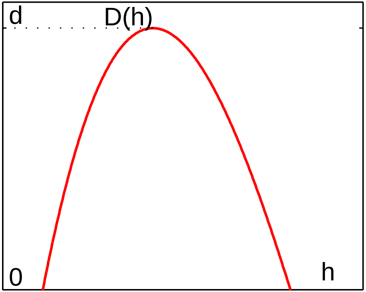


Multifractal Spectrum

- **Multifractal Spectre** $D(h)$:
  - Irregularity : Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set :

$$D(h) = \dim_H\{t : \ h(t) = h\}$$





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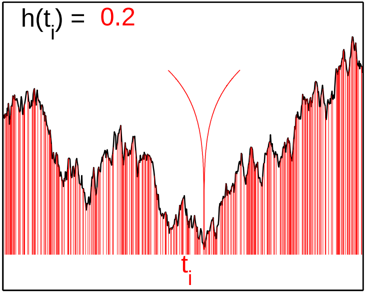

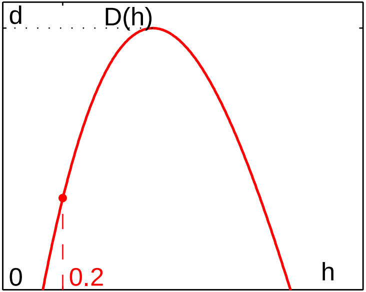

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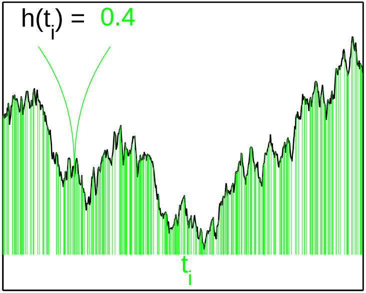

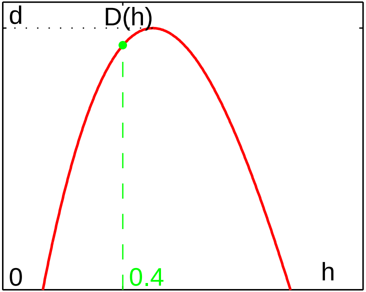

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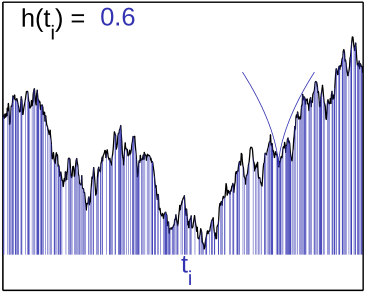

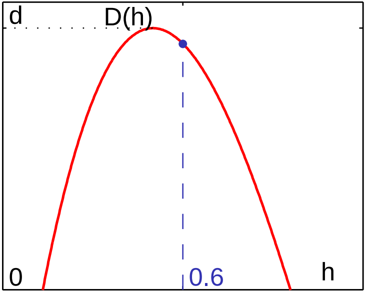

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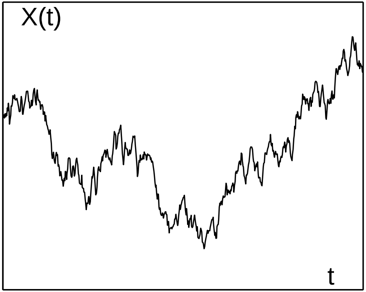
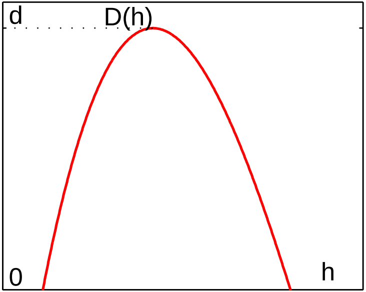
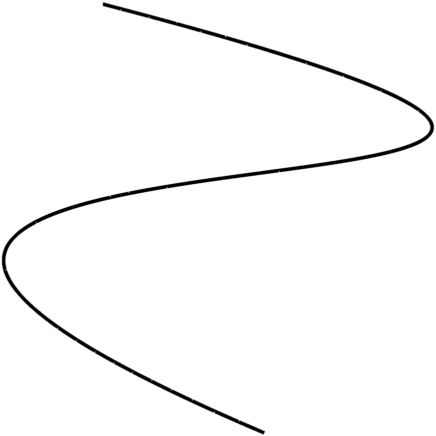
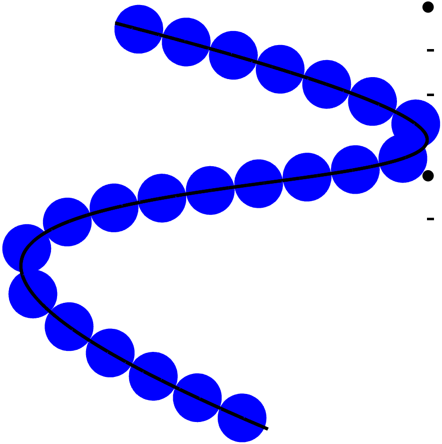




Dimension of a geometrical set

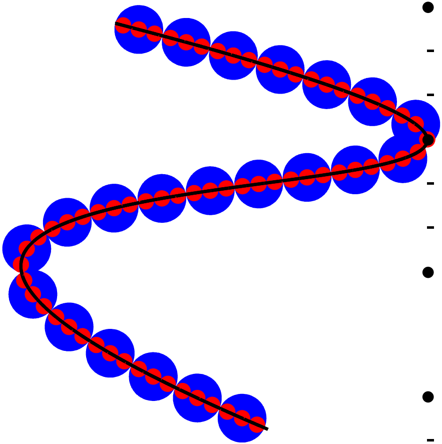


Euclidean dimension



- Let
  - $a (= 1)$ be the analysis scale,
  - N denote the number of covering boxes with size a,
- Then
  - Length is : $L = N \cdot a$

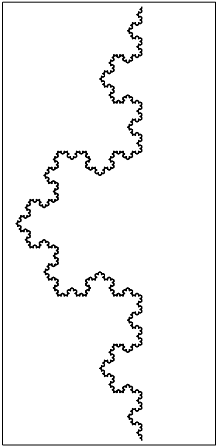
Euclidean dimension



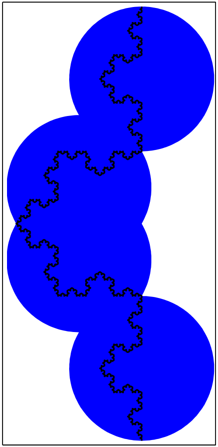
- Let
  - $a (= 1)$,
  - $a (= 1/3)$,
- hence,
  - $N = \frac{a}{a} \cdot N (= 3 \cdot N)$,
  - $L = N \cdot a = L = N \cdot a = L_0$,
- donc

  $L(a)$ does not depend on a nor on a!
- and
  - $L(a) = N(a) \cdot a = L_0$,
  - $\boxed{N(a) = L_0/a = L_0 \cdot a^{-1}}$.

Dimension of a geometrical set

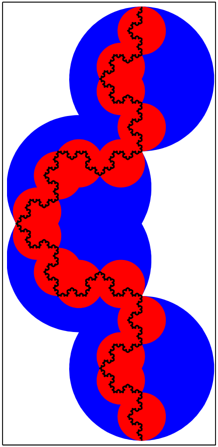


Fractal dimension



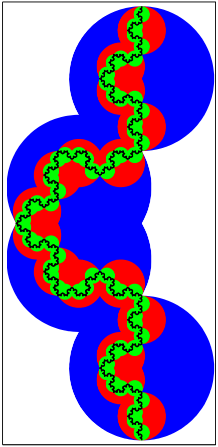
- Let
  - a, be the analysis scale
  - N denote the number of covering boxes with size a,
- Then
  - Length is : $L = N \cdot a$
- Here,
  - $a = 1/3$,
  - $N = 4$,

Fractal dimension



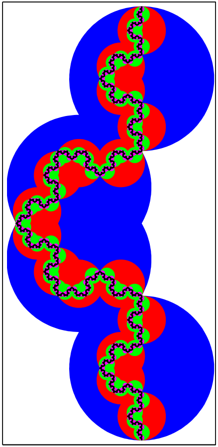
- Let
  - $a (= 1/3)$,
  - $a (= 1/9)$,
- Then,
  - $N = 4$,
  - $N = 16$,
- Hence
  - $L = N \cdot a \neq L = N \cdot a$! ,

Fractal dimension



- Let
  - $a (= 1/3)$,
  - $a (= 1/9)$,
  - $a = 1/27$,
- Then,
  - $N = 4$,
  - $N = 16$,
  - $N = 64$,
- donc
  - $L = N \cdot a \neq L = N \cdot a \neq L = N \cdot a$!

Fractal dimension



- One shows :
- $a(n) = (1/3)^n$,
- $N(n) = 4^n$,
- hence
- $L(a) = N(a) \cdot a$,
- $L(a)$ does depend on a !
- with,
- $\boxed{N(a) = a^{-D}}$,
- $L(a) = L_0 \cdot a^{1-D}$,
- D : fractal dimension,
- $1 < D < 2$,
- non integer = Frac-.

Haussdorf Dimension

- Intuition :

  Fractal dimension,
  Non integer extension of the natural *Euclidean* dimension, $0 \leq D \leq d$.
  Cover a set A with balls of size ϵ, Count how many you need $N(\epsilon)$.
  Assume a power law behaviour $N(\epsilon) \sim \epsilon^{-D}$.
  Define $D = \lim_{\epsilon \to 0} - \log N(\epsilon) / \log \epsilon$.

- Definition :

  $A \in \mathcal{R}^d$,
  $\epsilon > 0, R$ ϵ-covering of A with a countable collection of bounded sets A_i, $|A_i| \leq \epsilon$,
  $\delta \in [0, d], M_\epsilon^\delta(A) = \inf_R \left(\sum_i |A_i|^\delta \right), M^\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta(A)$,
  D is such that $\delta > D, M^\delta(A) = 0, \delta < D, M^\delta(A) = \infty$.

Thermodynamic analogy (Parisi-Frisch, 85)

Thermodynamic	Multifractal
- $Z_\beta(U) = \sum_k e^{-\beta E_k}$,	- $S(a,q) = \sum_k \|T_X(a,k)\|^q$ $S(a,q) = \sum_k e^{q \log \|T_X(a,k)\|}$
$U = \langle E_k \rangle = \partial \log Z_\beta / \partial \beta$	- $\|T_X(a,k)\| = a^{h_k}$, - $S(a,q) = \sum_k e^{q h_k \log a}$
- β	- q
- $E_k = \epsilon_k \delta V$,	- $h_k \log a$, - $S(a,q) = a^{\zeta(q)}$,
- $F = -\ln Z_\beta$	- $\zeta(q) \log a = \log S(a,q)$,
- Entropy : $F = U - S/\beta$ (Legendre transform)	- Spectrum : $D(h) = qh - \zeta(q)$ (Legendre transform)

◂ to MF Form.

Rényi entropy
Strange attractors and chaotic systems (Kadanoff, 75)

- Rényi entropy : $Z_\alpha(a) = \sum_k P_k(a)^\alpha$,
- Rényi information : $I_\alpha(a) = \log Z_\alpha(a)/(1-\alpha)$,
- Generalized dimensions : $D_\alpha = \lim_{a\to 0} I_\alpha(a)/(-\log a)$,

$\Rightarrow \quad (1-\alpha)D_\alpha = \lim_{a\to 0} \log Z_\alpha(a)/\log a \equiv \zeta(\alpha)$!

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