Orthonormal bases generated by Cuntz algebras

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Joint work with Dorin E. Dukay, and Gabriel Picioroaga

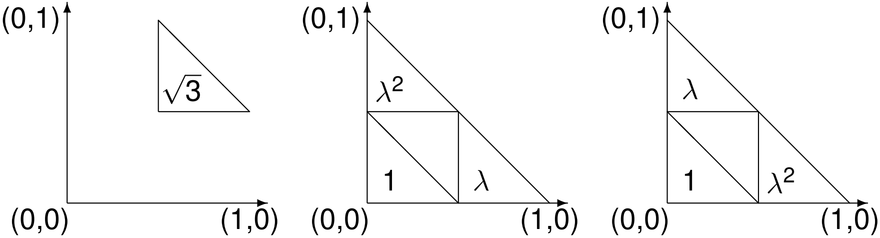
Motivation

- D.E. Dutkay and Palle E. T. Jorgensen, “Wavelets on fractals,” Rev. Mat. Iberoam., 22(1):131-180, 2006.
- J. D’Andrea, K. D. Merrill, and J. Packer, “Fractal wavelets of Dutkay-Jorgensen type for the Sierpinski gasket space,” Frames and operator theory in analysis and signal processing, vol. 451 Contemp. Math.: 69–88. Amer. Math. Soc., Providence, RI, 2008.

Example with complex coefficients ($\lambda = e^{\frac{2\pi i}{3}}$).

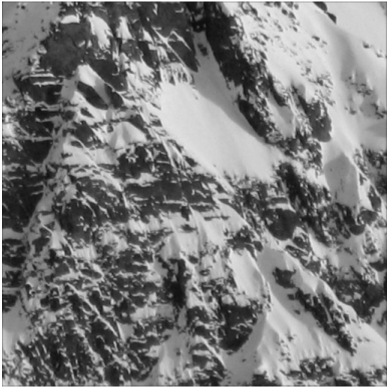
$$
\begin{aligned}
&\chi_S \\
\psi_1 &= D(T_{(1,1)}\chi_S) \quad \text{'gap' wavelet} \\
\psi_2 &= \tfrac{1}{\sqrt{3}} D(\chi_S + \lambda T_{(1,0)}\chi_S + \lambda^2 T_{(0,1)}\chi_S) \\
\psi_3 &= \tfrac{1}{\sqrt{3}} D(\chi_S + \lambda^2 T_{(1,0)}\chi_S + \lambda T_{(0,1)}\chi_S)
\end{aligned}
$$

$$
\begin{aligned}
\vec{v}_0 &= \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, 0\right) \\
\vec{v}_1 &= (0, 0, 0, 1) \\
\vec{v}_2 &= \left(\tfrac{1}{\sqrt{3}}, \tfrac{\lambda}{\sqrt{3}}, \tfrac{\lambda^2}{\sqrt{3}}, 0\right) \\
\vec{v}_3 &= \left(\tfrac{1}{\sqrt{3}}, \tfrac{\lambda^2}{\sqrt{3}}, \tfrac{\lambda}{\sqrt{3}}, 0\right)
\end{aligned}
$$



ψ_1 ψ_2 ψ_3

reconstruction using 30% of transform coefficients



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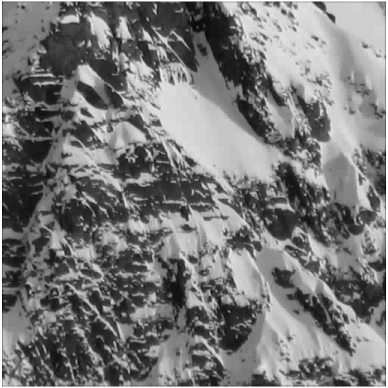
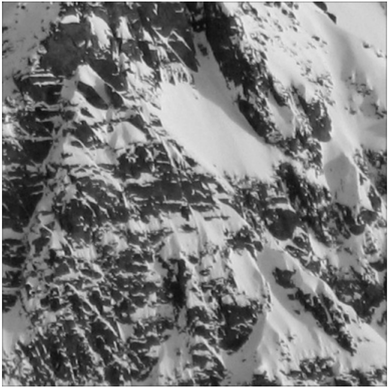


Haar filter

reconstruction using 30% of transform coefficients



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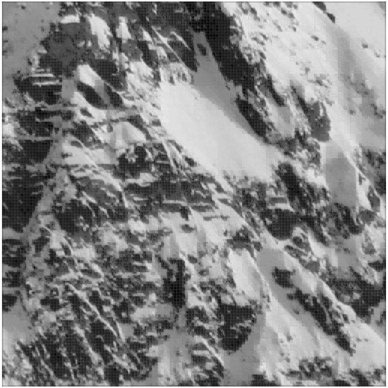
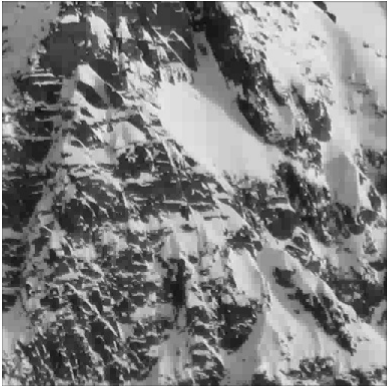
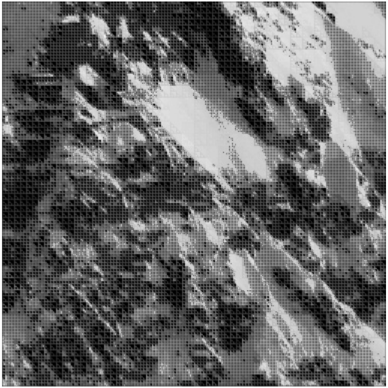


real SG filter

reconstruction using 10% of transform coefficients

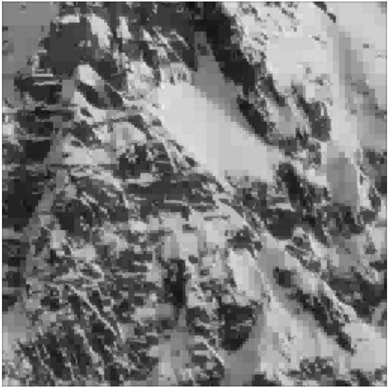


Haar filter

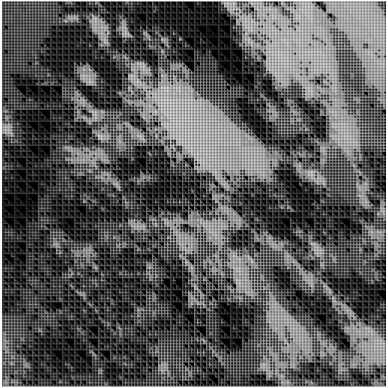


real SG filter

reconstruction using 3% of transform coefficients

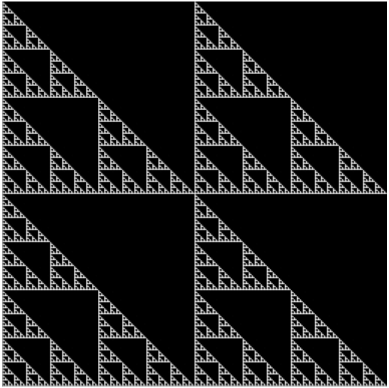


Haar filter

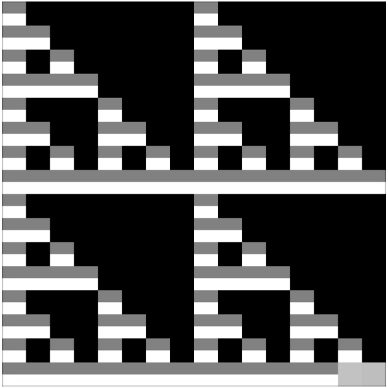


real SG filter

reconstruction using 0.1% of transform coefficients



real SG filter
same as original



Haar filter

Introduction

The Cuntz algebra $\mathcal{O}_N$, is the C^*-algebra generated by N isometries S_i, $i = 0, \ldots, N-1$ with the properties:

$$S_i^* S_j = \delta_{ij},\ i, j = 0, \ldots, N-1, \quad \sum_{i=0}^{N-1} S_i S_i^* = I. \tag{0.1}$$

Orthonormal wavelet bases are constructed from various choices of quadrature mirror filters (QMF). These filters are in one-to-one correspondence with certain representations of the Cuntz algebra.

Introduction

- The proposition shows representations of the Cuntz algebra are obtained from a choice of a quadrature mirror filter (QMF) basis (Definition 4).
- We then show how QMF bases can be constructed using some unitary matrix valued functions (Theorem 7). This gives us a large variety of representations of the Cuntz algebras, which we use to construct various orthonormal bases.
- We present a general criterion for a Cuntz algebra representation to generate an orthonormal basis. Specifically when applied to some affine iterated function systems, we obtain a construction of piecewise exponential bases on some Cantor fractal measures which extends a result of Dutkay and Jorgensen. In particular, we obtain piecewise exponential orthonormal bases on the middle third Cantor set which is known not to have any orthonormal bases of exponential functions.

QMF bases and representations of the Cuntz algebra

Definition

A *quadrature mirror filter (QMF)* for r is a function m_0 in $L^\infty(X, \mu)$ with the property that

$$\frac{1}{N} \sum_{r(w)=z} |m_0(w)|^2 = 1, \quad (z \in X) \tag{0.2}$$

QMF bases and representations of the Cuntz algebra-cont'd

Theorem

Let m_0 be a QMF for r. Then there exists a Hilbert space $\mathcal{H}$, a representation π of $L^\infty(X)$ on $\mathcal{H}$, a unitary operator U on $\mathcal{H}$ and a vector φ in $\mathcal{H}$ such that

QMF bases and representations of the Cuntz algebra-cont'd

1. **(Covariance)**

$$U\pi(f)U^* = \pi(f \circ r), \quad (f \in L^\infty(X)) \tag{0.3}$$

2. **(Scaling equation)**

$$U\varphi = \pi(m_0)\varphi \tag{0.4}$$

3. **(Orthogonality)**

$$\langle \pi(f)\varphi \, , \, \varphi \rangle = \int f \, d\mu, \quad (f \in L^\infty(X)) \tag{0.5}$$

4. **(Density)**

$$\overline{\text{span}} \left\{ U^{-n}\pi(f)\varphi : f \in L^\infty(X), n \geq 0 \right\} = \mathcal{H} \tag{0.6}$$

QMF bases and representations of the Cuntz algebra-cont'd

Definition

The system $(\mathcal{H}, U, \pi, \varphi)$ in Theorem 2 is called *the wavelet representation associated to the QMF m_0*.

To construct a multiresolution, for a wavelet representation, one needs a QMF basis.

Definition

A *QMF basis* is a set of N QMF's $m_0, m_1, \dots, m_{N-1}$ such that

$$\frac{1}{N}\sum_{r(w)=z} m_i(w)\overline{m_j}(w) = \delta_{ij}, \quad (i,j \in \{0,\dots,N-1\}, z \in X) \quad (0.7)$$

QMF bases and representations of the Cuntz algebra-cont'd

Next, we show how a QMF basis induces a representation of the Cuntz algebra.

Proposition

Let $(m_i)_{i=0}^{N-1}$ be a QMF basis. Define the operators on $L^2(X,\mu)$

$$S_i(f) = m_i f \circ r, \quad i = 0, \dots, N-1 \tag{0.8}$$

Then the operators S_i are isometries and they form a representation of the Cuntz algebra $\mathcal{O}_N$, i.e.

$$S_i^* S_j = \delta_{ij}, \quad i,j = 0, \dots, N-1, \qquad \sum_{i=0}^{N-1} S_i S_i^* = I \tag{0.9}$$

The adjoint of S_i is given by the formula

$$S_i^*(f)(z) = \frac{1}{N} \sum_{r(w)=z} \overline{m_i}(w) f(w) \tag{0.10}$$

QMF bases and representations of the Cuntz algebra-cont'd

Proof.

We compute the adjoint: take f, g in $L^2(X, \mu)$. We use the strong invariance of μ.

$$\langle S_i^* f\,,\, g\rangle = \int f \overline{m_i} \overline{g \circ r}\, d\mu = \int \frac{1}{N} \sum_{r(w)=z} \overline{m_i}(w) f(w) \overline{g}(z) d\mu(z)$$

Then (0.10) follows. The Cuntz relations in (0.9) are then easily checked with Proposition. $\square$

QMF bases and representations of the Cuntz algebra-cont'd

Every QMF basis generates a multiresolution for the wavelet representation associated to m_0. Since the ideas are simple and they are the same as in the classical wavelet theory.

QMF bases and representations of the Cuntz algebra-cont'd

Proposition

Let $(m_i)_{i=0}^{N-1}$ *be a QMF basis. Let* $(\mathcal{H}, U, \pi, \varphi)$ *be the wavelet representation associated to* m_0*. Define*

$$V_0 := \overline{\text{span}}\left\{\pi(f)\varphi : f \in L^\infty(X)\right\}, \qquad V_n = U^{-n}V_0, \quad n \in \mathbb{Z} \tag{0.11}$$

$$\psi_i = U^{-1}\pi(m_i)\varphi, \quad i = 1, \ldots, N-1 \tag{0.12}$$

$$W_i := \overline{\text{span}}\left\{\pi(f)\psi_i : f \in L^\infty(X)\right\} \tag{0.13}$$

QMF bases and representations of the Cuntz algebra-cont'd

Then

1. $\overline{\cup_{n\in\mathbb{Z}} V_n} = \mathcal{H}$
2. $V_1 = V_0 \oplus W_1 \oplus \cdots \oplus W_{N-1}$
3. If $\cap_{n\in\mathbb{Z}} V_n = \{0\}$ then

$$\bigoplus_{n\in\mathbb{Z}} U^n\,(W_1 \oplus \cdots \oplus W_{N-1}) = \mathcal{H}$$

QMF bases and representations of the Cuntz algebra-cont'd

If B is a finite set and $R^{-1}B$ has spectrum Λ, then the set $\{e_\lambda : \lambda \in \Lambda\}$ is a QMF basis, by Proposition. Then, with Proposition 5, the operators $S_\lambda f = e_\lambda f \circ r$ form a representation of the Cuntz algebra. Such representations were studied in.

QMF bases and representations of the Cuntz algebra-cont'd

Theorem

Fix $(m_i)_{i=0}^{N-1}$ a QMF basis. There is a one-to-one correspondence between the following two sets:

1. *QMF bases $(m_i')_{i=0}^{N-1}$*
2. *Unitary valued maps $A : X \to U_N(\mathbb{C})$*

Given a QMF basis $(m_i')_{i=0}^{N-1}$ the matrix A with entries

$$(1) \quad A_{ij}(z) = \frac{1}{N} \sum_{r(w)=z} m_i'(w)\overline{m}_j(w), \quad (z \in X, i, j = 0, \ldots, N-1)$$

is unitary.

QMF bases and representations of the Cuntz algebra-cont'd

Given a unitary-valued map $A : X \to U_N(\mathbb{C})$, the functions form a QMF basis

$$(2) \quad m_i'(z) = \sum_{j=0}^{N-1} A_{ij}(r(z))m_j(z), \quad (z \in X, i = 0, \dots N-1)$$

These correspondences are inverse to each other.

QMF bases and representations of the Cuntz algebra-cont'd

Proof.

The result requires some simple computations

$$\sum_{j=0}^{N-1} A_{ij}(z)\overline{A_{i'j}(z)} = \frac{1}{N^2}\sum_{j}\sum_{r(w)=z} m_i'(w)\overline{m_j(z)}\cdot \sum_{r(w')=z} \overline{m_{i'}'(w')\overline{m_j(w')}}$$

$$= \frac{1}{N^2}\sum_{w,w'} m_i'(w)\overline{m_{i'}'(w')}\cdot\sum_{j}\overline{m_j(w)}m_j(w')$$

$$= \frac{1}{N}\sum_{w,w'} m_i'(w)\overline{m_{i'}'(w')}\delta_{w,w'} = \delta_{ii'}$$

Note that we used the equality

$$\sum_{j}\overline{m_j(w)}m_j(w') = \delta_{ww'}$$

QMF bases and representations of the Cuntz algebra-cont'd

which follows from the fact that the matrix

$$\frac{1}{\sqrt{N}}\left[m_i(w)\right]_{w\in r^{-1}(z)}^{i=0,\dots N-1}$$

is unitary, which, in turn, is a consequence of the QMF property. Hence A is unitary.

If A is unitary, we check the QMF relations:

$$\frac{1}{N}\sum_{r(w)=z} m_i'(w)\overline{m_j'(w)}$$

$$=\frac{1}{N}\sum_w \sum_k A_{ik}(r(w))m_k(w)\sum_l \overline{A_{jl}(r(w))m_l(w)}=$$

$$\frac{1}{N}\sum_{k,l} A_{ik}(z)\overline{A_{jl}(z)}\sum_w m_k(w)\overline{m_l(w)}$$

$$=\sum_{k,l} A_{jk}(z)\overline{A_{jl}(z)}\delta_{kl}=\delta_{ij}$$

QMF bases and representations of the Cuntz algebra-cont'd

Hence $(m_i')_{i=0}^{N-1}$ is a QMF basis. The fact that the two correspondences are inverse to each other follows from the next computation:

$$\sum_j A_{ij}(r(z))m_j(z) = \sum_j \left(\frac{1}{N} \sum_{r(w)=r(z)} m_i'(w)\overline{m_j}(w) \right) m_j(z)$$

$$= \sum_{r(w)=r(z)} m_i'(w) \cdot \frac{1}{N} \sum_j \overline{m_j}(w) m_j(z)$$

$$= \sum_{r(w)=r(z)} m_i'(w)\delta_{wz} = m_i'(z)$$

Orthonormal bases generated by Cuntz algebras

A general criterion for a family generated by the Cuntz isometries an orthonormal basis.

Theorem

Let $\mathcal{H}$ be a Hilbert space and $(S_i)_{i=0}^{N-1}$ be a representation of the Cuntz algebra $\mathcal{O}_N$. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$ and $f : X \to \mathcal{H}$ a norm continuous function on a topological space X with the following properties:

Orthonormal bases generated by Cuntz algebras

1. $\mathcal{E} = \cup_{i=0}^{N-1} S_i \mathcal{E}$.
2. $\overline{\text{span}}\{f(t) : t \in X\} = \mathcal{H}$ and $||f(t)|| = 1$, for all $t \in X$.
3. There exist functions $\mathfrak{m}_i : X \to \mathbb{C}$, $g_i : X \to X$, $i = 0, \dots, N-1$ such that

$$S_i^* f(t) = \mathfrak{m}_i(t) f(g_i(t)), \quad t \in X. \tag{0.14}$$

4. There exist $c_0 \in X$ such that $f(c_0) \in \overline{\text{span}}\mathcal{E}$.
5. The only function $h \in \mathcal{C}(X)$ with $h \geq 0$, $h(c) = 1$, $\forall$ $c \in \{x \in X : f(x) \in \overline{\text{span}}\mathcal{E}\}$, and

$$h(t) = \sum_{i=0}^{N-1} |\mathfrak{m}_i(t)|^2 h(g_i(t)), \quad t \in X \tag{0.15}$$

are the constant functions.

Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$.

Piecewise exponential bases on fractals

We consider affine iterated function systems with no overlap. Let R be a $d \times d$ expansive real matrix, i.e., all the eigenvalues of R have absolute value strictly greater than 1.Let $B \subset \mathbb{R}^d$ a finite set such that $N = |B|$. Define the affine iterated function system

$$\tau_b(x) = R^{-1}(x + b) \quad (x \in \mathbb{R}^d, b \in B) \tag{0.16}$$

There exists a unique compact subset X_B of $\mathbb{R}^d$ which satisfies the invariance equation

$$X_B = \cup_{b \in B} \tau_b(X_B) \tag{0.17}$$

Piecewise exponential bases on fractals-cont'd

X_B is called the attractor of the iterated function system $(\tau_b)_{b\in B}$. Moreover X_B is given by

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k : b_k \in B \text{ for all } k \geq 1 \right\} \tag{0.18}$$

There is a unique probability measure μ_B on $\mathbb{R}^d$ satisfying the invariance equation

$$\int f d\mu_B = \frac{1}{N} \sum_{b\in B} \int f \circ \tau_b d\mu_B \tag{0.19}$$

for all continuous compactly supported functions f on $\mathbb{R}$. We call μ_B the invariant measure for the iterated function system (IFS) $(\tau_b)_{b\in B}$. μ_B is supported on the attractor X_B.

Piecewise exponential bases on fractals-cont'd

We say that the IFS has no overlap if
$\mu_B(\tau_b(X_B) \cap \tau_b'(X_B)) = \emptyset$ for all $b \neq b'$ in B.
Assume that the IFS $(\tau_b)_{b\in B}$ has no overlap. Define the map
$r : X_B \to X_B$

$$r(x) = \tau_b^{-1}(x), \text{ if } x \in \tau_b(X_B) \tag{0.20}$$

Then r is an N-to-1 onto map and μ_B is strongly invariant for r.
Note that $r^{-1}(x) = \{\tau_b(x) : b \in B\}$ for μ_B.a.e. $x \in X_B$.

Piecewise exponential bases on fractals-cont'd

Definition

Let L in $\mathbb{R}$, $|L| = N$, $R > 1$ such that L is a spectrum for the set $\frac{1}{R}B$. We say that $c \in \mathbb{R}$ is an *extreme cycle point* for (B, L) if there exists $l_0, l_1, \ldots, l_{p-1}$ in L such that, if $c_0 = c$, $c_1 = \frac{c_0+l_0}{R}, c_2 = \frac{c_1+l_1}{R} \ldots c_{p-1} = \frac{c_{p-2}+l_{p-2}}{R}$ then $\frac{c_{p-1}+l_{p-1}}{R} = c_0$, and $|m_B(c_i)| = 1$ for $i = 0, \ldots, p-1$ where

$$m_B(x) = \frac{1}{N}\sum_{b\in B} e^{2\pi ibx} \quad x \in \mathbb{R}.$$

Orthonormal bases generated by Cuntz algebras-cont'd

Definition

We denote by L^* the set of all finite words with digits in L, including the empty word. For $l \in L$ let S_l be given as in (0.8) where m_l is replaced by the exponential e_l. If $w = l_1 l_2 \dots l_n \in L^*$ then by S_w we denote the composition $S_{l_1} S_{l_2} \dots S_{l_n}$.

Orthonormal bases generated by Cuntz algebras-cont'd

Definition

We denote by L^* the set of all finite words with digits in L, including the empty word. For $l \in L$ let S_l be given as in (0.8) where m_l is replaced by the exponential e_l. If $w = l_1 l_2 \dots l_n \in L^*$ then by S_w we denote the composition $S_{l_1} S_{l_2} \dots S_{l_n}$.

Orthonormal bases generated by Cuntz algebras-cont'd

Theorem

Let $B \subset \mathbb{R}$, $0 \in B$, $|B| = N$, $R > 1$ and let μ_B be the invariant measure associated to the IFS $\tau_b(x) = R^{-1}(x + b)$, $b \in B$. Assume that the IFS has no overlap and that the set $\frac{1}{R}B$ has a spectrum $L \subset \mathbb{R}$, $0 \in L$. Then the set

$$\mathcal{E}(L) = \{S_w e_{-c} : c \text{ is an extreme cycle point for } (B, L), w \in L^*\}$$

is an orthonormal basis in $L^2(\mu_B)$. Some of the vectors in $\mathcal{E}(L)$ are repeated but we count them only once.

Orthonormal bases generated by Cuntz algebras-cont'd

Corollary

In the hypothesis of Theorem 8, if in addition $B, L \subset \mathbb{Z}$ *and* $R \in \mathbb{Z}$*, then there exists a set* Λ *such that* $\{e_\lambda : \lambda \in \Lambda\}$ *is an orthonormal basis for* $L^2(\mu_B)$*.*

Orthonormal bases generated by Cuntz algebras-cont'd

Example

We consider the IFS that generates the middle third Cantor set: $R = 3$, $B = \{0, 2\}$. The set $\frac{1}{3}\{0, 2\}$ has spectrum $L = \{0, 3/4\}$. We look for the extreme cycle points for (B, L).

We need $|m_B(-c)| = 1$ so $|\frac{1+e^{2\pi i 2c}}{2}| = 1$, therefore $c \in \frac{1}{2}\mathbb{Z}$. Also c has to be a cycle for the IFS $g_0(x) = x/3$, $g_{3/4}(x) = \frac{x+3/4}{3}$ so $0 \leq c \leq \frac{3/4}{3-1} = 3/8$. Thus, the only extreme cycle is $\{0\}$. By Theorem 8 $\mathcal{E} = \{S_w 1 : w \in \{0, 3/4\}^*\}$ is an orthonormal basis for $L^2(\mu_B)$. Note also that the numbers $e^{2\pi i \alpha(b,l,c)}$ are ± 1 because $2\pi i B \cdot L \subset \pi i \mathbb{Z}$.

Orthonormal bases generated by Cuntz algebras-cont'd

Walsh Bases: We will focus on the unit interval, which can be regarded as the attractor of a simple IFS and we use step functions for the QMF basis to generate Walsh-type bases for $L^2[0,1]$.

Example

The interval $[0,1]$ is the attractor of the IFS $\tau_0 x = \frac{x}{2}$, $\tau_1 x = \frac{x+1}{2}$, and the invariant measure is the Lebesgue measure on $[0,1]$. The map r defined is $rx = 2x \mathrm{mod} 1$. Let $m_0 = 1$, $m_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. It is easy to see that $\{m_0, m_1\}$ is a QMF basis. Therefore S_0, S_1 defined as in Proposition 5 form a representation of the Cuntz algebra $\mathcal{O}_2$.

Orthonormal bases generated by Cuntz algebras-cont'd

Theorem

Let $N \in \mathbb{N}$, $N \geq 2$. Let $A = [a_{ij}]$ be an $N \times N$ unitary matrix whose first row is constant $\frac{1}{\sqrt{N}}$. Consider the IFS $\tau_j x = \frac{x+j}{N}$, $x \in \mathbb{R}$, $j = 0, \ldots, N-1$ with the attractor $[0,1]$ and invariant measure the Lebesgue measure on $[0,1]$. Define

$$m_i(x) = \sqrt{N} \sum_{j=0}^{N-1} a_{ij} \chi_{[j/N,(j+1)/N]}(x)$$

Then $\{m_i\}_{i=0}^{N-1}$ is a QMF basis. Consider the associated representation of the Cuntz algebra $\mathcal{O}_N$. Then the set $\mathcal{E} := \{S_w 1 : w \in \{0, ...N-1\}^*\}$ is an orthonormal basis for $L^2[0,1]$.

Orthonormal bases generated by Cuntz algebras-cont'd

Proof.

We check the conditions in Theorem 8. Let $f(t) = e_t$, $t \in \mathbb{R}$.
To check (i) note that $S_0 1 \equiv 1$. (ii) is clear. For (iii) we compute:

$$S_k^* e_t = \frac{1}{N} \sum_{j=0}^{N-1} \overline{m_k}(\tau_j x) e_t(\tau_j x)$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{a_{kj}} e^{2\pi i t \cdot (x+j)/N}$$

$$= e^{2\pi i t \cdot x/N} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{a_{kj}} e^{2\pi i t \cdot j/N}$$

So (iii) is true with $\mathfrak{m}_k(t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{a_{kj}} e^{2\pi i t \cdot j/N}$ and $g_k(t) = \frac{t}{N}$. $\square$

Orthonormal bases generated by Cuntz algebras-cont'd

(iv) is true with $c_0 = 0$. For (v) take $h \in \mathcal{C}(\mathbb{R})$, $0 \leq h \leq 1$, $h(c) = 1$ for all $c \in \mathbb{R}$ with $e_c \in \overline{\text{span}}\mathcal{E}$ (in particular $h(0) = 1$), and

$$\begin{aligned} h(t) &= \sum_{k=0}^{N-1} |\mathfrak{m}_k(t)|^2 h(t/N) \\ &= h(t/N) \sum_{k=0}^{N-1} \frac{1}{N} |\sum_{j=0}^{N-1} a_{kj} e^{-2\pi i t \cdot j/N}|^2 \\ &= h(t/N) \cdot \frac{1}{N} ||Av||^2 \end{aligned}$$

where $v = (e^{-2\pi i t \cdot j/N})_{j=0}^{N-1}$. Since A is unitary, $||Av||^2 = ||v||^2 = N$. Then $h(t) = h(t/N^n)$. Letting $n \to \infty$ and using the continuity of h we obtain that $h(t) = 1$ for all $t \in \mathbb{R}$. Thus, Theorem 8 implies that $\mathcal{E}$ is an orthonormal basis.

Selected References

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