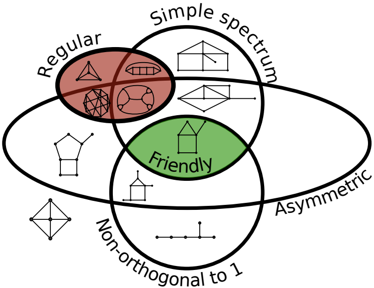
Recent algorithmic and theoretical advances on graph matching



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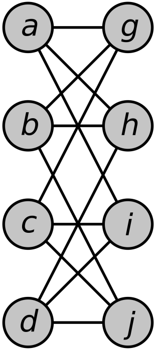
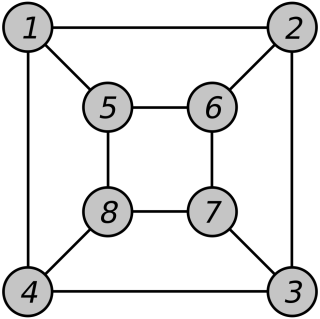
Joint work with G. Sapiro,
P. Sprechmann, J. Vogelstein, P. Musé,
V. Lyzinski, D. Fishkind and C.E. Priebe.

Graph Isomorphisms





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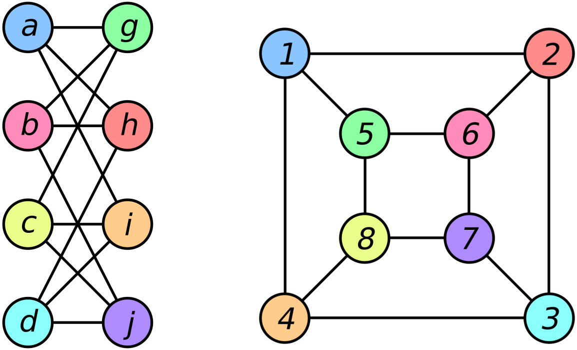

GIP/GMP/GAP

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- Graph Automorphism Problem (GAP):
  decide whether a graph has nontrivial automorphism group.

The Graph Matching Problem

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- "Best Matching"

$$\hat{P} = \arg\min_{P \in \mathcal{P}} ||AP - PB||_F^2$$

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$$\hat{P} = \arg\min_{P \in \underset{\mathcal{D}}{\cancel{\mathcal{P}}}} ||AP - PB||_F^2$$

Then project:

$$P^* = \arg\min_{P \in \mathcal{P}} ||P - \hat{P}||_F^2$$

Graph Matching Algorithms (GLAG) [F12]

Idea: instead of asking $AP \approx PB$, ask them to have (approximately) the same support

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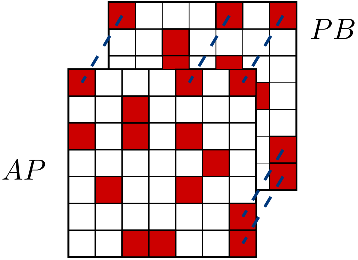

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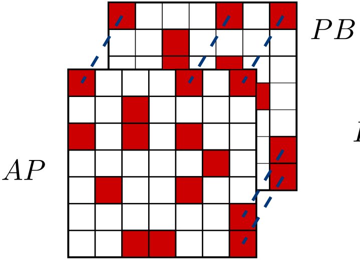

$$\tilde{P} = \arg\min_{P \in \mathcal{D}} \sum_{i,j} \left|\left| \left((AP)_{ij}, (PB)_{ij} \right) \right|\right|_2$$

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Specially useful for:

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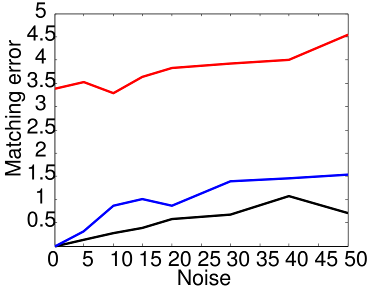
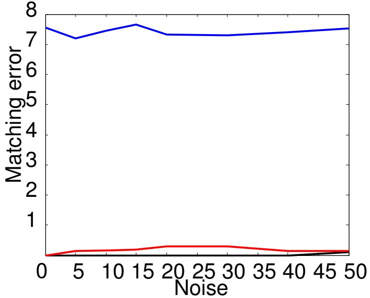
$$\min_{\substack{\Theta^A \succ 0 \\ \Theta^B \succ 0 \\ P \in \mathcal{P}}} \mathrm{tr}(S^A \Theta^A) - \log\det\Theta^A + \mathrm{tr}(S^B \Theta^B) - \log\det\Theta^B + \lambda \sum_{i,j} \left|\left| \left((\Theta^A P)_{ij}, (P\Theta^B)_{ij} \right) \right|\right|_2$$

Results for real graphs



(a) Electrical connection graph



(b) Chemical connection graph

Figure: Matching error for the C. elegans connectome.
Black: GLAG, blue: PATH, red: FAQ.

Graph Matching theory

Convex relaxation: When does it work?

$$P_o = \arg\min_{P \in \mathcal{P}} ||AP - PB||_F^2 \qquad (P_1)$$

$$\hat{P} = \arg\min_{P \in \mathcal{D}} ||AP - PB||_F^2 \qquad (P_2)$$

Graph Matching theory I (probabilistic)

Theorem ([L14])

Suppose A *and* B *are adjacency matrices for* ρ*-correlated Bernoulli*(Λ) *graphs, and there is an* $\alpha \in (0, 1/2)$ *such that* $\Lambda_{i,j} \in [\alpha, 1-\alpha]$ *for all* $i \neq j$*. Let* $P^* \in \mathit{\Pi}$*, and denote* $A' := P^* A P^{*T}$*.*

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a) *If the between graph correlation $\rho < 1$, then it almost always holds that $P^* \notin \arg\min_{D \in \mathcal{D}} \|A'D - DB\|_F$.*

b) *If $(1-\alpha)(1-\rho) < 1/2$, then it almost always holds that*

$$\arg\min_{D \in \mathcal{D}} -\langle A'D, DB \rangle = \arg\min_{P \in \Pi} \|A' - PBP^T\|_F = \{P^*\}.$$

Success rate in recovering P^*

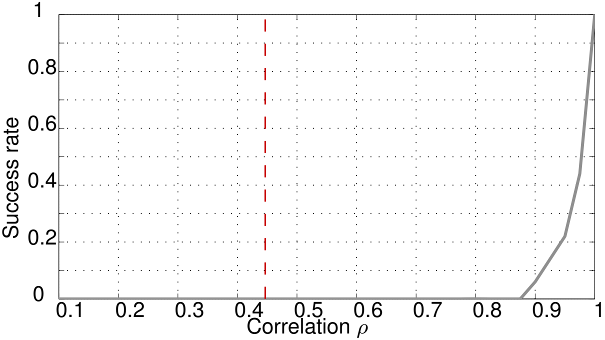

Figure: Non-convex optimization with different initializations: J (gray).

Success rate in recovering P^*

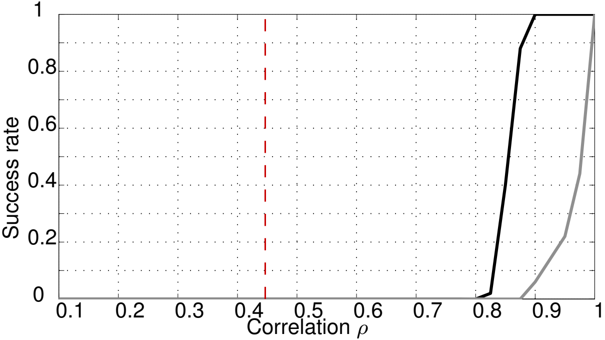

Figure: Non-convex optimization with different initializations: J (gray), convex relaxation in black.

Success rate in recovering P^*

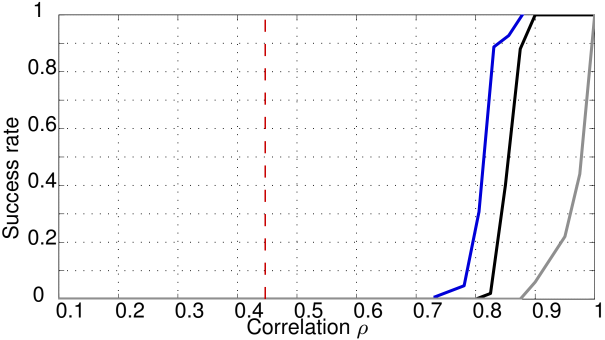

Figure: Non-convex optimization with different initializations: J (gray), convex relaxation in black, and GLAG in blue.

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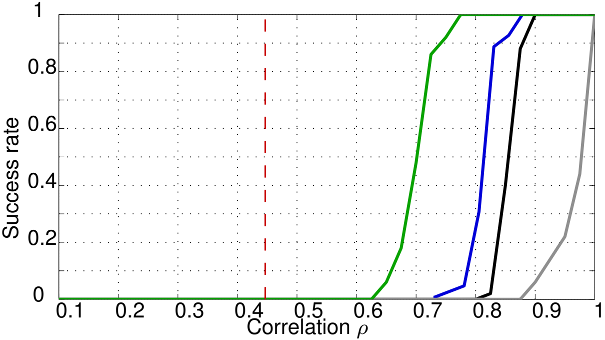

Figure: Non-convex optimization with different initializations: D^* (green), and J (gray), convex relaxation in black, and GLAG in blue.

Success rate in recovering P^*

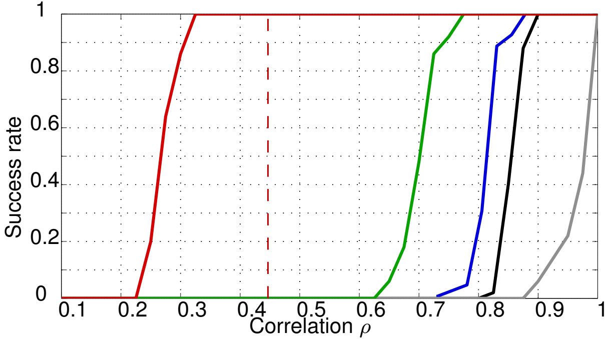

Figure: Non-convex optimization with different initializations: P^* (red), D^* (green), and J (gray), convex relaxation in black, and GLAG in blue.

Graph Matching theory II (deterministic)

Based on spectral properties

- eigenvalues multiplicity
- eigenvectors u s.t. $u^T \mathbf{1} = 0$

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Problems equivalent for “Friendly” graphs [A14]

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Theorem ([F14])

If A has no repeated eigenvalues (simple spectrum), and there are k eigenvectors u_i such that $u_i^T \mathbf{1} = 0$, each one of these vectors having at least $2k+1$ nonzero entries, then problems (P_1) and (P_2) are equivalent.

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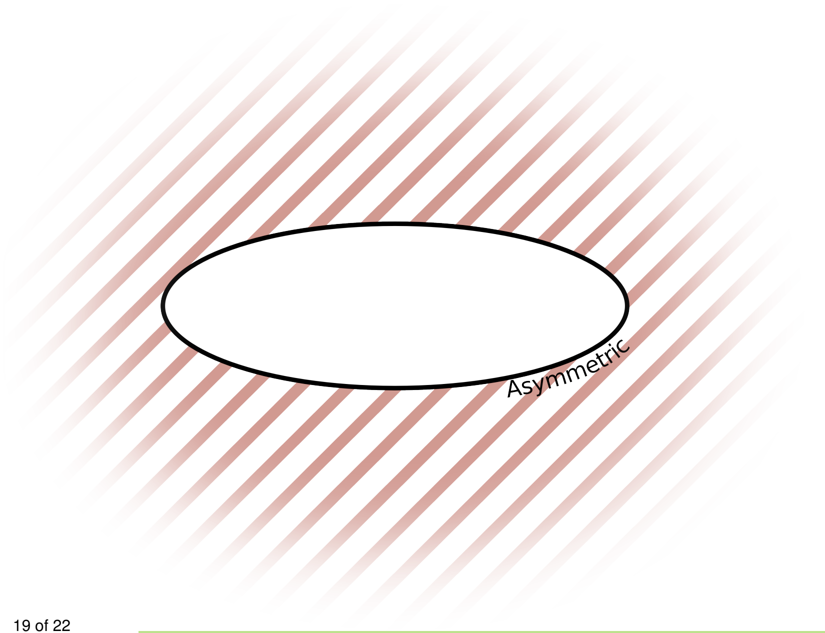
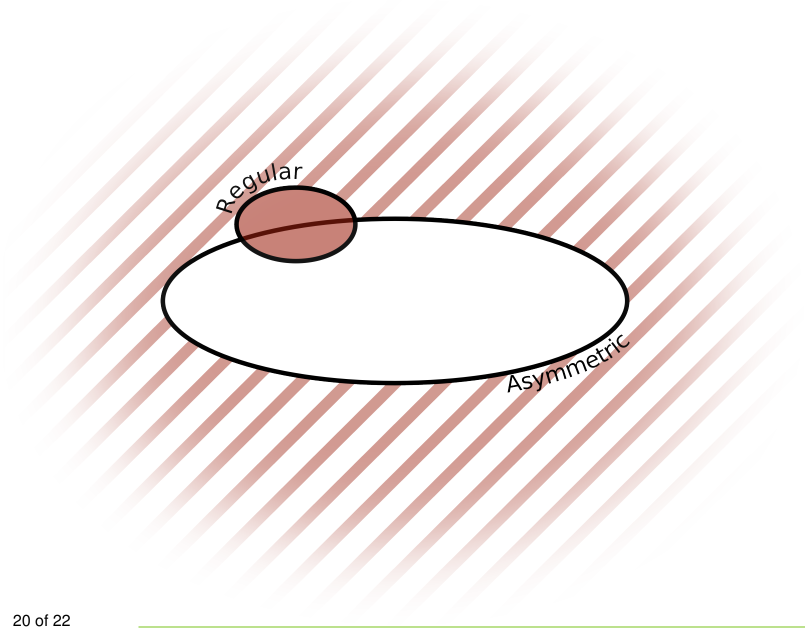
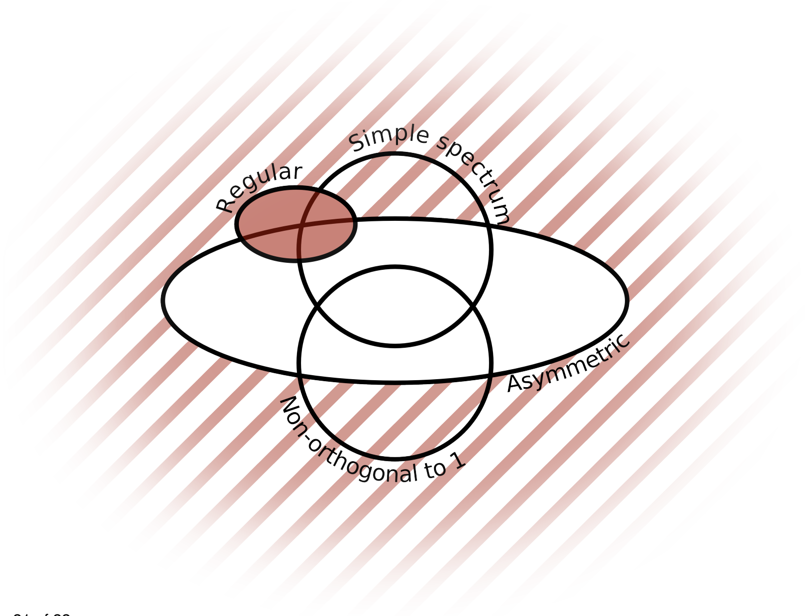
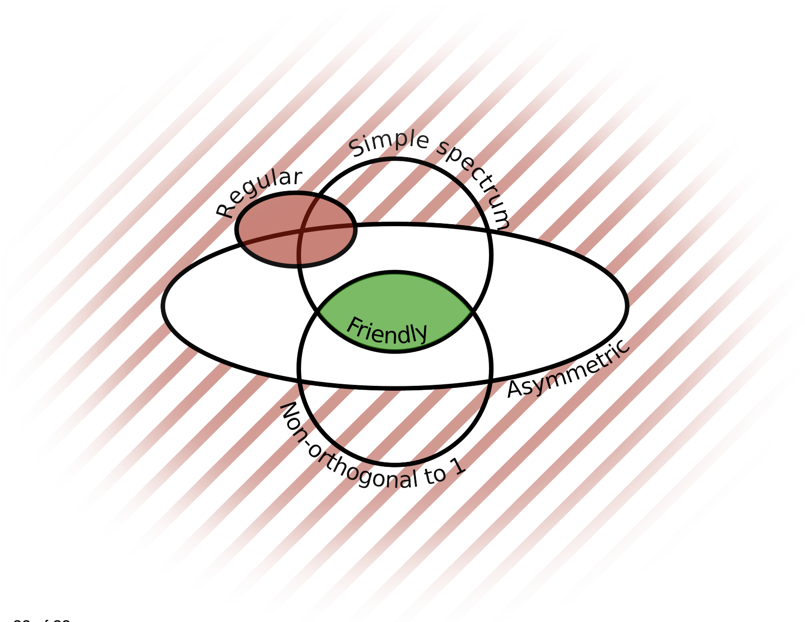
Corollary

In the same conditions as above, the automorphism group of the corresponding graph G_A is the trivial group.


Asymmetric


Regular
Asymmetric


Regular
Simple spectrum
Asymmetric
Non-orthogonal to 1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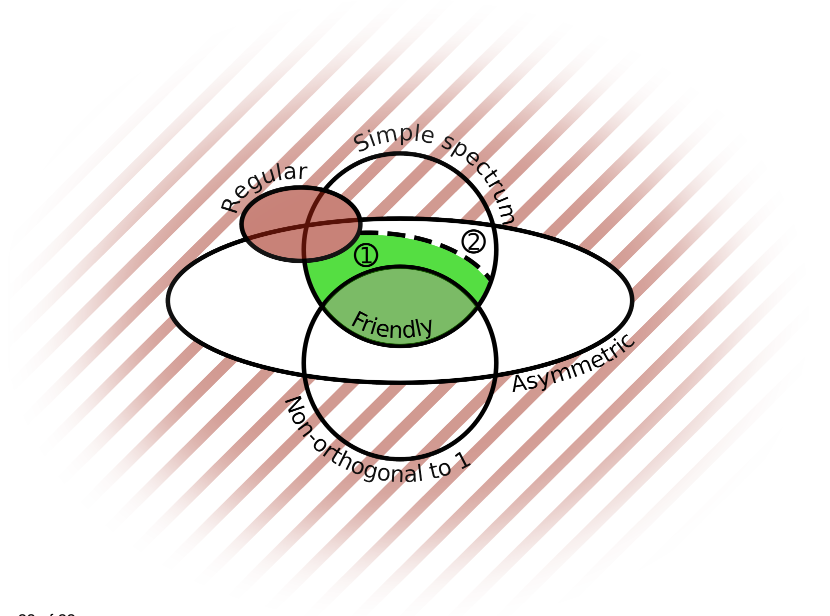
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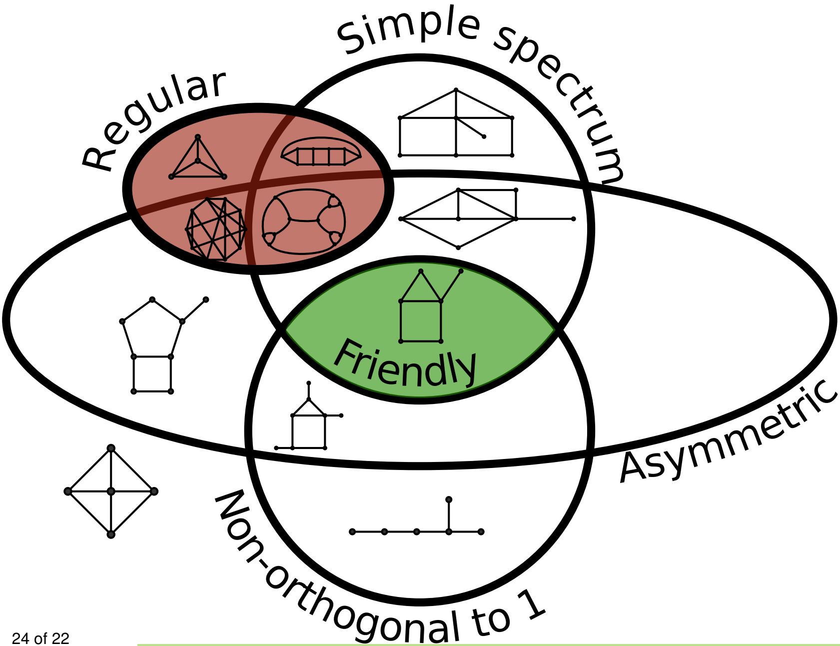
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Thank you!

M. Fiori, P. Sprechmann, J.T. Vogelstein, P. Musé, G. Sapiro.
Robust Multimodal Graph Matching: Sparse Coding Meets Graph Matching.
Advances in Neural Information Processing Systems 26 (NIPS 2013).

V. Lyzinski, D. Fishkind, M. Fiori, J.T. Vogelstein, C.E. Priebe, G. Sapiro.
Graph Matching: Relax at Your Own Risk.
arXiv preprint 2014 arXiv:1405.3133.

Y. Aflalo, A. Bronstein, R. Kimmel
Graph matching: relax or not?.
arXiv preprint 2014 arXiv:1401.7623..

M. Fiori, G. Sapiro.
On spectral properties for graph matching and graph isomorphism problems.
arXiv preprint 2014 arXiv:1409.6806.