Dynamical Sampling

Carlos Cabrelli
University of Buenos Aires and IMAS-CONICET

December, 2014

Dynamical Sampling

Carlos Cabrelli
University of Buenos Aires and IMAS-CONICET

December, 2014

Joint work with
Akram Aldroubi,
Ursula Molter
Sui Tang.

Dynamical Sampling

Carlos Cabrelli
University of Buenos Aires and IMAS-CONICET

December, 2014

Joint work with
Akram Aldroubi,
Ursula Molter
Sui Tang.

preprint: http://arxiv.org/pdf/1409.8333.pdf

What is Dynamical Sampling?

Let $x_0 \in \mathbb{C}^N$ be an unknown vector.

We only know a few samples (components) of x_0.

So, the information is not enough to reconstruct x_0.

What is Dynamical Sampling?

Let $x_0 \in \mathbb{C}^N$ be an unknown vector.

We only know a few samples (components) of x_0.

So, the information is not enough to reconstruct x_0.

Assume that A is a known operator acting on $\mathbb{C}^N$, (the evolution operator).

At time 1 we have $x_1 = Ax_0$ and at time 2 we have $x_2 = A^2 x_0$.

We are only allowed to get some few samples from the vectors $x_1, x_2, ...$

What is Dynamical Sampling?

Let $x_0 \in \mathbb{C}^N$ be an unknown vector.

We only know a few samples (components) of x_0.

So, the information is not enough to reconstruct x_0.

Assume that A is a known operator acting on $\mathbb{C}^N$, (the evolution operator).

At time 1 we have $x_1 = Ax_0$ and at time 2 we have $x_2 = A^2 x_0$.

We are only allowed to get some few samples from the vectors $x_1, x_2, ...$

Can we compensate the loss of information from $x = x_0$ having some extra but coarse information from the evolved signals $x_1, x_2, ...$?

What is Dynamical Sampling?

Let $x_0 \in \mathbb{C}^N$ be an unknown vector.

We only know a few samples (components) of x_0.

So, the information is not enough to reconstruct x_0.

Assume that A is a known operator acting on $\mathbb{C}^N$, (the evolution operator).

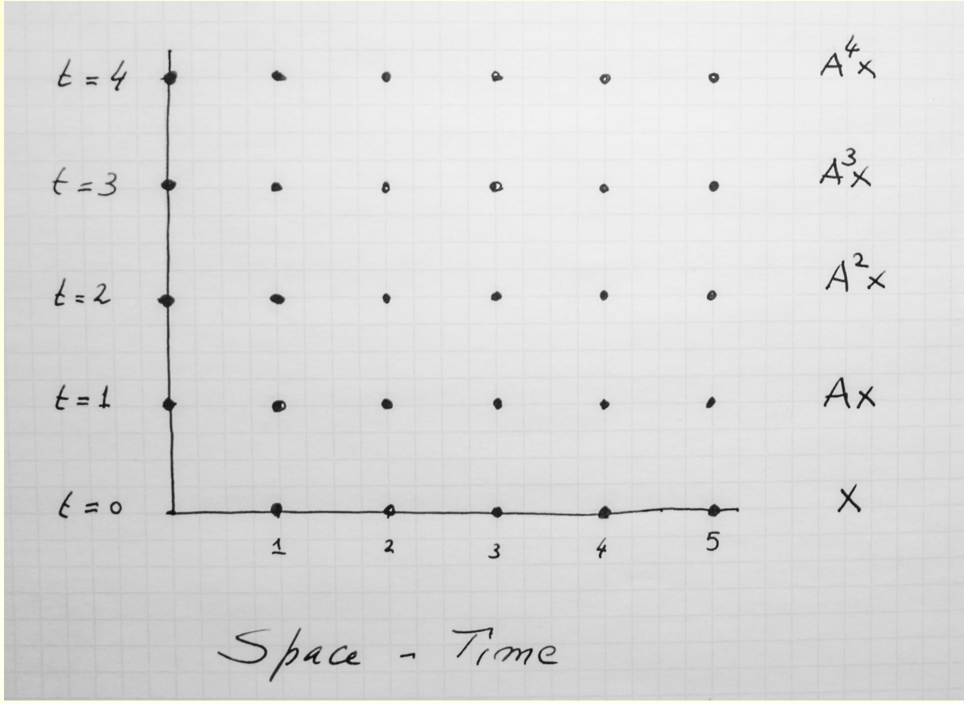
At time 1 we have $x_1 = Ax_0$ and at time 2 we have $x_2 = A^2 x_0$.

We are only allowed to get some few samples from the vectors $x_1, x_2, ...$

Can we compensate the loss of information from $x = x_0$ having some extra but coarse information from the evolved signals $x_1, x_2, ...$?

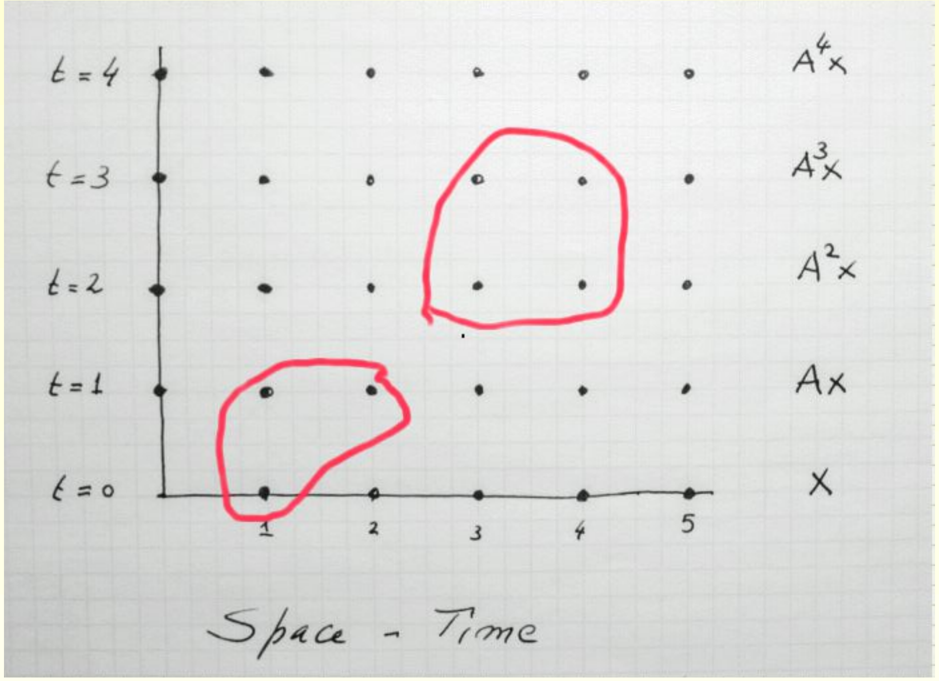
This is the main question of Dynamical sampling!!

Example



The space-time for $\mathbb{C}^5$.

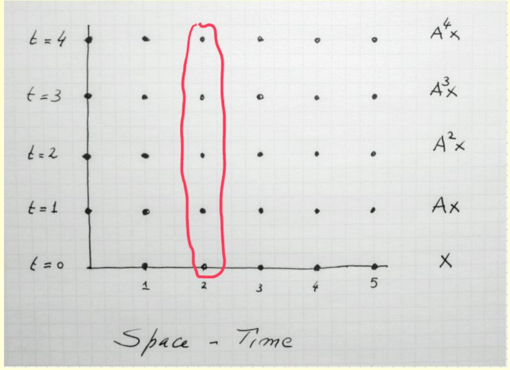
Example



The space-time for $\mathbb{C}^5$.

Consider the following matrix acting on $\mathbb{C}^5$.

$$A = \begin{pmatrix} 9/2 & 1/2 & -7 & 5 & -3 \\ 15/2 & 3/2 & -11 & 5 & -7 \\ 5 & 0 & -7 & 5 & -5 \\ 4 & 0 & -4 & 3 & -4 \\ 1/2 & 1/2 & -1 & 0 & 1 \end{pmatrix}$$

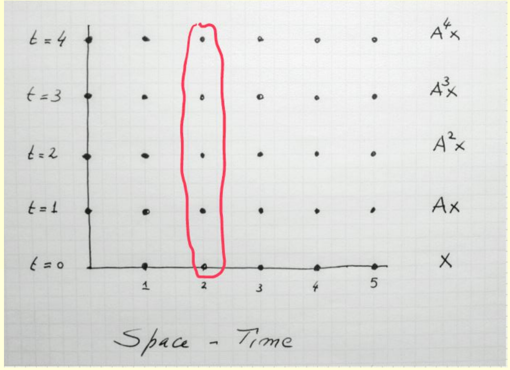


For the matrix A, any $f \in \mathbb{C}^5$ can be recovered from the data sampled at the single "spacial" point $i = 2$, i.e., from

$$Y = \{f(2), Af(2), A^2 f(2), A^3 f(2), A^4 f(2)\}.$$

Consider the following matrix acting on $\mathbb{C}^5$.

$$A = \begin{pmatrix} 9/2 & 1/2 & -7 & 5 & -3 \\ 15/2 & 3/2 & -11 & 5 & -7 \\ 5 & 0 & -7 & 5 & -5 \\ 4 & 0 & -4 & 3 & -4 \\ 1/2 & 1/2 & -1 & 0 & 1 \end{pmatrix}$$



For the matrix A, any $f \in \mathbb{C}^5$ can be recovered from the data sampled at the single "spacial" point $i = 2$, i.e., from

$$Y = \{f(2), Af(2), A^2f(2), A^3f(2), A^4f(2)\}.$$

However, if $i = 3$, i.e., $Y = \{f(3), Af(3), A^2f(3), A^3f(3), A^4f(3)\}$ the information is not sufficient to determine f. In fact if we do not sample at $i = 1$, or $i = 2$, the only way to recover any $f \in \mathbb{C}^5$ is to sample at all the remaining "spacial" points $i = 3, 4, 5$.

For example, $Y = \{f(i), Af(i) : i = 3, 4, 5\}$ is enough data to recover

f, but $Y = \{f(i), Af(i), ..., A^L f(i) : i = 3, 4\}$, is not enough information no matter how large L is.

Let A be the evolution operator acting in $\ell^2(I)$, $\Omega \subset I$ a fixed set of locations, and $\{l_i : i \in \Omega\}$ where l_i is a positive integer or $+\infty$.

Problem 1 *Find conditions on A, Ω and $\{l_i : i \in \Omega\}$ such that any vector $f \in \ell^2(I)$ can be recovered from the samples $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way.*

Writing $A^s f(i) =< A^s f, e_i >$ we can say that f can be recovered from $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way if and only if there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_2^2 \leq \sum_{i \in \Omega} |\langle A^s f, e_i \rangle|^2 \leq c_2 \|f\|_2^2. \tag{1}$$

That is

$$c_1 \|f\|_2^2 \leq \sum_{i \in \Omega} \sum_{j=0}^{l_i} |\langle f, A^{*j} e_i \rangle|^2 \leq c_2 \|f\|_2^2.$$

Thus we get

Lemma 1 *Every* $f \in \ell^2(I)$ *can be recovered from the measurements set* $Y = \{f(i), Af(i), \ldots, A^{l_i}f(i) : i \in \Omega\}$ *in a stable way if and only if the set of vectors* $\{A^{*j}e_i : \ i \in \Omega, \ j = 0, \ldots, l_i\}$ *is a frame for* $\ell^2(I)$.

The problem can be further reduced as follows: Let B be any invertible matrix with complex coefficients, and let Q be the matrix $Q = BA^*B^{-1}$, so that $A^* = B^{-1}QB$. Let b_i denote the ith column of B. Since a frame is transformed to a frame by invertible linear operators, we can just study when $\{Q^j b_i : i \in \Omega, \ j = 0, \ldots, l_i\}$ is a frame of $\mathbb{C}^d$.

This allows us to replace the general matrix A^* by a possibly simpler matrix and we have:

Lemma 2 *Every* $f \in \mathbb{C}^d$ *can be recovered from the measurement set* $Y = \{A^j f(i) : \ i \in \Omega, j = 0, \ldots, l_i\}$ *if and only if the set of vectors* $\{Q^j b_i : i \in \Omega, \ j = 0, \ldots, l_i\}$ *is a frame for* $\mathbb{C}^d$.

The Jordan Decomposition

V vector space over $\mathbb{C}$, dim$(V) = d$, $N : V \longrightarrow V$ a nilpotent operator.

$V = V_1 \oplus \cdots \oplus V_h$ the cyclic decomposition ($N_{|V_j}$ is cyclic on V_j.)

$\omega_s \in V_s$ a cyclic vector for each s. We associate to N the subspace:

$W_N =$ span$\{w_s : s = 1, .., h\}$.

The Jordan Decomposition

V vector space over $\mathbb{C}$, dim$(V) = d$, $N : V \longrightarrow V$ a nilpotent operator.

$V = V_1 \oplus \cdots \oplus V_h$ the cyclic decomposition ($N_{|V_j}$ is cyclic on V_j.)

$\omega_s \in V_s$ a cyclic vector for each s. We associate to N the subspace:

$W_N =$ span$\{w_s : s = 1, .., h\}$.

Now for a general transformation $T : V \longrightarrow V$ let

$m_T(x) = (x - \lambda_1)^{r_1} \ldots (x - \lambda_n)^{r_n}$ the minimal polynomial of T. with $\lambda_1, \ldots, \lambda_n$ distinct elements of $\mathbb{C}$.

Let $T_s = N_s + \lambda_s I$ be the restriction of T to $V_s =$ Ker$(T - \lambda_s I)^{r_s}$, $s = 1, \ldots, n$, and

N_s nilpotent on V_s with associate W_s.

Theorem. *Let* $\{b_i : i \in \Omega\}$ *be a set of vectors in* V*. Then,*

the set $\{b_i, Tb_i, \ldots, T^{l_i}b_i : i \in \Omega\}$ *is a frame of* V*,*

if and only if

$\{P_{W_s}b_i : i \in \Omega\}$ *is complete in* W_s *for each* $s = 1, \ldots, n$

Here r_i *is the degree of the* T*-annihilator of* b_i *and* $l_i = r_i - 1$*.*

Now, for a general matrix A, we can state:

Corollary. *Let A be a matrix, such that $A^* = B^{-1}JB$, where $J \in \mathbb{C}^{d\times d}$ is the Jordan matrix for A^*. Let $\{b_i : i \in \Omega\}$ be a subset of the column vectors of B, r_i be the degree of the J-annihilator of the vector b_i, and let $l_i = r_i - 1$.*
Then, every $f \in \mathbb{C}^d$ can be recovered from the measurement set $Y = \{(A^j f)(i) : \ i \in \Omega, j = 0, \ldots, l_i\}$ of $\mathbb{C}^d$ if and only if $\{P_s(b_i), i \in \Omega\}$ form a frame of W_s.

In other words, we will be able to recover f from the measurements Y, if and only if the Jordan-vectors of A^* (i.e. the columns of the matrix B that reduces A^* to its Jordan form) corresponding to Ω satisfy that their projections on the spaces W_s form a frame.

The infinite dimensional case

The setting for the general case is:

H will be Hilbert space that we can assume is $\ell^2(I)$ with $I = \mathbb{N}$.

The class of evolution operators will be

$$\mathcal{A} = \{A \in \mathcal{B}(\ell^2(\mathbb{N})) : A = A^*, \text{and } \exists \text{ a basis of } \ell^2(\mathbb{N}) \text{ of eigenvectors of } A\}.$$

So $A = B^*DB$ with D a diagonal operator with pure spectrum $\{\lambda_i : i \in \mathbb{N}\}$, and B a unitary operator.

Ω will be a finite subset of $\mathbb{N}$.

We want to find conditions on Ω and A in order to be able to recover every $f \in \ell^2(I)$ in an stable way from $Y = \{f(i), Af(i), ..., A^{l_i}f(i) : i \in \Omega\}$.

Because of the special form of A, we have that for any $f \in \ell^2(\mathbb{N})$ and $l = 0, \ldots$

$$< f, A^l e_j > = < f, B^* D^l B e_j > = < Bf, D^l b_j > \quad \text{and} \quad \|A^l\| = \|D^l\|.$$

It follows that:

$\mathcal{F}_\Omega = \left\{ A^l e_i : \; i \in \Omega, l = 0, \ldots, l_i \right\}$ is complete, (minimal, frame)

if and only if

$\left\{ D^l b_i : \; i \in \Omega, l = 0, \ldots, l_i \right\}$ is complete (minimal, frame).

Here $b_i = Be_i$.

Completeness

Theorem. *Let A be as above and $\Omega \subset \mathbb{N}$. The set $\mathcal{F}_\Omega = \{A^l e_i : i \in \Omega, l = 0, \ldots, r_i - 1\}$ is complete in $\ell^2(\mathbb{N})$ if and only if for each j, the set $\{P_j(b_i) : i \in \Omega\}$ is complete on the range E_j of P_j.*

Here r_i is the degree of the D-annihilator of b_i if such annihilator exists, or $r_i = \infty$.
P_j is the orthogonal projection onto the subspace associated to the eigenvalue λ_j.

In that case f is determined uniquely from the set

$$Y = \{f(i), Af(i), A^2 f(i), \ldots, A^{l_i} f(i) : i \in \Omega\}$$

i.e.

$$A^l f(i) = 0, \quad i \in \Omega, \; l = 1, ..., l_i \quad \text{then} \quad f = 0.$$

Minimality and Basis

Minimality and Basis

Theorem. *Let $A \in \mathcal{A}$ and let Ω be a non-empty subset of $\mathbb{N}$. If there exists $b_i,\ i \in \Omega$ such that $r_i = \infty$, then the set $\mathcal{F}_\Omega$ is not minimal.*

Minimality and Basis

Theorem. *Let $A \in \mathcal{A}$ and let Ω be a non-empty subset of $\mathbb{N}$. If there exists $b_i,\ i \in \Omega$ such that $r_i = \infty$, then the set $\mathcal{F}_\Omega$ is not minimal.*

(The proof of this result uses the well known Müntz-Szász theorem).

As an immediate corollary we get

Theorem. *Let $A \in \mathcal{A}$ and let Ω be a finite subset of $\mathbb{N}$. If $\mathcal{F}_\Omega = \{A^l e_i : i \in \Omega, l = 0, \ldots, l_i\}$ is complete in $\ell^2(\mathbb{N})$, then $\mathcal{F}_\Omega$ is not minimal in $\ell^2(\mathbb{N})$.*

Minimality and Basis

Theorem. *Let $A \in \mathcal{A}$ and let Ω be a non-empty subset of $\mathbb{N}$. If there exists $b_i,\ i \in \Omega$ such that $r_i = \infty$, then the set $\mathcal{F}_\Omega$ is not minimal.*

(The proof of this result uses the well known Müntz-Szász theorem).

As an immediate corollary we get

Theorem. *Let $A \in \mathcal{A}$ and let Ω be a finite subset of $\mathbb{N}$. If $\mathcal{F}_\Omega = \{A^l e_i :\ i \in \Omega, l = 0, \dots, l_i\}$ is complete in $\ell^2(\mathbb{N})$, then $\mathcal{F}_\Omega$ is not minimal in $\ell^2(\mathbb{N})$.*

So, $\mathcal{F}_\Omega = \{A^l e_i :\ i \in \Omega, l = 0, \dots, l_i\}$ **never is a basis !!**

Frames

Necessary conditions:

Frames

Necessary conditions:

If $\mathcal{F}_\Omega = \left\{A^l e_i : \; i \in \Omega, l = 0, \ldots, l_i\right\}$ is a frame, then

i) $\inf\{\|A^l e_i\|_2 : \; i \in \Omega, l = 0, \ldots, l_i\} = 0$. (Kadison-Singer conjecture)

Frames

Necessary conditions:

If $\mathcal{F}_\Omega = \left\{A^l e_i : \ i \in \Omega, l = 0, \ldots, l_i\right\}$ is a frame, then

i) $\inf\{\|A^l e_i\|_2 : \ i \in \Omega, l = 0, \ldots, l_i\} = 0$. (Kadison-Singer conjecture)

Let $A \in \mathcal{A}$ and let $\Omega \subset \mathbb{N}$ be a finite set. Then the unit norm sequence $\left\{\frac{A^l e_i}{\|A^l e_i\|_2} : \ i \in \Omega, l = 0, \ldots, l_i\right\}$ is not a frame.

Frames

Necessary conditions:

If $\mathcal{F}_\Omega = \big\{A^l e_i : \ i \in \Omega, l = 0, \ldots, l_i\big\}$ is a frame, then

i) $\inf\{\|A^l e_i\|_2 : \ i \in \Omega, l = 0, \ldots, l_i\} = 0$. (Kadison-Singer conjecture)

Let $A \in \mathcal{A}$ and let $\Omega \subset \mathbb{N}$ be a finite set. Then the unit norm sequence $\big\{\frac{A^l e_i}{\|A^l e_i\|_2} : \ i \in \Omega, l = 0, \ldots, l_i\big\}$ is not a frame.

ii) 1 or -1 should be a cluster point of the spectrum $\sigma_A = \{\lambda_j\}$.

Frames

Necessary conditions:

If $\mathcal{F}_\Omega = \{A^l e_i : \ i \in \Omega, l = 0, \ldots, l_i\}$ is a frame, then

i) $\inf\{\|A^l e_i\|_2 : \ i \in \Omega, l = 0, \ldots, l_i\} = 0.$ (Kadison-Singer conjecture)

Let $A \in \mathcal{A}$ and let $\Omega \subset \mathbb{N}$ be a finite set. Then the unit norm sequence $\{\frac{A^l e_i}{\|A^l e_i\|_2} : \ i \in \Omega, l = 0, \ldots, l_i\}$ is not a frame.

ii) 1 or -1 should be a cluster point of the spectrum $\sigma_A = \{\lambda_j\}$.

So know we will concentrate on the case $|\Omega| = 1$, that is when we have only one sampling point.

So we are looking for conditions on $b \in \ell^2(\mathbb{N})$ and $D = \text{diag}\{\lambda_j\}$ such that $\{D^l b : l = 0, \dots\}$ is a frame for $\ell^2(\mathbb{N})$.

Recall that 1 or -1 should be cluster points.

We also get in that case the following necessary conditions:

i) $|\lambda_k| < 1$ for all k

ii) $|\lambda_k| \to 1$.

iii) $|\{j \in \mathbb{N} : b(j) \neq 0\}| = +\infty$

iV) P_j have rank 1 for each $j \in \mathbb{N}$.

Existence of Frames

Theorem. *Let $D = \sum_j \lambda_j P_j$ be such that P_j have rank 1 for all $j \in \mathbb{N}$, and let $b \in \ell^2(\mathbb{N})$. Then $\{D^l b : l = 0, 1, \dots\}$ is a frame if and only if*

i) $|\lambda_k| < 1$ *for all* k.

ii) $|\lambda_k| \to 1$.

iii) $\{\lambda_k\}$ *satisfies Carleson's condition*

$$\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta, \qquad \text{for some } \delta > 0. \tag{2}$$

iv) $b_k = m_k \sqrt{1 - |\lambda_k|^2}$ *for some sequence* $\{m_k\}$ *satisfying* $0 < C_1 \leq |m_k| \leq C_2 < \infty$.

Existence of Frames

Theorem. *Let $D = \sum_j \lambda_j P_j$ be such that P_j have rank 1 for all $j \in \mathbb{N}$, and let $b \in \ell^2(\mathbb{N})$. Then $\{D^l b : l = 0, 1, \dots\}$ is a frame if and only if*

i) $|\lambda_k| < 1$ *for all* k.

ii) $|\lambda_k| \to 1$.

iii) $\{\lambda_k\}$ *satisfies Carleson's condition*

$$\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta, \qquad \text{for some } \delta > 0. \tag{2}$$

iv) $b_k = m_k \sqrt{1 - |\lambda_k|^2}$ *for some sequence* $\{m_k\}$ *satisfying* $0 < C_1 \leq |m_k| \leq C_2 < \infty$.

Condition (2) comes from Carleson's theorem on interpolating sequences in the Hardy space $H^2(\mathbb{D})$ on the open unit disk $\mathbb{D}$ in the complex plane. The connection was pointed us by J. Antezana.

Frames of the form $\{D^l b_i : i \in \Omega, l = 0 \ldots, l_i\}$ for the case when $|\Omega| \geq 1$ or when the projections P_j have finite rank but possibly greater than or equal to 1 can easily be found using the last theorem.